INDIVISIBILITIES, LOTTERIES, AND SUNSPOT EQUILIBRIA

Karl Shell

Cornell University

Randall Wright

Federal Reserve Bank of Minneapolis
and University of Pennsylvania

ABSTRACT——
We analyze economies with indivisible commodities. There are two reasons for
doing so. First, we extend and provide new insights into sunspot equilibrium
theory. Finite competitive economies with perfect markets and convex consumption
sets do not allow sunspot equilibria; these same economies with nonconvex
consumption sets do, and they have several properties that can never arise in
convex environments. Second, we provide a reinterpretation of the employment
lotteries used in contract theory and in macroeconomic models with indivisible
labor. We show how socially optimal employment lotteries can be decentralized
as competitive equilibria once sunspots are introduced.

1. INTRODUCTION AND SUMMARY

The allocation of resources in the presence of nonconvexities can be an important and complicated problem. Indeed, King Solomon made his name by proposing a mechanism to solve one such problem. In this paper, we analyze economies with indivisible commodities, with two main objectives. First, we extend and provide some new insights into theories of "sunspot equilibria," theories that examine how extrinsic uncertainty can affect a competitive economy's resource allocation process and welfare properties. Second, we provide a reinterpretation of "employment lotteries," devices that have been used in contract theory and in equilibrium macroeconomics to allocate resources in economies with indivisible labor.

In terms of its relationship to the sunspot literature, this work continues the program of characterizing environments in which extrinsic uncertainty plays a role. In convex economies, it is well known that: (1) finite economies with complete and unrestricted markets and competitive behavior do not allow equilibria in which sunspots matter; (2) allocations that depend nontrivially on sunspots are never Pareto optimal; (3) equilibria in economies without extrinsic uncertainty always reappear, once extrinsic uncertainty is introduced, as degenerate sunspot equilibria.[1] There has been less work on nonconvex settings. Cass and Polemarchakis (1988) show finite, competitive economies with complete, unrestricted

---

[1] See Cass and Shell (1983, 1989). It is also well known that sunspots can matter in some infinite horizon economies, including overlapping generations models (e.g., Shell, 1977, Azariadis 1981), and in economies with incomplete markets (Cass 1984), liquidity constraints (Woodford 1986), limited participation (Cass and Shell 1983), or imperfect competition (Peck and Shell 1985).

markets but nonconvex production sets cannot have nondegenerate sunspot equilibria. Guesnerie and Laffont (1987) and Pietra (1989) consider nonconvex preferences, and do have examples with nondegenerate sunspot equilibria but no degenerate equilibria, and show that these nondegenerate sunspot equilibria can be Pareto optimal.

We study finite competitive economies with complete and unrestricted markets, convex preferences and technology, but nonconvex consumption sets. We show that: (1) these economies can have nondegenerate sunspot equilibria; (2) sunspot equilibria in these economies can be Pareto optimal and can even dominate certainty equilibrium allocations; (3) equilibria in the economy without sunspots do not necessarily reappear as degenerate sunspot equilibria when extrinsic uncertainty is introduced. These contrast with results (1)-(3) above for convex economies, and are similar to the results for the case of nonconvex preferences. Additionally, in contrast to much of the existing literature, instead of prespecifying. the probability distribution of sunspots, we solve for it as part of our equilibrium concept and we analyze the "stability" of sunspot equilibria with respect to generalizations of the exogenous uncertainty and with respect to cooperative agreements among the agents.

These results led us to explore the connection between sunspots and the employment lotteries used in macroeconomics by Rogerson (1984, 1988), Hansen (1985), Greenwood and Huffman (1987, 1988), Hansen and Sargent (1988), Rogerson and Wright (1988), and others (see Prescott 1986 and Lucas 1987 for discussions of the relevance for modern business cycle theory). In these models, labor is indivisible and is allocated randomly by lotteries, as in versions of the Azariadis (1975) - Baily (1974) labor contract model that

assume indivisible labor or some other nonconvexity.[2] Furthermore, these lotteries are similar to those in the private information economies of Prescott and Townsend (1984a, 1984b), where opportunity sets can be nonconvex due to incentive constraints, and the nonconvex economies studied by Hyland and Zeckhauser (1979) and Pratt and Zeckhauser (1983).

This literature can be interpreted as studying optimal randomized allocations, or optimal employment lotteries. We demonstrate here how to decentralize these allocations as competitive equilibria with sunspots. In particular, in a version of the environment originally specified by Rogerson (1984, 1988), we support the optimal randomized allocation as a competitive equilibrium with complete contingent commodity markets and extrinsic uncertainty. It seems useful to make explicit this close relationship between lotteries and sunspot equilibria, especially in the context of a standard model like the indivisible labor economy. Furthermore, there is an advantage to supporting these allocations as sunspot equilibria, rather having agents use individual lotteries, as in Rogerson. The advantage is that our technique can work with a finite number of agents (since we do not need to appeal to any law of large numbers). One interpretation of this is that sunspots can act as a signalling device to coordinate individual actions as well as a randomization device to convexify opportunity sets.

---

[2] In spite of much early confusion in the labor contract literature, random layoffs do not require differences in risk aversion between employers and workers, nor do they require intrinsic uncertainty at all. The standard contract model does have stochastic shocks as well as differential attitudes towards risk, but the random layoffs result from nonconvexities and not from these assumptions. See Burdett and Wright (1989) for an up-to-date discussion.

The paper can be summarized as follows. In Section 2 we examine pure exchange. We show in a simple two agent example that nondegenerate sunspot equilibria exist and can Pareto dominate the certainty equilibrium allocation, and that the latter does not survive as a degenerate sunspot equilibrium once extrinsic uncertainty is introduced (Proposition 1). We generalize this to N agents and show how to construct sunspot equilibria with a minimal number of states (Proposition 2). We then look for equilibria with different distributions of the extrinsic uncertainty. There can be many distributions consistent with different equilibria with different welfare properties; but if we assume the distribution is continuous then there is at most one equilibrium (up to a relabeling). The allocation supported by this equilibrium is also the unique core allocation that survives replication (Proposition 3). In Section 3 we study the indivisible labor economy. It has a unique certainty equilibrium that is optimal with respect to the set of certainty allocations, but can be dominated in expected utility terms by an allocation with employment lotteries (Rogerson's result). We construct a nondegenerate sunspot equilibrium allocation that supports this allocation (Proposition 4), and also show how to reduce the distribution of extrinsic uncertainty to the minimal number of states. In Section 4 we conclude.[3]

_____

[3] We are for the most part here not concerned with questions of the existence or determinacy of certainty equilibria with indivisibilities; see Mas-Collel (1977) on these issues. We also neglect literatures on fair allocations with indivisible commodities, including Crawford and Heller (1979), Svenson (1983) and Maskin (1987), and on core allocations with indivisible commodities, including Shapley and Scarf (1974) and Quinzii (1984).

