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CAN THERE BE SHORT-PERIOD DETERMINISTIC CYCLES WHEN PEOPLE ARE LONG LIVED?

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ABSTRACT

This paper considers whether short-period deterministic cycles can exist in a class of stationary overlapping generations models with long- (but finite-) lived agents. It shows that if agents discount the future positively, then as life spans get large, nonmonetary cycles will disappear. Further, neither constant monetary steady states nor stationary monetary cycles can exist. It also shows that if agents discount the future negatively, then there are robust examples in which constant monetary steady states as well as stationary monetary cycles (with undiminished amplitude) can occur no matter how long agents live.

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I. Introduction

This paper investigates the occurrence of periodic deterministic cycles of short periods in stationary, pure exchange overlapping generations (OLG) models with long-lived agents. Since Gale's [1973] paper, it has been known that OLG models can possess cyclic steady-state equilibria in which interest rates and consumption allocations oscillate in a periodic fashion. This can happen even though the characteristics of the economy (people, preferences, and endowment patterns) are the same in each period. Grandmont's [1985] paper shows the variety of periodic cycles that can exist and how such cycles arise when the intertemporal elasticity of substitution is sufficiently smaller than unity.[1] This produces strong income effects (relative to substitution effects) from a change in the interest rate and leads to backward-bending offer curves in a graph of future versus current consumption. Grandmont argues that such endogenous cycles can be consistent with some observed business cycle relationships and that government policy can eliminate cycles and lead the economy to a constant (nonfluctuating) steady state. However, all of Grandmont's discussion is in the context of a two-period-lived agent OLG model, and hence all of the cycles in his model have periods greater than the agents' life spans. This has prompted the comment [Sims, 1986] that observed business cycles have periods much shorter than agents' life spans and that short-period cycles would either be unlikely to exist or be quantitatively insignificant in amplitude in OLG models with long-lived agents. The argument for this is presumably based on the incentive (due to concave utility functions) as well as the opportunity (since agents live many more periods relative to a cycle, they will overlap with many other generations) to avoid fluctuating lifetime consumptions.

The above argument, however, does not seem entirely convincing. It is true that agents who face a constant interest rate and fluctuating incomes would wish to smooth consumption. But when interest rates themselves are fluctuating, agents would not choose to smooth consumption, even when incomes are constant. It is therefore not obvious that such short-period cycles cannot exist, whatever their magnitude.

This paper considers the issue in a class of stationary, pure exchange OLG economies with long-lived agents. The method used is similar to that in Aiyagari [1987a]. I construct a sequence of OLG economies with longer and longer lived agents. Preferences are of the discounted-sum-of-utilities type with a fixed discount rate, and lifetime endowment patterns are quite arbitrary. Utility functions with small intertemporal elasticities of substitution which generate strong income effects (relative to substitution effects) are permitted. I fix attention on cycles of a given period and consider what happens as life spans become large. For simplicity, discussion is restricted to cycles of period two, but the method carries over for cycles of any period. In addition, and again for simplicity, the utility function is initially taken to be of the type with constant elasticity of substitution. This, I believe, brings out most clearly why such equilibria may or may not exist. It will be seen, however, that we can dispense with this simplification, too.

Equilibria in an OLG model can be of two types: (i) monetary, in which there is a fixed quantity of fiat (outside) money which is <u>positively valued</u> in terms of goods, and (ii) nonmonetary, in which fiat money is <u>absent</u> or, equivalently, has <u>zero value</u> in terms of goods. Therefore, periodic cycles can also be of either of these two types. With two-period-lived agents and only one good at each date (essentially Grandmont's [1985] model), the only nonmonetary equilibrium is autarkic; hence, there can be no nonmonetary cycles. However, as shown by Gale [1973] and Grandmont [1985], monetary cycles can exist with the right preferences and endowment patterns. With many-period-lived agents, there may in general be periodic cycles of both types. (Some examples are given later, in note 4).

I first consider the case in which agents have a positive utility discount rate. In an earlier paper [Aiyagari, 1987a], I showed that constant monetary steady states do not exist for any large length of life (denoted T). A necessary condition for the existence of a monetary cycle is that there exist a constant monetary steady-state equilibrium. An immediate implication is that monetary cycles of any period cannot exist for any T large. Therefore, for this case, I focus on short-period nonmonetary cycles and show that these, too, must disappear (i.e., cannot exist) as T becomes sufficiently large. Note that this is stronger than asserting only that the amplitude of cycles goes to zero as agents live longer.

Since much of the discussion of cycles has taken place in the context of monetary equilibria, we need to allow for the existence of (at least) constant monetary steady states when agents have long lifetimes. This leads to a consideration of the case in which agents have a negative discount rate. In such a case, it is possible to construct robust examples such that both constant monetary steady states and cyclical monetary equilibria (with undiminishing amplitude) exist no matter how long agents live. Thus, the intuition referred to earlier seems valid for the case of a positive discount rate but does not seem so when the discount rate is negative.

This result may have implications for the existence of stationary "sunspot" equilibria, in which prices and consumption allocations fluctuate stochastically even though preferences and endowments are nonrandom (see

Azariadis [1981] and Cass and Shell [1983]). Spear [1984] shows that these too arise due to strong income effects. Azariadis and Guesnerie [1986] show that sunspot equilibria following a two-state Markov process arise if and only if there are two-period deterministic cycles. This connection, however, is not pursued here due to the difficulties inherent in analyzing stochastic steady states in OLG models with more than two-period-lived agents [Aiyagari, 1987a].[2] Another implication would be for endogenous fluctuations in asset prices. The analysis suggests that in positive discount rate OLG economies, endogenous cyclical fluctuations (of short period) in asset prices unrelated to dividend fluctuations would not occur.

The differences between my analysis and that of Woodford [1986] should be noted. Woodford works with an infinitely lived agents model with borrowing constraints and shows that sunspot equilibria (as well as deterministic cycles)[3] can arise. However, the assumptions under which a sequence of generations behaves as a single infinitely lived agent (altruistic preferences, perfect credit markets, and operative transfers) are rather stringent. Moreover, the same set of assumptions rules out the existence of monetary equilibria so that one needs to resort to ad hoc "frictions" to generate monetary equilibria. The existence of such frictions may be incompatible with an infinitely lived agents representation of the underlying sequences of overlapping generations. I therefore stick to the pure OLG framework of Grandmont [1985] while letting agents have longer life spans.

The rest of this paper is organized as follows. Section II describes the model and exhibits the (nonexistence of cycles for all large T) result for the case of a positive discount rate. Section III contains a discussion of monetary cycles when the discount rate is negative. And Section IV concludes. The Appendix shows that constant monetary steady states can exist for all large T when the discount rate is negative but cannot exist for any large T when the discount rate is positive.

