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Gambling for Redemption and Self-Fulfilling Debt Crises*

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ABSTRACT

We develop a model for analyzing the sovereign debt crises of 2010–2013 in the Eurozone. The government sets its expenditure-debt policy optimally. The need to sell large quantities of bonds every period leaves the government vulnerable to self-fulfilling crises in which investors, anticipating a crisis, are unwilling to buy the bonds, thereby provoking the crisis. In this situation, the optimal policy of the government is to reduce its debt to a level where crises are not possible. If, however, the economy is in a recession where there is a positive probability of recovery in fiscal revenues, the government also has an incentive to smooth consumption and increase debt. Our exercise identifies conditions on fundamentals for which the incentive to smooth consumption dominates, giving rise to a situation where governments optimally "gamble for redemption," running fiscal deficits and increasing their debt, thereby increasing their vulnerability to crises.

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1. Introduction

This paper develops a model for analyzing the sovereign debt crises of 2010–2013 in Greece, Ireland, Italy, Portugal, and Spain. During this period, the yields on bonds issued by the governments of these countries had substantial spreads over the yields on German bonds, as seen in the data presented in figure 1. Arellano *et al*. (2012) provide a timeline and detailed discussion of these events. In our model, we interpret the spread as capturing the probability that investors assign to a government defaulting on its debt.

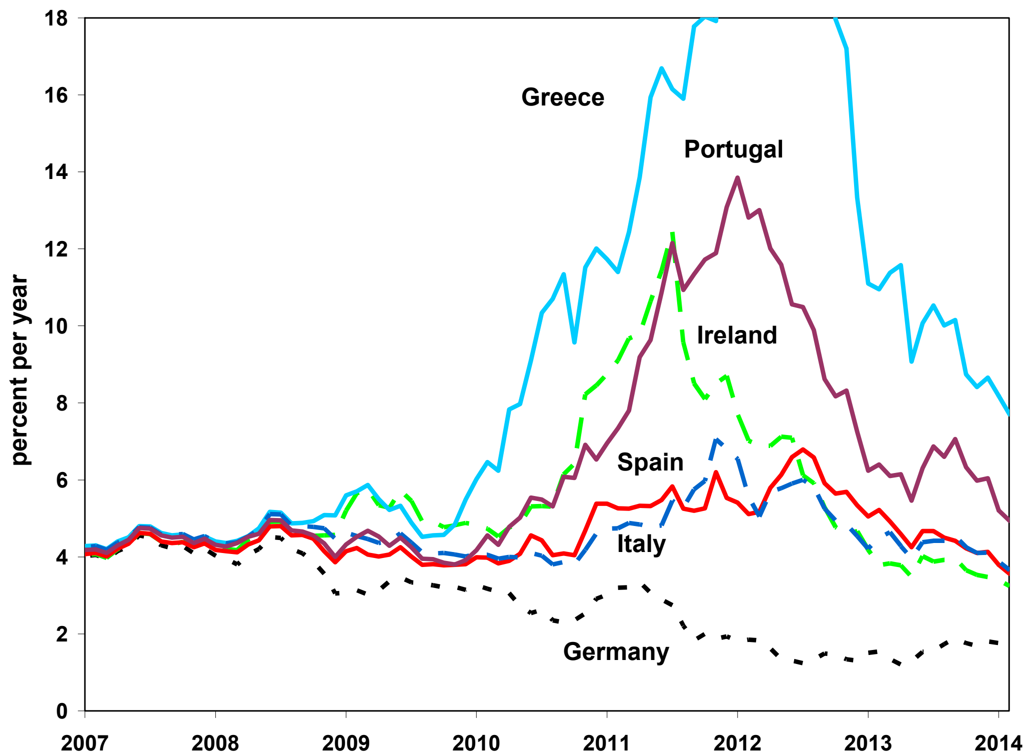

Figure 1: Harmonized long-term interest rates on government bonds in selected Eurozone countries

Note: Interest rate on Greek bonds was 29.24 percent per year in February 2012.