2. INDIVISIBILITIES AND SUNSPOTS IN PURE EXCHANGE

There are K goods and the commodity space is $\mathbb{R}_+^K$. However, some of the goods may be *indivisible*. To say x_k is indivisible means that it must either be consumed in a single unit or not at all, $x_k \in \{0,1\}$.[4] If we label goods so that the first J are divisible, the consumption set for each consumer is given by $X = \mathbb{R}_+^J \times \{0,1\}^{K-J}$. There is a set I of consumers. Sometimes I is finite and sometimes it is a continuum in the economies studied below, and $\alpha(i)$ is a measure defined on I describing the distribution of agents. The preferences of individual i are described by a strictly increasing, strictly concave, von Neumann – Morgenstern utility function, $U^i: X \to \mathbb{R}$. His endowment is given by $e^i \in \mathbb{R}_+^K$, but note that we do *not* necessarily assume that $e^i \in X$. Thus, consumers may be endowed with and may trade fractional claims on indivisible goods, even though they can only consume integer quantities.[5] There is no intrinsic uncertainty (i.e., preferences and endowments are nonstochastic).

An *allocation* (x^i) is a list of consumption points for all consumers, and is said to be *feasible* if $x^i \in X$ for all i and $\int x^i d\alpha(i) \leq \int e^i d\alpha(i)$. A feasible allocation is said to be *Pareto optimal with respect to X* if there

---

[4]More generally, one could assume an indivisible good can be consumed in any integer quantity, $x_k \in \{0,1,2,\ldots\}$. The results are similar.

[5]Thus, e^i is in the commodity space but not necessarily in the consumption set. An alternative formulation that delivers virtually the same results is to assume U(x) is a step function of each indivisible good (a form of local satiation). Under this interpretation, it does not matter if X actually restricts indivisible goods to integers or not and, therefore, we could insist that endowments belong to X without changing the results.

does not exist an alternative feasible allocation $(\hat{x}^i)$ such that $U^i(\hat{x}^i) \geq$ $U^i(x^i)$ for all i, with strict inequality for a set of agents with positive measure. A *Walrasian equilibrium* (WE) is an allocation and a price vector $p \in \mathbb{R}_+^K$, normalized so that $\Sigma_k p_k = 1$, satisfying: (a) for all i, x^i maximizes $U^i(x)$ over X subject to $p \cdot x \leq p \cdot e^i$, and (b) $\int x^i d\alpha(i) \leq \int e^i d\alpha$ (feasibility).[6]

We concentrate for now on some examples with one indivisible good, x, so that X = {0,1}. In fact, we can demonstrate the basic message in the case of two consumers (N = 2) with $e^1 = e^2 = 1/2$. This economy has a unique WE, with $x^1 = x^2 = 0$, which yields utilities $U^i = 0$ if we normalize $U^i(0)$ to zero. This is *not* Pareto optimal with respect to X; it is dominated by giving x = 1 to one of the agents and nothing to the other.[7] Furthermore, consider randomizing over the allocations that are optimal with respect to X, by forming the *lottery*

$$(x^1, x^2) = \begin{cases} (1,0) & \text{with prob } \pi_1 \\ \\ (0,1) & \text{with prob } \pi_2 \end{cases} \qquad (2.1)$$

where $\pi_1 \in [0,1]$ and $\pi_2 = 1-\pi_1$. The expected utilities generated by this lottery are $EU^i = \pi_i U^i(1)$, for i = 1,2, which exceeds the utilities generated by the Walrasian mechanism. We say that the randomized allocation

---

[6] Note that we do not require markets to clear exactly here, in the sense that the feasibility or market clearing condition may hold with strict inequality for some good with a positive price, simply because the economy cannot possibly consume everything when the aggregate endowment of some indivisible good is not exactly an integer. Very little hinges on this technicality for our purposes.

[7] The First Welfare Theorem does not hold because it requires that at least one good be divisible; see, e.g., Quirk and Saposnik (1968), p. 134.

Pareto dominates the WE allocation, although not with respect to X, since it is not actually an element of X.

The fact that the extrinsic uncertainty introduced by the lottery has a role here leads us to consider equilibria with sunspots. Returning to the general model, we introduce extrinsic uncertainty by way of a probability space (S,Σ,π), where S is a set of states s representing *sunspot activity*, Σ is a σ-algebra of subsets called events, and π is a probability measure. By the definition of extrinsic uncertainty, preferences and endowments do not depend on s, although in principle, an agent's behavior might. We model this by reformulating commodity space as the set of π-measurable functions of the state, x: $S \to \mathbb{R}^J$, that are bounded in the essential supremum norm. Let this space be denoted by Z. The consumption set is the set of these functions such that $x(s) \in X$ for all s.

In particular, consumer i chooses such a function $x^i(\cdot)$ to solve the following problem,

$$\text{maximize } EU^i = \int_S U^i[x^i(s)]d\pi(s)$$

$$\text{subject to } \int_S \tilde{p}(s)x^i(s)d\pi(s) \le \int_S \tilde{p}(s)e^i d\pi(s) \equiv W^i,$$

(2.2)

where W^i is wealth, and $\tilde{p}$ is a measurable function with the following interpretation. For any set $A \in \Sigma$, $\int_A \tilde{p}_k(s)d\pi(s)$ is the cost of one unit of good k to be delivered just in case event A occurs. If s has a density function, $\varphi(s)$, then we write the budget constraint as $\int p(s)x^i(s)ds \le W^i$, where $p(s) \equiv \tilde{p}(s)\varphi(s)$, and the kth component $p_k(s)$ is precisely the price of good k in state s. Similarly, if $S = \{s_1,s_2,\ldots\}$ is discrete, we write the budget constraint as $\Sigma_j p(s_j)x^i(s_j) \le W^i$, where $p(s_j) \equiv \tilde{p}(s_j)\pi(s_j)$.[8]

---

[8]Despite the intuitive nature of this formulation, there are some technical

A feasible allocation for the economy with sunspots is a list $[x^i(\cdot)]$ with $x^i(\cdot) \in Z$ for each i, such that $\int x^i(s)d\alpha(i) \leq \int e^i d\alpha(i)$ with probability 1. It is said to be *degenerate* if, for all i, $x^i(s) = x^i$ with probability 1; in other words, if the allocation is essentially independent of the state. It is *nondegenerate* otherwise. We sometimes abuse terms slightly and identify an allocation for the economy without uncertainty with a degenerate allocation in the more general economy; i.e., $x^i \in X$ is identified with $x^i(\cdot) \in Z$, where $x^i(s) = x^i$ for all s. An allocation $[x^i(\cdot)]$ is *Pareto optimal with respect to* Z if there does not exist another feasible allocation $[\hat{x}^i(\cdot)]$ such that $\int U^i[\hat{x}^i(s)]d\pi(s) \geq \int U^i[x^i(s)]d\pi(s)$ for all i, with strict inequality for a set of agents with positive measure. A *sunspot*

---

issues that need to be dealt with when S is not finite dimensional. The standard way to define a price system in an infinite dimensional commodity space Z, is by a continuous linear functional $v: Z \rightarrow \mathbb{R}$. Then a *valuation equilibrium* is a feasible allocation (z^i), $z^i \in Z$ for all i, together with a price system v, such that every i maximizes $u^i(z^i)$ over Z subject to $v(z^i) \leq v(e^i)$. An *inner product representation* for v is a vector $\tilde{p}$ in the dual space of Z, such that $v(z) = \tilde{p} \cdot z$ for every $z \in Z$, with the natural interpretation as the price vector. Our commodity space Z is the space of measurable functions bounded in the essential supremum norm; thus, $\tilde{p}$ should be an element of the dual space of Z, the set of measurable functions bounded in the L_1 norm, such that

$$v(x) = \int \tilde{p}(s)x(s)d\pi(s) \text{ for all } x \in Z.$$

Although it is not true for all economies, our economies satisfy conditions that guarantee such a representation exists for any valuation equilibrium; hence, we only consider inner product prices in what follows. See Bewley (1972), Prescott and Lucas (1972), or the discussion in Stokey, Lucas and Prescott (1989).

equilibrium (SE) is an allocation together with a pricing function $\tilde{p}(\cdot)$, normalized so that $\int \Sigma_k \tilde{p}_k(s) d\pi(s) = 1$, satisfying: (a) for all i, $x^i(\cdot)$ solves (2.2), and (b) feasibility. A SE is degenerate if the implied allocation is degenerate, and nondegenerate otherwise.

