Samuelson's Pure Consumption Loans Model
With Constant Returns-to-Scale Storage

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My main purpose here is to prove two propositions:[1]

__Proposition I__: There exists at least one fixed-supply fiat-money

perfect foresight competitive equilibrium (PCE) if and only if there

exists at least one nonoptimal, nonfiat-money PCE.

__Proposition II__: If there exists at least one fixed-supply fiat-money

PCE, then there exists at least one that is optimal.

In Section 1 I describe the economy to be studied. In Section 2

I describe necessary and sufficient conditions for the existence of

nonmonetary and monetary equilibria.[2] In Section 3, in something of

digression, I discuss uniqueness and so-called "stability" questions.

And, finally, in Section 4 I derive the optimality results needed for

propositions I and II.

1.   The Economy to be Studied

The model is of a discrete-time, one-good economy. At any

date t, the population consists of $N(t)$ young (or age 1), the members of

generation t, and $N(t-1)$ old (or age 2), the members of generation

t-1. Each young person at t maximizes $u[c^h(t)]$; $c^h(t) = (c_1^h(t), c_2^h(t))$

where $c_j^h(t)$ is age j consumption of member h of generation t. The

arguments of u are assumed to be superior goods, and for positive

values of its arguments, u is assumed to be twice differentiable with

strictly convex upper contours. Each old person at t maximizes $c_2^h(t-1)$.

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[1] These were stated slightly differently as general conjectures
in Wallace.

[2] Unless otherwise noted, we use the word equilibrium to
denote a PCE.

Each young person is endowed at t with one unit of the consumption good. The good may be exchanged, consumed, or stored; if $k \geq 0$ units are stored, the result is xk units of t+1 consumption where $x > 0$. We assume that $N(t)/N(t-1) = n > 0$ for all t.

We will study the evolution of this economy from some arbitrary initial date, labelled t=1 for convenience. In the aggregate, the t=1 old, the members of generation 0, are endowed with $K(0) \geq 0$ units of the consumption good and with M(1) units of fiat money.

For all t, M(t), the post-transfer time t stock of money, obeys

$$M(t) = zM(t-1), \quad z > 0.$$

The time t transfer (or tax), $(z-1)M(t-1)$, is divided equally at time t among the N(t-1) members of generation t-1. The handouts are fully anticipated and are viewed as lump-sum, as not dependent on saving or portfolio behavior.

2. Equilibria

Let p(t) be the price of a unit of fiat money at time t in units of time t consumption. Then, letting $c(t) = (\ldots, c^h(t), \ldots)$, $k(t) = (\ldots, k^h(t), \ldots)$, $m(t) = (\ldots, m^h(t), \ldots)$ be the vectors of generation t's lifetime consumption, time t storage, and time t money purchases, respectively, an equilibrium is a sequence (c(t-1), k(t), m(t), p(t)); t=1,2, ... that is consistent with

    (a) $c^h(t)$, $m^h(t)$, and $k^h(t)$ being optimal for the perfect foresight competitive choice problem of the young to be described below;

(b) $c_2^h(0)$ being maximal for the (trivial) competitive choice

problem of the current old; and

(c) $M(t) = N(t)m^h(t)$,

it being taken for granted in (c) that $m^h(t) = m^{h'}(t)$ for all h and h'

in generation t.

2.1   The Choice Problem of the Young

The young choose $c^h(t) \geq 0$, $k^h(t) \geq 0$, and $m^h(t)$ to maximize

$u[c^h(t)]$ subject to

(1) $c_1^h(t) + k^h(t) + p(t)m^h(t) - 1 \leq 0$

(2) $c_2^h(t) - xk^h(t) - p(t+1)[m^h(t)+(z-1)M(t)/(N(t))] \leq 0$

for either $p(t) = p(t+1) = 0$ or $p(t) > 0$ and $p(t+1) > 0.$[3/]

It follows that the optima satisfy (1) and (2) with equality

and

(3) $u_1 - \ell_1^h \leq 0$ with = if $c_1^h(t) > 0$

(4) $u_2 - \ell_2^h \leq 0$ with = if $c_2^h(t) > 0$

(5) $-\ell_1^h + x\ell_2^h \leq 0$ with = if $k^h(t) > 0$

(6) $-\ell_1^h p(t) + \ell_2^h p(t+1) = 0$

where ℓ_j^h is the nonnegative multiplier associated with constraint j and

where, by our nonsatiety assumption about u and the boundedness of the

feasible $c^h(t)$, $\ell_j^h > 0$ in any equilibrium.