We model the government as setting its expenditure-debt policy optimally, given a probability of a recovery in fiscal revenues. In doing so, the government can optimally choose to "gamble for redemption," increasing its debt and exposing the economy to increasing vulnerability to speculative attacks. We provide a theory of sovereign debt crises in which both borrowers and lenders behave optimally, but where countries borrow so much, and lenders are willing to lend them that much, as to make a default unavoidable. Our theory contrasts alternative explanations based on misperceptions or other forms of irrationality.

Our paper provides a theory for analyzing government behavior in the face of fiscal pressures and a tool for testing the implications of alternative policy responses. In our analysis, as in that of Cole and Kehoe (1996, 2000), we characterize in a simple Markov structure the time consistent policy of a strategic government that is faced with nonstrategic bondholders.

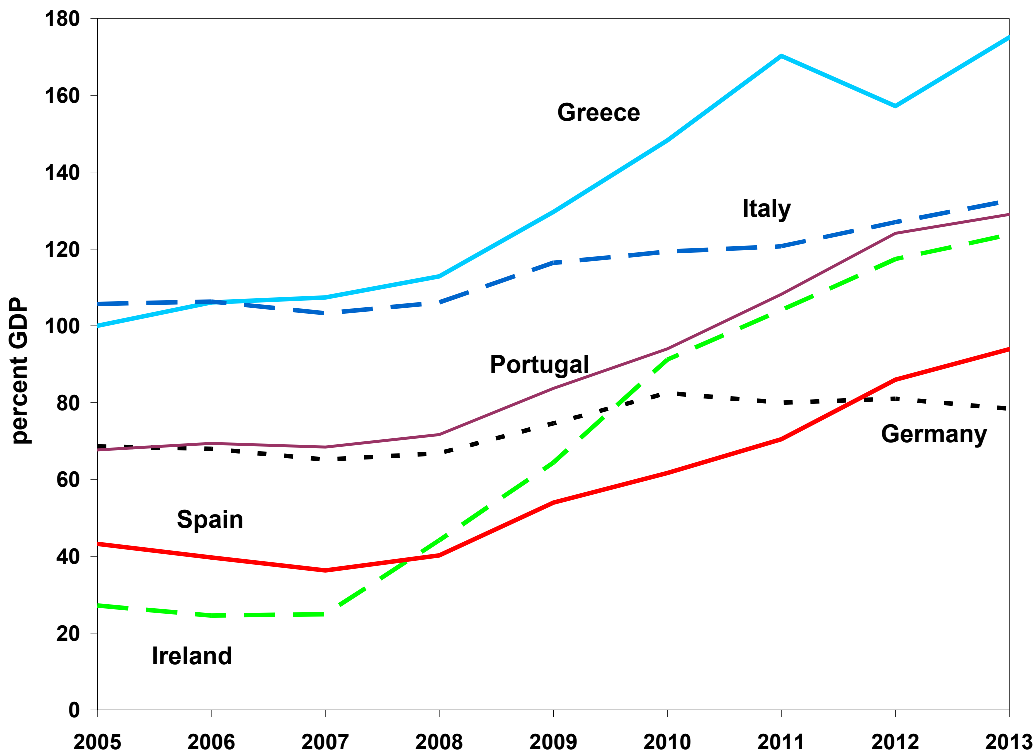

Figure 2: Government debt in selected European countries

The worldwide recession that began in 2008 and the policies intended to overcome it generated large government budget deficits and increases in government debt over the entire developed world. Figure 2 plots debt-to-GDP ratios for the most troubled European economies — the so-called PIIGS: Portugal, Ireland, Italy, Greece, and Spain — as well as Germany. (We ignore Cyprus, which experienced a banking crisis rather than a debt crisis of the sort we analyze here.) These data are at odds with the theory developed by Cole and Kehoe (1996, 2000), who argue that the optimal policy of a government that faces a positive probability of a self-fulfilling debt crisis is to run fiscal surpluses to pay down its debt.

In our model, the crucial element that drives a government to risk suffering a self-fulfilling debt crisis is the drop in government revenues that occurs as the result of a recession. Figure 3 shows that the worldwide recession that started in 2008 was still ongoing in Greece, Italy, Portugal, and Spain in 2013. Notice that the drops in real government revenues (deflated by the GDP deflator) — presented in figure 4 — are also large.

