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BANKING IN COMPUTABLE GENERAL EQUILIBRIUM ECONOMIES: TECHNICAL APPENDICES I AND II

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ABSTRACT

Following are the technical appendixes for "Banking in Computable General Equilibrium Economies" by Javier Díaz-Giménez, Edward C. Prescott, Terry Fitzgerald, and Fernando Alvarez, in *Journal of Economic Dynamics and Control* 16 (1992), 533–59. Technical Appendix I, by Fernando Alvarez, describes the procedures used to construct the balance sheets reported in Tables 1 and 2 in page 536 and 537 of the paper. Technical Appendix II, by Terry Fitzgerald, describes the computational procedures used in this paper.

*We thank, without implicating for the errors that may remain, the extremely valuable suggestions of Ed Prescott, Javier Díaz-Giménez, and Steve Parente.

Technical Appendix I

by

Fernando Alvarez

Introduction

This appendix describes the procedures used to construct the balance sheets reported in Tables 1 and 2 (pages 536 and 537 of the article). Given the nature of our model, we organize the data so that they are consistent with the sectors that are present in our model economy and the ownership relationships among these sectors. Of prime importance is how much each sector owes to the other sectors.

For our study the sectors of interest are the household, banking, and government sectors. In Tables 1 and 2 of the paper we report the balance sheets for a consolidated Household Sector and a consolidated Corporate Sector for the U.S. economy. These are the two sectors that correspond most closely to the sectors in our model economy. An important difference is that in our model economy the Corporate Sector does not own or rent physical capital. Instead all the capital is in the Household Sector and is used in household production. We emphasize that our definition of the Household Sector is not the standard one used in the National Income and Product Accounts. Our definition is consistent with the concept of a household that follows from the theoretical framework employed in the paper. In particular, it is the budget constraints and the ownerships of the assets which define the units. Thus if a household owns and operates a small business, such as a farm, this business is consolidated with that household in our framework. Corporations and governments, on the other hand, are distinct legal entities which own assets and issue liabilities.

Our primary source of information is the set of balance sheets produced by the Flow of Funds Division of the Board of Governors of the Federal Reserve System. The balance sheets in Tables

1 and 2 in the paper differ from the balance sheets in the Flow of Funds data set in two main respects: (i) the sectors covered, and (ii) the criteria used to classify assets and liabilities.

Appendix I is organized as follows. In the first section we describe the framework that the Flow of Funds Division uses to organize their data. In the second section we describe in detail the framework that we use to organize the data. In the third section we add a sector to the Flow of Funds system, obtaining what we refer to as the expanded Flow of Funds system. The fourth section describes the procedure used to obtain our system of balance sheets using the expanded Flow of Funds system of balance sheets. In the fifth section we describe how to obtain Tables 1 and 2 from our system of balance sheets. A series of notes describing technical details not covered in the main text conclude the appendix.

1. The Flow of Funds Framework

Our primary source of data is the *Balance Sheets For the U.S. Economy 1949–1990,* September 1991, Board of Governors of the Federal Reserve System, Flow of Funds Division. This publication reports year-end balance sheets for the following five sectors for the 1949–1990 period. These sectors are:

- Households, Personal Trust, and Nonprofit Organizations,
- Farm Noncorporate Business,
- Nonfarm Noncorporate Business,
- Nonfinancial Corporate Business, and
- Private Financial Institutions.

The balance sheet for each of these five sectors lists the stock of various tangible and financial assets held by the sectors, the stock of various liabilities issued by each sector, and each sector's net worth. Tangible assets are partitioned into subsets defined by their physical characteristics (for example, land, plant, and equipment). Financial assets are partitioned into equity and debt assets. Financial assets and liabilities are partitioned into subsets defined by their contractual characteristics (for example, deposits, trade credit, mortgages, and bank loans).

We use the following notation to describe the Flow of Funds balance sheets. Let i and j denote any two sectors, r denote tangible asset types, and k denote contractual arrangements types. Then the sector i balance sheet is described by:

T_{ir}: type r tangible assets owned by sector i,

A_{ik}: type k debt assets owned by sector i,

E_{ik}: type k equities owned by sector i,

L_{ik}: type k liabilities issued by sector i, and

NW_i: net worth of sector i,

for all tangible assets type r and all contractual arrangements types k. The net worth of a sector is defined as total assets minus total liabilities.

2. Our Framework

Our system of balance sheets, unlike the Flow of Funds system, is closed. By closed we mean the following: (i) all debt assets of a sector in the system are liabilities of sectors in the system, (ii) all liabilities of a sector in the system are debt assets of sectors in the system, (iii) all equity holdings of a sector in the system are issued by sectors in the system, and (iv) all the equities issued by a sector in the system are held by sectors in the system. Our system consists of a balance sheet for seven sectors. These are the five sectors listed above for which the Flow of Funds provides balance sheets, plus two sectors that we add in order to have a closed system. These two additional sectors are a consolidated U.S. Government Sector and a Rest of the World Sector. Our Government Sector is the financial part of the consolidated governmental units, which is the part that is relevant for the study in the article. It corresponds roughly to the consolidated Federal Reserve System and treasuries of the various levels of government. Our Rest of the World sector is a fictional sector that we create to deal with the assets held by U.S. residents issued by foreigners and with the liabilities issued by foreign residents held by U.S. residents.

We use the following notation to describe our system of balance sheets. Let i and j denote any of these seven sectors and r the tangible asset types. We use lowercase letters for the entries of our balance sheets (for the Flow of Funds we use capital letters). In our framework the sector i balance sheet is described by:

t_{ir}: type r tangible assets owned by sector i,

a_{ij}: debt assets owned by sector i and issued by sector j,

e_{ij}: equity assets owned by sector i and issued by sector j,

l_{ij}: liabilities issued by sector i and owned by sector j, and

nw_i: net worth of sector i,

for all tangible assets type r and for all seven sectors j. We also define net worth as total assets minus total liabilities.

We emphasize that there are two main differences between our system of balance sheets and the Flow of Funds system. The first is that our system is complete. The second is that we partition the financial assets and liabilities in subsets defined by the issuer-owner relationship (the ij's and ji's). The Flow of Funds balance sheets partition the financial assets and liabilities in subsets defined by the type of securities or type of contractual arrangements (the k's) used to create the corresponding assets and liabilities.

3. Expanding the Flow of Funds System: Adding a Rest of the World Sector

To deal with the foreign assets and liabilities we create a fictional sector that we refer to as the Rest of the World Sector. The Flow of Funds reports the Net Foreign Assets. These are the holdings of foreign financial assets by U.S. residents (that is, American households, businesses, and government) and the holdings of U.S. financial assets by foreigners. The Flow of Funds reports these financial assets holdings, partitioning them into subsets defined by their contractual characteristics.