[3/] Obviously, any other nonnegative p(t) sequence could not be
an equilibrium.

2.2 Nonmonetary Equilibrium

By definition, $p(t) = 0$ for all t in such an equilibrium. This implies that (3)-(5) hold with equality. So, letting $v(c^h(t)) \equiv u_1/u_2$ be the marginal rate of substitution function, (3)-(5) give us

(7) $v[c^h(t)] = x.$

Since, by the superiority of $c_j^h(t)$, $v_1 < 0$ and $v_2 > 0$, there is a unique $k^h(t)$ for which (7) holds. (To "prove" this, use (1) and (2) to write the $c_j^h(t)$ as functions of $k^h(t)$.)

2.3 Monetary Equilibria

By definition, $p(t) > 0$ for all t in such an equilibrium. Then, by (3)-(6), we have

(8) $v[c^h(t)] = p(t+1)/p(t) \geq x$

where the inequality is implied by (5) and (6).

We want to prove the following:

(9)   if $xz/n > 1$, then there does not exist a
           monetary equilibrium;

(10)   if $xz/n \leq 1$, then there exists at least one
           purely monetary equilibrium.

<u>Proof of (9)</u>. Suppose to the contrary. Then, by the rule generating $M(t)$ and the requirement that $M(t) = N(t)m^h(t)$, we have

(11) $\dfrac{p(t+1)}{p(t)} = \dfrac{M(t+1)p(t+1)}{zM(t)p(t)} = \dfrac{N(t+1)m^h(t+1)p(t+1)}{zN(t)m^h(t)p(t)} = \dfrac{nq^h(t+1)}{zq^h(t)}$

where $q^h(t) \equiv p(t)m^h(t)$. Then, by the inequality part of (8), we have

(12) $q^h(t+1)/q^h(t) \geq xz/n > 1$.

But a monetary equilibrium is among other things a positive bounded q^h sequence. No such sequence can satisfy (12).

Proof of (10). By definition, a purely monetary equilibrium has $k^h(t) = 0$ for all h and $t \geq 1$. In such an equilibrium, by the first part of (8) and (1), (2), and (11), we have

(13) $v[1-q(t),q(t+1)n] = [q(t+1)/q(t)](n/z)$.

If there is a $q\epsilon(0,1)$ such that $q(t) = q(t+1) = q$ satisfies (13), we will have proved (10), because then the inequality part of (8) is implied by $xz/n \leq 1$. But the existence (and uniqueness) of such a q, denoted $q*$, is trivial. Let $v*(q) \equiv v(1-q,nq)$. Then $v*$ is continuous, (strictly increasing) with $0 = \lim v*(q)$ as $q \to 0$ from above and $\infty = \lim v*(q)$ as $q \to 1$ from below.

3.   Uniqueness and So-Called Stability of Monetary Equilibria

This section is a digression in the sense that nothing in it is needed for the proofs of propositions I and II.

3.1 Uniqueness of Monetary Equilibrium

In a sense, it seems beside the point to ask about the uniqueness of monetary equilibrium because we already know there is nonuniqueness of one sort: there is always a nonmonetary equilibrium. But the question here is whether there are positive q sequences other than $q(t) = q*$ for all t that satisfy (8) and are bounded above by 1.

A positive q sequence satisfies (13), part of the condition for a purely monetary equilibrium, if and only if

(14) $q(t) = H[q(t+1)]$

where H is defined on $(0,\infty)$, has a unique fixed point q^*, is such that $0 < H < 1$, and

$$H' = [nv_2 q(t) - n/z]/[v - v_1 q(t)]$$

where $H'(q^*) < 1$.

Two kinds of nonconstant q sequences could conceivably be purely monetary equilibria. First, if $0 = \lim H(x)$ as $x \to 0$ from above, then there exist positive q sequences that converge to zero that satisfy (14) and, presumably, (8) also. The second kind involves sequences that cycle. Thus, if there exists $\delta > 0$ such that $0 < q^* - \delta = H(q^*+\delta)$ and $1 > q^* + \delta = H(q^*-\delta)$ with $(q^*-\delta)/(q^*+\delta) \geq xz/n$, then a q sequence that oscillates between $q^* + \delta$ and $q^* - \delta$ is a purely monetary equilibrium.