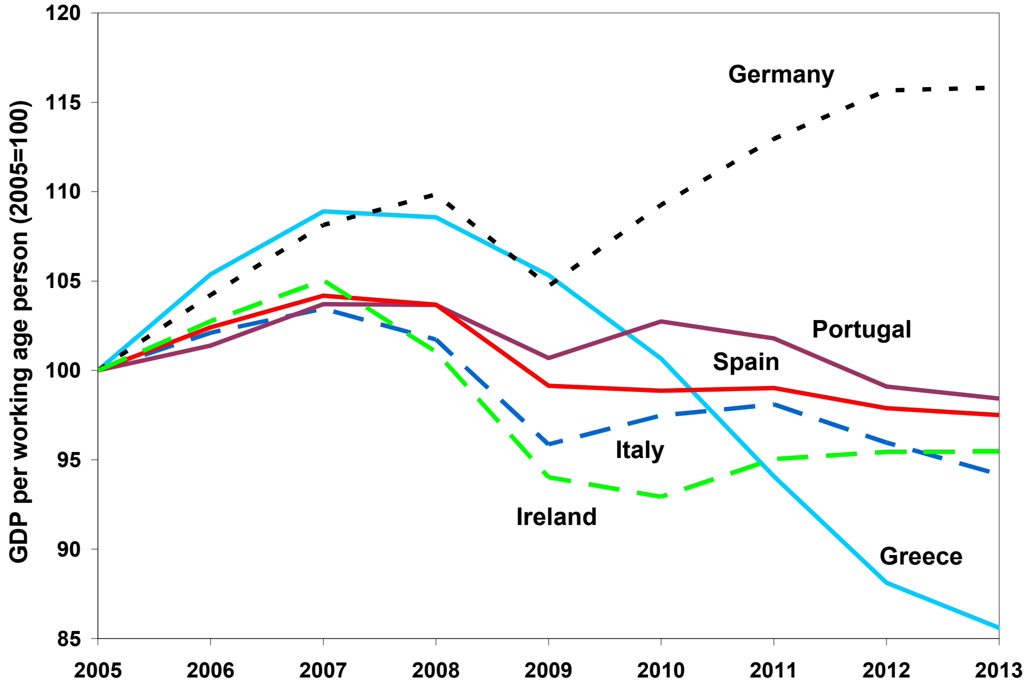

Figure 3: Real GDP in selected Eurozone countries

It is worth pointing out that according to the Kehoe-Prescott (2002) definition of a great recession — that real GDP per working-age person falls 20 percent below a balanced growth path of 2 percent per year — Greece was already in a great depression by 2011 and deeply in it by 2013, and by 2013 Ireland and Italy were on the edge of being in great depressions.