We use the Net Foreign Assets to create a balance sheet for our fictional Rest of the World Sector. We organize this balance sheet in the same way as the other five balance sheets reported by the Flow of Funds. The U.S. financial assets held by foreigners are defined as the Rest of the World assets. The foreign assets held by U.S. residents are classified into two groups: debt assets and equity assets. These debt assets are defined as the Rest of the World liabilities. We define the Rest of the World net worth to be its assets minus its liabilities. (For more specific information about this procedure, see Technical Note 1.)

The expanded Flow of Funds system of balance sheets is not closed. The Flow of Funds data set does not contain a balance sheet for the consolidated U.S. Government Sector. This is not a severe problem. If we have all the sectors' balance sheets except one, the elements of this missing balance sheet are implied by the elements of the other balance sheets.

4. Obtaining our System of Balance Sheets from the Expanded Flow of Funds System

This section describes the procedure used to obtain our balance sheets system using the expanded Flow of Funds system of balance sheets. The first step is to estimate the t_{ir}, a_{ij}, and e_{ij} using the expanded Flow of Funds T_{ir}, A_{ij}, E_{ij}, L_{ij}, and NW_i. The second step is to estimate the l_{ij}

using the a_{ij}. The final step is to compute the nw_i using the t_{ir}, a_{ij}, e_{ij}, l_{ij}, and the definition of net worth.

Tangible Assets

The t_{ir} (type r tangible assets owned by sector i) are set equal to the T_{ir} (type r tangible assets owned by sector i) for all tangible asset types r, and for all sectors i different from the Rest of the World and Government Sectors. Since we do not have comparable data for the (net) tangible capital of the Rest of the World Sector, we set its tangible capital equal to zero. Since we consider only the financial part of the consolidated governmental units, the tangible asset of the Government Sector is zero.

Debt Assets

In this section we describe the procedure used to compute the a_{ij} (debt assets owned by sector i and issued by sector j). First we deal with sectors other than the Government.

The Nongovernmental Sectors i

We let A_{ijk} denote the stock of the contractual type k debt assets owned by sector i and issued by sector j, and s_{ijk} denote the proportion of the contractual type k debt assets owned by sector i and issued by sector j. Note that $\Sigma_j s_{ijk} = 1$, and that $A_{ik} = \Sigma_j A_{ijk}$. These elements are related as follows:

$$A_{ijk} = s_{ijk} \cdot A_{ik}. \tag{I.1}$$

The Flow of Funds system reports the A_{ik} but does not report the A_{ijk} or s_{ijk}. Some assumption is necessary if we are to estimate the A_{ijk} from the Flow of Funds data. We assume that

$$s_{ijk} = s_{jk} \tag{I.2}$$

for all i. Given this assumption we can use the fact that

$$s_{jk} = L_{jk} \Big/ \left(\sum_i L_{ik} \right) \tag{I.3}$$

to compute the s_{jk} from the Flow of Funds data.

Finally, to determine the a_{ij} we aggregate the A_{ijk}

$$a_{ij} = \sum_k A_{ijk}. \tag{I.4}$$

This completes the specification of the procedure used to estimate the a_{ij} for the nongovernmental sectors i.

The Government Sector

Let i = g denote the Government sector. We estimate the a_{gj} directly from the reported liabilities of sector j in the Flow of Funds balance sheets. For each sector j, we add up all liabilities that we identify as being owned by the Government sector to obtain our estimate of a_{gj}.

Equity Assets

In this section we describe the procedure used to estimate the e_{ij} (equities owned by sector i and issued by sector j). First we deal with sectors other than the Government.

The Nongovernmental Sectors i

The procedure for equity assets is very similar to the procedure for debt asset; hence the description of our procedure for equity assets will be abbreviated.

We use q_{ijk} to denote the proportion of type k equities owned by sector i and issued by sector j. We make the assumption that the proportions q_{ijk} are independent of i. Thus $q_{ijk} = q_{jk}$. As in the case of debt assets, the Flow of Funds data set does not report the q_{ijk}. Hence, we must estimate them.

A feature that simplifies the computations is that each sector issues only one type of equity assets (see Technical Note 2 for the one exception). Given that each sector issues only one type of equity, then the k index (type of equity according to contractual characteristic) and the j index (issuer of the equity) correspond to each other.

Thus,

$$q_{jk} = \begin{cases} 1 & \text{if j is the sector that has issued the share of equity,} \\ 0 & \text{otherwise.} \end{cases} \tag{I.5}$$

Using the q_{jk} we estimate the e_{ij} as follows:

$$e_{ij} = \sum_k q_{jk} \cdot E_{ik}. \tag{I.6}$$

The Government Sector

For the Government Sector we assume that it does not own any other sector equities, nor does it issue any equity type.

Liabilities

In this section we describe the procedure used to estimate the l_{ij} (liabilities issued by sector i and owned by sector j). We estimate the l_{ij} using our previous estimate of a_{ji}, and by requiring that the claims between sectors be consistent, i.e., that one sector's liabilities correspond to another sector's debt assets. With these assumptions our estimates of the l_{ij} is equal to a_{ij} for all i and j. Thus

$$l_{ij} = a_{ji}. \tag{I.7}$$

It is important to note that our treatment of the mutual fund shares is different from the Flow of Funds system treatment. We follow the Flow of Funds conventions and consider the holdings of mutual fund shares as an equity asset of the sector that holds them. We deviate from the Flow of Funds conventions in that we do not consider the mutual fund shares as a liability of the Private Financial Institutions Sector. (See Technical Note 3.)

Our estimates of the total liabilities for a sector may differ from the total reported in the Flow of Funds data set. (For a discussion of the reasons for these discrepancies, see Technical Note 4.)

Net Worth

In this section we compute the nw_i (net worth of sector i) as total assets minus total liabilities. Thus,

$$nw_i = \sum_r t_{ir} + \sum_j a_{ij} - \sum_j l_{ij}. \tag{I.8}$$

Our net worth estimates for the Nonfinancial Corporate Business Sector and the Private Financial Institutions Sector may differ from the total value of the equities issued by these sectors. This is mainly due to the fact that in the Flow of Funds balance sheets, equity holdings are reported at market value but the net worth is defined as a residual category being the difference between total assets and total liabilities. (For more details on this issue, see Technical Note 5.)

5. Producing Tables 1 and 2: Consolidation of Sectors

Tables 1 and 2 in the article report balance sheets for consolidated sectors. Table 1 presents the balance sheets for what we refer to as the consolidated "Household Sector." This is the consolidation of the Households, Personal Trust, and Nonprofit Organizations Sector, the Farm Noncorporate Business Sector, and the Nonfarm Noncorporate Business Sector. Table 2 presents the balance sheets for what we refer to as the consolidated "Corporate Sector." This is the consolidation of the

Nonfinancial Corporate Business, the Private Financial Institutions, and the Rest of the World Sectors.

