Federal Reserve Bank of Minneapolis
Research Department Staff Report 203

Revised November 1997

The Optimum Quantity of Debt[†]

S. Rao Aiyagari
University of Rochester

Ellen R. McGrattan [‡]
Federal Reserve Bank of Minneapolis

ABSTRACT

We find that the welfare gains to being at the optimum quantity of debt rather than the current U.S. level are small, and, therefore, concerns regarding the high level of debt in the U.S. economy may be misplaced. This finding is based on a model of a large number of infinitely-lived households whose saving behavior is influenced by precautionary saving motives and borrowing constraints. This model incorporates a different role for government debt than is found in standard models, and it captures different cost-benefit trade-offs. On the benefit side, government debt enhances the liquidity of households by providing an additional means of smoothing consumption and by effectively loosening borrowing constraints. On the cost side, the implied taxes have adverse wealth distribution and incentive effects. In addition, government debt crowds out capital via higher interest rates and lowers per capita consumption.

Keywords: Government debt; Precautionary saving; Borrowing constraints
JEL classification: E6; H6

[†] We received helpful comments from V. V. Chari, Ed Prescott, and an anonymous referee. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

[‡] Corresponding author, Research Department, Federal Reserve Bank of Minneapolis, 90 Hennepin Avenue, Minneapolis, Minnesota, 55480, erm@ellen.mpls.frb.fed.us, phone: 612-204-5523, fax: 612-204-5515.

1. Introduction

The level, time path, and type of government debt are important issues in fiscal policy. In this paper we undertake the normative exercise of calculating the optimum quantity of risk-free public debt and the welfare costs of being at levels other than the optimum. These calculations are done with a model parameterized to match various other features of the U.S. economy.[1] We compare the optimum quantity to the debt/GDP ratio for the U.S. economy over the post-second world war period. For our benchmark parameterization, we find that the observed average debt/GDP ratio for the U.S. economy is optimal, and the welfare costs of being at the U.S. level are, therefore, zero. Furthermore, for a wide range of parameter values, the welfare gains to being at the optimum rather than the U.S. level are trivially small. Therefore, concerns regarding the high level of debt in the U.S. economy, and possibly in other economies, may be misplaced.

The model we use consists of a large number of infinitely-lived households whose saving behavior is influenced by precautionary saving motives and borrowing constraints. The households supply labor elastically and are subject to a proportional income tax. This model incorporates a different role for government debt than standard models do and captures different trade-offs between the benefits and costs of varying the quantity of debt. On the benefit side, in our model government debt enhances the liquidity of households by providing an additional means of smoothing consumption (in addition to claims to capital) and by effectively loosening borrowing constraints.[2] When the interest rate is raised, government debt makes assets both less costly to hold and more effective in smoothing consumption. On the cost side, the implied taxes have adverse wealth distribution and incentive effects. In addition, government debt crowds out capital via higher interest rates,

[1] Our analysis of the optimum quantity of debt is closely related to Bewley's (1980, 1983) analysis of the optimum quantity of money, for which he used a pure exchange model with a finite number of consumers. Our analysis is richer and is quantitatively oriented.

[2] A recent paper by Jappelli and Pagano (1994) analyzes the welfare effects of borrowing constraints in an overlapping generations model.

and it lowers per capita consumption.

The role of government debt and its welfare effects in our model may be contrasted with its role and welfare effects in the standard, deterministic, representative-agent growth model. In the latter model, if lump-sum taxes are permitted, then there is no role for government debt, and, hence, there are no welfare effects. With distorting taxes there is a role for government debt as a means of smoothing tax distortions over time. Optimal debt policy in such a model (see Barro 1979; Chamley 1985, 1986) generally implies that the steady-state level of debt depends on the initial level of debt. If sufficient capital levies are available at the initial date, then the optimum quantity of debt is the negative of the present value of government consumption, evaluated using the undistorted sequence of interest rates. This permits setting all distorting taxes to zero, thereby avoiding losses from such taxes. Government consumption is financed by the interest earned on public assets. Thus in the standard, representative-agent growth model the optimum quantity of debt either depends on some unknown initial conditions or is indeterminate.

As we will see, our model has some similarities with the overlapping generations model; in particular, the crowding out of capital by debt arises in both models. However, the role of debt in the two models is somewhat different. In the overlapping generations model, government debt is equivalent to lump-sum redistributions across generations that are revenue-neutral, and such redistributions affect equilibrium allocations and interest rates. In our model, due to the implicit altruism among generations, such lump-sum redistributions are completely neutral. Thus our model abstracts from concerns for the intergenerational distribution. More precisely, while we permit intergenerational transfers to be negative as well as positive, we do not permit the family as a whole to carry negative financial assets from one period to the next. Thus our analysis differs from the analysis of models with nonnegativity constraints on bequests and the effects of public debt in such models. (See Cukierman and Meltzer 1989 for an analysis of nonnegativity constraints on bequests.)

The class of models we consider consists of variants of the deterministic growth model that are modified to include a large number of individuals subject to uninsured, idiosyncratic shocks to their labor productivities. Though there is no aggregate uncertainty in these models, there is individual uncertainty due to the absence of insurance markets. This is the feature that generates precautionary saving. Further, there is ex-post heterogeneity among individuals; in the steady state there is a distribution of individuals according to asset holdings and earnings.

As Aiyagari (1994b) has argued, these models have many empirically plausible implications; furthermore, they are quite attractive for quantitative analysis.[3] When insurance markets are complete, our model collapses to the representative-agent growth model which facilitates comparison between the two frameworks. Most aspects of our model can be parameterized in exactly the same way that the representative-agent growth model has been parameterized for quantitative analyses of growth and business cycles. The only additional aspect of our model that is not found in a representative-agent growth model is the stochastic process that governs the idiosyncratic labor productivity shocks. This process can be parameterized by using empirical microeconomic studies of earnings behavior. Thus, with the exception of the process that governs the idiosyncratic labor productivity shocks, the parameters of the two models could be exactly the same.

We parameterize our model using data for the U.S. economy and compute an estimate of the optimum quantity of debt. Our estimate of the optimum debt/GDP ratio is equal to 2/3 which is also the level of debt to GDP for the post-war U.S. economy. Therefore, the welfare gain to being at the optimum is zero. We consider various perturbations of our key parameters to determine the sensitivity of these results. In many cases, the observed debt/GDP ratio for the U.S. economy actually falls below the optimum. However, in no

[3] Bewley (undated) and Laitner (1979, 1992) were among the first to analyze such models. Aiyagari (1994a) uses such a model to evaluate the quantitative significance of precautionary saving.

cases that we consider do we find significant gains to being at the optimum level rather than the U.S. level. Thus, from the perspective of welfare analysis, although debt plays both a positive and negative role, their relative magnitudes may be of minor practical importance.

The rest of this paper is organized as follows. In Section 2 we describe a version of our model with lump-sum taxes and inelastic labor. We start with this specification, even though our benchmark model has proportional taxes and elastic labor, because this specification permits us to do several things. First, and most importantly, it is relatively easier to provide intuition for the workings of the model with lump-sum taxes and inelastic labor than to provide it for the model with proportional taxes and elastic labor. Second, assuming lump-sum taxes and inelastic labor permits us to separate the wealth distribution effect of taxes from the incentive and insurance effects that arise when taxes are proportional. After analyzing the model with lump-sum taxes and inelastic labor, we describe, in Section 3, the model with a proportional income tax and elastic labor. In Section 4 we describe how we choose the parameter values for the benchmark model. The results are described in Section 5. In Section 6 we discuss the robustness of our results for some alternative parameter values. Section 7 concludes.

2. A Growth Model With Uninsured Idiosyncratic Shocks

In this section we describe an augmented version of the model in Aiyagari (1994a) – augmented to permit growth and to include government debt, lump-sum taxes, and government consumption.