We review a few facts about convex economies (where X is convex and U^i strictly concave), all of which are easy to prove. First, in a convex economy a nondegenerate allocation is never Pareto optimal with resect to Z, since it is dominated by the degenerate allocation $\hat{x}^i(s) = \int x^i(s) d\pi(s)$ for all s and for all i. An implication is that, in any convex economy for which the First Welfare Theorem holds, there cannot exist nondegenerate SE. If the First Welfare Theorem does not hold (say, for some of the possible reasons mentioned in footnote 1), there may exist nondegenerate SE, but they are not optimal. Finally, in a convex model, if the allocation (x^i) and price p constitute a WE for the economy without uncertainty, then we can always construct a degenerate SE by setting $x^i(s) = x^i$ for all s and i, and $\tilde{p}(s) = p$ for all s.

Consider again the trivial economy with X = {0,1}, N = 2 and $e^1 = e^2 = 1/2$. Introduce a little extrinsic uncertainty by assuming there are exactly two states of possible sunspot activity, S = {s_1, s_2}, with $\pi_j = \pi(s_j)$. Problem (2.2) then becomes

$$\text{maximize } EU^i = \pi_1 U^i[x^i(s_1)] + \pi_2 U^i[x^i(s_2)]$$

$$\text{subject to } p(s_1)x^i(s_1) + p(s_2)x^i(s_2) \leq 1/2,$$

(2.3)

where $p(s_j)$ is the price of the good in state s_j, as discussed above, normalized so that $p(s_1) + p(s_2) = 1$. Then, the following results hold.

Proposition 1. In the economy with $X = \{0,1\}$, $N = 2$, and $e^1 = e^2 = 1/2$, we have: (a) If $\pi_1 \neq \pi_2$ then SE do not exist. (b) If $\pi_1 = \pi_2$ then there are exactly two SE, with prices $p(s_1) = p(s_2) = 1/2$ and one of the following two allocations

$$[x^1(s_1), x^1(s_2)] = (1,0) \text{ and } [x^2(s_1), x^2(s_2)] = (0,1)$$

$$(2.4)$$

$$[x^1(s_1), x^1(s_2)] = (0,1) \text{ and } [x^2(s_1), x^2(s_2)] = (1,0),$$

which are simply relabelings of the same outcome. (c) All SE are nondegenerate, and in particular, the WE allocation cannot be supported as a SE. (d) The SE are Pareto optimal with respect to Z and dominate the WE allocation.

Proof. For any prices the budget set of each agent must contain either $[x(s_1), x(s_2)] = (1,0)$ or $(0,1)$. Feasibility entails $\Sigma_i x^i(s) \leq 1$ for all s. These two observations imply that any SE must involve one of the two allocations in (2.4), and this proves (c). Suppose the first allocation in (2.4) is a SE; if Mr. 1 is to demand $(1,0)$ we must have

$$\pi_1 U^1(1) + (1-\pi_1)U^1(0) \geq \pi_1 U^1(0) + (1-\pi_1)U^1(1).$$

This implies $[U^1(1)-U^1(0)](1-2\pi_1) \leq 0$, or $\pi_1 \geq 1/2$. Similarly, if Mr. 2 is to demand $(0,1)$ we must have $\pi_1 \leq 1/2$, and so $\pi_1 = 1/2$. The same is true for the other allocation in (2.4), and this verifies (a). Given these results, the allocations in (2.4) in fact solve problem (2.3) for both agents if and only if $p(s_1) = p(s_2)$, which proves (b). Finally, the statements in (d) are obvious from our earlier discussion of lotteries and their welfare properties. ∎

This example is interesting because it contrasts with the results for convex economies outlined above. In convex economies, WE always reappear as SE, so that result (c) could not have held. Result (d) could not have held in a convex economy, where SE are never Pareto optimal; SE are not only optimal here, they dominate the WE allocation. Also, results (a) and (b) go beyond the existing literature in that, instead of taking the probability distribution of extrinsic uncertainty as given, we have gone some way towards deriving what that distribution must be in order for SE to exist (given two states, here they have to be equiprobable).

To pursue these issues further, we begin to generalize this example. Continue to assume $X = \{0,1\}$, but now let there be $N < \infty$ agents, with homogeneous endowments $e^i = e$. There is no loss in generality to assuming $e < 1$.[9] The unique WE again entails $x^i = 0$, and utilities $U^i(x^i) = 0$ for all i. Let $n = \text{int}(Ne)$ be the integer part of the aggreate endowment. If $n = 0$ the WE is optimal. If $n \geq 1$, however, then one can generalize Proposition 1 to show that there exists a SE with N equiprobable states and constant prices supporting an allocation with $x^i = 1$ in exactly n states and $x^i = 0$ in the remaining $N-n$ states, for each consumer i. This SE is optimal, and dominates the WE. However, rather than N states, we prefer to construct a SE with as few states as possible.

To this end, let n^*/N^* reduce n/N to its lowest terms (e.g., 10/4 reduces to 5/2). Let there be N^* equiprobable states and constant prices,

_____

[9]More generally, let i's endowment be $y^i + e$, where y^i is the integer part and e is the fractional part that is common across agents, and let his utility function be $u^i : \{0,1,..\} \to \mathbb{R}$. Then, we can let $e^i = e$ and define a new utility function $U^i : \{0,1\} \to \mathbb{R}$ by $U^i(x) = u^i(y^i + x)$ to get exactly the model in the text.

$p(s_j) = 1/N^*$. Agent i has wealth $W^i = e$, which means that the greatest number of units he can afford is n^* (because n^*+1 units cost more than e). Given he consumes n^* units, strict concavity implies he maximizes utility by consuming exactly 1 unit in n^* states and 0 units in the remaining N^*-n^* states. Therefore, to construct a SE we need only choose an allocation with two properties: (a) each agent i receives $x^i = 1$ in n^* states and $x^i = 0$ in the rest, so that he is maximizing utility subject to his budget constraint, and (b) in each state the fraction n^*/N^* of agents receive $x^i = 1$ while the rest receive 0, so that markets clear. One way to choose such an allocation is to use a square matrix of size N^*, denoted $[a_{ij}]$, with the property that each element is either 0 or 1, all columns sum to n^*, and all rows sum to n^*. Then, for each consumer $i = 1,2,...N^*$, we set $x^i(s_j) = a_{ij}$, while for consumers $i = N^*+1,...,$ we simply reproduce the allocation of the first N^*.