We now describe the procedure used to consolidate the balance sheets of any two sectors in our framework. The extension of this procedure to consolidate three or more sectors is straightforward. Suppose sectors i and j are the sectors to be consolidated. Let s denote any sector different from i and j. Let c denote the new sector, which is the consolidation of sectors i and j.

The type r tangible assets of sector c are

$$t_{cr} = t_{ir} + t_{jr}. \tag{I.9}$$

We define the debt assets owned by, equities owned by, and liabilities issued by the consolidated sector c by simply adding up the two corresponding quantities for sectors i and j, for each *other* sector s. Then we compute a_{cs}, e_{cs}, and l_{cs} for each sector s as

$$a_{cs} = a_{is} + a_{js}, \tag{I.10}$$

$$e_{cs} = e_{is} + e_{js}, \tag{I.11}$$

$$l_{cs} = l_{is} + l_{js}. \tag{I.12}$$

We define the consolidated sector c net worth as the sector c total assets minus sector c total liabilities. Formally this is

$$nw_c = \sum_r t_{cr} + \sum_s a_{cs} + \sum_s e_{cs} - \sum_s l_{cs}. \tag{I.13}$$

This procedure produces a consolidated sector c that has assets and liabilities net of intra-sectorial claims. By this we mean

- the consolidated sector does not hold any debt assets issued by any of the sectors in the consolidation,
- the consolidated sector does not hold any equity assets issued by any of the sectors in the consolidation, and
- none of the consolidated sector liabilities are held by any of the sectors in the consolidation.

Our motivation for producing consolidated balance sheets with assets and liabilities net of intrasectorial claims is that in our model economy there are no intrasectorial claims. In our model economy all the household borrowing and lending, and all the government borrowing and lending is intermediated by banks. For the U.S. economy, the household's nonintermediated borrowing and lending is in fact small. The stock of debt assets issued by units in either the Households, Personal Trust, and Nonprofit Organizations Sector or in the Farm and Noncorporate Business Sector and held by other units in either of these sectors is less than one percent of GNP in 1959, 1975, and 1986, the years for which we report balance sheets.

Technical Notes

Technical Note 1

In our framework we impose the following *consistency requirement:*

- Debt assets of one sector correspond to the liabilities of some other sector.
- Equity assets represent claims to residual profits. They have no corresponding liability.

The second part of the consistency requirement implies the following treatment for the liabilities of the Rest of the World Sector. The equity assets issued by foreigners and held by U.S. residents are not reported as liabilities of the Rest of the World Sector. As equity assets they are claims against the profits of the Rest of the World Sector and therefore have no corresponding liability. These equity assets are foreign corporate securities and U.S. direct foreign investment abroad.

Technical Note 2

Recall that an equity is defined (indexed) by its contractual characteristics (k index) and by its issuer (j index). For most of the equity types (for example, noncorporate business equity, mutual fund shares, and foreign direct investment abroad), the formula specified in the subsection *equity assets* applies, since each equity type is issued by only one sector. For these equity types there is a one-to-one correspondence between the k and j indices. For example, noncorporate equity is only issued by the Noncorporate Business Sector. The exception to that formula is corporate shares. The corporate shares holdings of a sector can be partitioned into those issued by the Nonfinancial Corporate Business Sector, the Private Financial Institutions Sector, and the Rest of the World Sector.

Thus for k equal to corporate shares, we use

$$q_{jk} = NW_j / \left(\sum_i NW_i \right), \tag{I.14}$$

j being Nonfinancial Corporate Business, Private Financial Institutions, and the Rest of the World Sectors, and where the summation is over the NW's of these three sectors.

Recall that these NW_i's are net worth reported by the Flow of Funds. The net worth of the Private Financial Institutions Sector is computed using the Flow of Funds total liabilities, which include an entry for the value of the mutual fund shares (see Technical Note 3 on this point).

Technical Note 3

In the Flow of Funds balance sheets, mutual funds are part of the Private Financial Institutions Sector. The Private Financial Institutions financial assets reported in the Flow of Funds balance sheets include the financial assets that belong to the mutual funds. The Flow of Funds balance sheet for the Private Financial Institutions Sector also include a liability of the same value, denoted as "mutual fund shares." We do not consider the mutual fund shares as a liability of the Private Financial Institutions Sector, but as a part of the net worth of this sector. This is a consequence of our consistency requirement that equity assets do not have a corresponding liability, (see Technical Note 1 on this point). Consequently we follow the Flow of Funds conventions and consider the holdings of mutual funds as an equity asset of the sector that holds them. But we deviate from the Flow of Funds convention in *not* considering mutual fund shares as liabilities of the Private Financial Institutions Sector.

Technical Note 4

The total liabilities for a sector in our system of balance sheets may differ from the total liabilities for that sector in the Flow of Funds system. There are two main reasons for these differences.

The first is that our treatment of mutual fund shares in the balance sheet of the Private Financial Institutions Sector is different from the treatment in the Flow of Funds system (see Technical Note 3 on this issue).

The second is that the procedure used to compute the liabilities of each sector is based on our consistency requirement (see Technical Note 1). There are three ways in which our procedure may produce total liabilities for a sector different from the total liabilities reported by the Flow of Funds.

- To compute the asset of each sector we use the s_{ijk} and we make the assumption that the s_{ijk} are the same for all sectors i. Our procedure uses the assets of all the sectors to compute each sector's liabilities. Consequently deviations from this assumption may produce different total liabilities.
- There is no balance sheet for the Government Sector in the Flow of Funds system. Our procedure uses the assets of the Government Sector to compute the other sectors' liabilities. Consequently, errors in the computation of the Government Sector assets may produce different total liabilities.
- The assets and liabilities of the Flow of Funds system may not satisfy our consistency requirement. In this case, even if our assumption on s_{ijk} is satisfied and our assets for the Government Sector are correct, our procedure may produce different total liabilities.

Technical Note 5

The Flow of Funds equity holdings are reported at market value. The net worth of the Private Financial Institutions Sector and the net worth of the Nonfinancial Corporate Business Sector are defined as total assets minus total liabilities. There are two sources of discrepancy between the market value of equity and the net worth. The first discrepancy is errors in the Flow of Funds system. In particular, not all assets and liabilities are measured and some assets and liabilities are not measured at market value. The second discrepancy is that our definition and the Flow of Funds' definition of liabilities differ. (See Technical Note 4.)

To reconcile these differences we introduce a new category which we refer to as imputed unassigned net liabilities. We define this new category as the difference between the market value of equity held and the estimated net worth. This category includes unmeasured liabilities minus unmeasured assets as well as part of any estimation errors.