Our economy has a continuum of infinitely-lived agents of measure unity who receive idiosyncratic shocks to their labor productivities and supply one unit of labor inelastically. Let e_t denote an individual's labor productivity, and suppose that this productivity is i.i.d. across agents and follows some Markov process over time. We normalize per capita labor productivity to unity so that $E(e_t) = 1$. The assumption that all agents face the same stochastic process for productivity can be relaxed somewhat without affecting our main results.[4] We also assume that there are no aggregate shocks. We describe the balanced growth path of such an economy without insurance markets but with trading in risk-free assets, namely, capital and government debt. Along the balanced growth path there will be fluctuations in an individual's consumption, income, and wealth, but per capita variables will be growing at constant rates, and cross-sectional distributions (relative to per capita values) will be constant over time.

There is a neoclassical aggregate production function, $Y_t = F(K_t, z_t N_t)$, where Y_t is per capita output, K_t is per capita capital, N_t is per capita labor input, and z_t is a measure of labor-augmenting, exogenous, technical progress in period t. Note that per capita labor input equals unity in equilibrium. We assume that $z_t = z(1+g)^t$, where g is the rate of technical progress. Also, capital is assumed to depreciate at the constant geometric rate δ. When we assume competitive product and factor markets, the wage rate w_t and the interest rate r_t are then given by

$$w_t = z_t F_2(K_t, z_t), \tag{1}$$

$$r_t = F_1(K_t, z_t) - \delta. \tag{2}$$

Note that in a balanced growth equilibrium, w_t, Y_t, and K_t will be growing at the rate g, whereas the interest rate will be constant (i.e., $r_t = r$ for all t).

We now describe consumer behavior. Let c_t, a_t, and T_t denote an individual's consumption in period t, an individual's assets at the beginning of period t, and the lump-sum tax in period t, respectively. The consumer derives utility in period t from consumption in period t; this utility is given by $c_t^{1-\nu}/(1-\nu)$, where $\nu > 0$ is the relative risk aversion

[4] We could assume that there are several groups of agents with permanent skill-level differences and that the groups are distinguished by the average level of productivity. If the preferences are homothetic, the technology has constant returns, and the distribution of shocks is the same for all groups, then each group of agents is just a scaled version of any other. The equilibrium interest rate and quantities will be the same as in the case in which there is only one group of agents.

coefficient. The typical consumer starts at date zero with some initial assets a_0 and some initial productivity shock e_0. The consumer chooses stochastic processes for consumption and asset holdings in order to maximize the expected discounted sum of utilities of consumption subject to the sequence of budget constraints and nonnegativity constraints, i.e.,

$$\max_{\{c_t, a_{t+1}\}} E\left[\sum_{t=0}^{\infty} \beta^t c_t^{1-\nu}/(1-\nu) \,\middle|\, a_0, e_0\right],$$

$$\text{subject to} \qquad c_t + a_{t+1} \leq (1+r)a_t + w_t e_t - T_t, \quad t \geq 0,$$

$$c_t \geq 0, \quad a_t \geq 0, \quad t \geq 0.$$

Note that the constraint $a_t \geq 0$ rules out borrowing. If $r < g$, then some limit on borrowing must be imposed in order for the consumer's problem to be well-defined. If $r < g$, then the present value of earnings is infinite (almost surely), and without a borrowing limit, nothing prevents the consumer from running a Ponzi scheme. When there is a limit on borrowing it is possible to have a steady state with $r < g$, which is very reminiscent of overlapping generations models.

Let G_t denote per capita government consumption. We assume that government consumption equals a fixed fraction of per capita output. In other words, we do not explicitly model the output-enhancing or welfare-enhancing role of government consumption. Our assumption regarding government consumption may be thought of as an approximation to the way government consumption would be chosen if it were endogenous and had either an output- or welfare-enhancing role. Let B_t denote per capita government debt. Then the government budget constraint is given by

$$G_t + rB_t = B_{t+1} - B_t + T_t.$$

Let A_t denote per capita assets held by consumers. In a balanced growth equilibrium of this economy we must have

$$A_t = K_t + B_t. \tag{3}$$

It will be convenient to transform the model into a stationary form. Toward this end let $k = K_t/Y_t$, $\tilde{w} = w_t/Y_t$, $\tilde{c}_t = c_t/Y_t$, $\tilde{a}_t = a_t/Y_t$, $\tau = T_t/Y_t$, $b = B_t/Y_t$, and $\bar{a} = A_t/Y_t$. Note that, in a balanced growth equilibrium, T_t, B_t, and A_t will also be growing at the rate g.

We can now rewrite the model as follows. First, we divide through the consumer's budget constraint by Y_t and rewrite the consumer's preferences by using the definition of $\tilde{c}_t$. Thus, the consumer's problem becomes

$$\max_{\{\tilde{c}_t, \tilde{a}_{t+1}\}} \; E\left[Y_0^{1-\nu} \sum_{t=0}^{\infty} [\beta(1+g)^{1-\nu}]^t \, \tilde{c}_t^{1-\nu}/(1-\nu) \middle| \tilde{a}_0, e_0 \right],$$

$$\text{subject to} \quad \tilde{c}_t + (1+g)\tilde{a}_{t+1} \leq (1+r)\tilde{a}_t + \tilde{w}e_t - \tau, \quad t \geq 0, \tag{4}$$

$$\tilde{c}_t \geq 0, \quad \tilde{a}_t \geq 0, \quad t \geq 0.$$

For some parameterizations of the model, we require more restrictive borrowing constraints, namely, $\tilde{a}_t \geq \max(0, \underline{a})$ with $\underline{a} = (-\tilde{w}e_{\min} + \gamma + (r-g)b)/(r-g)$, in order to ensure that consumption is nonnegative in all states of the world. However, for all of the parameterizations that we consider in Sections 5 and 6, $\underline{a} < 0$.

We divide through the government budget constraint by Y_t so that it becomes

$$\gamma + (r-g)b = \tau, \tag{5}$$

where $\gamma = G_t/Y_t$. We divide through the asset market equilibrium condition by Y_t to get

$$\bar{a} = k + b. \tag{6}$$

By using equation (2) we can express K_t/z_t as a function of r; hence we can express k $(= K_t/Y_t)$ as a function of r, say $\kappa(r)$. Further, using (1) we can express $\tilde{w}$ $(= w_t/Y_t)$ as a function of r, say $\omega(r)$. Thus we have

$$k = \kappa(r), \tag{7}$$

$$\tilde{w} = \omega(r). \tag{8}$$

The balanced growth equilibrium of the original economy corresponds to the steady state of the transformed economy, which is characterized by an interest rate r^* that satisfies

$$\bar{\alpha}(r;\gamma,b,g) = \kappa(r) + b, \tag{9}$$

where $\bar{\alpha}(r;\cdot)$ represents the per capita assets desired by consumers (relative to per capita output) as a function of the interest rate, and where $\kappa(r)+b$ is the per capita supply of assets (relative to per capita output) expressed as a function of the interest rate. Equation (9) is obtained in the following way. The solution to the consumer's problem yields a decision rule for asset accumulation: $\tilde{a}_{t+1} = \alpha(\tilde{a}_t, e_t; r, \gamma, b, g)$. This decision rule can be used, together with the Markov process for the labor productivity shock (e_t), to calculate the stationary joint distribution of assets and the productivity shock, denoted by $H(\tilde{a}, e; r, \gamma, b, g)$. This stationary distribution then implies an expression for per capita assets, $\bar{a} = \int\int \tilde{a}\, dH = \bar{\alpha}(r;\gamma,b,g)$. This is the left side of (9). The right side of (9) is obtained from (6) by noting that $k = \kappa(r)$.