It is obvious what it takes to rule out these possibilities. Thus,

(15)   if H is bounded away from zero and $|H'| < 1$,
         (a condition implied by the $c_j(t)$ being gross sub-
         stitutes), then $q(t) = q^*$ for all t is the unique
         purely monetary equilibrium.[4]

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[4] For a proof see Kareken and Wallace.

There is another way to rule out nonstationary paths. Each member of generation t for $t \geq 1$, must forecast the real saving behavior of members of generation t+1. (Since the evolution of the money supply is known, this is enough to find p(t+1) which is what members of generation t want to know.) But given the structure of the economy, each member of generation t could easily have the view that each of next period's young will save in real terms what the current young save. In other words, if each young person acting like a competitor responds to the calling out of an arbitrary current price $\bar{p}(t) > 0$, say, with a money demand based on p(t+1) satisfying $M(t)\bar{p}(t) = M(t+1)p(t+1)$, then such demands imply $q(t) = q^*$ for all t.

But even these conditions do not imply uniqueness of a monetary

equilibrium. Thus, it is easy to verify that if $xz/n = 1$, then for any

$\hat{q}\varepsilon(0,q^*)$, $q(t) = \hat{q}$ and $k^h(t) = q^* - \hat{q}$ for all t is a monetary equilibrium.

3.2 The "Hahn Problem"

Frank Hahn and others have posed the following "stability"

question of models like the one we have been examining: Given $q(1)\varepsilon(0,1)$

but otherwise arbitrary, does the implied q sequence satisfying (13)

converge to q^*? The answer, very generally, is no. Indeed, we state

the following answer. If H satisfies the hypotheses of (15), a class

that includes very innocent specifications like u Cobb Douglas, then the

answer is no. Indeed, to have a model that implies an affirmative

answer to Hahn's question is to have a model that implies that any

$q(1)\varepsilon(0,1)$ is the first element of a perfect foresight equilibrium path

that converges to q^*. In other words, to have such a model is to have

one that says nothing about what happens in the first period.[5]

For this and other reasons, I do not regard Hahn's question as

a reasonable way to pose a stability question.

3.3 Arbitrary Convergent Expectations[6]

It may be of interest to note that there is a different sort

of stability question that might be posed. Thus, suppose that (i) gen-

eration 1 acts on the basis of an arbitrary positive point forecast of

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[5] For other discussions of the "Hahn Problem," see Hahn,
Sargent and Wallace, and Burmeister and Long. For Hahn stability, H^{-1}
must exist and satisfy $|(H^{-1})'| < 1$.

[6] This is a simple application of Lucas's "stability" discussion.

$p(2)$, denoted $\bar{p}(2)$, and that (ii) succeeding generations act on the basis of a point forecast formed "adaptively."

We now show that if the "adaptive" scheme and initial conditions are chosen carefully, then $|H'| < 1$ (one of the hypotheses of (15)) implies the existence of a monetary equilibrium path that converges to q^* if $xz/n \leq 1$.

Let the point forecast of $p(t+2)$ for generation $t+1 \geq 2$ be given by

$$(16) \qquad \bar{p}(t+2)(z/n) = \lambda \bar{p}(t+1) + (1-\lambda)p(t+1).$$

Multiplying by $M(t+2) = zM(t+1)$ and letting $\bar{q}(t) = \bar{p}(t)M(t)$ for all $t > 1$, we have

$$(17) \qquad \bar{q}(t+2) = \lambda \bar{q}(t+1) + (1-\lambda)q(t+1) \qquad \text{all } t \geq 1.$$

But as the reader can verify by retracing the derivation of (13), in a purely monetary equilibrium under this scheme,

$$(18) \qquad q(t+1) = H[\bar{q}(t+2)] \qquad \text{all } t \geq 1$$

where this is the H function of (14).

Now substitute the RHS of (18) into (17) to get

$$(19) \qquad \bar{q}(t+2) - \lambda \bar{q}(t+1) - (1-\lambda)H[\bar{q}(t+2)] = 0 \qquad \text{all } t \geq 1.$$

If $|H'| < 1$, then $1-(1-\lambda)H' > 0$ and (19) is equivalent to

$$(20) \qquad \bar{q}(t+2) = \bar{H}[\bar{q}(t+1)]$$

where $\bar{H}' = \lambda/[1-(1-\lambda)H'] \varepsilon (0,1)$.