II. Deterministically Cycling Steady States

The model used is a simplified version of the one in Aiyagari [1987a] without any intragenerational heterogeneity. Consider a stationary OLG economy with one representative agent per generation who lives for T periods. At any given date there are T agents of different generations indexed by their current age s, which runs from 1 (for the newly born) to T (for the about to die). If we let c(s) be the consumption of an agent at age s, then a newly born agent has preferences given by $\sum_{s=1}^{T}\beta^{s-1}U(c(s))$, where $0 < \beta < 1$ and $U(c) = (c^{1-\alpha}-1)/(1-\alpha)$, $\alpha > 0$. Note that α is the (absolute value of the) elasticity of marginal utility and α^{-1} is the (intertemporal) elasticity of substitution. Lifetime endowments are given by $\{w_s, s=1,2,\ldots,T\}$. These endowments are viewed as truncations (at T) from a given infinite sequence $\{w_s\}_{s=1}^{\infty}$ which is taken to be nonnegative, bounded, and bounded away from zero. As we increase T, we get a sequence of OLG economies with longer and longer lived agents.[4]

The strategy for showing that cycles cannot exist when T is large is proof by contradiction. We start by assuming that a cyclic steady state exists and derive its implications for consumption and endowment patterns and preferences. These implications are shown to be contradictory as soon as T becomes large. We consider two-period cycles in detail; the method, however, extends to cycles of any fixed period.[5]

Let r_t be the interest rate from t to t + 1 and let $(\ldots,r_1,r_2,r_1,r_2,\ldots)$ be a two-period cycle in interest rates with $r_1 > r_2$. Let the discount factors be $\gamma_i = (1+r_i)^{-1}$ so that $\gamma_1 < \gamma_2$. Due to stationarity and the focus on steady states, we only need to consider two types of agents. Let Agent 1 be the one who faces the sequence $r_1, r_2, \ldots$ over his lifetime and let $c^1(s)$, $s = 1, 2, \ldots, T$, be his lifetime consumptions. Agent 2 faces the sequence $r_2, r_1, \ldots$ and let $c^2(s)$, $s = 1, 2, \ldots, T$, be his lifetime consumptions. Let

$$\tau = \begin{cases} (T-2)/2, & \text{if } T \text{ is even} \\ (T-1)/2, & \text{if } T \text{ is odd} \end{cases}$$

and

$$k = \begin{cases} 1, & \text{if } T \text{ is even} \\ 0, & \text{if } T \text{ is odd.} \end{cases}$$

The agents solve the following optimization problems.

<u>Agent 1</u>:

(1) $$\max \sum_{s=1}^{T} \beta^{s-1} U(c^1(s)), \text{ subject to}$$

$$c^1(1) + \gamma_1 c^1(2) + \gamma_1\gamma_2 c^1(3) + \gamma_1^2\gamma_2 c^1(4) + \ldots + (\gamma_1\gamma_2)^\tau \gamma_1^k c^1(T)$$

$$= w_1 + \gamma_1 w_2 + \gamma_1\gamma_2 w_3 + \gamma_1^2\gamma_2 w_4 + \ldots + (\gamma_1\gamma_2)^\tau \gamma_1^k w_T.$$

<u>Agent 2</u>:

(2) $$\max \sum_{s=1}^{T} \beta^{s-1} U(c^2(s)), \text{ subject to}$$

$$c^2(1) + \gamma_2 c^2(2) + \gamma_2\gamma_1 c^2(3) + \gamma_2^2\gamma_1 c^2(4) + \ldots + (\gamma_2\gamma_1)^\tau \gamma_2^k c^2(T)$$

$$= w_1 + \gamma_2 w_2 + \gamma_2\gamma_1 w_3 + \gamma_2^2\gamma_1 w_4 + \ldots + (\gamma_2\gamma_1)^\tau \gamma_2^k w_T.$$

Market clearing: Let $W^T = \sum_{s=1}^{T} w_s$ be the aggregate endowment which is constant over time. Due to stationarity it is enough to look at market clearing at two consecutive dates, when the interest rates are r_1 and r_2.

The market-clearing conditions yield

$$\text{(3a)}\qquad c^1(1) + c^1(3) + c^1(5) + \ldots + c^1(T-k) + c^2(2) + c^2(4) + c^2(6) + \ldots + c^2(T-1+k) = W^T$$

$$\text{(3b)}\qquad c^2(1) + c^2(3) + c^2(5) + \ldots + c^2(T-k) + c^1(2) + c^1(4) + c^1(6) + \ldots + c^1(T-1+k) = W^T.$$

Utility maximization implies that for any agent i,

$$\text{(4)}\qquad \frac{c^i(s+1)}{c^i(s)} = \begin{cases} x_1 \equiv (\beta/\gamma_1)^{1/\alpha}, & \text{if } r_t = r_1 \\ x_2 \equiv (\beta/\gamma_2)^{1/\alpha}, & \text{if } r_t = r_2, \end{cases}$$

and obviously, $x_1 > x_2$.

A. Assuming T Is Odd

At this point we assume that T is an odd number. Using (4) in the market-clearing conditions (3), we can solve for consumptions. Let

$$A = 1 + x_1x_2 + \ldots + (x_1x_2)^{(T+1)/2}$$

$$B = 1 + x_1x_2 + \ldots + (x_1x_2)^{(T-1)/2}$$

$$\Delta = A^2 - x_1x_2B^2 = A^2 - B(A-1) = A(A-B) + B > 0.$$

We then have the following solutions for consumptions:

$$\text{(5a)} \qquad \frac{c^1(1)}{W^T} = \frac{(A-x_2B)}{\Delta}$$

$$\text{(5b)} \qquad \frac{c^2(1)}{W^T} = \frac{(A-x_1B)}{\Delta}$$

$$\text{(6a)} \qquad \frac{c^1(1) - c^2(1)}{W^T} = \frac{B(x_1-x_2)}{\Delta} > 0$$

$$\text{(6b)} \qquad \frac{c^1(2) - c^2(2)}{W^T} = \frac{x_1c^1(1) - x_2c^2(1)}{\sum w_s} = \frac{(x_1-x_2)A}{\Delta} > 0.$$

It then follows that

$$\text{(7a)} \qquad \frac{c^1(s) - c^2(s)}{W^T} = \frac{B(x_1-x_2)}{\Delta}(x_1x_2)^{(s-1)/2} > 0, \text{ s odd}$$

$$\text{(7b)} \qquad \frac{c^1(s) - c^2(s)}{W^T} = \frac{A(x_1-x_2)}{\Delta}(x_1x_2)^{(s-2)/2} > 0, \text{ s even.}$$

In general, we conclude that

$$\text{(8)} \qquad c^1(s) > c^2(s), \text{ for all s.}$$

That is, Agent 1, who is born when the interest rate is high, must have a uniformly higher lifetime consumption profile as compared to Agent 2.

When can this happen? I now illustrate the role of a low intertemporal elasticity of substitution (high α) in generating large income effects (relative to substitution effects) and backward-bending offer curves which can lead to the desired effect on the consumption profiles of the two agents. For this purpose, it is convenient to rewrite the optimization problems of the two agents in the following manner, which takes advantage of separability.