The matrix $[a_{ij}]$ can always be constructed.[10] Figure 1a shows the case $N^* = 3$ and $n^* = 1$, where Mr. 1 consumes 1 unit in state s_1, Mr. 2 consumes 1 unit in s_2, and Mr. 3 consumes 1 unit in s_3. Figure 1b shows the case $N^* = 3$ and $n^* = 2$, where Mr. 1 consumes 1 unit in states s_1 and s_3, etc. The general discussion is summarized as follows:

Proposition 2. The economy with $X = \{0,1\}$ and N consumers with $e^i = e < 1$ for all i has a unique WE with $x^i = 0$ for all i. Let $n = int(Ne)$ and let n^*/N^* reduce n/N to lowest terms. Then, this economy has a SE with N^*

---

[10] The algorithm is as follows: Begin with $a_{ij} = 0$ for all i,j. If $n^* \geq 1$ then change a_{ij} from 0 to 1 if $i = j$; if $n^* \geq 2$ then also change a_{ij} from 0 to 1 if $i = j+1$ modulo N^*; if $n^* \geq 3$ then also change a_{ij} from 0 to 1 if $i = j+2$ modulo N^*; and so on. This is known as the *method of circulants* in combinatorical analysis.

states, $\pi(s_j) = p(s_j) = 1/N^*$, and an allocation where $x^i(s_j) = 1$ in n^* states and $x^i(s_j) = 0$ in $N^* - n^*$ states for all i. If $n^* \geq 1$, then the SE is optimal with respect to Z and dominates the WE, and the WE does not reappear as a SE.

The next step is to consider heterogeneous endowments. Suppose $X = \{0,1\}$, $N = 2$, and $0 < e^1 < e^2$ with $e^1 + e^2 = 1$. Assume $S = \{s_1, s_2\}$ with $\pi_j = \pi(s_j)$, and consider the Edgeworth box in Figure 2, with the endowment point e on the diagonal. Clearly any SE must have the price line going through e and also through either the point $A = (1,0)$ or the point $B = (0,1)$. The former case is shown and implies $p(s_2)/p(s_1) = e^2/e^1$. At these prices, Mr. 1 necessarily chooses point A, while A is also in the demand correspondence of Mr. 2 if and only if $\pi_1 \leq 1/2$. Thus, for any $\pi_1 \leq 1/2$, there is a SE with prices $[p(s_1), p(s_2)] = (e^1, e^2)$ and allocation

$$[x^1(s_1), x^1(s_2)] = (1,0) \quad \text{and} \quad [x^2(s_1), x^2(s_2)] = (0,1).$$

Symmetrically, for any $\pi_1 \geq 1/2$, there is a SE with the prices and the allocation reversed.

The point of this example is that there can be many different SE, with different values of π_1 and, therefore, with different expected utilities. This contrasts with our earlier results, where the equality of endowments delivered a uniform distribution of states as the unique distribution consistent with SE. Furthermore, notice that here the equilibrium with $\pi_1 = 1/2$ has both agents consuming $x^i = 1$ with the same probability and therefore receiving the same expected utility, even if e^1 is very small compared to e^2. Mr. 2 starts with more, so why doesn't he end up with more? One answer is that with $N = 2$, a lottery of the form (2.1) is in the core for any π_1.

However, if we replicate this economy, the lottery with $\pi_1 = 1/2$ may no longer be in the core. For instance, say $e^2 = m/M$ is a rational number. Then, any coalition of size M type-2 agents could hold its own lottery, where each member receives $x = 1$ with probability $m/M = e^2 > 1/2$.

At the extreme, suppose there is a continuum of agents with unit mass, and let α_t be the fraction of type t, $t = 1,2,...T < \infty$, where each type t agent has endowment e^t and $\Sigma_t \alpha_t e^t = 1$. Then, any coalition of type t agents with positive measure could hold a lottery in which each member received $x^i = 1$ with probability e^t, and, therefore, any core allocation must have $prob(x^i=1) \geq e^t$ for almost all type t agents. At the same time, feasibility means that total consumption cannot exceed the total endowment, so the set of agents for whom $prob(x^i) > e^i$ must be null. We conclude that the core consists of randomized allocations in which $prob(x^i=1) = e^i$ for almost all i. This seems intuitively like what an equilibrium should be; but recall from the above discussion that there generally can be many SE with different probability distributions. We claim, however, that unless $prob(x^i=1) = e^i$, the SE will not be "stable" with respect to the introduction of other probability distributions for sunspot activity.

To illustrate this, let I be the set of agents and let their endowments satisfy $e^i \in [0,1]$ and $\int e^i d\alpha(i) = 1$. Now assume that s is a *continuous* random variable with density $\varphi(s)$. Problem (2.2) then becomes

$$\text{maximize} \int_S U^i [x^i(s)] \varphi(s) ds \qquad (2.5)$$
$$\text{subject to} \int_S p(s) x^i(s) ds \leq e^i,$$

where p(s) is the price of a unit of the good in state s. Unless p(s) = $\varphi(s)$ for almost all s, agents will switch their consumption from states with $p(s) > \varphi(s)$ to those with $p(s) < \varphi(s)$ to get more utility at the same cost,

and markets could not clear. This means that equilibrium requires $p(s) = \varphi(s)$, in which case problem (2.5) is solved by setting $x^i(s) = 1$ for all s in any set of measure e^i. Any partition S_i, with $\text{prob}(S_i) = e^i$ for all i, generates a SE with prices $p(s) = \varphi(s)$ and an allocation described by $x^i(s) = 1$ if and only if $s \in S_i$.

Hence, with continuous sunspots, there is a unique (up to a relabeling) SE. Summarizing, we have:

Proposition 3. Suppose $\int e^i d\alpha(i) = 1$. Let s be continuous with density $\varphi(s)$. Then, up to a relabeling, there is a unique SE, and it has the following properties: (a) $p(s) = \varphi(s)$ for all s, and (b) $\text{prob}(x^i) = e^i$ for almost all i. The corresponding allocation is the unique core allocation that necessarily survives replication. With a finite distribution for s, there can be other SE, with allocations such that $\text{prob}(x^i) \neq e^i$.

We close this section with an example involving two goods, one divisible and the other indivisible: $X = \mathbb{R}_+ \times \{0,1\}$. Let I be a continuum of homogeneous consumers with unit mass. Suppose $e^i = (1/2, 1/2)$ and $U^i(x_1, x_2) = u(x_1) + u(x_2)$ for all i, where $u(0) = 0$. Then, in WE, exactly half of the consumers receives $(x_1^i, x_2^i) = (1,0)$ while the other half receives $(0,1)$. Note that in contrast to our earlier examples, the WE allocation here *is* Pareto optimal with respect to X, and yields utilities $U^i = u(1)$ for all i. Yet it is obvious the lottery that gives each agent

$$(x_1^i, x_2^i) = \begin{cases} (1/2, 0) & \text{with prob } 1/2 \\ \\ (1/2, 1) & \text{with prob } 1/2 \end{cases}$$

yields *greater* expected utility (by strict concavity).