Technical Appendix II

by

Terry Fitzgerald

In this appendix we describe the computational procedures used in the paper. First we present some theory and an algorithm for solving the optimality equation for a finite-state, discounted dynamic program. Next we describe our procedure for computing candidate price processes and a candidate allocation which satisfy all the equilibrium conditions except possibly one. If an equilibrium exists, these candidate elements are the unique equilibrium for the policy arrangement. Third we describe our procedure for testing whether the candidate satisfies the final equilibrium condition. Finally we describe the procedure for computing the welfare effects of changing policy arrangements. Throughout this appendix we refer to equation numbers and use notation from the paper.

1. Finite-State, Discounted Dynamic Programming: Some Theory and an Algorithm

In this section we describe our algorithm for solving the optimality equation for a finite-state, discounted dynamic program and discuss the theory underpinning the algorithm. The section is organized as follows. First we present a prototype structure of the optimality equation for a finite-state, discounted dynamic program. Next we describe the standard solution procedure. Third we describe a class of iterative solution procedures and show that if the initial point satisfies a particular condition, these iterative schemes converge to the solution of the optimality equation. We then present an algorithm for computing an initial point which satisfies the particular condition. Finally we present an algorithm for finding an approximate solution to the optimality equation.

The Prototype Optimality Equation

A prototype structure of the optimality equation for a finite-state, discounted dynamic program takes the form

$$v(x,z) = \max_{d,y}\left\{F(d,x,y,z) + \beta \sum_{z' \in Z} Q(z,z')v(y,z')\right\} \tag{II.1}$$

subject to

$$(d,y) \in \Gamma(x,z) \quad \text{all } (x,z) \in X \times Z$$

where

x is the endogenous state;

z is the exogenous state;

y is the next period's endogenous state;

z' is the next period's exogenous state;

d contains the decision variables that are not components of y;

X is a finite set of possible values for x and y;

Z is a finite set of possible values for z;

D is a finite set of possible values for d;

$\Gamma(x,z)$ is a nonempty correspondence for each $(x,z) \in X \times Z$;

$Q(z_1,z_2)$ are the transition probabilities $\Pr[z' = z_2 | z = z_1]$;

F is a real-valued bounded return function;

β is a discount factor belonging to $(0,1)$.

Remark 1. One principle used in defining the constraint correspondence Γ for any dynamic programming problem is to make the correspondence as small as possible. Often the economics of a problem implies that certain restrictions must be satisfied at any optimal policy.

Imposing these additional restrictions reduces the sets $\Gamma(x,z)$ over which one must search and typically reduces the computational cost of solving the problem.

We use Π to denote the set of feasible Markov policy rules; that is,

$$\Pi \equiv \{\pi: X \times Z \rightarrow X \times D \mid \pi(x,z) \in \Gamma(x,z) \quad \text{for all } (x,z) \in X \times Z\}.$$

We make use of the fact that $\pi(\cdot,\cdot)$ can be represented as two functions $\pi_y(\cdot,\cdot)$ and $\pi_d(\cdot,\cdot)$, where π_y: $X \times Z \rightarrow X$ and π_d: $X \times Z \rightarrow D$. Also, let

$$V \equiv \{v: X \times Z \rightarrow R\}.$$

Then any value function $v \in V$ can be represented as a point in R^q, where q is the number of points in $X \times Z$.

It will be useful to write the optimality equation in a more concise language using the following operators. The operator S: $V \times \Pi \rightarrow V$ is defined by

$$S(v,\pi)(x,z) \equiv F[\pi_d(x,z),x,\pi_y(x,z),z] + \beta \sum_{z' \in Z} Q(z,z')v[\pi_y(x,z),z'].$$

The operator P: $V \rightarrow \Pi$ is defined by

$$P(v) \equiv \underset{\pi \in \Pi}{\operatorname{argmax}}\ S(v,\pi).$$

Remark 2. A problem arises if the argmax defining P is not unique. In such cases there must be a selection rule for P to be a function. We suppress the dependency of P on this selection rule in order to simplify notation. In practice such ties almost never occur and are not an issue of practical importance.

The operator T: $V \rightarrow V$ is defined by

$T(v) \equiv S[v,P(v)]$.

The optimality equation (II.1) can now be written $v = T(v)$. Solving the optimality equation means finding a $v \in R^q$ which is a fixed point of T. It is well known that there exists a unique fixed point of T. This unique fixed point is denoted v^* throughout this appendix.

The Standard Solution Procedure

One procedure for constructing the fixed point, v^*, of T is to use the method of successive approximations. Given any $v_0 \in R^q$, this procedure produces a sequence $\{v_j\}$ by computing $v_{j+1} = T(v_j)$ for $j = 0, 1, 2, \ldots$. Since the space V is complete under the sup norm and T is a contraction with modulus β, we know that $\{v_j\}$ converges uniformly to v^* in the sup norm for any $v_0 \in R^q$. The difficulty with using this algorithm for our problem is that it requires excessive computer time.

Throughout the remainder of this appendix if a norm is not explicitly stated, the sup norm is assumed. Let $\|\cdot\|$ denote the sup norm.

A Class of Solution Procedures

In this subsection we define a class of operators for which v^* is a fixed point. Each operator along with the method of successive approximations defines a solution procedure. We found that given an initial point $w_0 \in R^q$, some solution procedures in this class dramatically reduced the computer time required to approximate v^* to a given degree of accuracy, compared to the standard solution procedure.

This subsection is organized as follows. First we define a class of operators. Second we show that if the initial point $w_0 \in R^q$ satisfies $T(w_0) \geq w_0$, then any operator from this class of operators and the method of successive approximations produces a sequence which converges uniformly, monotonically, and geometrically to v^*. Third we show that this convergence result continues to hold when the class of operators is generalized.

A Class of Operators

These operators are indexed by positive integers k and are denoted T_k: $V \rightarrow V$. The operators are defined recursively by

$$T_k(v) \equiv S[T_{k-1}(v),P(v)] \quad \text{for } k = 1, 2, 3, \ldots \text{ where } T_0(v) = v, \text{ and } T_1 = T.$$

Remark 3. The T operator is the member of this class with $k = 1$. Thus the standard solution procedure falls within the class of procedures.

Remark 4. The operator $S(\cdot,\pi)$: $V \rightarrow V$ is a contraction mapping and is monotone for any $\pi \in \Pi$.

Remark 5. $T_k(v)$ is the value of following policy P(v) for k periods given that the $k + 1$ period state is valued according to v.

Remark 6. The T_k operators are not in general monotone when $k \geq 2$. Consequently, standard monotonicity arguments cannot be used when dealing with sequences generated using T_k.