For Agent 1: Let

(9) $$V_1(e^1(1)) = \max \sum_{s \text{ odd}}^{T} \beta^{s-1} U(c^1(s)), \text{ subject to}$$

$$\sum_{s \text{ odd}}^{T} (\gamma_1\gamma_2)^{(s-1)/2} c^1(s) = e^1(1)$$

(10) $$V_2(e^1(2)) = \max \sum_{s \text{ even}}^{T-1} \beta^{s-2} U(c^1(s)), \text{ subject to}$$

$$\sum_{s \text{ even}}^{T-1} (\gamma_1\gamma_2)^{(s-2)/2} c^1(s) = e^1(2)$$

(11) $$\max V_1(e^1(1)) + \beta V_2(e^1(2)), \text{ subject to}$$

$$e^1(1) + \gamma_1 e^1(2) = \tilde{w}_1 + \gamma_1 \tilde{w}_2,$$

where

(12a) $$\tilde{w}_1 = \sum_{s \text{ odd}}^{T} (\gamma_1\gamma_2)^{(s-1)/2} w_s$$

(12b) $$\tilde{w}_2 = \sum_{s \text{ even}}^{T-1} (\gamma_1\gamma_2)^{(s-2)/2} w_s.$$

For Agent 2: The problem is rewritten in the same way as for Agent 1, except $c^1(s)$, $e^1(1)$, $e^1(2)$ are replaced by $c^2(s)$, $e^2(1)$, $e^2(2)$, and in equation (11), γ_1 is replaced by γ_2. Obviously, we need only consider the two-period optimization problem

(13) $$\max V_1(e_1) + \beta V_2(e_2), \text{ subject to}$$

$$e_1 + \gamma e_2 = \tilde{w}_1 + \gamma \tilde{w}_2$$

because this yields

(14a) $\quad (e^1(1),e^1(2)) = (e_1,e_2)\big|_{\gamma=\gamma_1}$

(14b) $\quad (e^2(1),e^2(2)) = (e_1,e_2)\big|_{\gamma=\gamma_2}.$

The requirement on consumption profiles derived earlier in (8) then implies that we must have

(15) $\quad e^1(1) > e^2(1), \quad e^1(2) > e^2(2).$

This follows from (9) and (10) and the corresponding problems for the second agent.

Looking at (13) and (14), and keeping in mind that $\gamma_1 < \gamma_2$, we see that this can only happen if the offer curve (in a graph of e_2 versus e_1) is positively sloped and e_1 is a gross complement for e_2; that is, e_1 falls as γ rises in the relevant neighborhood. This in turn requires that the excess demand for good 2 be positive ($e_2-\tilde{w}_2>0$) and that the elasticity of marginal utility for $V_2(\cdot)$ be sufficiently greater than one (in absolute value), so that the situation is as shown in Figure 1.

Formally, it is easy to verify that

(16) $$\frac{de_1}{d\gamma} = \frac{V_2'[1-\alpha_2(e_2-\tilde{w}_2)/e_2]}{\Delta'}$$

(17) $$\frac{de_2}{d\gamma} = \frac{-V_1'[1+\alpha_1(\tilde{w}_1-e_1)/e_1]}{\beta\Delta'}$$

(18) $$\frac{de_2}{de_1} = \frac{V_1'[1+\alpha_1(\tilde{w}_1-e_1)/e_1]}{\beta V_2'[\alpha_2(e_2-\tilde{w}_2)/e_2-1]},$$

where

$$\Delta' = \frac{V_1''\gamma^2}{\beta} - V_2'', \quad \alpha_1 = -\frac{e_1V_1''}{V_1'} > 0, \quad \alpha_2 = -\frac{e_2V_2''}{V_2'} > 0.$$

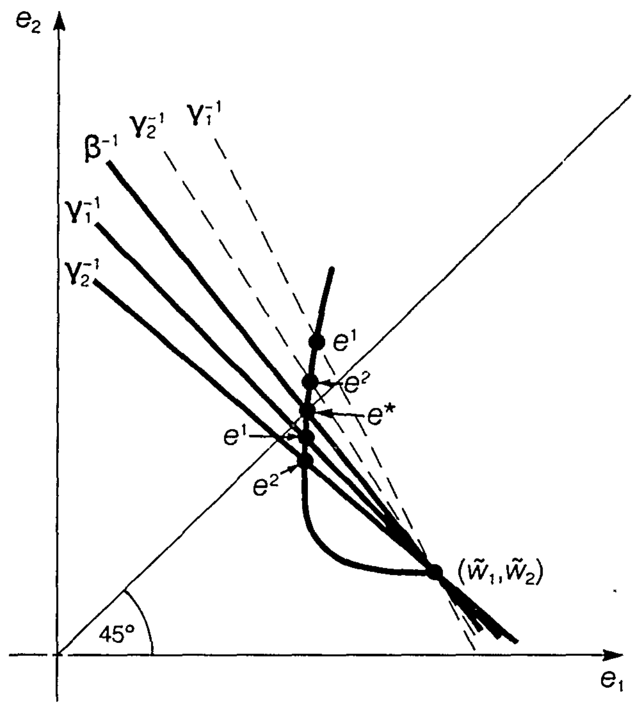

Figure I

Nonmonetary Cycles ($\beta < 1$) When Endowments Are Larger in Odd Periods of Life

The parameters α_1 and α_2 will be inherited by $V_1(\cdot)$ and $V_2(\cdot)$, respectively, from $U(\cdot)$ via (9) and (10). In fact, for the case of constant elasticity of marginal utility, it is easily seen that $\alpha_1 = \alpha_2 = \alpha$. As is also obvious from Figure 1, to get $e_2 - \tilde{w}_2 > 0$, we need $\tilde{w}_1$ to be significantly larger than $\tilde{w}_2$. This, together with an α sufficiently larger than one, may generate cycles.

It is possible to get a rough idea of magnitudes as follows. First, it is not difficult to show (along the lines of Aiyagari [1987a]) that as T gets large, both γ_1 and γ_2 converge to β.[6] Further, the functions $V_1(\cdot)$ and $V_2(\cdot)$ are nearly identical for large T. The only reason for any difference between them is that we took T to be an odd number so that the definition of $V_1(\cdot)$ contains one additional term as compared to $V_2(\cdot)$. But this difference will tend to zero for large T.[7] It then follows from (13) and (14) that as T gets large,

$$e^1(1) \approx e^1(2), \quad e^2(1) \approx e^2(2). \tag{19}$$

Therefore, we have

$$e_2 \approx (\tilde{w}_1 + \beta\tilde{w}_2)/(1+\beta). \tag{20}$$

Plugging (20) into (16), we get the condition

$$1 - \alpha(e_2-\tilde{w}_2)/e_2 < 0,$$

which requires

$$\tilde{w}_2 < [(\alpha-1)/(\alpha+\beta)]\tilde{w}_1, \tag{21}$$

where (approximately) we have, using (12),

$$\tilde{w}_1 \approx \sum_{s\ \text{odd}}^{\infty} \beta^{s-1} w_s, \quad \tilde{w}_2 \approx \sum_{s\ \text{even}}^{\infty} \beta^{s-2} w_s. \tag{22}$$

This requires (in addition to $\alpha>1$) that the endowment streams be larger in odd periods of life as compared to even periods in the above (present value) sense. For example, if $\alpha = 2$ and $\beta \approx 1$, we need $\tilde{w}_2 < 1/3\ \tilde{w}_1$. Such a requirement may not seem odd in the context of a two-period-lived agent OLG model, but it does seem a little strange in the context of a many-period-lived agent OLG model. This consideration in itself may be deemed sufficient to make cycles seem unlikely. We will see, however, that cycles can be ruled out independently of the pattern of lifetime endowments as well as independently of the elasticity of substitution parameter.