Following the reasoning of the earlier examples, we could support this randomized allocation as a SE, which is optimal with respect to Z, with two equiprobable states and prices $p_1(s_j) = p_2(s_j)$ for each state s_j. The fact that the SE is optimal here is more striking because the First Welfare Theorem implies the WE is optimal with respect to X (while, e.g., in Proposition 1, the SE dominated a WE that was not even optimal within the set of nonrandomized allocations). This economy has some other interesting properties, but we do not pursue them because things are quite similar in the perhaps more familiar model studied in the next section, the indivisible labor economy of Rogerson (1984, 1988). One important point to note, however, is that the above example works perfectly well when the set I contains an even finite number of agents rather than a continuum.

3. EMPLOYMENT LOTTERIES AND SUNSPOT EQUILIBRIA

The consumption set is now $X = \mathbb{R}_+ \times \{0,1\}$. There is a continuum of homogeneous consumers distributed uniformly on $[0,1]$. Their preferences are described by the utility function $U(x_1, x_2)$, where we write $(x_1, x_2) = (c, L)$ with the interpretation that the consumption good c is divisible while leisure L is not. It simplifies the presentation to assume U is continuously differentiable with respect to c and that $U_1(c, L) \to \infty$ as $c \to 0$ for all L. The endowment point is $e = (0,1)$. There is a representative firm, with production set $Y = \{y \in \mathbb{R}_+^2 : y_1 \leq f(y_2)\}$. We write $(y_1, y_2) = (q, h)$. Assume the production function is twice continuously differentiable, with $f' > 0$, $f'' \leq 0$, and $f'(h) \to \infty$ as $h \to 0$. All consumers share equally in the ownership of the firm and any profit that is earned is distributed back to them equally as dividends. As above, there is no intrinsic uncertainty: preferences, endowments, and technology are all nonstochastic.

We refer to this model as the *indivisible labor economy*. A feasible allocation is a consumption – leisure pair for each $i \in I$, $x^i = (c^i, L^i) \in X$, and a production plan for the firm, $y = (q, h) \in Y$, satisfying $\int L^i di + h = 1$ and $\int c^i di = q$. It is Pareto optimal with respect to X if there is no feasible alternative that dominates it in the obvious sense. A Walrasian equilibrium is an allocation $[(x^i), y]$ together with prices for output and labor, denoted (p, w), such that: (a) for all i, x^i maximizes U(x) over X subject to $pc + wL \leq w + \Pi$, where Π is profit; (b) y maximizes $\Pi = pq - wh$ over Y; and (c) market clearing.

In WE, each agent will have either $x^i = (\Pi, 1)$ or $(w + \Pi, 0)$. Let μ be the measure of agents that choose the latter option; for obvious reasons, they are called *employed* while the others are called *unemployed*. If $\mu \in (0, 1)$ then $U(\Pi, 1) = U(w + \Pi, 0)$. It is easy to show that there exists a unique WE (up to a relabeling). It is Pareto optimal with respect to X by the First Welfare Theorem. Let W^* be the common utility of consumers in WE. Rogerson's insight was to construct a randomized allocation (or lottery) in which each consumer receives $(c^i, L^i) = (c_0, 0)$ with probability μ and $(c^i, L^i) = (c_1, 1)$ with probability $1 - \mu$, which yields expected utility $V = \mu U(c_0, 0) + (1 - \mu) U(c_1, 1)$. Given the results in the previous section, it should be no surprise that the application of such a randomization device can be useful in this economy. In particular, consider the social planner's problem of maximizing V by choosing μ, c_0, and c_1, subject to the feasibility constraint $\mu c_0 + (1 - \mu) c_1 \leq f(\mu)$ and the constraint $\mu \leq 1$ (nonnegativity constraints can be ignored, given our curvature assumptions).

Let λ and β be the multipliers on the resource constraint and the constraint $\mu \leq 1$. Then, the solution to the planner's problem is fully characterized by

$$U(c_0, 0) - U(c_1, 1) + \lambda[f(\mu) - c_0 + c_1] = \beta$$

$$\mu U_1(c_0, 0) - \mu\lambda = 0$$

$$(1-\mu)U_1(c_1, 1) - (1-\mu)\lambda = 0 \qquad (3.1)$$

$$f(\mu) - \mu c_0 - (1-\mu)c_1 = 0,$$

plus $\mu \leq 1$ and $\beta(1-\mu) = 0$. Let (μ^*, c_0^*, c_1^*), along with $(\bar{\lambda}^*, \beta^*)$, be the solution and V^* the implied level of expected utility. As long as $\mu^* < 1$, we "typically" have $V^* > W^*$, and the lottery improves welfare even though the WE is optimal with respect to X.[11]

Our goal now is to *decentralize* the planner's randomized allocation. Rogerson (1988) discusses the possibility of supporting randomized allocations as equilibria of a mechanism in which each individual "chooses a lottery where with probability $[\mu]$ they work ... and with probability $[1-\mu]$ they don't." This means that individual wage income will be uncertain and, therefore, it "is assumed that the individual can purchase insurance ... contingent on the outcome of the lottery." We will use a more conventional mechanism, with *contingent commodity markets* rather than individual lotteries and insurance contracts. This is not only more standard, it also has one substantive advantage. When Rogerson lets his individuals choose lotteries, he must appeal to a law of large numbers to guarantee that the probability of working chosen by each agent equals the actual number who end up working. As illustrated by the examples in the previous section, our equilibrium concepts works perfectly well with a small (finite) number of agents. One interpretation of this is that sunspots can act not only as a

---

[11]Utility functions of the class $U = u(c+v(L))$, where $u(\cdot)$ and $v(\cdot)$ are increasing, concave functions, are the only ones that entail $V^* = W^*$.

randomizing mechanism, but also a coordinating mechanism.[12]

As in the previous section, we introduce sunspots by way of a probability space (S,Σ,π). Consumer i choose a measurable, bounded function of the state, x^i: $S \to X$, to solve

$$\text{maximize } EU = \int_S U[c(s),L(s)]d\pi(s) \qquad (3.2)$$

$$\text{subject to } \int_S [p(s)c(s)+w(s)L(s)]d\pi(s) \leq \Pi + \int_S w(s)d\pi(s),$$

where Π denotes profit, $p(s)$ is the price of $c(s)$ and $w(s)$ is the price of $L(s)$. Similarly, the firm chooses a function y: $S \to Y$, to solve

$$\text{maximize } \Pi = \int_S [p(s)q(s)-w(s)h(s)]d\pi(s). \qquad (3.3)$$

A sunspot equilibrium is an allocation $[x^i(\cdot),y(\cdot)]$, together with price functions $[p(\cdot),w(\cdot)]$, satisfying: (a) for all i, $x^i(\cdot)$ solves (3.2); (b) $y(\cdot)$ solves (3.3); and (c) $\int L^i(s)di + h(s) = 1$ and $\int c^i(s)di = q(s)$ for all s. It is nondegenerate, Pareto optimal etc. if the obvious conditions hold.