Remark 7. When $k \geq 2$, obtaining $T_k(v)$ for a given v is computationally more expensive than obtaining T(v). The computational savings of using an operator with $k \geq 2$ depends upon the problem at hand and the selected k. For our problem the time required for one T_k iteration was roughly $1 + 0.0002 \times (k-1)$ times the time required for one T iteration.

Remark 8. Let $T_\infty \equiv \lim_{n\rightarrow\infty} T_k$. Given an initial point, the solution procedure using T_∞ corresponds to what Bertsekas (1976, p. 245) calls the *policy iteration algorithm* and what Sargent (1987, p. 47) calls the *Howard policy-improvement algorithm.*

Convergence Properties of Sequences Generated by a Solution Procedure

We next present a theorem stating that for any $k \geq 1$, if the initial point $w_0 \in R^q$ satisfies $T(w_0) \geq w_0$, then T_k and the method of successive approximations produces a sequence which converges uniformly, monotonically, and geometrically to v^*. Before stating the theorem, we provide two lemmas which allow the proof of the theorem to be made concise.

Given an operator T_k with $k \geq 1$ and an initial point $w_0 \in R^q$, define the sequence $\{w_n\}$ as $w_{n+1} = T_k(w_n)$ for $n = 0, 1, 2, \ldots$.

Lemma 1. If $T(w_n) \geq w_n$, then

(i) $T(w_{n+1}) \geq w_{n+1}$, and

(ii) $w_{n+1} \geq T(w_n)$.

Proof. By assumption $w_n \leq T(w_n)$. The monotonicity of $S(\cdot,\pi)$ and the definitions of the T_i imply

$$w_n \leq T(w_n) = T_1(w_n) \leq T_2(w_n) \leq \ldots \leq T_k(w_n) = w_{n+1} \leq S[w_{n+1},P(w_n)] \qquad \text{(II.2)}$$

by the induction argument. This establishes (ii).

By the definition of T,

$$S[w_{n+1},P(w_n)] \leq T(w_{n+1}).$$

This, along with (II.2), establishes (i). □

Lemma 2. If $T(w_0) \geq w_0$, then $w_n \leq v^*$ for all n.

Proof. The properties of T guarantee that if $T(v) \geq v$, then $v \leq v^*$. Lemma 1 implies that $T(w_n) \geq w_n$. Hence $w_n \leq v^*$. □

THEOREM. *If $T(w_0) \geq w_0$, then*

(i) $\{w_n\}$ is a nondecreasing sequence,

(ii) $\{w_n\}$ converges uniformly to v^*, and

(iii) $\|w_n - v^*\| \leq \|w_n - T(w_n)\| / (1-\beta)$ *for* $n = 0, 1, 2, \ldots.$

Proof.

(i) Lemma 1 implies (i) directly.

(ii) Lemmas 1 and 2 imply $T(w_n) \leq w_{n+1} \leq v^*$. These inequalities imply

$$\|v^* - w_{n+1}\| \leq \|v^* - T(w_n)\|. \tag{II.3}$$

The fact that T is a contraction with modulus β implies

$$\|v^* - T(w_n)\| \leq \beta \|v^* - w_n\|. \tag{II.4}$$

Combining (II.3) and (II.4) we have for all n

$$\|v^* - w_{n+1}\| \leq \beta \|v^* - w_n\|.$$

This proves (ii).

(iii) The fact that T is a contraction with modulus β implies (iii) directly. □

Given an operator T_k with $k \geq 1$ and an initial point w_0, define the sequences $\{w_n\}$ and $\{v_n\}$ as follows: $w_{n+1} = T_k(w_n)$ and $v_{n+1} = T(v_n)$ for $n = 0, 1, 2, \ldots$ where $v_0 = w_0$.

PROPOSITION. *If $T(w_0) \geq w_0$, then*

$$\|v^* - w_n\| \leq \|v^* - v_n\| \quad \textit{for } n = 0, 1, 2, \ldots.$$

The proof is straightforward. This proposition states that for certain initial points and for any fixed number of steps, any solution procedure in the class of solution procedures is guaranteed to result in a value function at least as close to v* in the sup norm as the value function obtained by the standard solution procedure.

A More General Class of Operators

We now describe a more general class of operators which includes the T_k operators, and we discuss the convergence properties of sequences generated by operators in this class using the method of successive approximations. The operator that we used in our algorithm for approximating v* is an element of this more general class of operators.

Nowhere in the lemmas or in the theorem do we use the fact that k is fixed for all iterations in a solution procedure. Hence, the theorem can be rewritten to incorporate the possibility that k varies across iterations. Moreover, the value of k at each iteration can be a function of previously computed elements. In our algorithm, for example, the value of k at iteration n is set to 20 if $P(w_n) \neq P(w_{n-1})$; when $P(w_n) = P(w_{n-1})$, we pick k so that $T_k(w_n)$ and $T_{k-1}(w_n)$ are sufficiently "close." We found that an algorithm using this operator and the method of successive approximations required less computer time to approximate v* to a given degree of accuracy than algorithms using the fixed k operators.

An Initial Point for a Solution Procedure

For each operator in the general class of operators we have discussed, the method of successive approximations is guaranteed to converge to v* if the initial point $w_0 \in R^q$ satisfies $T(w_0) \geq w_0$. (For T_1, convergence is guaranteed from any initial point.) Here we provide an algorithm for constructing such a w_0.

Let $b \equiv \min F(d,x,y,z)/(1-\beta)$ where the minimum is over all elements in the graph of correspondence Γ. For any π let $z_{n+1}(\pi) = S[z_n(\pi),\pi]$ where $z_0(\pi) = b$. The sequence $\{z_n(\pi)\}$ is increasing. By definition of T, $T[z_n(\pi)] \geq z_{n+1}(\pi) \geq z_n(\pi)$. Thus, $w_0 = z_n(\pi)$ has the property $T(w_0) \geq w_0$.

Remark 9. The sequence $z_n(\pi)$ is the value of following policy π for n periods and then receiving terminal return b at stage $n + 1$.

An Algorithm to Solve the Optimality Equation

In this subsection we develop an algorithm for computing a sequence of functions whose limit is the solution to the optimality equation. First we develop the convergence criterion used to determine when to stop the iterations associated with any solution procedure. We then present the algorithm.

The Convergence Criterion

We now describe the criterion used to determine when to stop iterating. The iterations are continued until the difference between v_n and $T(v_n)$ is sufficiently small. Part of the definition of the algorithm is this measure of difference. The measure D: $V \times V \rightarrow \mathbf{R}_+$ need not be a metric, but must have the following properties: i) $D(v,w) = 0$ if and only if $v = w$; ii) if $D(v,w)$ is sufficiently small, then $P(v) = P(w)$; iii) if $\lim_{n\to\infty} \|v_n - v\| = 0$, then $\lim_{n\to\infty} D(v_n,v) = 0$.