From (6), and in view of note 6, we see that $\left(c^1(1)-c^2(1)\right)/\left(c^1(2)-c^2(2)\right) = B/A \to 1$ as $T \to \infty$. It must then follow from (9) and (10) that $\left(e^1(1)-e^2(1)\right)/\left(e^1(2)-e^2(2)\right) \to 1$ as $T \to \infty$. By looking at Figure 1 and noting that γ_1, $\gamma_2 \to \beta$, we conclude that the slope of the offer curve at $\gamma = \beta$ is

$$\left.\frac{\partial e_2}{\partial e_1}\right|_{\gamma=\beta} \to 1 \text{ as } T \to \infty.$$

However, this can be seen to be impossible because at $\gamma = \beta$, $V_1' \approx V_2'$ and $e_1 \approx e_2 \approx (\tilde{w}_1+\beta\tilde{w}_2)/(1+\beta)$. Hence, from (18), we get that

$$\left.\frac{\partial e_2}{\partial e_1}\right|_{\gamma=\beta} = \frac{1 + \left[\alpha\beta(\tilde{w}_1-\tilde{w}_2)/(\tilde{w}_1+\beta\tilde{w}_2)\right]}{\beta\left\{\left[\alpha(\tilde{w}_1-\tilde{w}_2)/(\tilde{w}_1+\beta\tilde{w}_2)\right]-1\right\}} = 1 + \frac{1+\beta}{\beta\left\{\left[\alpha(\tilde{w}_1-\tilde{w}_2)/(\tilde{w}_1+\beta\tilde{w}_2)\right]-1\right\}},$$

which is strictly greater than and bounded away from one.

B. Assuming T Is Even

We now briefly look at the case when T is even. Substituting (4) in the market-clearing conditions (3), we have

$$\left(c^1(1) + x_2 c^2(1)\right)\left[1 + x_1 x_2 + \ldots + (x_1 x_2)^{T/2}\right] = W^T$$

$$\left(c^2(1) + x_1 c^1(1)\right)\left[1 + x_1 x_2 + \ldots + (x_1 x_2)^{T/2}\right] = W^T.$$

The above two equations imply that $c^1(1) + x_2 c^2(1) = c^2(1) + x_1 c^1(1)$ and hence that $c^1(1)(1-x_1) = c^2(1)(1-x_2)$. Therefore, either x_1, $x_2 < 1$ or x_1, $x_2 > 1$. Further, we have

$$\frac{c^1(1)}{W^T} = \frac{1 - x_2}{1 - (x_1 x_2)^{(T/2)+1}}$$

$$\frac{c^2(1)}{W^T} = \frac{1 - x_1}{1 - (x_1 x_2)^{(T/2)+1}}.$$

Consider what happens if x_1, $x_2 > 1$. Then

(23a) $$\frac{c^1(1) - c^2(1)}{W^T} = \frac{x_1 - x_2}{1 - (x_1 x_2)^{(T/2)+1}} < 0$$

(23b) $$\frac{c^1(2) - c^2(2)}{W^T} = \frac{x_1 c^1(1) - x_2 c^2(1)}{W^T} = \frac{x_1 - x_2}{1 - (x_1 x_2)^{(T/2)+1}} < 0$$

and, in general, $c^1(s) < c^2(s)$ for all s. From (9) and (10) we then see that this requires $e^1(1) < e^2(1)$ and $e^1(2) < e^2(2)$. Moreover, note that $\gamma_1 < \gamma_2 < \beta$ and $V_1(\cdot) \equiv V_2(\cdot)$. In terms of the offer curve, the situation must look like that depicted in Figure 2 (see the dashed budget lines).

This again requires that the offer curve be positively sloped in a neighborhood of $\gamma = \beta$, as shown. This will take a high α and a low $\tilde{w}_1$ relative to $\tilde{w}_2$; that is, endowments should be relatively larger in even periods of life compared to odd periods. However, the same argument used previously can be used again to eliminate these cycles for all sufficiently large T. From (23), $\left(c^2(1) - c^1(1)\right)/\left(c^2(2) - c^1(2)\right) = 1$. This requires that

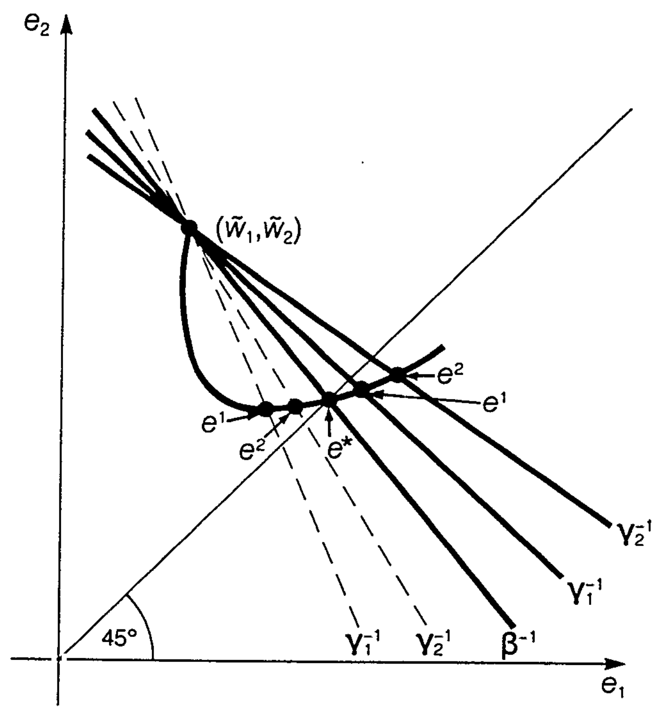

Figure II

Nonmonetary Cycles ($\beta < 1$) When Endowments Are Larger in Even Periods of Life

$(e^2(1)-e^1(1)/(e^2(2)-e^1(2)) \to 1$ as $T \to \infty$, which cannot happen since the offer curve (this time) has a slope that is strictly less than and bounded away from one as T gets large.

Lastly, consider the case x_1, $x_2 < 1$. Then we have $c^1(1) - c^2(1) = c^1(2) - c^2(2) > 0$, and this time $c^1(s) > c^2(s)$ for all s. From (9) and (10) this requires that $e^1(1) > e^2(1)$ and $e^1(2) > e^2(2)$. We also have $\gamma_2 > \gamma_1 > \beta$ and $V_1(\cdot) \equiv V_2(\cdot)$. The offer curve picture must look as shown in Figure 1 (see the solid budget lines). Again, the same argument used before (namely the slope of the offer curve at $\gamma=\beta$) leads to the elimination of these cycles.

Thus, in all cases, cycles cannot survive large T.

C. Extension to Other Utility Functions

For simplicity we restrict attention to two-period cycles. Following Aiyagari [1987a] we assume that the elasticity of substitution is bounded and bounded away from zero.[8] We let T be even so that there are an equal number of odd and even periods in an agent's life. This makes the functions $V_1(\cdot)$ and $V_2(\cdot)$ in (9) and (10) identical. Utility maximization now implies that

$$\left.\frac{U'(c^1(s+1))}{U'(c^1(s))}\right|_{s\ \text{odd}} = \left.\frac{U'(c^2(s+1))}{U'(c^2(s))}\right|_{s\ \text{even}} = \frac{\gamma_1}{\beta}$$

$$\left.\frac{U'(c^1(s+1))}{U'(c^1(s))}\right|_{s\ \text{even}} = \left.\frac{U'(c^2(s+1))}{U'(c^2(s))}\right|_{s\ \text{odd}} = \frac{\gamma_2}{\beta}.$$

It is easy to show either that γ_1 and γ_2 both exceed β or that they are both less than β. For suppose to the contrary that $\gamma_1 < \beta < \gamma_2$. Then it follows from above that

$$c^1(1) < c^1(2),\ c^1(3) < c^1(4),\ \ldots,\ c^1(T-1) < c^1(T)$$

$$c^2(1) > c^2(2),\ c^2(3) > c^2(4),\ \ldots,\ c^2(T-1) > c^2(T).$$

These inequalities are inconsistent with the market-clearing conditions (3).