In Fig. 1 we show how the steady-state interest rate is determined; the steady state is marked IM (for "incomplete markets"). The crucial feature of this picture is that $\bar{\alpha}(r;\gamma,b,g)$ tends to infinity as r approaches λ $(\equiv (1+g)^{\nu}/\beta - 1)$ from below. The intuition is as follows. When r equals λ the consumer would like to maintain a smooth marginal utility of consumption profile. This can be seen by looking at the following version of the consumer's Euler equation:

$$\tilde{c}_t^{-\nu} \geq [(1+r)/(1+\lambda)] E_t \tilde{c}_{t+1}^{-\nu}, \quad (= \text{ if } \tilde{a}_{t+1} > 0).$$

However, since there is some probability of receiving a long sequence of low labor productivity shocks, the only way for the consumer to maintain a smooth marginal utility of consumption profile would be to have an infinitely large amount of assets. Note also that under incomplete markets, due to the precautionary motive and to borrowing constraints, the consumer will hold assets over and above the credit limit in order to buffer earnings shocks, even when r is less than λ. This would not be the case under complete markets.

Under complete markets the consumer can fully insure against the idiosyncratic labor productivity shock and effectively eliminate any uncertainty in individual earnings. The asset demand function in this case is described by the dotted line in Fig. 1, which leads to the steady state marked CM (for "complete markets"). This is the usual result; the capital stock satisfies the modified golden rule, i.e., $F_1(K_t, z_t) - \delta = r = \lambda$.

It follows that the interest rate with incomplete markets is lower – and, hence, the capital stock is higher – than with complete markets. Further, increases in government debt are neutral with complete markets but not with incomplete markets. In general, we may expect an increase in the debt/GDP ratio b to raise the interest rate and thereby crowd out some capital. To see this last point more clearly, and to appreciate the role of government debt in loosening borrowing constraints, it is useful to rewrite the consumer's budget constraint by substituting for taxes τ from (5) into (4), and by defining $a_t^* = \tilde{a}_t - b$. This leads to the following form of the consumer's budget constraint:

$$\tilde{c}_t + (1+g)a_{t+1}^* \leq (1+r)a_t^* + \omega(r)e_t - \gamma, \quad a_t^* \geq -b.$$

The steady-state equilibrium condition in the asset market is now given by $\bar{\alpha}^*(r; \gamma, b, g) = \kappa(r)$, where $\bar{\alpha}^*$ is defined in a way analogous to $\bar{\alpha}(\cdot)$, and where $\bar{\alpha}^*(\cdot) \equiv \bar{\alpha}(\cdot) - b$. In this formulation government debt only enters the consumer's borrowing constraint. Higher levels of b in effect loosen the consumer's borrowing constraint and reduce the consumer's average asset holdings (net of government debt). In other words, when the consumer is allowed to borrow more, he can smooth consumption by relying on loans and need not hold a large quantity of assets to buffer earnings shocks. Thus $\bar{\alpha}^*(\cdot)$ is decreasing in b, which explains why the interest rate rises and why the capital stock falls with an increase in b.

Thus, in this framework, the interest rate is not pinned down solely by the growth rate and by preference parameters. It responds to policy variables such as debt, taxes, and spending. Moreover, it is possible to have steady states in which the interest rate is less than the growth rate, implying dynamic inefficiency – a result that is familiar from the overlapping generations model. If taxes were proportional, rather than lump-sum, then both the before-tax interest rate and also the after-tax interest rate would vary with policy variables.

What about the effect of increases in the quantity of debt on welfare? The welfare criterion we use is

$$\Omega = \int \int V(a, e)\, dH(a, e), \tag{10}$$

where $V(a, e)$ is the optimal value function and where H is the steady-state joint distribution of assets and productivity. This criterion for optimality may be deemed reasonable for several reasons. First, it can be thought of as a utilitarian social welfare function. Second, it can be thought of as steady-state ex-ante welfare, i.e., welfare of a typical consumer before he realizes his initial assets and the productivity shock which are assumed to be drawn from the steady-state joint distribution H. The third reason is as follows. For the sake of exposition only, let us interpret an infinitely-lived household as a sequence of altruistically-linked, one-period-lived generations. Let U_t be the utility of a generation that depends on its own consumption as well as the utility of the next generation; specifically, $U_t = c_t^{1-\nu}/(1-\nu) + \beta E_t U_{t+1}$. It follows that, under the optimal consumption and asset accumulation program, $U_t = V(a_t, e_t)$. Let the welfare criterion be $\Omega' = \text{plim}_{t\to\infty}(1/T)\sum_{t=0}^{T-1} U_t$, where "plim" stands for probability limit. This welfare criterion weights the utilities of all generations equally. It can now be seen that $\Omega' = \Omega$. Intuitively, this is because, in any allocation that converges to a steady state, infinitely many generations of households are within an arbitrarily small neighborhood of the steady state, whereas only finitely many generations of households are outside such a neighborhood of the steady state. This argument can easily be extended to altruistically-linked generations of households who live for an arbitrary number of periods instead of just one period. A fourth justification for this welfare criterion is that it is computationally tractable for the class of models being used. Methods for computing equilibrium paths from arbitrary initial conditions are still being developed (see Krusell and Smith 1994), due to the difficulties involved in tracking the wealth distribution as an endogenous state variable. Fifth, this welfare criterion is commonly used in models of this type (e.g., Imrohoroglu 1989, 1992; Hansen and Imrohoroglu 1992; Alvarez et al. 1992; and Mehrling 1993).

There are several effects present, making it difficult to analytically determine the effect of an increase in the quantity of debt on welfare. First, an increase in debt increases the return on assets, thus making them less costly for the consumer to hold. The opportunity cost of assets to the consumer is λ, and the closer the interest rate is to λ, the less costly it is to hold assets and the more effective assets are in enabling the consumer to smooth consumption.[5]

The second effect on welfare of raising the level of government debt arises from the lump-sum taxes levied to pay interest on government debt. This effect can be understood in two ways. First, by using the utilitarian interpretation of the welfare criterion we may note that lump-sum taxes have distributional effects. They are relatively more onerous for individuals with low assets and low earnings than for individuals with high assets and high earnings.[6] Since the marginal value of assets is decreasing in the amount of assets held, this tends to lower welfare. Second, by using the ex-ante interpretation of the welfare criterion we may note that, for a particular household, due to the uncertainty in earnings, lump-sum taxes exacerbate the percentage variability in after-tax earnings; they thereby lead to greater consumption variability. Hence welfare is lowered.

The third effect on welfare of raising the level of government debt is the crowding out of capital and the consequent reduction in per capita consumption that is familiar

[5] Note that there is a limit to how high an interest rate can be supported as an equilibrium by increases in b. This limit is strictly below λ. See Aiyagari (1994a) for a detailed explanation.

[6] Kehoe, Levine, and Woodford (1990) and Mehrling (1993) emphasize this point. With distorting taxes there are also incentive and insurance effects in addition to the distributional effect. The insurance effect is a positive role for distorting taxes in these types of environments, as is emphasized by Eaton and Rosen (1980) and by Varian (1980).

from the overlapping generations framework. The intuition for why this effect arises in the present framework is as follows. First, note that the borrowing constraint in our framework effectively shortens a household's horizon and makes an infinitely-lived household's behavior similar to that of a sequence of finitely-lived households. Each time the borrowing constraint binds, the household is in the same situation as a newly born household in an overlapping generations model. Formally, the stochastic process for a household's assets is a renewal process, with zero assets being the renewal point. The average number of periods between successive visits to zero assets may be regarded as a measure of the average life span. If the dates at which the borrowing constraint binds were exogenous instead of endogenous, then the household's optimization problem could be broken up into a sequence of optimization problems by truncating the preferences and the budget constraint at each such date. Second, in the overlapping generations model note that when the debt level is higher, older households experience an increase in consumption, whereas younger households experience a reduction in consumption. This leads younger households to cut their saving (net of government debt), i.e., their saving rises by less than the increase in government debt. Therefore, capital is crowded out and the interest rate rises. (See Woodford 1990.) In our framework, a household with a high probability of being constrained in its borrowing is like an older household in an overlapping generations model; a household with a low probability of being constrained in its borrowing is like a younger household in an overlapping generations model. This similarity between the present framework and the overlapping generations framework explains the crowding out effect. The crowding out of capital reduces consumption and welfare.