Remark 10. Part iii of the Theorem can be used to compute an upper bound on the distance in the sup norm between v^* and any approximation to v^*.

The Algorithm

Step 1. Choose an $\epsilon > 0$.

Choose a $k \geq 1$.

Step 2. Compute $v = \min F(d,x,y,z)/(1-\beta)$.

Choose a $\pi \in \Pi$.

Choose an integer $n \geq 0$.

Step 3. Repeat the following n times:

Compute $v = S(v,\pi)$.

Step 4. Compute $\pi' = P(v)$.

Compute $v' = T(v)$.

Step 5. If $D(v,v') \leq \epsilon$, go to Step 11.

Step 6. Compute $v' = T_k(v)$.

Step 7. If $\pi \neq \pi'$, then:

Set $\pi = \pi'$.

Set $v = v'$.

Go to Step 4.

Endif.

Step 8. Set $v = v'$.

Compute $v' = S(v,\pi')$.

Step 9. If $D(v,v') > \epsilon$, go to Step 8.

Step 10. Set $\pi = \pi'$.

Go to Step 4.

Step 11. Stop.

Remark 11. One problem is the choice of ϵ. A general principle is to set ϵ small enough so that further reductions in ϵ have a negligible effect on the statistics of the model's generated data which are of interest.

2. Computing a Candidate for an Equilibrium

Generically there is at most one equilibrium for a policy arrangement. The computational procedure that we use solves for the unique set of elements that satisfy all the conditions for this set to be an equilibrium except for one. The additional requirement for the set to be an equilibrium is that the stochastic process on government consumption be nonnegative with probability one. In this section we explain how to compute this candidate set of elements.

The Inflation Rate and Interest Rate Processes

Obtaining the candidate inflation rate and interest rate processes is straightforward. The equilibrium conditions require that the processes on the inflation rate, $e(z)$, and the nominal interest rate on government debt, $i(z)$, be equal to the corresponding elements of the policy arrangement, $\epsilon(z)$ and $\iota(z)$. The interest rate processes on deposits and loans are then determined by equations (8) and (9). The inflation rate and interest rate processes are required for the household to have a well-defined dynamic programming problem.

The Household Policy

We now explain our procedure for computing an approximate solution to the optimality equation for the household's finite-state, discounted dynamic program. First we specify the optimality equation which we solved. Second we state our measure of difference used in the convergence criterion for our solution procedure. Third we describe the details of the algorithm we used to approximate the solution to the household's optimality equation. Finally we discuss ways to reduce the computational cost of solving the optimality equation.

The Household's Optimality Equation

We now specify the household's optimality equation for which we find an approximate solution. We use the principle of making the constraint correspondence as small as possible. To do this we exploit three facts about any optimal policy: (i) equations (20) and (22) will hold with equality; (ii) households will never simultaneously hold deposits and loans; and (iii) households will never buy and sell capital in the same period. These facts imply that for any optimal policy the decision variables c, d, ℓ, x^d, and x^s can each be written as a function of variables a, k, s, z, a′, k′, n. Using these functions, the optimality equation can be written

$$v(a,k,s,z) = \max_{n,a',k'} F(n,a,k,a',k',s,z) + \beta \sum_{s',z'} v(a',k',s,z)\pi[(s',z') \mid (s,z)] \tag{II.5}$$

subject to

$$(n,a',k') \in \Gamma(a,k,s,z).$$

This optimality equation maps into the prototype structure (II.1) defined earlier, where $d = n$, $x = (a,k)$, $y = (a',k')$, and $z = (s,z)$.

The constraint correspondence Γ is not well-defined until the grids on A and K are specified. We set $K = \{0,3\}$. We chose an A grid with 800 evenly spaced points. The lower bound on A was chosen to be the maximum amount a bank would be willing to lend to a household who holds the maximum stock of capital in the finite set K. Hence the lower bound on A depends upon K and ϕ, and for our economies equals -2.7. The spacing was selected so that 0.0 was in the grid and the upper bound of the grid was slightly greater than 6.0. This upper bound was selected through experimentation and was subject to the restriction that no household's optimal policy rule be affected by the upper bound. Using a finer grid over the same range had little effect on our results but increased the computational costs significantly.

The Convergence Criterion

To make use of the algorithm provided earlier for finding an approximate solution to an optimality equation, we must first define our measure of difference, D, to be used in the convergence criterion. We define D to be

$$D(v,w) \equiv \max_i \left| [v(i) - w(i)]/v(i) \right|,$$

where v(i) is the i^{th} element of the vector v. For our calibrated economies this measure of difference is well-defined since the return function is bounded above by a strictly negative number.

The motivation for this measure of difference is the following. First, a property of our economies is that scaling the unit of account by λ results in the solution to the optimality equation being scaled by $\lambda^{\alpha(1-\psi)}$. With our measure of difference the number of iterations required to meet the convergence criterion is invariant to such scalings. Second, percent changes are economically meaningful statistics.

The Algorithm

Next we provide details on how we implement the algorithm. In Step 1 we chose $\epsilon = 1.0 \times 10^{-9}$ and $k = 20$. We found that the policy rules had converged well before the convergence criterion was met.

Steps 2 and 3 in the algorithm compute an initial value function v which satisfies $T(v) \geq v$ and delivers this v and an initial policy rule π to Step 4. We experimented with three different procedures for computing this (v,π) pair. These procedures are ranked by the total computational time required to approximate the solution to (II.5).

- The slowest procedure is to set $n = 0$ in Step 2. In this case the choice of π is irrelevant.
- A faster procedure is to select a "reasonable" π and set $n = 1{,}000$, where "reasonable" was determined by the economics of our problem.
- We found that a still faster procedure is the following. First we used our algorithm with $\epsilon = 1.0 \times 10^{-9}$, $k = 20$, and $n = 1{,}000$ to find an approximate solution to a more restricted problem, that is, a problem with a smaller constraint correspondence. Search costs in the maximization steps are relatively small for the more restricted problem. We then used the approximate solution to this restricted problem along with its approximate optimal policy rule as the (v,π) pair delivered to Step 4. This v satisfied $T(v) \geq v$.

Two Computational Insights

We exploited two insights to reduce the computer time required to approximate the solution to the household's optimality equation. First, the Markov chain on (s,z) has the property that $\Pr[s' \in \{1,2\} \mid s = 3] = 0$. Because of this property, the optimality equation has a special structure. Evaluating v(a,k,3) requires only the value function of the retired households ($s = 3$) and does not depend on the value function of the working-age households ($s \in \{1,2\}$). We exploit this fact in our algorithm. First we compute the value function of the retired households. We then compute the value function of the working-age households, which requires the already computed value function of the retired households. Using this two-step procedure rather than the straightforward one-step procedure reduced the computer time required to solve the optimality equation by 32 percent.