Next, it is easy to show that either $c^1(s) > c^2(s)$ for all s or that $c^1(s) < c^2(s)$ for all s. This follows because

$$\frac{U'\left(c^1(s+2)\right)}{U'\left(c^1(s)\right)} = \frac{U'\left(c^2(s+2)\right)}{U'\left(c^2(s)\right)} = \frac{\gamma_1\gamma_2}{\beta^2}, \text{ for all s.}$$

If $c^1(1) > c^2(1)$, then $c^1(3) > c^2(3)$, $c^1(5) > c^2(5)$, and so on, until $c^1(T-1) > c^2(T-1)$. We cannot have $c^1(2) < c^2(2)$ because this implies that $c^1(4) < c^2(4)$, $c^1(6) < c^2(6)$, and so on, until $c^1(T) < c^2(T)$, which is inconsistent with market clearing. Therefore, we must have $c^1(s) > c^2(s)$ for all s. Similarly, if $c^1(1) < c^2(1)$, then $c^1(s) < c^2(s)$ for all s.

It follows from (9) and (10) and the analogous problems for Agent 2 that we must have either $\{e^1(1)>e^2(1)$ and $e^1(2)>e^2(2)\}$ or $\{e^1(1)<e^2(1)$ and $e^1(2)<e^2(2)\}$. From (13) and (14) this leads to the conclusion that the offer curve must be positively sloped. There are four possible situations, as shown in Figures 1 and 2 (and depicted by solid or dashed budget lines).

As in the case of constant elasticity of substitution, here too it is not difficult to show that $\gamma_1\gamma_2 \to \beta^2$ and hence that $\gamma_1, \gamma_2 \to \beta$. (The latter follows because either $\gamma_1 < \gamma_2 < \beta$ or $\beta < \gamma_1 < \gamma_2$.) Therefore, each of the $e^i(s) \to e^* = (\tilde{w}_1+\beta\tilde{w}_2)/(1+\beta)$ where $(\tilde{w}_1,\tilde{w}_2)$ is given by (22). The offer curves in the figures are therefore drawn in a small neighborhood of e^*. As before, the important facts about the offer curves are the following: In Figure 1,

$$\left.\frac{de_2}{de_1}\right|_{e^*} > 1,$$

whereas in Figure 2,

$$\left.\frac{de_2}{de_1}\right|_{e^*} < 1.$$

These follow from (18) because at e^*, $V_1' = V_2'$ and $\alpha_1 = \alpha_2$.

First, consider the situation in Figure 1. For all T sufficiently large, $e^1(2) - e^2(2) \geq (1+\varepsilon)(e^1(1)-e^2(1))$ for some ε positive. In view of (9) and (10) and the analogous problems for Agent 2, we have that $c^1(2) - c^2(2) > c^1(1) - c^2(1)$, $c^1(4) - c^2(4) > c^1(3) - c^2(3)$, and so on, until $c^1(T) - c^2(T) > c^1(T-1) - c^2(T-1)$. These inequalities are inconsistent with the market-clearing conditions (3). Next, consider the situation in Figure 2, which is exactly the opposite because it implies that for all large T, $e^2(2) - e^1(2) \leq (e^2(1)-e^1(1))/(1+\delta)$ for some δ positive. This implies that $c^2(2) - c^1(2) < c^2(1) - c^1(1)$, $c^2(4) - c^1(4) < c^2(3) - c^1(3)$, and so on, until $c^2(T) - c^1(T) < c^2(T-1) - c^1(T-1)$, which again contradicts market clearing. Thus, from both situations, such two-period cycles cannot persist as T gets large; as a result, such cycles must disappear.

One objection to the specification of preferences considered here could be that the period utility function $U(\cdot)$ is the same in every period. Grandmont [1985] discusses the possibility of cycles in terms of the elasticity of marginal utility being relatively high for old agents compared to young agents. Quite aside from how to separate the young from the old when people live many (as opposed to only two) periods, our specification with an identical $U(\cdot)$ across periods and constant elasticity may not capture this. Even if the elasticity were nonconstant, it may not be helpful because consumption at every age converges to the same value (permanent income) as T becomes large (see note 6). Thus, in the limit, the elasticity would be equalized across any two (fixed) periods.[9]

One possible way of handling this, while still focusing on two-period cycles, is to alter preferences as follows. For i = 1, 2, agent i maximizes $\sum_{s=1}^{T} \beta^{s-1}\{(c^i(s))^{1-\alpha_s}-1\}/(1-\alpha_s)$, where α_s equals α_1 if s is odd and equals α_2 if s is even. Thus, agents have different elasticities of marginal utilities in odd, as opposed to even, periods of life. It then follows from (9) and (10) that

$$V_1(e) = K_1(T)e^{1-\alpha_1}/(1-\alpha_1), \quad V_2(e) = K_2(T)e^{1-\alpha_2}/(1-\alpha_2). \tag{24}$$

Equations (13) and (14) now indicate the sense in which this is comparable to a two-period-lived agent problem with different elasticity coefficients for the young and the old. The young in one generation face γ_1 while those in the next face γ_2, and so on. In fact the analogy can be made a lot closer. We can solve (9) and (10) and the analogous problems for Agent 2 to obtain $c^i(s)$. Substituting these in the market-clearing conditions (3), we have

$$e^1(1)A_1B_1 + e^2(2)A_2B_2 = W^T \tag{25a}$$

$$e^2(1)A_1B_1 + e^1(2)A_2B_2 = W^T \tag{25b}$$

where

$$A_1 = \left\{\sum_{s\ \text{odd}} (\gamma_1\gamma_2)^{(s-1)/2}(\beta^2/\gamma_1\gamma_2)^{(s-1)/2\alpha_1}\right\}^{-1}$$

$$A_2 = \left\{\sum_{s\ \text{even}} (\gamma_1\gamma_2)^{(s-2)/2}(\beta^2/\gamma_1\gamma_2)^{(s-2)/2\alpha_2}\right\}^{-1}$$

$$B_1 = \sum_{s\ \text{odd}} (\beta^2/\gamma_1\gamma_2)^{(s-1)/2\alpha_1}, \quad B_2 = \sum_{s\ \text{even}} (\beta^2/\gamma_1\gamma_2)^{(s-2)/2\alpha_2}.$$

Now consider the case $\gamma_1\gamma_2 = \beta^2$ and $w(s) = w_1$ for s odd and $w(s) = w_2$ for s even. Then the market-clearing conditions (25) reduce to

(26a) $\quad e^1(1) + e^2(2) = \tilde{w}_1 + \tilde{w}_2$

(26b) $\quad e^2(1) + e^1(2) = \tilde{w}_1 + \tilde{w}_2.$

Together with (13) and (14), this is exactly analogous to a two-period-lived agent model. However, equations (13), (14), and (26) together imply that $\gamma_1\gamma_2 = 1$, which is a contradiction. This can be seen as follows. Using (14) in the budget constraint for (13), we have

$$\gamma_1 = \frac{\tilde{w}_1 - e^1(1)}{e^1(2) - \tilde{w}_2}, \quad \gamma_2 = \frac{\tilde{w}_1 - e^2(1)}{e^2(2) - \tilde{w}_2}.$$

Now use (26a) and (26b) to substitute for the numerator and the denominator, respectively, in the expression for γ_1 to see that $\gamma_1 = \gamma_2^{-1}$. Therefore, such a two-period cycle cannot exist.