It is immediate that the unique fixed point of H is the unique fixed point of $\bar{H}$ and that (20) implies monotone convergence of the $\bar{q}$ sequence to q^* with the initial condition $\bar{q}(2) > 0$. And convergence of the $\bar{q}$ sequence to q^* implies convergence of the q sequence to q^*. (See (18), for example.)

But we are not done. Equation (18) holds in a purely monetary equilibrium. But the existence of a purely monetary equilibrium requires that the forecasted gross return on money, $\bar{p}(t+1)/p(t)$, be at least as great as x, or that

$$(21) \qquad \bar{p}(t+1)/p(t) = [\bar{q}(t+1)/q(t)](n/z) \geq x.$$

If $q(t)$ is given by (18), then requirement (21) is equivalent to

$$(22) \qquad \bar{q}(t+1)/H[\bar{q}(t+1)] \geq xz/n.$$

If $|H'| < 1$, then $\bar{q}(t+1) < q^*$ implies $\bar{q}(t+1)/q(t) < 1$, while $\bar{q}(t+1) \geq q^*$ implies $\bar{q}(t+1)/q(t) \geq 1$. Thus, even if $xz/n \leq 1$, (22) may fail to hold if $\bar{q}(2) < q^*$. And if $xz/n > 1$, then (22) must fail for some value of t.

If (22) does not hold, then for any $\bar{q}(t+1) > 0$, the second part of (21) holds with equality; that is,

$$(23) \qquad q(t) = \bar{q}(t+1)(n/zx).$$

Then by (17)

$$\bar{q}(t+2) = \lambda\bar{q}(t+1) + (1-\lambda)\bar{q}(t+2)(n/zx)$$

or

$$(24) \qquad \bar{q}(t+2) = \bar{q}(t+1)\lambda/[1-(1-\lambda)n/zx].$$

If $xz/n < 1$, then the $\bar{q}$ sequence given by (24) is exponential increasing, so we are assured that there exists some t such that $\bar{q}(t) \geq q^*$. If $xz/n > 1$, then (24) is exponential decreasing implying that the $\bar{q}$ and q sequences approach zero.

Assembling these results we have the following: if $|H'| < 1$; if successive generations act on the basis of point forecasts formed according to (16) for some arbitrary $\bar{p}(2) > 0$; and

(i)   if $xz/n < 1$, then $\lim q(t) = q^*$,

(ii)   if $xz/n = 1$ and $\bar{q}(2) \geq q^*$, then $\lim q(t) = q^*$,

(iii) if $xz/n = 1$ and $\bar{q}(2) < q^*$, then $q(t) = \bar{q}(2)$ and
       $k^h(t) = q^* - \bar{q}(2)$ for all t, and

(iv)   if $xz/n > 1$, then $\lim q(t) = 0$.

This result stands as another reason not to take seriously Hahn's version of the stability question.

4.   Optimality

We first prove two propositions:

(25)   if $x > n$, then any equilibrium allocation
          is optimal;

(26)   if $x \leq n$, then the nonmonetary equilibrium
          allocation is nonoptimal.

These propositions together with the existence results, (9) and (10), imply proposition I.

4.1 Proof of (25)

The technology imposes the following feasibility condition: for all $t \geq 1$ and with $K(0) \geq 0$ given by an initial condition

(27) $\qquad C_1(t) + K(t) + C_2(t-1) \leq N(t) + xK(t-1); \quad t \geq 1$

where $C_1(t)$ is total time t consumption of generation t, $C_2(t-1)$ is total time t consumption of generation t-1, $K(t) \geq 0$ is time t output stored, and $K(t-1) \geq 0$ is output carried over from t-1.

We let "^" denote an equilibrium allocation and "—" a feasible Pareto superior (P.S.) allocation. We show that the assumed existence of the latter gives rise to a contradiction.

Without loss of generality, we assume that the "—" allocation satisfies (27) with equality and that for all $t \geq 1$, $v[\overline{c}^h(t)] = v[\overline{c}^{h'}(t)]$ for all h and h' in generation t. (Given an allocation P.S. to the "^" allocation that does not satisfy these conditions, one can easily construct the "—" allocation that is P.S. to the former and, hence, to the latter.)

We will prove in detail that $\overline{K}(t) = \hat{K}(t)$ for all t and will then refer the reader to published results on pure exchange economies for the rest.

Suppose $\overline{K}(t) \neq \hat{K}(t)$ for some t. Then there is a smallest $t \geq 1$ at which this happens. We first rule out

A first departure of the form $\overline{K}(t) > \hat{K}(t)$.