The net effect of these considerations on welfare is unclear.[7] It seems that determining

[7] Note that in a pure exchange setting only the first two effects are present, and there is no effect on per capita consumption arising from changes in the capital stock. In the pure exchange case, Bewley (1980) suggested that setting $r = \lambda$ would achieve a welfare maximum. Later (Bewley 1983) he realized that, in general, this policy is not even feasible. He showed that the supremum of interest rates that can be supported as equilibria is strictly less than λ. In general, because of the distribution effect of taxes, it need not be optimal to choose the level of government debt to achieve the supremum of feasible interest rates. See Kehoe, Levine, and Woodford (1990) and Mehrling (1993).

the welfare effects of government debt and finding its optimum quantity analytically are rather difficult, even in the simpler case of lump-sum taxes and inelastic labor supply. Therefore, we use computational methods to address this question. As our benchmark we use a model with a proportional income tax and an elastic labor supply. This model is described in the next section. In the section after that we describe how this model is parameterized. Appendix A contains some details on how the steady state is computed.

3. The Benchmark Model

In the previous section we described the workings of a model with lump-sum taxes and inelastic labor. While these features are useful for building intuition, they are not adequately realistic characterizations of most actual economies for the purpose at hand. Actual tax systems distort labor supply and saving decisions, and they also provide some insurance; these features are likely to be important for determining the optimum quantity of debt. In this section we describe our benchmark model, which has a proportional income tax and an elastic labor supply. This model is, we hope, a more adequate characterization of actual economies.

Let τ_y be a proportional income tax levied on the sum of labor income and interest income. The after-tax wage and the interest rate are denoted by $\bar{w}_t$ and $\bar{r}$, respectively, where

$$\bar{w}_t = (1-\tau_y)z_t F_2(K_t, z_t N) = (1-\tau_y)w_t, \tag{11}$$

$$\bar{r} = (1-\tau_y)(F_1(K_t, z_t N) - \delta) = (1-\tau_y)r, \tag{12}$$

with N denoting per capita effective labor input. Note that we have dropped the time subscript on the interest rate and on the per capita labor input, since both of these variables will be constant in a balanced growth equilibrium.

Let the total time endowment for the household be normalized to unity, and let ℓ_t denote leisure time. The household's period utility is now specified as $(c_t^{\eta}\ell_t^{1-\eta})^{1-\mu}/(1-\mu)$.

The household's budget constraint and nonnegativity constraints now take the form

$$c_t + a_{t+1} \leq \bar{w}_t e_t(1-\ell_t) + TR_t + (1+\bar{r})a_t, \quad c_t \geq 0,\, 0 \leq \ell_t \leq 1,\, a_t \geq 0,$$

where TR_t are lump-sum transfer payments.

The government budget constraint takes the form

$$G_t + TR_t + rB_t = B_{t+1} - B_t + \tau_y(w_t N + rA_t). \tag{13}$$

The asset market equilibrium condition is the same as before, namely, (3). In addition, we require that the labor market clear, i.e.,

$$N = Ee_t(1-\ell_t),$$

where the expectation is taken with respect to the steady-state distribution.

As before, it is convenient to transform the model into a stationary form. This will also help in explaining the parameterization that we choose later. By following the same procedure as in the previous section, we can rewrite the consumer's problem as

$$\max_{\{\tilde{c}_t, \ell_t, \tilde{a}_{t+1}\}} E\left[Y_0^{\eta(1-\mu)} \sum_{t=0}^{\infty} [\beta(1+g)^{\eta(1-\mu)}]^t (\tilde{c}_t^{\eta}\ell_t^{1-\eta})^{1-\mu}/(1-\mu) \middle| \tilde{a}_0,\, e_0\right],$$

$$\text{subject to} \quad \tilde{c}_t + (1+g)\tilde{a}_{t+1} \leq (1+\bar{r})\tilde{a}_t + \bar{w}e_t(1-\ell_t) + \chi, \quad t \geq 0,$$

$$\tilde{c}_t \geq 0, \quad 0 \leq \ell_t \leq 1, \quad \tilde{a}_t \geq 0, \quad t \geq 0,$$

where $\bar{w}$ is now defined as $(1-\tau_y)w_t/Y_t$ and where TR_t/Y_t equals χ, a constant. For some parameterizations of the benchmark model, we require more restrictive borrowing constraints, namely, $\tilde{a}_t \geq \max(0, \underline{a})$ with $\underline{a} = (-\bar{w}e_{\min} - \chi)/(\bar{r} - g)$, in order to ensure that consumption is nonnegative in all states of the world. For all of the parameterizations that we consider in Sections 5 and 6, $\underline{a} < 0$.

By substituting for per capita assets from the asset market clearing condition into the government budget constraint (13), and by using the definitions of w_t and r together with the first-degree homogeneity of the production function, we can rewrite (13) as

$$G_t + TR_t + \bar{r}B_t = B_{t+1} - B_t + \tau_y(F(K_t, z_t N) - \delta K_t). \tag{14}$$

We now divide through (14) by Y_t so that it becomes

$$\gamma + \chi + (\bar{r} - g)b = \tau_y(1 - \delta k), \tag{15}$$

where $k = K_t/Y_t$.

The asset market equilibrium condition (3) can be rewritten in the form of (6) by dividing through Y_t. Note that by using equation (12) we can express $K_t/(z_t N)$ as a function of r, and, hence, we can express k $(= K_t/Y_t)$ as a function of r that is the function $\kappa(r)$ appearing in (7). Further, by using (8) we can express $w_t N/Y_t$ as a function of r. Thus we have

$$\bar{w} = (1 - \tau_y)\omega(r)/N. \tag{16}$$

The steady state of this economy is characterized by an interest rate r^* and per capita effective labor input N^*, which solve

$$\bar{\alpha}(r, N; \gamma, b, g, \chi) = \kappa(r) + b,$$

$$\bar{\varphi}(r, N; \gamma, b, g, \chi) = N.$$

Here $\bar{\alpha}(r, N; \cdot)$ represents the per capita assets desired by consumers (relative to per capita output) as a function of the interest rate and the per capita effective labor input; $\kappa(r) + b$ is the per capita supply of assets (capital plus government debt) relative to per capita output, expressed as a function of the interest rate; and $\bar{\varphi}(r, N; \gamma, b, g, \chi)$ is the per capita effective labor supplied by households.

Note that for given values of $(r, N, \gamma, b, g, \chi)$ the government budget constraint can be used to find the income tax rate τ_y. Thus $\bar{w}$ and $\bar{r}$ are known. The household's problem can now be solved to obtain decision rules for assets and leisure. The decision rule for assets can be used to determine the stationary distribution of assets and, thereby, per capita assets desired by households and effective labor supplied by households. The values of r and N are varied until equilibrium obtains in the asset and labor markets. In Appendix A we describe the algorithm used to compute the decision rule for assets and the stationary distribution of assets.

The welfare standard we use is the one given by (10). This is converted as follows into a welfare measure in units of consumption. The welfare measure is the percent increase in benchmark consumption at every date and state (with leisure at every date and state held at benchmark values) that leads to the same value of the welfare standard as its value at any alternative debt/GDP ratio.

We now describe how the above model is parameterized.

4. Parameterization of the Benchmark Model

The model period is specified to be one year. The production function is specified as Cobb-Douglas, with θ denoting the capital share parameter. The stochastic process for e_t is specified as follows. The natural logarithm of e_t is assumed to be a first-order, autoregressive process with a serial correlation coefficient ρ and a standard deviation σ. After specifying ρ and σ, we use the procedure of Tauchen (1986) to approximate the autoregression of $\log(e_t)$ with a first-order Markov chain that has seven states. Thus the parameters for the Markov process are the values for the seven states and a probability transition matrix that we denote by π. Table 1 in Aiyagari (1994a) shows that this approximation is very good. In addition to the technology parameters, we must choose parameter values for preferences (β, η, μ), the depreciation rate (δ), the growth rate (g), transfers (χ), and government consumption (γ).