It may, however, be possible to get nonmonetary cycles with $\gamma_1\gamma_2 \to \beta^2$ and $\alpha_1 \neq \alpha_2$. In this case, equations (25) reduce to

$$e^1(1)(\lim 2B_1/T) + e^2(2)(\lim 2B_2/T) = \tilde{w}_1 + \tilde{w}_2$$

$$e^2(1)(\lim 2B_1/T) + e^1(2)(\lim 2B_2/T) = \tilde{w}_1 + \tilde{w}_2.$$

However, since $\alpha_1 \neq \alpha_2$, it is possible to have $\lim(2B_1/T) \neq \lim(2B_2/T)$, and hence $[e^1(2)-e^2(2)]/[e^1(1)-e^2(1)]$ need not converge to one.[10] Thus in this case, having $\alpha_1 \neq \alpha_2$ may permit such cycles to persist (but with amplitude going to zero) even as T tends to infinity.

III. Monetary Cycles

Grandmont's [1985] paper is concerned solely with monetary cycles. Therefore, in this section I indicate under what conditions there can be equilibria with valued fiat money and discuss if there can be monetary cycles under those conditions. As shown in Aiyagari [1987a], the assumption of a positive discount rate ($\beta<1$) rules out a constant monetary steady state for all large T (also see the Appendix), which is a prerequisite for obtaining monetary cycles. Thus, the cycles analyzed in Section II are nonmonetary cycles. The Appendix shows that constant monetary steady states <u>can</u> exist for all large T if the discount rate is negative.

If the specification of a negative utility discount rate for agents seems odd, one alternative would be to adopt the scenario in Aiyagari [1987b]. There, the discount rate was taken to be positive, but population was assumed to be growing. It was shown that a constant monetary steady state exists if and only if the discount rate is less than the growth rate of population, <u>provided the elasticity of substitution is sufficiently larger than unity.</u>[11] The proviso clearly works against the possibility of getting monetary cycles. Therefore, even though it seems appealing to interpret a negative discount rate as corresponding to a situation where the discount rate (while positive) is less than the growth rate, we do not interpret it as such here. Instead, we exhibit robust examples of periodic monetary cycles when $\beta > 1$. In a two-period monetary cycle, $\gamma_1\gamma_2 = 1$ because the price level p (the inverse of the value of money) must be alternating between two values, p_1 and p_2, at successive dates. The gross rate of return on money must then be alternating between the values $p_1/p_2(=\gamma_1^{-1})$ and $p_2/p_1(=\gamma_2^{-1})$. Therefore equations (25) reduce to equations (26) where

$$\tilde{w}_1 = \sum_{s\ \mathrm{odd}} w(s) \text{ and } \tilde{w}_2 = \sum_{s\ \mathrm{even}} w(s).$$

This happens because $A_1B_1 = A_2B_2 = 1$. Note that we do <u>not</u> require w(s) to be constant over s odd (or s even). If $\alpha_1 = \alpha_2 = \alpha$ and <u>T is even</u>, then the functions $V_1(\cdot)$ and $V_2(\cdot)$ are identical (see note 7) and utility maximization implies

$$e^1(2)/e^1(1) = (\beta/\gamma_1)^{1/\alpha}, \quad e^2(2)/e^2(1) = (\beta/\gamma_2)^{1/\alpha} = (\beta\gamma_1)^{1/\alpha}.$$

Substituting the above in (26), we have

$$e^1(1)\left(1-(\beta/\gamma_1)^{1/\alpha}\right) = e^2(1)\left(1-(\beta\gamma_1)^{1/\alpha}\right).$$

Since $\beta > 1 > \gamma_1$, we must have $\beta > \gamma_1^{-1} > 1 > \gamma_1$ for both $e^1(1)$ and $e^2(1)$ to be positive. Further, the budget constraints are

$$e^1(1) + \gamma_1 e^1(2) = \tilde{w}_1 + \gamma_1\tilde{w}_2$$

$$e^2(1) + \gamma_1^{-1}e^2(2) = \tilde{w}_1 + \gamma_1^{-1}\tilde{w}_2.$$

Multiplying the second constraint by γ_1 and adding the two, we have

$$\left(e^1(1)+e^2(2)\right) + \gamma_1\left(e^1(2)+e^2(1)\right) = (1+\gamma_1)(\tilde{w}_1+\tilde{w}_2).$$

It follows that one of the equations in (26) is redundant. It is straightforward to compute the demand functions for $e^1(1)$ and $e^2(2)$ and to use (26a) to obtain

$$\frac{\tilde{w}_1 + \gamma_1\tilde{w}_2}{1 + \beta^{1/\alpha}\gamma_1^{\,1-(1/\alpha)}} + \frac{(\beta\gamma_1)^{1/\alpha}(\tilde{w}_1+\gamma_1^{-1}\tilde{w}_2)}{1 + \beta^{1/\alpha}\gamma_1^{\,-[1-(1/\alpha)]}} = \tilde{w}_1 + \tilde{w}_2.$$

Positive solutions for $\tilde{w}_1$ and $\tilde{w}_2$ will exist, provided

$$1 - \gamma_1 < \beta^{1/\alpha}\left[\gamma_1^{\,1/\alpha} - \gamma_1^{\,1-(1/\alpha)}\right] < \beta^{2/\alpha}(1-\gamma_1).$$

This requires an α of at least 2. In fact, it requires

$$1 < \beta^{1/\alpha}\left[\gamma_1^{\,1/\alpha} - \gamma_1^{\,1-(1/\alpha)}\right]/(1-\gamma_1) < \beta^{1/\alpha}\left[1-(2/\alpha)\right]$$

and therefore $\beta > [1-(2/\alpha)]^{-\alpha}$. This implies incredibly large values of either α or β, or both. For example, if $\alpha = 3$, then $\beta > 27$ or $\beta > 7.39$ even if $\alpha = \infty$. However, robust examples of two-period cycles (for all large T even) do exist. For instance, choose $\alpha = 10$, $\beta = 20$, $\gamma_1 = 0.99$, $\gamma_1^{-1} = 1.01$, $w(s) = 0.097$ for s odd, and $w(s) = 0.903$ for s even. This is a stationary monetary cycle that persists with constant amplitude for all large T (even). Graphically, the situation is as shown in Figure 3, with $(\tilde{w}_1, \tilde{w}_2)$ increasing along a ray through the origin as T increases.

One reason why such large values of α and β are required may be that we imposed $\alpha_1 = \alpha_2 = \alpha$. If we allow α_1 and α_2 to differ, then there is an extra degree of freedom which may expand the set of robust examples.[12] It should be noted that in these monetary cycles with β exceeding one, the offer curve is positively sloped but consumption in even periods $[e(2)]$ is a gross complement for consumption in odd periods $[e(1)]$. This requires [from (16)-(18)] a sufficiently small $\tilde{w}_1$ relative to $\tilde{w}_2$ and a sufficiently large α_1. This happens because we took T to be even.