Being a first departure, it follows from (27) with equality that either (a) $\overline{C}_2(t-1) < \hat{C}_2(t-1)$ or (b) $\overline{C}_1(t) < \hat{C}_1(t)$ or both.

Case (a): This is easy. Since t is the first departure of $\{\overline{K}\}$ from $\{K\}$, we have for i=1, 2, ..., t-1

(28) $\qquad \overline{C}_1(t-i) + \overline{C}_2(t-i-1) = \hat{C}_1(t-i) + \hat{C}_2(t-i-1).$

But by (a) and the properties of "—," $\overline{C}_1(t-1) > \hat{C}_1(t-1)$. One, then, proceeds backwards from t-1 to t-2 and so on using (28) to conclude that $\overline{C}_2(0) < \hat{C}_2(0)$, a contradiction.

<u>Case (b)</u>: This is more demanding. Under "—," the members of generation t must have more second-period consumption than under "^." And since $v[\hat{c}^h(t)] \geq x$, the extra storage does not produce enough. Therefore, it follows by (27) at equality that

$$(29) \qquad \hat{C}_1(t+1) + \hat{K}(t+1) - [\bar{C}_1(t+1)+\bar{K}(t+1)] \equiv N(t+1)d(t+1) > 0.$$

We now show by induction that the d sequence is increasing and unbounded. Since $[\hat{C}_1(t+1)+\hat{K}(t+1)]/N(t+1) \leq 1$, this will rule out $\bar{K}(t) > K(t)$ under case (b).

For the induction step, we use (29) as an initial condition and consider the following problem:

Choose c(t+1)--an allocation for members of generation t+1--to minimize $C_2(t+1)$ subject to

$$(30) \qquad \hat{C}_1(t+1) + \hat{K}(t+1) - [C_1(t+1)+\bar{K}(t+1)] \geq N(t+1)d(t+1)$$

$$(31) \qquad u[c^h(t+1)] \geq u[\hat{c}^h(t+1)].$$

Since $\bar{c}(t+1)$ is feasible for this problem--i.e., satisfies (31) and (30) (see (29))--we have $\bar{C}_2(t+1) \geq \tilde{C}_2(t+1)$, where "~" denotes solution values for this minimization problem. Before we use this inequality, though, we want to derive a convenient expression for $\tilde{C}_2(t+1)$ in terms of d(t+1).

It is easily verified that there is a unique solution to this minimization problem that satisfies (30) and (31) with equality and, since $\hat{c}^h(t+1) = \hat{c}^{h'}(t+1),$[7]

$$(32) \qquad \hat{c}_1^h(t+1) - \tilde{c}_1^h(t+1) = d(t+1) + \Delta(t+1), \quad \text{all } h$$

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[7] We could get by with $\hat{c}^h(t+1) \neq \hat{c}^{h'}(t+1)$. See Kareken and Wallace.

where

$$\Delta(t+1) = [\overline{K}(t+1) - \hat{K}(t+1)]/N(t+1).$$

But, in general, along a contour of u, $c_2^h = g(c_1^h)$ where $g' = -u_1/u_2 = -v$ and $g'' > 0$. Therefore, applying the intermediate value theorem to g, we have

$$(33) \qquad g(c_1^h) = g(\hat{c}_1^h) + (\hat{c}_1^h - c_1^h)[-g'(\hat{c}_1^h) + f_{\hat{c}}(\hat{c}_1^h - c_1^h)]$$

where the function $f_{\hat{c}}$, whose argument is $(\hat{c}_1^h - c_1^h)$, is strictly increasing and such that $f_{\hat{c}}(0) = 0$.

Now since $\tilde{c}^h(t)$ and $\hat{c}^h(t)$ are on the same contour of u, we may use (32) and (33) to write

$$(34) \qquad \tilde{c}_2^h(t+1) = \hat{c}^h(t+1) +$$

$$[d(t+1)+\Delta(t+1)]\{v[\hat{c}^h(t+1)]+f_{\hat{c}(t+1)}[d(t+1)+\Delta(t+1)]\}$$

or since

$$\overline{C}_2(t+1) \geq \tilde{C}_2(t+1) = N(t+1)\tilde{c}_2^h(t+1)$$

$$(35) \qquad [\overline{C}_2(t+1) - \hat{C}_2(t+1)]/N(t+1) \geq$$

$$[d(t+1)+\Delta(t+1)]\{v[\hat{c}^h(t+1)]+f_{\hat{c}(t+1)}[d(t+1)+\Delta(t+1)]\}.$$