---

[12]Prescott and Townsend (1984a, 1984b) discuss decentralization of optimal randomized allocations in their private information economies, where the objects being traded are lotteries over points in commodity space. They suggest (1984b, p. 18) the possibility of of supporting these allocations as decentralized equilibria with allocations indexed by "a naturally occurring random variable that is unrelated to preferences and technology" that can be interpreted as our sunspot activity; but this is never explicitly carried out. Upon pursuing this to fruition, one sees that an advantage of sunspots is that they not only randomize but also coordinate activity, which means that economies with finite populations can take advantage of convexification without appealing to the law of large numbers.

Proposition 4. In the indivisible labor economy, the planner's randomized allocation can be supported as a nondegenerate SE.

Proof. We will construct a particular SE with s distributed uniformly on [0,1]. Let $p(s) = 1$ for all s, and let $w(s) = f'(\mu^*)$, where μ^* is the employment rate chosen as the solution to the planner's problem. This immediately implies from the profit maximization condition, $f'[h(s)] = w(s)$, that $h(s) = \mu^*$ for all s. Consider consumer i. Let $S_0 = \{s \in S: L(s) = 0\}$ and $S_1 = \{s \in S: L(s) = 1\}$, and let $\hat{\mu} = \text{prob}(S_0)$. Problem (3.2) can then be rewritten (ignoring the superscript i)

$$\text{maximize EU} = U\int_{S_0}[c(s),0]ds + \int_{S_1}U[c(s),1]ds$$

$$\text{(3.4)}$$

$$\text{subject to} \int_{S_0}c(s)ds + \int_{S_1}c(s)ds + (1-\hat{\mu})f'(\mu^*) = \Pi + f'(\mu^*)$$

after substituting $p(s)$ and $w(s)$. By strict concavity, the solution to (3.4) involves setting $c(s) = \hat{c}_0$ for all $s \in S_0$ and $c(s) = \hat{c}_1$ for all $s \in S_1$. Problem (3.4) therefore further reduces to

$$\text{maximize EU} = \hat{\mu}U(\hat{c}_0,0) + (1-\hat{\mu})U(\hat{c}_1,1)$$

$$\text{(3.5)}$$

$$\text{subject to} \hat{\mu}\hat{c}_0 + (1-\hat{\mu})\hat{c}_1 - \hat{\mu}f'(\mu^*) = f(\mu^*) - \mu^*f'(\mu^*)$$

after also inserting $\Pi = f(\mu^*) - \mu^*f'(\mu^*)$.

Notice that the only feature of $L(s)$ that matters for this problem is $\hat{\mu} = \text{prob}[L(s)=0]$ (consumers only care about the number of states, and not the names of states, in which they work). Hence, all that is really necessary to solve (3.5) is to choose $\hat{c}_0$, $\hat{c}_1$, and $\hat{\mu}$. Letting λ and β be the multipliers on the budget constraint and $\hat{\mu} \leq 1$, first order conditions are

$$U(\hat{c}_0,0) - U(\hat{c}_1,1) + \hat{\lambda}[f(\mu^*) - \hat{c}_0 + \hat{c}_1] = \hat{\beta}$$

$$\hat{\mu}U_1(\hat{c}_0,0) - \hat{\mu}\hat{\lambda} = 0$$

$$(1-\hat{\mu})U_1(\hat{c}_1,1) - (1-\hat{\mu})\hat{\lambda} = 0 \qquad (3.6)$$

$$f(\mu^*) + (\hat{\mu}-\mu^*)f'(\mu^*) - \hat{\mu}\hat{c}_0 - (1-\hat{\mu})\hat{c}_1 = 0,$$

plus $\hat{\mu} \le 1$ and $\bar{\beta}(1-\hat{\mu}) = 0$. Comparing (3.6) with (3.1), we see that problem (3.5) is in fact solved by setting $\hat{\mu} = \mu^*$, $\hat{c}_0 = c_0^*$, and $\hat{c}_1 = c_1^*$ (along with $\hat{\beta} = \beta^*$ and $\hat{\lambda} = \lambda^*$). In other words, the consumer's demand correspondence includes the employment probability and consumption the planner chooses.

All that remains is to construct $[L^i(s)]$ with two properties: (a) $\int L^i(s)ds = 1-\mu^*$ for all i, so that each individual works in exactly μ^* states, and (b) $\int L^i(s)di = 1-\mu^*$ for all s, so that there are exactly μ^* individuals working in each state. Given $\mu = \mu^*$, define $[L^i(s)]$ by:

$$\text{if } s \le 1-\mu \text{ then } L^i(s) = \begin{cases} 0 & \text{if } i\in[1-\mu-s,1-s] \\ 1 & \text{otherwise;} \end{cases}$$

$$\text{if } s > 1-\mu \text{ then } L^i(s) = \begin{cases} 1 & \text{if } i\in[1-s,2-\mu-s] \\ 0 & \text{otherwise.} \end{cases}$$

This is illustrated in Figure 3a for $\mu = 1/3$, from which it is clear that $L^i(s)$ integrates to $1-\mu$ both horizontally for each i and vertically for each s. This completes the proof. ∎

The set of sunspot states in the equilibrium constructed in the proof is $S = [0,1]$; but this is not necessarily the minimal set that can be used. As in the previous section, we can also construct a SE with as few states as possible. If μ is a rational number, let n^*/N^* reduces μ to its lowest terms. Then, there is a SE with N^* equiprobable states, where each individual works in n^* of them and enjoys leisure in the rest. This is

shown in Figure 3b, again for the case $\mu = 1/3$. There are three states, S = $\{s_1, s_2, s_3\}$. Each consumer works in 1 of the three states, and each state has 1/3 of the consumers working. The outcome is equivalent to that with a continuum of states, as in Figure 3a, but whenever μ is a rational number we can economize on the number of states and, therefore, on the number of contingent commodity markets needed to decentralize the planner's allocation as a SE. If μ is irrational, an infinite number of states and markets are required to support the planner's allocation exactly.

Finally, we point out that, as in Section 2, there are several features of this economy that are interesting from the perspective of the sunspot literature. In the convex version of this economy, SE do not exist, and any allocation that depends nontrivially on extrinsic uncertainty is inefficient. Here there is a nondegenerate SE, it is Pareto optimal, and it dominates the (certainty) WE allocation, even though the latter is optimal with respect to the set of nonrandomized allocations. And the WE does not reappear as a degenerate SE since, except for the case of very special utility functions (or corner solutions), the WE allocation does not solve the first order conditions (3.6).

4. CONCLUDING REMARKS

This paper has explored the role of extrinsic uncertainty in economies with indivisible commodities. It was demonstrated that nonconvex consumption sets imply a potential role for lotteries, and that these lotteries are closely related to the concept of sunspot equilibria. In our models, sunspot equilibria can be Pareto optimal and can dominate certainty allocations (even when these allocations are optimal within the set of nonstochastic outcomes). We also showed for this class of models that not

all sunspot equilibria are equally plausible: some are not stable with respect to cooperative coalition formation, and some are not stable with respect to changes in the probability distribution of extrinsic uncertainty. The extent to which these "stability" issues are important in the convex economies studied in the literature is an interesting open question.