If T is odd, then the functions $V_1(\cdot)$ and $V_2(\cdot)$ are not identical because, as noted earlier, the definition of $V_1(\cdot)$ contains an additional term in the budget constraint compared to that of $V_2(\cdot)$. Since $\gamma_1\gamma_2 = 1$ and $\beta > 1$, we see from note 7 that

$$V_2(e)/V_1(e) \equiv k(T) \to 1/\beta^2.$$

As the discussion following equation (15) shows, the offer curve must be positively sloped and e(1) must be a gross complement for e(2). From (16)-(18) this requires a large $\tilde{w}_1$ relative to $\tilde{w}_2$ and a large α_2. From utility maximization we have

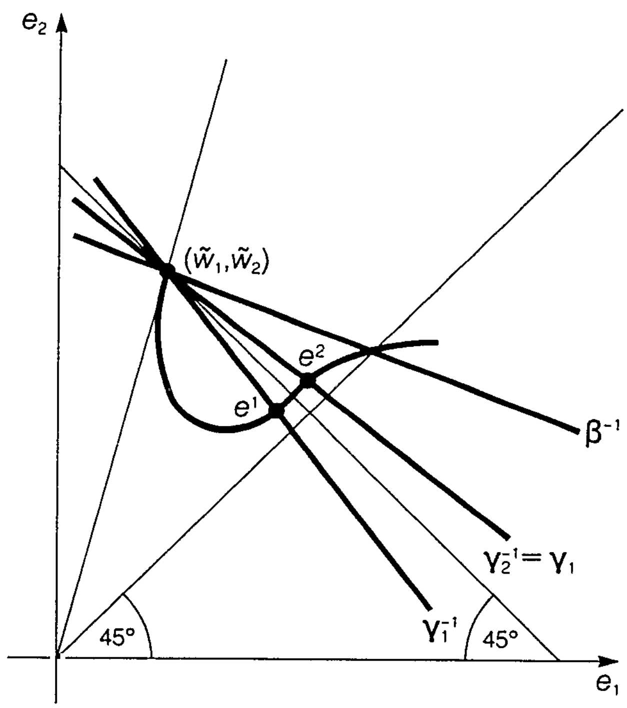

Figure III

Monetary Cycles ($\beta > 1$) for T Even

$$V_1'(e_1)/\beta V_2'(e_2) = \begin{cases} \gamma_1^{-1} & \text{for Agent 1} \\ \gamma_1 & \text{for Agent 2.} \end{cases}$$

Hence, we have

$$e^1(2)/e^1(1) = (\beta k(T)/\gamma_1)^{1/\alpha}, \quad e^2(2)/e^2(1) = (\beta k(T)\gamma_1)^{1/\alpha}.$$

This, together with market clearing, then requires that $e^2(2) < e^2(1)$ and $e^1(2) < e^1(1)$ so that we want $\beta k(T) < \gamma_1 < 1 < \gamma_1^{-1}$. Since $k(T)$ is converging to $1/\beta^2$ and $\beta > 1$, the situation is as graphed in Figure 4, which is similar to the case for a two-period-lived agent model. Robust examples of cycles corresponding to Figure 4 can easily be constructed, as shown in Grandmont [1985]. It should be noted that the offer curve in this case crosses the 45° line at a gross interest rate equal to β (and not β^{-1}) because $V_1'(e)/V_2'(e)$ is converging to β^2.

Of interest is that the type of endowment patterns which generate cycles (of period two) for T even do not generate cycles for T odd, and vice versa. When T is even, the total endowment in odd periods of life has to be much smaller than in even periods; when T is odd, the converse is required. The most important aspect of the examples is clearly that monetary cycles can persist with undiminished amplitude. There is no tendency towards damping as there is in the case of a positive discount rate.

IV. Conclusions

The main results of this study are as follows:

(i) In stationary no-growth OLG models where agents have a positive discount rate ($\beta<1$) and a sufficiently long (but finite) life span T, periodic cyclical nonmonetary steady states of short period cannot exist. Constant monetary steady states do not exist for any T suffi-

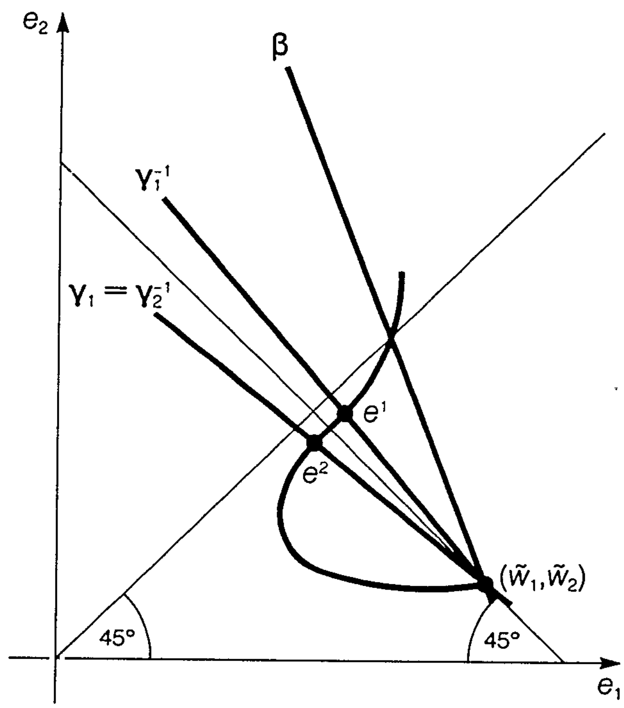

Figure IV

Monetary Cycles ($\beta > 1$) for T Odd

ciently large and, consequently, monetary cycles of any period cannot exist. Nonmonetary cycles may exist and persist if agents exhibit systematically oscillating (over, say, odd and even periods of life) patterns of elasticities of marginal utilities and endowments.

(ii) If agents exhibit a negative discount rate ($\beta>1$), then constant monetary steady states can exist for all T and cyclical monetary steady states also can exist <u>and be undamped</u>, given suitable preferences and lifetime patterns of endowments. An example in which a two-period cycle can arise would be one in which agents exhibit a systematically oscillating pattern of endowments (and possibly, but not necessarily, of preferences) as in result (i) above.

Thus, the comment of Sims [1986], referred to in the Introduction, seems reasonable when agents have a positive discount rate, but does not seem so otherwise. I conclude that the case for periodic deterministic cycles (and possibly also for stationary "sunspot" equilibria) of periods much shorter than the life spans of agents is weak in a class of OLG models with sufficiently long-lived agents who discount the future positively.