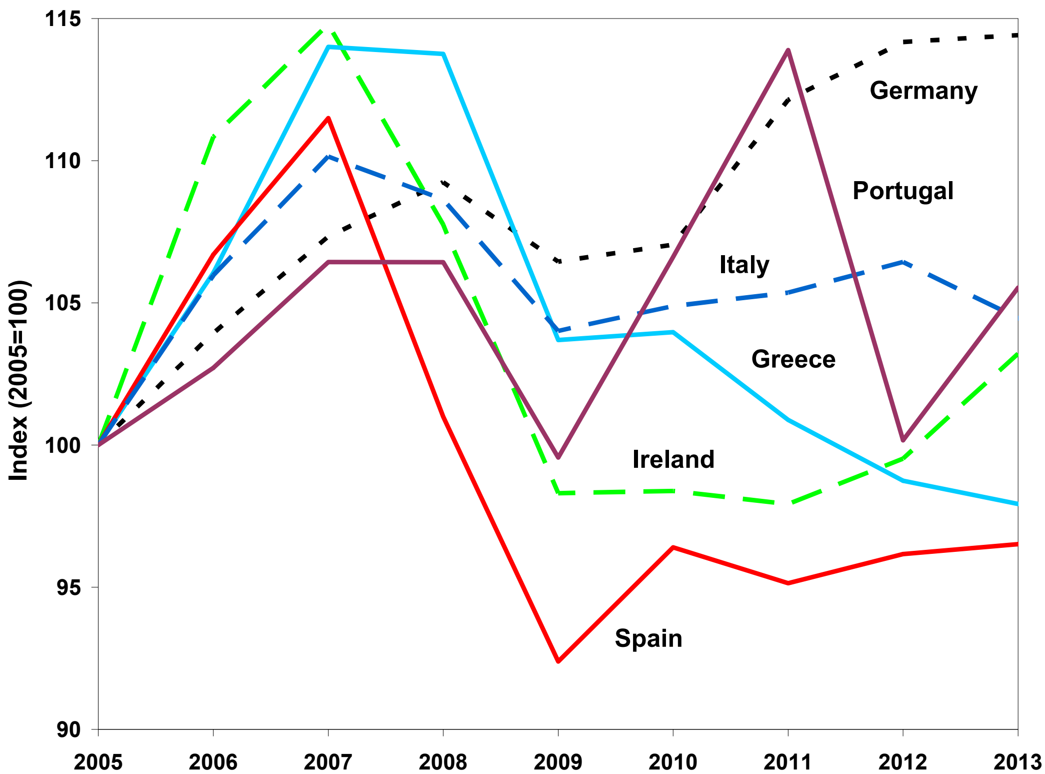

Figure 4: Real government tax revenues in selected Eurozone countries

Our analysis shows that — under certain conditions, which correspond to parameter values and the fundamentals of the economy — it is optimal for the government to "gamble for redemption." By this we refer to a situation where fundamentals are such that the incentive for consumption smoothing dominates the incentive to reduce debt and, as a result, governments optimally increase debt, thereby increasing their vulnerability to a debt crisis. Indeed, the government strategy follows a martingale gambling strategy that sends the economy into the crisis zone if the recovery does not happen soon enough. Under other conditions on fundamentals, however, the government gradually reduces the level of debt to exit the crisis zone and avert the possibility of a liquidity crisis, as in Cole and Kehoe (1996, 2000).

The data in figures 1 and 5 indicate that the governments of the PIIGS continued to borrow even as the spreads on their debt indicated the danger of self-fulfilling debt crises. A relevant question in terms of interpreting the data, though, is from whom these countries were borrowing. Arellano et al. (2012) argue that in 2012 the PIIGS faced a sudden stop of private credit, and debt continued increasing only because public institutions (mostly the European Central Bank) replaced private lenders. In that sense, these economies experienced a debt crisis in the summer of 2012 and were — at least partially — bailed out by a public institution. The analysis in this paper would then apply only to the period before the bailout. See Conesa and Kehoe (2014) and Roch and Uhlig (2016) for an analysis of the role of external bailouts in this type of economic model.