But since the "—" and "^" allocations satisfy (27) at equality, we have

$$\hat{C}_1(t+2) + \hat{K}(t+2) - [\overline{C}_1(t+2)+\overline{K}(t+2)] \equiv N(t+2)d(t+2) =$$

$$\overline{C}_2(t+1) - \hat{C}_2(t+1) - xN(t+1)\Delta(t+1)$$

or

(36) $d(t+2) = [\bar{c}_2(t+1) - \hat{c}_2(t+1)]/N(t+1)n - x\Delta(t+1)/n.$

Then using (35)

(37) $d(t+2) \geq d(t+1)v[\hat{c}^h(t+1)]/n + \Delta(t+1)\{v[\hat{c}^h(t+1)]-x\}/n +$

$[d(t+1)+\Delta(t+1)]f_{\hat{c}(t+1)}[d(t+1)+\Delta(t+1)]/n.$

The RHS consists of a sum of three terms. The last term has the form $xf(x)/n$ which is nonnegative for any x by the properties of f. The second term is also nonnegative since $v[\hat{c}^h(t+1)] \geq x$ with strict equality if $\Delta(t+1) < 0$. (If $\Delta(t+1) > 0$, then $\hat{K}(t+1) > 0$.) Thus, (37) implies

(38) $d(t+2) \geq d(t+1)v[\hat{c}^h(t+1)]/n \geq d(t+1)(x/n).$

Thus, the d sequence is bounded below by a strictly increasing exponential, and, hence, is unbounded.

Next we quickly rule out

A first departure of the form $\bar{K}(t) < \hat{K}(t)$.

If there is such a "—" allocation, then by (27) either

(39) $\bar{c}_1(t+1) + \bar{K}(t+1) < \hat{c}_1(t+1) + \hat{K}(t+1)$

or

(40) $\bar{c}_2(t) < \hat{c}_2(t)$

or both. If (39) holds, we have an initial condition for the induction proof just given. If (40) holds, then $\bar{c}_1(t)$ must exceed $\hat{c}_1(t)$ by more than $x[\hat{K}(t)-\bar{K}(t)]$ because $\bar{K}(t) < \hat{K}(t)$ implies $\hat{K}(t) > 0$ and hence $v[\hat{c}^h(t)] = x$. But, then, we can work backwards as under case (a) above.

We have now proved that if there is a "–" allocation, $\overline{K}(t) = \hat{K}(t)$ for all t. Therefore, by (27) at equality, any such "–" allocation satisfies

$$(41) \qquad \overline{C}_1(t) + \overline{C}_2(t-1) = \hat{C}_1(t) + \hat{C}_2(t-1) \quad \text{for all } t \geq 1.$$

Then, since $v[\hat{c}^h(c)] \geq x > n$, one can derive a contradiction from assuming that somebody is strictly better off under "–" than under "^." To prove this the reader can either adapt the case (a) and (b) arguments above or can consult the proof in Kareken and Wallace, which itself is similar to the case (a) and case (b) arguments made above.

4.2 Proof of (26)

This is easy. All we do is find a feasible P.S. allocation. An obvious candidate is the allocation for the purely monetary equilibrium with z=1. Two remarks establish that this allocation is P.S. to the nonmonetary equilibrium allocation.

1) Under the nonmonetary equilibrium, the young of all generations choose $c^h(t)$ from a budget set that is a subset of their choice set under the purely monetary equilibrium.

2) Total consumption of the current old is greater under a monetary equilibrium than under a nonmonetary equilibrium.

4.3 Proof of Proposition II

By (10), it is enough to prove that a purely monetary equilibrium with z=1 is optimal when it exists—namely, when $x \leq n$. We only outline a proof that follows the proof of (25).

First, one rules out the existence of a feasible P.S. allocation with a first date $t \geq 1$ at which $\overline{K}(t) > 0$. In such a proof one gets to

an expression like (37), but in this case with $\Delta(t+1) \geq 0$. And since $v(\hat{c}^h) = n$, one must use the fact that the relevant f function in the third term on the RHS of (37) is strictly increasing. Then, any feasible P.S. allocation—i.e., any "—" allocation—satisfies (41) and one can proceed as there indicated.

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