All of the parameter values that we use are based on post-second world war data for the United States. The growth rate g is set equal to 1.85 percent per year, which is the annual growth rate of per capita gross domestic product. The ratio of government purchases to GDP over the postwar sample is, on average, 21.7 percent. Thus we set $\gamma = 0.217$. The ratio of government transfers to GDP χ is set equal to its postwar average of 8.2 percent. We set b equal to $2/3$ for our benchmark economy. This figure is an average of the U.S. federal plus U.S. state debt divided by gross domestic product over the postwar period.

We set labor's share of income equal to 70 percent. Assuming that proprietors' income plus the indirect business tax is divided between capital and labor according to their shares in income, the estimate of 70 percent is found by dividing compensation of employees by gross domestic product, less proprietors' income and less the indirect business tax. Thus we set θ equal to 0.3.

Values of the parameters ρ, σ, μ, and β are chosen so that they are in line with other estimates in the literature, and so that the equilibrium interest rate r is roughly consistent with the data. In equilibrium, r is equal to $\theta/k - \delta$. If we assume that the capital stock is composed of both fixed private capital and the stock of durables owned by consumers, then the capital/output ratio k is approximately 2.5. Because we want an estimate of the return on private capital, we subtract compensation to government employees from total product. We also exclude imputed net interest and business transfers from total product. Our estimate of the depreciation rate δ is 0.075. This figure is consistent with the postwar average depreciation of fixed private capital plus consumer durables. These values imply an interest rate equal to 4.5 percent. Note, however, that the equilibrium capital/output ratio and, hence, the equilibrium interest rate that we compute for the model will depend on all of our parameter choices.

In parameterizing the benchmark economy, we set ρ equal to 0.6 and σ equal to 0.3. These parameter values for the earnings process are in the range of estimates in the literature. (See Aiyagari (1994a) for a discussion of these estimates.) For the risk-aversion parameter μ there is a wide range of estimates in the literature. We use a value of 1.5, which implies slightly more curvature in utility than the logarithmic function. This choice is somewhat arbitrary, but in Section 6 we discuss how it affects the results. Finally, we set the discount factor β equal to 0.991 (which implies a discount factor of 0.988 in the transformed economy). This value was chosen to achieve similar interest rates in the model and in the data. Smaller values of β lead to higher rates of interest.

Given values for g, γ, χ, b, θ, k, δ, and an estimate of the elasticity of labor supply (holding constant the marginal utility of consumption), we can back out an estimate of η. The labor elasticity is given by $(1-\eta(1-\mu))(1-N)/(\mu N)$. Note that $N = 1 - E(e_t \ell_t)$. If we assume an interior solution, then the household's first-order condition for the labor-leisure choice yields $\ell_t = (1-\eta)\tilde{c}_t/(\eta \bar{w} e_t)$. Therefore, $E(e_t \ell_t) = (1-\eta)E(\tilde{c}_t)/(\eta \bar{w})$, with $\bar{w}$ given by (16). The resource constraint for the economy implies the following expression for per capita consumption: $E(\tilde{c}_t) = 1 - \gamma - (g+\delta)\kappa(r)$. By using the Cobb-Douglas form of the production function we can see that $\omega(r) = 1-\theta$ and $\kappa(r) = \theta/(r+\delta)$. This leads to the following expression for N: $N = 1/(1+AB)$, where $A = (1-\eta)/[\eta(1-\tau_y)(1-\theta)]$ and $B = 1-\gamma-\theta(g+\delta)/(r+\delta)$. If we use a labor elasticity of 2 percent, then the value of η is 0.328. We also consider a lower value of the elasticity when we check the robustness of our results.

To summarize, we use the following parameterization for our benchmark economy:

$$\begin{aligned} &g = 0.0185, \quad \chi = 0.082, \quad \delta = 0.075, \quad \rho = 0.6, \quad \mu = 1.5, \\ &\gamma = 0.217, \quad \theta = 0.3, \quad \eta = 0.328, \quad \sigma = 0.3, \quad \beta = 0.991. \end{aligned} \tag{17}$$

In the next section we describe our findings for the parameter values in (17). In the section after that, we discuss the robustness of our findings to changes in the values for ρ, σ, μ, β, and the elasticity of labor supply. These parameters could potentially have a large effect on our results, and there is not sufficient consensus on their values.

5. Results

Our main findings are reported in Fig. 2. Fig. 2 shows the graphs of the welfare gain, the before- and after-tax interest rates, the income tax rate, and total hours versus the debt/GDP ratio for our benchmark parameters. The plot of welfare shows that, for the benchmark model, the optimum quantity of debt is about equal to the average debt/GDP ratio for the United States over the post-second world war period.[8] That is, the optimum quantity of debt for our baseline parameters is 2/3.[9] Thus the positive role that debt plays in our model – that of enhancing liquidity – exactly balances the negative role of crowding out private capital and distorting labor supply and saving decisions through higher taxes.

Because we use the U.S. level of debt as our implicit benchmark, the welfare gain to being at the optimum is zero. Note, however, that the welfare profile displayed in Fig. 2 is very flat. For example, the loss to being at a debt/GDP ratio of zero rather than 2/3 is only 0.08 percent of consumption. The before-tax interest rate at the optimum is approximately 4.5 percent, which is the rate that we calculated for the data. The after-tax interest rate is 2.8 percent, with the income tax rate equal to 37.6 percent. This value of the income tax rate is roughly consistent with Lucas' (1990) figures for labor and capital income tax rates. He takes the labor income tax rate to be 40 percent and the capital income tax rate to be 36 percent. We have a high figure for the income tax rate in our model because we lump all consumption taxes into the labor income tax; these taxes affect the same consumption/leisure margin. This is also the case in Lucas (1990). Notice that the tax rate in Fig. 2 does not rise monotonically with debt; this is because changes in the level of debt induce changes in the before- and after-tax interest rates. An increase in debt raises the after-tax interest rate, which then lowers the capital/output ratio and, hence, raises

---

[8] We compute statistics for the cases $b = 2/3$ and $b = i/10$, where the i's are integers.

[9] If all transfers are treated as proportional to income, so that χ is set to zero, then the results change as follows. The optimal debt/GDP ratio is 1.3, and the welfare gain to being at the optimum is 0.072 percent of consumption. The optimal interest rate is about 4.7 percent, and the optimal income tax rate is about 29 percent.

the ratio of net income to gross income. As a result, there is an increase in the tax base, which has a negative effect on the tax rate. This is the reason why the tax rate changes nonmonotonically with debt. Thus, in our framework, tax policies may be consistent with more than one level of debt or may be infeasible. In the standard, representative-agent growth model, the after-tax interest rate is independent of the level of debt, and the tax rate increases monotonically with debt. This is why, in the representative-agent model, one can interpret the exercise of changing the debt as being equivalent to the exercise of changing the income tax rate. In the last frame of Fig. 2 we plot the aggregate hours of work. At the optimum, the fraction of time devoted to work is 28 percent.

We can assess the consequences of the incentive and insurance effects of the income tax by comparing our benchmark model to one with a lump-sum tax. With a lump-sum tax the only opposing forces on the optimum quantity of debt are its liquidity-enhancing effect and its capital crowding-out effect. In particular, let us assume that the parameters are those given in (17), with β set equal to 0.971 to ensure that the equilibrium interest rate is 4.5 percent, and with χ set equal to $-\gamma - (r - g)b$. In this case the optimum quantity of debt is approximately 1.4 times GDP. Although this is considerably higher than the current U.S. level of debt, the welfare gain to being at the optimum rather than the U.S. level is only 0.22 percent of consumption. The before-tax interest rate under lump-sum taxation is 4.6 percent, which is only slightly higher than our benchmark rate of 4.5 percent. Under lump-sum taxation the fraction of time devoted to work is somewhat higher than under an income tax: 38 percent versus 28 percent.