The second insight was to recognize that the return function is evaluated many times when computing T(v). We tabulated the values of $U_1(c,k',\tau-n,s)$ for each (k',n,s) triplet over a fine grid on c. This allows us to replace a function evaluation with a linear interpolation of grid points—computations which require significantly less computer time than evaluating the function. We make the c grid fine enough so that our results are unaffected by using this approximation to the utility function. Our grid on c has 10,000 evenly spaced grid points in the interval [0.00001, 3.0]. Using this tabulation procedure rather than computing the utility function directly reduced the computer time required to solve the optimality equation by 75 percent.

Given that $F(n,a,k,a',k',s,z)$ is defined over a finite set, an alternative procedure which reduces search costs even further is to tabulate F over all feasible points. This procedure was not practical due to excessive storage requirements.

The Banking and Government Policies

The values of aggregate quantities including the values of the candidate banking and government policies can be computed given a distribution over household types, y, an economy-wide shock, z, and the previously computed household policy using market-clearing conditions, equations (24) through (28). For the banking policy, $\ell_b(y,z)$ and $d_b(y,z)$ are computed using equations (26) and (28). Since we only consider economies with $\iota(z) > 0$, the bank's problem implies that $r_b(y,z) = \rho(z)d_b(y,z)$. Since bank profits are zero, $b_b(y,z) = d_b(y,z) - r_b(y,z) - \ell_b(y,z)$. For the government policy, $g(y,z)$, $b_g(y,z)$, and $r_g(y,z)$ are computed using equations (24), (25), and (27).

The Law of Motion for the Measure of Agent Types

The final candidate element is the law of motion for the measure of agent types. We explain how to simulate this law of motion to generate aggregate time series. At the beginning of any period, let y denote the distribution of households across types (a,k,s) and let z denote the current-period realization of the economy-wide shock. Let a prime (′) denote next period's value, so that y' and z' are next period's distribution of households and economy-wide shock, respectively. Note that y has the property that $y(a,k,4) = 0$ (that is, there are no dead households). In computing next period's distribution, y', we will make use of an intermediate distribution $\hat{y}$.

Step 1. Computing z'.

Draw z' according to the probabilities $\pi_z(z'|z)$ using a random number generator.

Step 2. Computing $\hat{y}$ from y.

Algebraically, we evaluate

$$\hat{y}(a',k',s') = \sum_{a,k,s \in B(a',k',z)} y(a,k,s)\pi_s(s'|s,z'),$$

where

$$B(a',k',z) = \{(a,k,s)\colon\ a' = a'(a,k,s,z),\ k' = k'(a,k,s,z)\}.$$

Computationally, this is done as follows:

Stage 1. Set $\hat{y}(a,k,s) = 0$ for all a, k, s.

Stage 2. For each (a,k,s), do

$a' = a'(a,k,s,z)$

$k' = k'(a,k,s,z)$

For each s', do

$$\hat{y}(a',k',s') = \hat{y}(a',k',s') + \pi_s(s'|s,z')y(a,k,s).$$

In general $\hat{y}(a,k,4) \neq 0$, where $\hat{y}$ can be interpreted as the distribution of household types next period before the dead households are replaced with an equal measure of newborn households.

Step 3. Computing y' from $\hat{y}$.

Compute dead $= \sum_{a',k'} \hat{y}(a',k',4)$.

Set $y'(a',k',s') = \hat{y}(a',k',s')$ for all a', k', s'.

Set $y'(a',k',4) = 0$ for all a', k'.

Set $y'(0,0,s') = \Psi(s')$ dead $+ y'(0,0,s')$ for $s' \in \{1,2\}$.

In Step 3 we use the fact that newborn households all enter with $a = k = 0$. The probability that a household is born in state s' is given by $\Psi(s')$, where $\Psi(s') = 0$ for $s' \in \{3,4\}$. This fits in with the notation in the paper as follows:

$$\psi_{a',k',s'} = \begin{Bmatrix} \Psi(s')\text{ dead} & \text{if } a' = 0,\ k' = 0 \\ 0 & \text{otherwise} \end{Bmatrix}.$$

3. Testing Whether the Candidate is an Equilibrium

In this section we test whether the candidate price processes and the candidate allocation computed in the previous section are an equilibrium for the policy arrangement. If an equilibrium exists for the policy arrangement, then the computed candidate is the unique equilibrium. The candidate is an equilibrium if the stochastic process on government consumption, g(y,z), is non-negative with probability one. When $n_z = 1$, the case with no aggregate uncertainty, this condition can be verified through simulation since the equilibrium path of an economy converges to a unique steady state with a fixed distribution of household types. For $n_z \geq 2$, the process on g is not deterministic. In theory one would need to check that $g(y_t,z_t)$ remained nonnegative for all possible paths for z. In practice we selected a diverse set of z paths and then checked that $g(y_t,z_t)$ remained nonnegative for all paths in this diverse set of paths. Included in this set were paths which seemed most likely to make g negative, such as paths where z is constant for many periods and then changes value. For the experiments reported in the paper, all of the z paths considered resulted in $g(y_t,z_t)$ being much larger than zero for all t.

4. Computing the Welfare Effects of Changing Policy Arrangements

In this section we describe our procedure for computing the welfare effects associated with switching from one policy arrangement to another. First, we describe how we compute household wealth. We then explain how to compute the household's utility value of private and public consumptions for a given policy arrangement. Lastly, we explain how to compute our aggregate welfare measure M, and we describe our procedure for decomposing the welfare effects of a policy switch into public and private effects.

Household Wealth

Computing household wealth relies upon a method for computing our measure of the human capital of a household, h(s,z). Since equation (31) can be written recursively as

$$h(s,z) = w(s,z)\sigma(s) + \beta \sum_{s',z'} \pi[(s',z') \mid (s,z)]h(s',z'),$$

we computed an approximation to h by simply plugging in an initial guess h_0 and iterating to obtain a sequence of h_t's that converge to h. Note that $h(s,z) = 0$ for $s \in \{3,4\}$. W(a,k,s,z) is computed using (30).

The Utility Value of Private and Public Consumptions

The household's utility value of private consumptions, which we denote v_1, under a given policy arrangement is given by the value function solved for in the household's dynamic programming problem. Computing the household's utility value of public consumptions, which we denote v_2, is more complicated because it depends upon the current and future distributions of y. We computed an approximation to $v_2(s,y,z|\pi)$. We describe the procedure used to approximate $v_2(s,y,z|\pi)$ in four steps.

Step 1. Computing the sequence $\bar{g}_t \equiv E[g_t(y_t,z_t)|y_0,z_0,\pi]$.