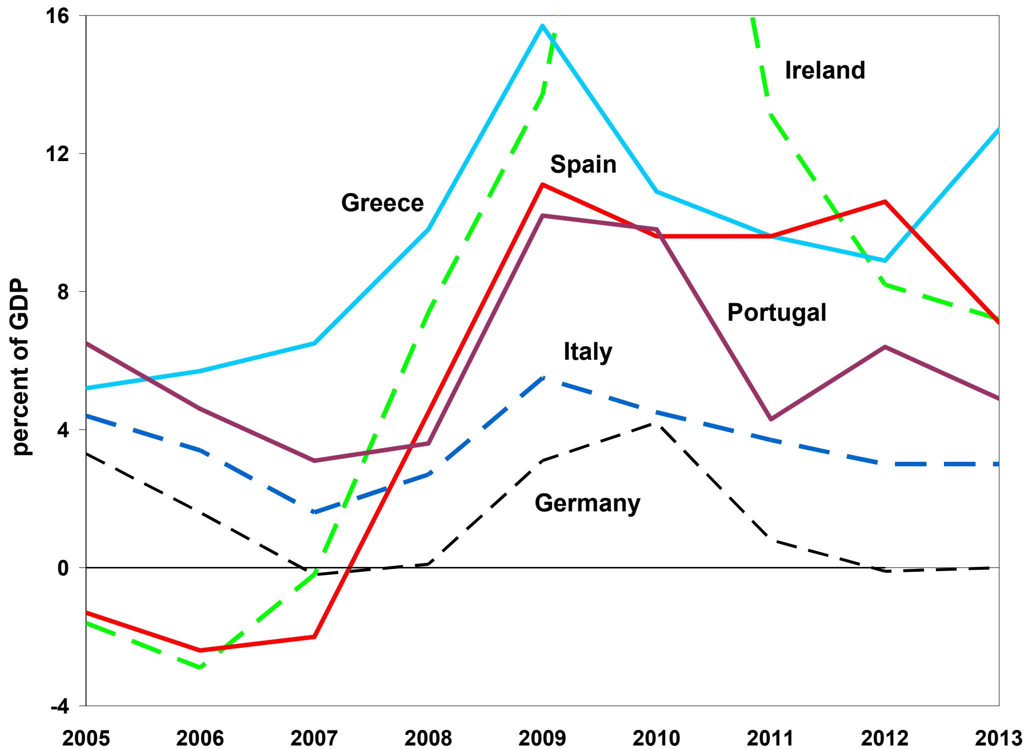

Figure 5: Net government borrowing in selected Eurozone countries
Note: Net government borrowing in Ireland in 2010 was 30.6 percent of GDP.

In our model, not running down debt, or running it up until default is unavoidable, can be part of the optimal strategy under some circumstances. In contrast, Reinhart and Rogoff (2009) argue that some countries fail to adjust and are vulnerable to a potential crisis because both the governments and their lenders are fooling themselves into thinking that "this time is different." As such, a country's vulnerability to a crisis would be the result of self-delusion and lack of rationality. In contrast to this view, we provide a model in which such apparently irrational behavior can be an optimal response to fundamentals by both borrowing governments and lenders that perfectly understand the risks of a crisis.

This paper is most closely related to those of Cole and Kehoe (1996, 2000). Aguiar and Amador (2014) provide a general overview of the literature on debt crises. Similar frameworks have been used to analyze currency crises following Calvo (1988). Cole and Kehoe provide a dynamic stochastic general equilibrium model of a country subject to the possibility of a self-fulfilling debt crisis in every period. The substantial difference between their framework and ours is that, in their framework, debt crises are liquidity crises that are due solely to the inability to roll over debt. As such, a decisive action by a third party providing a loan or bailout would be enough to avert the problem. Indeed, Cole and Kehoe (1996, 2000) intend their model as an analysis of the financial crisis in Mexico in 1994–1995, and, on that occasion, the decisive intervention of the Clinton administration on January 31, 1995, was enough to end the crisis. European Union rescue packages for Greece, Ireland, and Portugal have not had the same healing properties. In fact, quite the opposite seems to be the case, with the spreads on bonds, relative to Germany's, rising after the announcements and initial implementations of the rescue packages. This result suggests more fundamental solvency problems than those present in a standard liquidity crisis of the type studied in Cole and Kehoe, as discussed by Chamley and Pinto (2011) for the Greek case. Our model accommodates this issue.

The model we propose extends the Cole-Kehoe analysis to incorporate a severe recession of uncertain recovery. By doing that, we are incorporating a motive for consumption smoothing as in Arellano (2008) and Aguiar and Gopinath (2006), who focus on default incentives on international borrowing over the business cycle, but do not allow for self-fulfilling debt crises. For a recent analysis quantifying the importance of belief-driven fluctuations in sovereign debt markets, see Bocola and Dovis (2016). For studies of individual, rather than sovereign, default of unsecured debt, see Chatterjee et al. (2007) and Livshits et al. (2007). In our model, there is a trade-off between the benefits of consumption smoothing and the increased vulnerability associated with increasing the level of debt. Our quantitative results relate this trade-off — and whether we should observe gambling for redemption as an optimal response — to fundamentals of the economy such as the severity of a recession, the likelihood of a recovery, the existing stock of debt, and so on.