Extrinsic uncertainty, self-fulfilling prophecies, animal spirits, and related phenomena have been thought for some time to have a role in macroeconomics. It has even been suggested that they may be a contributing factor to problems like inefficiency and unemployment. Here we have presented models in which extrinsic uncertainty certainly does have a role to play in the allocation of economic resources, and a role in the determination of unemployment in particular. But, far from reducing or inhibiting the competitive mechanism's welfare properties, extrinsic uncertainty actually leads to more efficient outcomes here, due to the ability of sunspots to both convexify opportunity sets and to coordinate individual actions.

References

Azariadis, Costas (1975) "Implicit Contracts and Underemployment Equilibria," *Journal of Political Economy* 83, 1183-1202.

Azariadis, Costas (1981) "Self-Fulfilling Prophecies," *Journal of Economic Theory* 25, 380-396.

Baily, Martin N. (1974) "Wages and Unemployment Under Uncertain Demand," *Review of Economic Studies* 41, 37-50.

Bewley, Truman F. (1972) "Existence of Equilibria in Economies with Infinitely Many Commodities," *Journal of Economic Theory* 4, 514-540.

Burdett, Kenneth and Wright, Randall (1989) "Unemployment Insurance and Short-Time Compensation: The Effect Layoffs, Hours per Worker and Wages," *Journal of Political Economy*, in press.

Cass, David (1984) "Sunspots and Incomplete Financial Markets: The Leading Example," in G. Feiwell, ed., *Joan Robinson and Modern Economic Theory*.

Cass, David and Polemarchakis, Herekles (1988) "Convexity and Sunspots: A Comment," CARESS Working Paper #33-06R.

Cass, David and Shell, Karl (1983) "Do Sunspots Matter?" *Journal of Political Economy*.

Cass, David and Shell, Karl (1989) "Sunspot Equilibria in an Overlapping Generations Model with an Idealized Contingent-Commodities Market," in W. Barnett, J. Geweke, and K. Shell, eds., *Chaos, Sunspots, Bubbles, and Nonlinearity*. Cambridge University Press.

Crawford, Vince P. and Heller, Walter P. (1979) "Fair Division with Indivisible Commodities," *Journal of Economic Theory* 21, 10-27.

Guesnerie, R. and Laffont J. (1987) "Notes on Sunspot Equilibria in Finite Horizon Models," mimeo., Paris.

Greenwood, Jeremy and Huffman, Gregory (1987) "A Dynamic Equilibrium Model of Inflation and Unemployment," *Journal of Monetary Economics* 19, 203-228.

Greenwood, Jeremy and Huffman, Gregory (1988) "On Modeling the Natural Rate of Unemployment in Economies with Indivisible Labor," *Canadian Journal of Economics* 21, 587-609.

Hansen, Gary (1985) "Indivisible Labor and the Business Cycle," *Journal of Monetary Economics* 16, 309-328.

Hansen, Gary D. and Sargent, Thomas J. (1988) "Straight Time and Overtime in Equilibrium," *Journal of Monetary Economics* 21, 281-308.

Hyland, A. and Zeckhauser, R. J. (1979) "The Efficient Allocation of Individuals to Positions," *Journal of Political Economy* 87, 293–314.

Lucas, Robert E. Jr. (1987) *Models of Business Cycles*, Oxford: Basil Blackwell.

Mas–Colell, Andreu (1977) "Indivisible Commodities and General Equilibrium Theory," *Journal of Economic Theory* 16, 443–456.

Maskin, E. S. (1987) "On the Fair Allocation of Indivisible Goods," in G. Faiwel, ed., *Arrow and the Foundations of the Theory of Economic Policy*, London: MacMillan.

Peck, James and Shell, Karl (1985) "Market Uncertainty: Sunspot Equilibria in Imperfectly Competitive Economies," CARESS Working Paper 85–21.

Pietra, T. (1989) "Existence of Sunspot Equilibria in Financial Economies with Non–Sunspot Equilibria," mimeo., Rutgers University.

Pratt, J. W. and Zeckhauser, R. J. (1983) "The Fair and Efficient Division of the Winsor Family Silver," mimeo., Harvard University.

Prescott, Edward C. (1986) Theory Ahead of Business Cycle Measurement," *Federal Reserve Bank of Minneapolis Quarterly Review*, Fall 86, 9–22.

Prescott, Edward C. and Lucas, Robert E., Jr., (1972) "A Note on Price Systems in Infinite Dimensional Space," *International Economic Review* 13, 416–422.

Prescott, Edward C. and Townsend, Robert M. (1984a) "Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard," *Econometrica* 52, 21–45.

Prescott, Edward C. and Townsend, Robert M. (1984b) "General Competitive Analysis in an Economy with Private Information," *International Economic Review* 25, 1–20.

Quirk, J. and Saposnik, R. (1968) *Introduction to General Equilibrium Theory and Welfare Economics*. McGraw–Hill: New York.

Quinzii, M. (1984) "Core and Competitive Equilibria with Indivisibilities," *International Journal of Game Theory* 13, 41–60.

Rogerson, Richard (1984) Topics in the Theory of Labor Markets, unpublished Ph.D. dissertation, University of Minnesota.

Rogerson, Richard (1988) "Indivisible Labor, Lotteries and Equilibrium," *Journal of Monetary Economics* 21, 3–16.

Rogerson, Richard and Wright, Randall (1988) "Involuntary Unemployment in Economies with Efficient Risk Sharing," *Journal of Monetary Economics* 22, 501–513.

Shapley, L. S. and Scarf, H. E. (1974) "On Cores and Indivisibilities," *Journal of Monetary Economics* 1, 23-37.

Shell, Karl (1977) "Monnaie et Allocation Intertemporelle," mimeo.

Stokey, Nancy L., Lucas, Robert E., and Prescott, Edward C. (1989) *Recursive Methods in Economic Dynamics*, Cambridge, Mass.: Harvard University Press.

Svenson, Lars-Gunnar (1983) "Large Indivisibilities: An Equilibrium Analysis with Respect to Price Equilibria and Fairness," *Econometrica* 51, 939-954.

Woodford, Michael (1986) "Stationary Sunspot Equilibria in a Finance Constrained Economy," *Journal of Economic Theory* 40, 128-137.