6. Robustness of the Results

In this section we focus on the robustness of our main results to changes in the following parameters: the serial correlation coefficient of the earnings shock ρ, the standard deviation of the earnings shock σ, the risk-aversion coefficient μ, the elasticity of labor supply (which we alter by changing η), and the discount factor β. In our model, these parameters govern households' precautionary saving. By either exposing the household to greater earnings risk or by making the household more sensitive to risk, these parameters influence the average amount of assets that households desire to hold in order to buffer earnings shocks. Thus these parameters affect the optimum quantity of debt and the welfare gain.

Consider the demand function for assets $\bar{\alpha}$ in Fig. 1. For a given interest rate, decreases in the variability of earnings (σ) or the persistence of earnings (ρ) shift the demand function to the left. Thus, as ρ or σ are decreased, the fact that debt can be used to smooth consumption becomes less important. As a result, the optimum quantity of debt is lower when ρ or σ is lower.

For μ and η, the effects are less clear. An increase in μ implies that individuals are more risk-averse and desire a smoother consumption profile. This has a positive effect on saving. On the other hand, an increase in μ implies a decrease in the effective discount factor. This has a negative effect on saving. Thus the effect of a change in μ on the optimum quantity of debt is ambiguous. The effect of changing η on the optimum quantity of debt is also ambiguous. If μ exceeds 1, an increase in η implies a decrease in the effective discount factor and has a negative effect on saving. The lower is saving, the less important is debt for smoothing consumption. On the other hand, for the parameter values that we consider, an increase in η implies a less elastic labor supply, so that the distortionary taxes needed to finance interest payments on the debt are less costly.

We now consider the quantitative effects of changes in ρ, σ, μ, and η. We simultaneously adjust β to ensure that the equilibrium interest rate is equal to 4.5 percent when $b = 2/3$. The target of a 4.5 percent interest rate is roughly consistent with U.S. data. To determine how robust our results are to different assumptions about capital and its return, we also adjust β with all other parameters set equal to the values in (17). Because the optimum for the benchmark model is equal to the U.S. debt/GDP ratio, we only report quantitative results for perturbations yielding an optimal level lower than that of the benchmark model. We find similar results (in the opposite direction) when we consider perturbations yielding an optimal level higher than that of the benchmark model.

Our first experiment involves changing ρ. Consider the parameterization of equation (17) with ρ equal to 0.5 rather than 0.6 and with β equal to 0.9916 rather than 0.991. With ρ equal to 0.5 there is less persistence in the earnings process and, hence, the optimal debt/GDP ratio should be less than 2/3. In this case, we find that the optimal debt/GDP ratio is 0.5, the optimal interest rate is 4.4 percent, the optimal tax rate is 37.4 percent, and the gain to being at the optimum is only 0.01 percent of consumption.

Consider the parameterization of equation (17), with σ equal to 0.25 rather than 0.3 and with β equal to 0.9921 rather than 0.991. With σ equal to 0.25, the variability of earnings is lower than in our benchmark economy. Thus we would expect a lower debt/GDP ratio than 2/3. For this parameterization we find that the optimal debt/GDP ratio is approximately equal to 0.2, the optimal interest rate is 4.3 percent, and the optimal tax rate is 37.1 percent. The welfare gain, in this case, is small – only 0.034 percent of consumption.

Next, consider changing the value of μ. If we set μ equal to 2 and β equal to 0.9942, then the optimum quantity of debt is 0.4 times GDP and the welfare gain is again small – only 0.01 percent of consumption. The optimal interest and tax rates are similar to the benchmark optimum; they are 4.4 percent and 37.3 percent, respectively. If we set μ equal to 3 and β equal to 1, then the optimum quantity of debt is 0.2 times GDP, while the welfare gain is 0.04. The optimal interest and tax rates are in this case 4.3 percent and 37.1 percent, respectively. For both changes in μ, we find that the welfare gain is small and the differences in the tax rates and interest rates for the two experiments are small.

For the benchmark parameterization given in (17), with $\eta = 0.328$, the elasticity of labor supply is equal to 2. Many estimates of the labor supply elasticity fall below this value. If we set η equal to 0.512, then the implied elasticity of labor is 1. To ensure that the equilibrium interest rate is 4.5 percent when $b = 2/3$, we also set $\beta = 0.994$. In this case, the optimum quantity of debt is 0.2 times GDP, but the gain to being at the optimum is only 0.029 percent of consumption. The interest rate, tax rate, and total hours of work at the optimum are 4.3 percent, 37.1 percent, and 45.8 percent, respectively.

In the above experiments we adjusted the discount factor along with ρ, σ, μ, or η in order to maintain an equilibrium before-tax interest rate of 4.5 percent when $b = 2/3$. To see how our results change when the benchmark equilibrium interest rate changes, we adjusted only the discount factor. In particular, we set the discount factor equal to 0.9806 in order to achieve a before-tax interest rate of 6 percent. The higher interest rate may be viewed as corresponding to a lower benchmark capital/output ratio. In this case the optimal debt/GDP ratio is -0.5, which is significantly lower than our benchmark optimum of 2/3. In terms of welfare, however, the change in the discount factor does not have a large effect. The gain is only 0.48 percent of consumption. In this case, the interest rate and tax rate at the optimum are 5.2 percent and 35.4 percent, respectively.

The above experiments are illustrative for several reasons. First, changes in our key parameters that lead to lower optimal debt/GDP ratios do not yield large changes in welfare. Second, changes in our key parameters that lead to lower optimal debt/GDP ratios result in higher benchmark interest rates than the 4.5 percent that we calculated for the data. Third, obtaining negative optimal debt/GDP ratios requires benchmark interest rates that are significantly higher than 4.5 percent. Fourth, there are reasonable parameterizations that imply that the current U.S. debt/GDP ratio may actually be below the optimum.

7. Conclusions

In this paper we have calculated the optimum quantity of debt for a model that is parameterized to mimic certain features of the U.S. economy. In our model the optimum quantity of debt will be high if debt is effective in smoothing out consumption over the lifetime of an individual. The optimum quantity of debt will be low if debt crowds out capital and, therefore, lowers consumption. Or, it will be low if the incentive effects of higher distortionary taxes are important. Although our estimate of the optimum quantity of debt is equal to the average level in the post-war U.S. economy, the welfare function is very flat; the welfare costs to being at levels other than the U.S. level are small for a wide range of debt/GDP ratios. For some perturbations of our benchmark parameters, we find a significant change in our estimate of the optimum quantity of debt. But, in all cases that we investigate, the welfare effects are small.

Appendix A

In this appendix we provide some details on computing the equilibrium for the benchmark economy.[10] Here we focus on the two main tasks: computing the optimal asset holdings α and computing the distribution of assets H. Both of these tasks are accomplished by applying a finite element method.

In the case of the decision rules, we first derive the first-order conditions for the following optimization problem:

$$\max_{\{\tilde{c}_t,\ell_t,\tilde{a}_{t+1}\}} E\left[\sum_{t=0}^{\infty}\tilde{\beta}^t\left\{\frac{(\tilde{c}_t^{\eta}\ell_t^{1-\eta})^{1-\mu}}{1-\mu}+\frac{1}{3}\zeta(\min(\tilde{a}_t,0)^3+\min(1-\ell_t,0)^3)\right\}\middle|\tilde{a}_0,e_0\right],$$

$$\text{subject to}\quad \tilde{c}_t+(1+g)\tilde{a}_{t+1}\le(1+\bar{r})\tilde{a}_t+\bar{w}e_t(1-\ell_t)+\chi,$$

where $\tilde{\beta}=\beta(1+g)^{\eta(1-\mu)}$. Notice that we have changed the optimization problem described in Section 3 by using penalty functions for two sets of inequality constraints ($\tilde{a}_t\ge 0$ and $\ell_t\le 1$). The remaining inequality constraints ($\tilde{c}_t\ge 0$ and $\ell_t\ge 0$) do not bind for any of our parameter choices, so we ignore them when computing the optimal decisions. The first-order conditions imply that the Euler residual must be equal to 0 for all $x\in[0,x_{max}]$ and all i. The residual is given by

$$\begin{aligned}R(x,i;\alpha) &= \eta(1+g)c(\ell^*(x,i;\alpha))^{\eta(1-\mu)-1}\ell^*(x,i;\alpha)^{(1-\eta)(1-\mu)}\\&\qquad -\tilde{\beta}\{\sum_j\pi_{i,j}\,\eta(1+\bar{r})c(\ell^*(\alpha(x,i),j;\alpha))^{\eta(1-\mu)-1}\\&\qquad \cdot\,\ell^*(\alpha(x,i),j;\alpha)^{(1-\eta)(1-\mu)}+\zeta\min(\alpha(x,i),0)^2\},\end{aligned}$$

where $\pi_{i,j}$ is the probability of transiting from productivity level i to productivity level j, $\ell^*(x,i;\alpha)$ is the solution to the nonlinear equation $f(\ell)=0$, and x_{max} is such that no $x>x_{max}$ would be chosen by the consumer. The functions $c(\cdot)$ and $f(\cdot)$ are defined as