Our model establishes conditions under which a debt crisis can occur, and how that possibility shapes the government's optimal behavior, but is silent about why at a particular point in time a crisis might or might not occur. Indeed, once the government is in the crisis zone — and we show under what conditions a government would find it optimal to enter this zone — a potential crisis is triggered by a non-fundamental random variable: a sunspot. A debt crisis is one of the two potential equilibria in the crisis zone, and it happens as a sudden event. In contrast, Lorenzoni and Werning (2013) propose an alternative theory with multiple equilibria where debt crises are slow-moving events.

We solve the model numerically and show that gambling for redemption is optimal for moderate levels of initial debt. With high initial levels of debt, gambling for redemption is only optimal whenever the recession is severe. We also discuss the implications of the existence of bonds at different maturities. Our basic finding is that, as maturity increases, the incentives to gamble increase even for intermediate levels of debt. Indeed, in the limit with infinitely lived bonds, rollover crises are not possible.

2. General model

The model has a structure similar to that of Cole and Kehoe (1996, 2000). The major innovation is that output is stochastic, introducing a motive for consumption smoothing in the spirit of Aiyagari (1994) and Huggett (1993). To focus attention on this motive for consumption smoothing, we have chosen to simplify the Cole-Kehoe model by eliminating the representative household's consumption-investment choice. Allowing for private investment would be conceptually straightforward, but technically burdensome.

The state of the economy in every period, $s = (B, a, z_{-1}, \zeta)$, is the level of government debt B, whether or not the private sector is in normal conditions $a = 1$ or in a recession $a = 0$, whether default has occurred in the past $z_{-1} = 0$ or not $z_{-1} = 1$, and the value of the sunspot variable ζ. The country's GDP is

$$y(a,z) = A^{1-a} Z^{1-z} \bar{y}\,, \tag{1}$$

where $1 > A, Z > 0$. Before period 0, $a = 1$, $z = 1$. In period 0, a unexpectedly becomes $a_0 = 0$ and GDP drops from $y = \bar{y}$ to $y = A\bar{y} < \bar{y}$. In every period t, $t = 1, 2, \ldots$, a_t becomes 1 with probability p, $1 > p > 0$. Once $a = 1$, it stays equal to 1 forever. The drop in productivity by the factor Z is the country's default penalty. Once $z = 0$, it stays equal to 0 forever. Here the default penalty occurs in the same period as the crisis. We keep the structure of the income shocks deliberately simple to deliver in the simplest possible way the incentive to smooth consumption even in periods of vulnerability to lenders' panics. In contrast, most of the default literature assumes richer dynamics, such as an AR(1) process of income, that allow for a rich analysis of the business cycle properties of deficits, debt, and bond spreads. See Aguiar and Gopinath (2006) and Arellano (2008).

Figure 6 illustrates a possible evolution of the country's GDP over time. In terms of the crises in the Eurozone, we can think of $t = 0$ as 2008.