Appendix

Here I show that constant monetary steady states ($\gamma_1=\gamma_2=1$) can occur for all large T if $\beta > 1$, whereas they cannot occur for any large T if $\beta < 1$. With $\gamma_1 = \gamma_2 = 1$, the budget constraint and the market-clearing condition are identical and give

$$\sum_s c(s) = \sum_s w(s).$$

In the above equation, as well as throughout this appendix, sums over s are taken from 1 through T. An expression for the per capita desired assets of the population is given by [Aiyagari, 1987a]

$$a_T = \frac{1}{T} \sum_{s=1}^{T} s\big(c(s)-w(s)\big).$$

If an equilibrium with positively valued fiat money exists, then a_T must be positive because in a monetary steady state, the per capita value of real money balances must equal a_T. Suppose that the elasticity of substitution is constant and equal to α^{-1}. Then

$$U'(c_s)/\beta U'(c_{s+1}) = 1 \text{ implies } c_{s+1} = c_s \beta^{1/\alpha}.$$

From the budget constraint we have

$$\sum w(s) = \sum c(s) = c(1) \sum \beta^{(s-1)/\alpha}.$$

Therefore, the expression for a_T becomes

$$a_T = \frac{1}{T} \sum sc(1)\ \beta^{(s-1)/\alpha} - \frac{1}{T} \sum sw(s)$$

$$= \frac{1}{T} \frac{(\sum s\beta^{(s-1)/\alpha})\sum w(s)}{\sum \beta^{(s-1)/\alpha}} - \frac{1}{T} \sum sw(s).$$

Therefore,

$$a_T = \sum w(s) \left\{ \frac{\sum s\beta^{(s-1)/\alpha}}{T \sum \beta^{(s-1)/\alpha}} - \frac{\sum sw(s)}{T \sum w(s)} \right\}.$$

Now suppose that $\beta > 1$. Then the above expression for a_T will be positive for all large T, provided that $\sum sw(s)/T \sum w(s)$ is bounded away from one (it is always less than one). This is because the first term in braces is converging to one. If, for instance, w(s) is constant, then the second term converges to 1/2. This remains true if w(s) is constant separately over odd and even s. It follows that a constant monetary steady state will exist for all large T for a wide pattern of lifetime endowments. This result holds even if the elasticity of marginal utility fluctuates over odd and even periods of life. We simply have to separately consider sums over odd and even s.

In contrast, if $\beta < 1$, the expression for a_T will be negative for all large T. This is because the first term in braces is converging to zero, whereas the second term is bounded away from zero. Therefore, when $\beta < 1$, there cannot be positively valued fiat money equilibria for any large T.

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Footnotes

[1]Grandmont also shows how it is possible to have deterministic fluctuations which are <u>aperiodic</u>; i.e., the pattern of fluctuations never repeats, nor does it converge to a periodic cycle. In this paper, I only consider periodic fluctuations.

[2]These problems are similar to those in Spear [1985].

[3]See note 6, p. 132 in Woodford [1986].

[4]That cycles can arise in this framework is easily shown by examples. A two-period cycle with two-period-lived agents occurs when $\alpha = 3$, $\beta < 1/27$, $w_1 = 1$, $w_2 = 0$. This cycle is characterized by interest rates r_1 and r_2 where $(1+r_1)^{-1} = \beta/x_1^3$, $(1+r_2)^{-1} = \beta/x_2^3$, and x_1, x_2 = $\{[1-\beta^{1/3}]\pm[(1-\beta^{1/3})^2-4\beta^{2/3}]^{1/2}\}/2$. Note that this is a monetary cycle; nonmonetary cycles do not exist in this two-period-lived agent setup. However, if we have four-period-lived agents, then a two-period nonmonetary cycle occurs when $\alpha = 10$, $\beta = 0.95$, $w_1 = 0.041$, $w_2 = 0.04$, $w_3 = 0.259$, $w_4 = 0.66$. This is characterized by $(1+r_1)^{-1} = 0.055$ and $(1+r_2)^{-1} = 0.059$. A two-period monetary cycle can occur when $\alpha = 28.35$, $\beta = 0.004225$, $w_1 = 0.9544$, $w_2 = w_3 = 0$, $w_4 = 0.0456$. This is characterized by $(1+r_1)^{-1} = 0.42345$ and $(1+r_2)^{-1} = 2.3616$. Any resemblance of these examples to reality is purely coincidental!

[5]An important but unanswered question is whether a sufficiently large value $\hat{T}$ exists such that for all T exceeding $\hat{T}$ there are no cycles with periods less than some fraction (possibly unity) of T. Note that this is a different question than whether, for each cycle of some given fixed period, a sufficiently large T exists beyond which cycles with that period cannot exist.

[6]In Aiyagari [1987a] attention was restricted to constant steady states ($\gamma_1=\gamma_2=\gamma$), but within-generation heterogeneity was allowed. It was shown that (a) every sequence of equilibrium γ's converges to β as T gets large, (b) consumption at any fixed age s, converges to permanent income evaluated using β, and (c) monetary steady states do not exist for any T sufficiently large. In the present context, suppose that $c^1(1)$ and $c^2(1)$ remain bounded and bounded away from zero as T gets large. From (5) this implies that A/T and B/T are bounded and bounded away from zero. This immediately implies that $x_1x_2 \to 1$ and further that $(x_1x_2)^T$ is bounded. Therefore, A/B converges to 1. It must then follow that x_1, $x_2 \to 1$. Otherwise, either $c^1(1)$ or $c^2(1)$ will become negative for some finite T. Therefore, both γ_1 and γ_2 converge to β. Note that this argument only shows that the amplitude of cycles must go to zero as T gets large; it does not bear on whether such equilibria can exist.

[7]Direct computation from (9) and (10) shows that $V_1(e) = k_1(T)e^{(1-\alpha)}/(1-\alpha)$ and $V_2(e) = k_2(T)\, e^{(1-\alpha)}/(1-\alpha)$, where

$$k_1(T) = \left\{ \sum_{s \text{ odd}} (\gamma_1\gamma_2)^{(s-1)/2} (\beta^2/\gamma_1\gamma_2)^{(s-1)/2\alpha} \right\}^{\alpha}$$

$$k_2(T) = \left\{ \sum_{s \text{ even}} (\gamma_1\gamma_2)^{(s-2)/2} (\beta^2/\gamma_1\gamma_2)^{(s-2)/2\alpha} \right\}^{\alpha}.$$

Therefore, if $\gamma_1\gamma_2 \to \beta^2 < 1$, then $k_1(T)$, $k_2(T) \to (1-\beta^2)^{-\alpha}$.

[8]Boundedness away from zero is sufficient for interest rates to converge to $(1-\beta)/\beta$. Boundedness above also guarantees that consumptions converge to permanent income.

[9]The convergence of consumptions is not uniform. While it is true that $c_T(s)$ converges to y, say, for each fixed s (as T gets large), it is not true that $c_T(T)$ or $c_T(T-1)$ converges to y. See Aiyagari [1987a].

[10]As an example, suppose that $(\beta^2/\gamma_1\gamma_2)_T = (1+a/T)^{\alpha_1\alpha_2}$.

[11]This conclusion is robust to alternative specifications of growth in the aggregate endowment. One alternative is to assume that the lifetime endowment vector of each generation is some multiple of the lifetime endowment vector of the previous generation. This corresponds to assuming that labor productivity is vintage (i.e., generation) specific. Equivalently, each new generation is endowed with larger human capital. In order to obtain steady states, it is also necessary to assume that the period utility function is of the constant elasticity type. A second alternative is to assume that the cross-section vector of endowments across ages at each date is some multiple of the vector at the previous date. This corresponds to labor of all types becoming uniformly more productive over time due, say, to increasing productivity of new physical capital. The statement in the text holds true for either of these alternative specifications of growth.

[12]This extra degree comes at the expense of a rather strange specification of preferences alternating over odd and even periods of life, in addition to similarly alternating endowments.