[10] For more details on the computational methods used here, see McGrattan (1996) and our technical appendix, which is available from the authors upon request.

follows:

$$
\begin{aligned}
c(\ell) &= (1+\bar{r})x + \bar{w}e(i)(1-\ell) + \chi - (1+g)\alpha(x,i), \\
f(\ell) &= (1-\eta)c(\ell)^{\eta(1-\mu)}\ell^{(1-\eta)(1-\mu)-1} - \zeta \min(1-\ell, 0)^2 \\
&\quad - \bar{w}e(i)\eta c(\ell)^{\eta(1-\mu)-1}\ell^{(1-\eta)(1-\mu)},
\end{aligned}
$$

where $e(i)$ is the earnings shock in state i.

The computational task, therefore, is to find an approximation for $\alpha(x,i)$ – say $\alpha^h(x,i)$ – which implies that $R(x,i;\alpha^h)$ is approximately equal to zero for all x and i. We do this by applying a finite element method. In particular, we do the following. First, we choose some discretization of the domain of our functions. Since only x is continuous, we need to specify some partition of [0,x_{max}]. We refer to each subinterval of x as an element. On each element we choose a set of basis functions for approximating α. Local interpolations are assembled to construct a globally-defined piecewise approximation. In our case we choose linear basis functions for all elements, e.g.,

$$\alpha^h(x,i) = \psi_j^i N_j(x) + \psi_{j+1}^i N_{j+1}(x)$$

on the element $[x_j, x_{j+1}]$, where

$$N_j(x) = \frac{x_{j+1} - x}{x_{j+1} - x_j}, \quad N_{j+1}(x) = \frac{x - x_j}{x_{j+1} - x_j}.$$

This choice is motivated by related test problems described in the technical appendix, which is available from the authors upon request. Notice that $\alpha^h(x_j, i) = \psi_j^i$ and $\alpha^h(x_{j+1}, i) = \psi_{j+1}^i$. If we consider the approximation globally, we need to compute the asset holdings for all points x_j and all productivity levels i (i.e., ψ_j^i for all i and j). We choose these values for ψ_j^i by setting the weighted residual equal to zero, i.e.,

$$\int R(x,i;\alpha^h) N_j(x) dx = 0, \tag{18}$$

for all i and j. In effect, we solve a problem that has the following form: find $\vec{\psi}$ such that $h(\vec{\psi}) = 0$, where $\vec{\psi}$ is the vector of coefficients that we are searching over and $h(\cdot)$ is the system of equations in (18).

Asset holdings in the next period are a function of asset holdings in this period and of the earnings shock. Therefore, one can show that the invariant cumulative distribution function must satisfy

$$H(x,i) = \sum_j \pi_{j,i} H(\alpha^{-1}(x,j),j) I(x \geq \alpha(0,j)), \tag{19}$$

where I is an indicator function (e.g., $I(x > y)$ is equal to one if $x > y$ and zero otherwise). To compute H, we again apply the finite element method with linear basis functions. In this case the residual is the difference between the right and left-hand sides of (19). For this problem we do not have to worry about inequality constraints directly. But we do have to deal with them indirectly. If inequality constraints bind in the consumer's problem, then the decision functions for low productivity levels will be set equal to zero for some interval $[0, x^*]$. This implies a mass point at $x = 0$. It also implies that mass points will exist throughout the distribution; these occur at the states traversed prior to reaching the zero-asset position. The mass points in the distribution imply that the solution to (19) has discontinuities, possibly at a countably infinite number of points. In the technical appendix we show that the finite element method generates a good approximation, even when such discontinuities occur.

References

Aiyagari, S.R., 1994a, Uninsured idiosyncratic risk and aggregate saving, The Quarterly Journal of Economics 109, 659-84.

Aiyagari, S.R., 1994b, Macroeconomics with frictions, Federal Reserve Bank of Minneapolis Quarterly Review, Summer, 24-32.

Alvarez, F., J. Diaz-Giménez, T. Fitzgerald, and E.C. Prescott, 1992, Banking in computable general equilibrium economies, Journal of Economic Dynamics and Control 16, 533-59.

Barro, R.J., 1979, On the determination of the public debt, Journal of Political Economy 87, 940-71.

Bewley, T., no date, Interest bearing money and the equilibrium stock of capital, mimeo.

Bewley, T., 1980, The optimum quantity of money, in: J.H. Kareken and N. Wallace, eds., Models of monetary economies (The Federal Reserve Bank of Minneapolis, Minneapolis) 169-210.

Bewley, T., 1983, A difficulty with the optimum quantity of money, Econometrica 51, 1485-504.

Chamley, C., 1985, Efficient taxation in a stylized model of intertemporal general equilibrium, International Economic Review 26, 451-68.

Chamley, C., 1986, Optimal taxation of capital income in general equilibrium with infinite lives, Econometrica 54, 607-22.

Cukierman, A. and A.H. Meltzer, 1989, A Political theory of government debt and deficits in a neo-Ricardian framework, American Economic Review 79, 713-32.

Eaton, J. and H.S. Rosen, 1980, Labor supply, uncertainty, and efficient taxation, Journal of Public Economics 14, 365-74.

Hansen, G. D. and A. Imrohoroglu, 1992, The role of unemployment insurance in an economy with liquidity constraints and moral hazard, Journal of Political Economy 100, 118-42.

Imrohoroglu, A., 1989, Costs of business cycles with indivisibilities and liquidity constraints, Journal of Political Economy 97, 1364-83.

Imrohoroglu, A., 1992, The welfare cost of inflation under imperfect insurance, Journal of

Economic Dynamics and Control 16, 79-91.

Jappelli, T. and M. Pagano, 1994, The welfare effects of liquidity constraints, Working paper, Instituto di Studi Economici, Napoli, Italy.

Kehoe, T.J., D. Levine, and M. Woodford, 1992, The optimum quantity of money revisited, in P. Dasgupta, D. Gale, O. Hart, and E. Maskin, eds., The economic analysis of markets and games: Essays in honor of Frank Hahn (MIT Press, Cambridge MA) 501-26.

Krusell, P. and A.A. Smith, 1994, Income and wealth heterogeneity, aggregate fluctuations, and the representative agent, Working paper, University of Rochester.

Laitner, J., 1979, Bequests, golden-age capital accumulation and government debt, Economica 46, 403-14.

Laitner, J., 1992, Random earnings differences, lifetime liquidity constraints, and altruistic intergenerational transfers, Journal of Economic Theory 58, 135-70.

Lucas, R.E. Jr., 1990, Supply-side economics: An analytical review, Oxford Economic Papers 42, 293-316.

McGrattan, E.R., 1996, Solving the stochastic growth model with a finite element method, Journal of Economic Dynamics and Control 20, 19-42.

Mehrling, P., 1995, A note on the optimum quantity of money, Journal of Mathematical Economics 24, 249-258.

Tauchen, G., 1986, Finite state Markov-chain approximations to univariate and vector autoregressions, Economics Letters 20, 177-81.

Varian, H.R., 1980, Redistributive taxation as social insurance, Journal of Public Economics 14, 49-68.