	S_1	S_2	S_3
1	1	0	0
2	0	1	0
3	0	0	1

FIGURE 1a: N* = 3 AND n* = 1

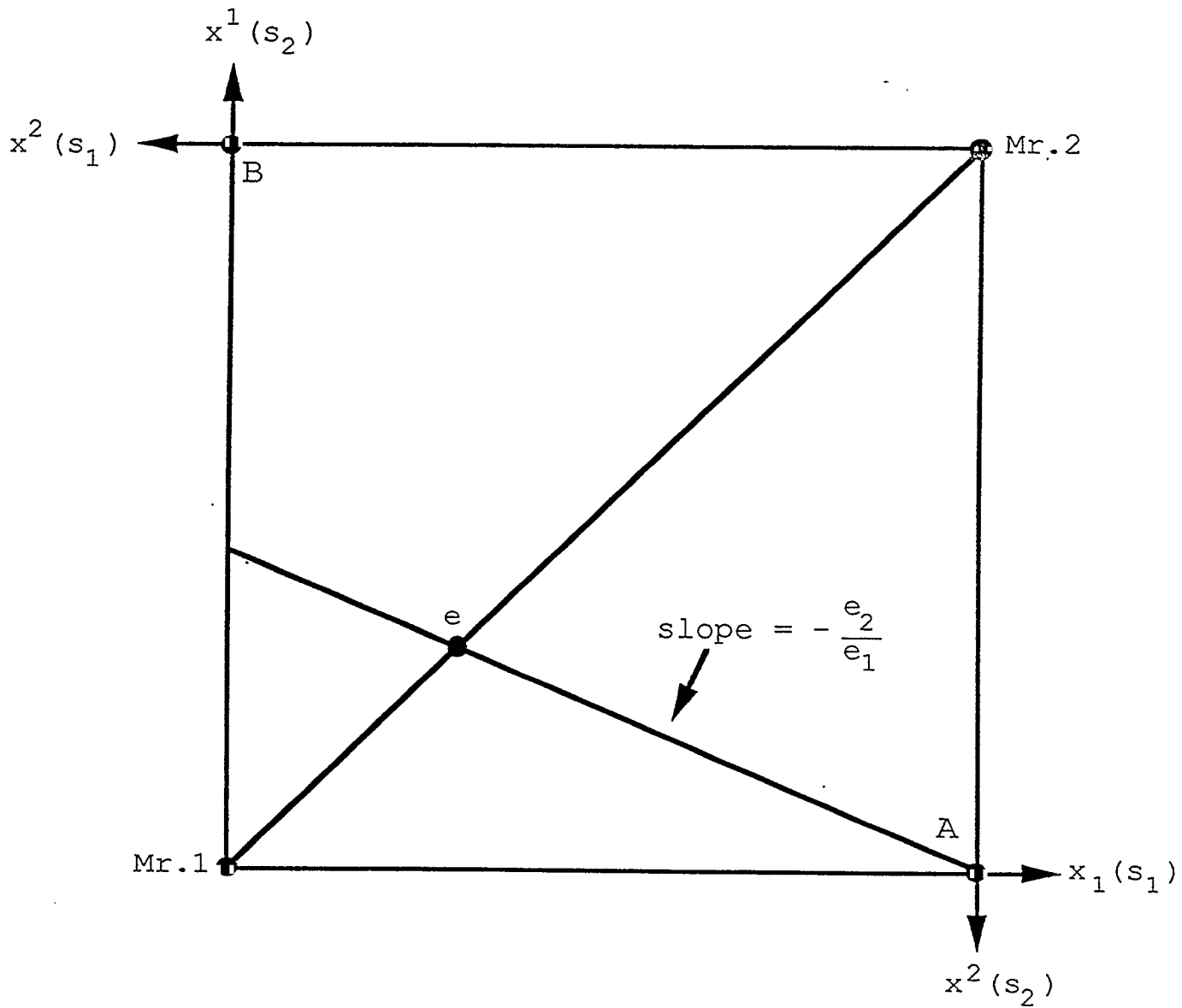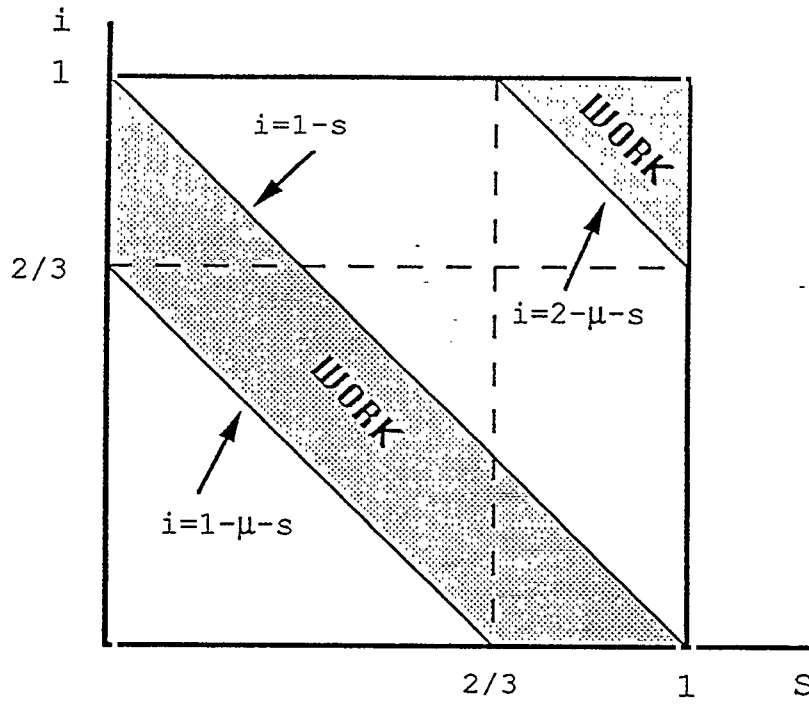|	S_1	S_2	S_3
1	1	0	1
2	1	1	0
3	0	1	1

FIGURE 1b: N* = 3 AND n* = 2
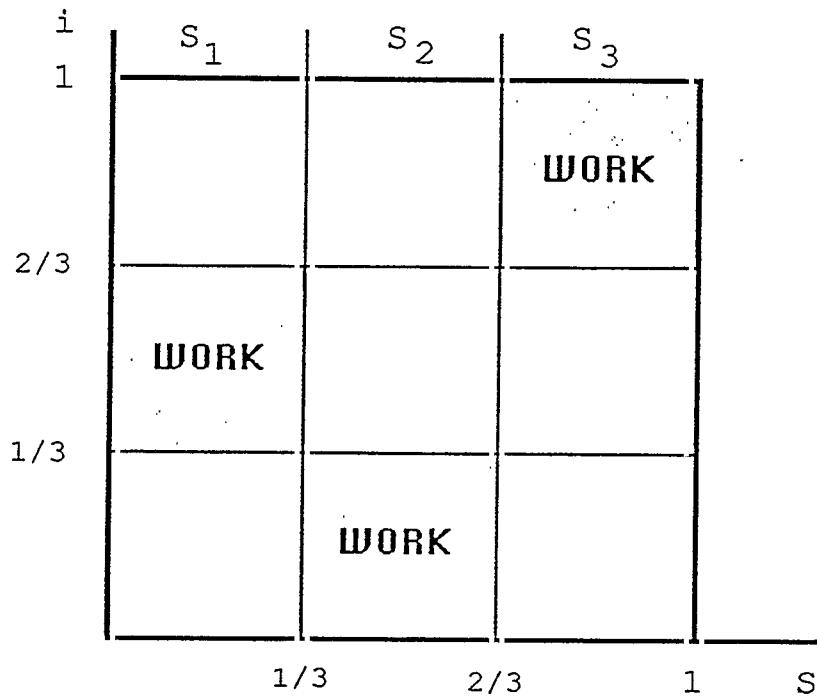
FIGURE 2

FIGURE 3a: CONTINUUM OF STATES SUPPORTS $\mu = 1/3$

FIGURE 3b: THREE STATES SUPPORTS $\mu = 1/3$