Let $i \equiv (a,k,s)$. Let (y_0,z_0) denote the current aggregate state and π the policy arrangement given the initial economy-wide state (y_0,z_0). Let $x_t(i,z)$ denote the expected measure of households of type i with economy-wide shock z at date t. We compute x_{t+1} from x_t in two stages using an intermediate vector $\hat{x}$, where $\hat{x}$ can be interpreted as the expected distribution of household types in period $t + 1$ before the dead households are replaced with an equal measure of newborn households. We define x_0 as follows:

$$x_0(i,z) = \left\{\begin{matrix} y_0(i) & \text{if } z = z_0 \\ 0 & \text{if } z \neq z_0 \end{matrix}\right\}.$$

<u>Stage 1</u>. *Computing* $\hat{x}$ *from* x_t.

Set $\hat{x}(i,z) = 0$ for all i, z.

For each (i,z), do

$a' = a'(i,z)$

$k' = k'(i,z)$.

For each (s',z'), do

$\hat{x}(i',z') = \hat{x}(i',z') + \pi[(s',z')|(s,z)]x_t(i,z)$.

Stage 2. *Computing x_{t+1} from $\hat{x}$.*

Compute $D(z') = \sum_{a',k'} \hat{x}(a',k',4,z')$, for all z'.

Set $x_{t+1}(i',z') = \hat{x}(i',z')$ for all i', z'.

Set $x_{t+1}(a',k',4,z') = 0$ for all a', k', z'.

Set $x_{t+1}(0,0,s',z') = \Psi(s')D(z') + x_{t+1}(0,0,s',z')$, for all s', z'.

We require that the measure of households who die in state z' be replaced by an equal measure of households born in state z'.

Let $g_{iz} \equiv n(i,z)w(s,z) + x^s(i,z) - c(i,z) - x^d(i,z) - d(i,z)\eta_D - \ell(i,z)\eta_L - \mu k'(i,z)$. We compute a $\hat{g}_t$ sequence from the x_t sequence as follows:

$$\hat{g}_t = \sum_{i,z} g_{iz} x_t(i,z).$$

In work in progress by Alvarez and Fitzgerald, it is shown that $\bar{g}_t = \hat{g}_t$.

Step 2. Computing the sequence $\phi_t(s) \equiv E[\sigma(s_t)|s = s_0]$.

Since s follows a Markov chain, this is a standard exercise. We note that a computationally efficient method to compute $\phi_t(s)$ is as follows. Let

$\gamma_1 \equiv \text{prob}[s' \in \{1,2\} | s \in \{1,2\}]$.

$\gamma_2 \equiv \text{prob}[s' = 3 | s = 3]$.

For $s = 3$,

$$\phi_t(s) = \gamma_2^t \quad \text{for } t = 0, 1, 2, \ldots.$$

For $s \in \{1,2\}$,

$$\phi_t(s) = 1 \quad \text{for } t = 0.$$

$$\phi_t(s) = \gamma_1\phi_{t-1}(s) + (1-\gamma_1)\gamma_2^{t-1} \quad \text{for } t = 1, 2, \ldots.$$

Step 3. Approximating $E[g_t^{\alpha(1-\psi)} | y_0 = y, z_0 = z, \pi]$.

We approximate $g_t^{\alpha(1-\psi)}$ with a first-order Taylor series about $\bar{g}_t$, $g_t^{\alpha(1-\psi)} \approx \bar{g}_t^{\alpha(1-\psi)} + \alpha(1-\psi)\bar{g}_t^{\alpha(1-\psi)-1}(g_t - \bar{g}_t)$. Consequently,

$$E[g_t^{\alpha(1-\psi)} | y_0, z_0, \pi] \approx \bar{g}_t^{\alpha(1-\psi)}$$

since $E[g_t - \bar{g}_t | y_0, z_0, \pi] = 0$.

Step 4. Computing an approximation of $v_2(s,y,z | \pi)$.

Notice that we can rewrite our definition of v_2 in (32) as

$$v_2(s,y,z | \pi) \equiv \sum_t \beta^t \left[\frac{\delta_g}{(1-\psi)} \right] E[\sigma(s_t) | s_0] E[g_t^{\alpha(1-\psi)} | y_0, z_0, \pi]$$

since $\sigma(s_t)$ and g_t are independent for our calibrated economies.

Using the results from Steps 1–3 above, we have

$$v_2(s,y,z | \pi) \approx \sum_t \beta^t \left[\frac{\delta_g}{(1-\psi)} \right] \phi_t(s) \bar{g}_t^{\alpha(1-\psi)}.$$

Since $\beta < 1$ and the remaining elements in the sum are bounded, this sum converges. We compute $\Sigma_{t=0}^{T} \beta^t [\delta_g/(1-\psi)] \phi_t(s) \bar{g}_t^{\alpha(1-\psi)}$ for a large T.

The Welfare Measure

Finally, we describe how to compute our measure of the welfare effects associated with switching from one policy arrangement to another. We also describe our procedure for decomposing the welfare effects into the effects due to changes in private consumptions and the effects due to changes in public consumptions.

To start, let π_0 denote the current policy arrangement, (y,z) the current aggregate state, and π_1 the alternative policy arrangement. We make the following definitions:

$$i \equiv (a,k,s)$$

$$W(i,z) \equiv a + k + h(s,z)$$

$$V(i,z,y|\pi) \equiv v_1(i,z|\pi) + v_2(s,y,z|\pi)$$

$$\lambda(i,z,y|\pi_0,\pi_1) \equiv \left[\frac{V(i,z,y|\pi_1)}{V(i,z,y|\pi_0)}\right]^{1/\alpha(1-\psi)}.$$

The welfare measure of the effects of a switch from policy arrangement π_0 to π_1 is the wealth-weighted average of $\lambda(i,z,y|\pi_0,\pi_1) - 1$, and is denoted by M.

$$M(y,z,\pi_0,\pi_1) \equiv \frac{\sum_i W(i,z)y(i)[\lambda(i,z,y|\pi_0,\pi_1) - 1]}{\sum_i W(i,z)y(i)}.$$

To get a sense of how much of our welfare measure M is due to changes in private consumptions and how much is due to changes in public consumptions, we define a method of decomposing the effects. We define our private welfare measure, M_1, as follows:

$$\mu(i,z,y,\pi_1,\pi_0) \equiv \frac{[v_1(i,z,y|\pi_1) - v_1(i,z,y|\pi_0)]}{[V(i,z,y|\pi_1) - V(i,z,y|\pi_0)]}$$

$$M_1(y,z,\pi_0,\pi_1) \equiv \frac{\sum_i W(i,z)y(i)[\lambda(i,z,y)|\pi_0,\pi_1) - 1]\mu(i,z,y,\pi_1,\pi_0)}{\sum_i W(i,z)y(i)}.$$

The public welfare measure is defined by replacing v_1 with v_2 in the definition of μ.

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