Woodford, M., 1990, Public debt as private liquidity, American Economic Review Papers and Proceedings 80, 382-88.

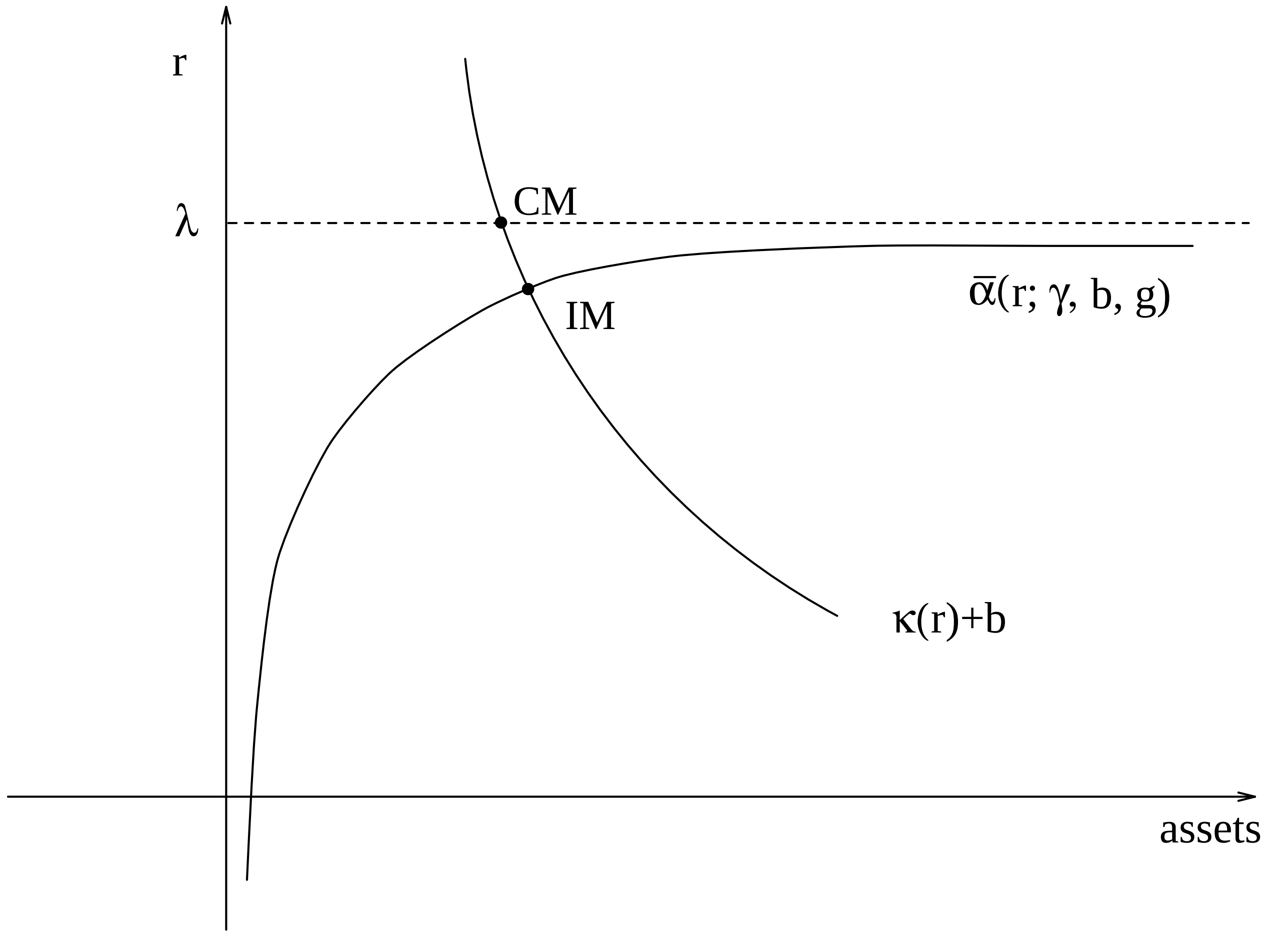

Fig. 1. Interest rate determination.

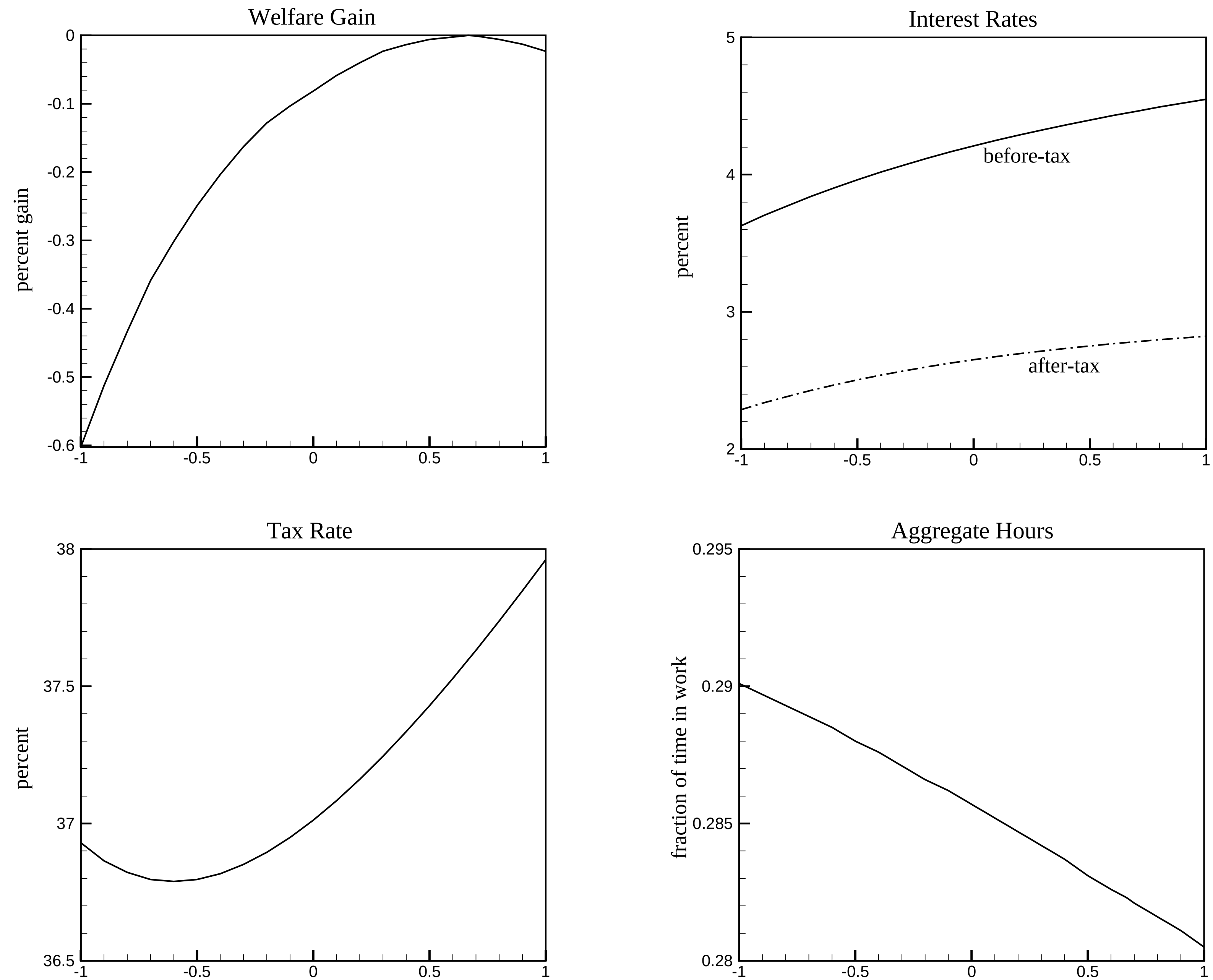

Fig. 2. Welfare gain, interest rates, tax rate, and aggregate hours versus debt/GDP ratio (x–axis) for the benchmark economy.