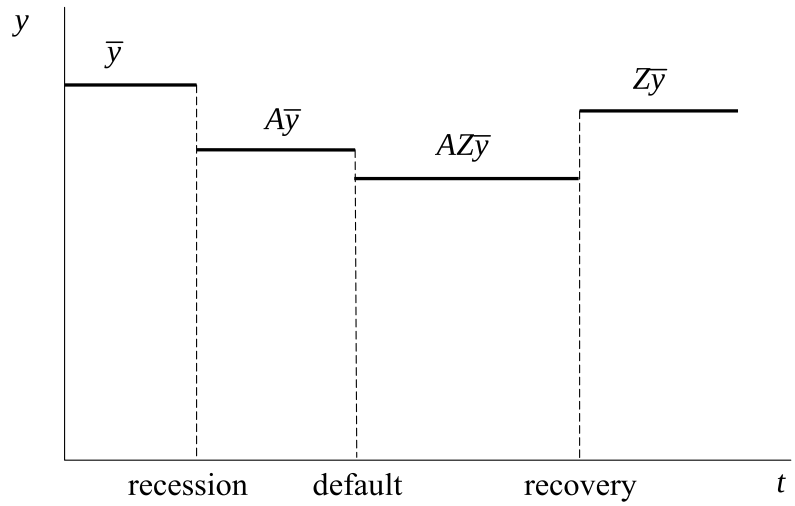

Figure 6: A possible time path for GDP

Government tax revenue is $\theta y(a,z)$, where we assume, as do Cole and Kehoe (1996, 2000) to keep things simple, that the tax rate θ is fixed. Conesa *et al*. (2017) study the case where the government can choose the tax rate θ. Given that there is no consumption-investment choice, the consumption of the representative household is

$$c(a,z) = (1-\theta)y(a,z). \tag{2}$$

The government offers B' in new bonds for sale and chooses whether or not to repay the debt becoming due, B. The government's budget constraint is

$$g + zB = \theta y(a,z) + q(B',s)B', \tag{3}$$

where $q(B',s)$ is the price that international bankers pay for B', g is government expenditure, and $z \in \{0,1\}$ is a binary variable that denotes the government decision to default or repay.

In every period, ζ is drawn from the uniform distribution on $[0,1]$. If $\zeta > 1-\pi$, international bankers expect a crisis to occur and do not lend to the government if such a crisis would be self-fulfilling. This allows us to set the probability of a self-fulfilling crisis at an arbitrary value π, $1 \geq \pi \geq 0$, if the level of debt is high enough for such a crisis to be possible. The timing within each period is like that in Cole and Kehoe (1996, 2000):

1. The shocks a and ζ are realized, the aggregate state is $s = (B, a, z_{-1}, \zeta)$, and the government chooses how much debt B' to sell.

2. Each of a continuum of measure one of international bankers chooses how much debt b' to purchase. In equilibrium, $b' = B'$.

3. The government makes its default decision z, which determines y, c, and g.

The three crucial elements of this timing are as follows. First, the government faces a time consistency problem because, when offering B' for sale, it cannot commit to repaying B. Second, since all uncertainty has been resolved at the beginning of the period, there is perfect foresight in equilibrium within the period and, in particular, international bankers do not lend if they know the government will default. Third, whether or not a crisis occurs during the period depends on B, whereas if no crisis occurs, the price of new bonds depends only on B'.

Given this timing, we can reduce the government's problem choosing c, g, B', z to solve

$$\begin{aligned} V(s) = \max \quad & u(c,g) + \beta E V(s') \\ \text{s.t. } & c = (1-\theta) y(a,z) \\ & g + zB = \theta y(a,z) + q(B',s)B' \\ & z = 0 \text{ if } z_{-1} = 0 . \end{aligned} \tag{4}$$

Here $z = 1$ is the decision not to default, and $z = 0$ is the decision to default.

In general, we assume that, for any B such that $A\bar{y} - B$ is an element of the feasible set of levels for government expenditures g,

$$u_g((1-\theta)A\bar{y}, \theta A\bar{y} - B) > u_g((1-\theta)\bar{y}, \theta\bar{y} - B) . \tag{5}$$

In other words, the marginal social benefit of government spending is higher during a recession than it is in normal times. This assumption provides the government with the incentive to transfer resources into the current period during a recession from future periods in which the economy has recovered. It is satisfied by any concave utility function separable in c and g. It is also satisfied by functions such as $\log(c+g-\overline{c}-\overline{g})$.

2.1. Bond prices

International bankers are risk neutral with discount factor β so that the bond prices $q(B',s)$ are determined by the probability of default in the next period. There is a continuum of measure one of bankers. Each solves the dynamic programming problem

$$
\begin{gathered}
W(b,B',s) = \max x + \beta E W(b',B'',s') \\
x + q(B',s)b' = w + z(B',s,q(B',s))b \qquad (6) \\
x \geq 0,\ b \geq -A.
\end{gathered}
$$

The constraint $b \geq -A$ eliminates Ponzi schemes, but A is large enough so that the constraint does not otherwise bind. We assume that the banker's endowment of consumption good w is large enough to rule out corner solutions in equilibrium. We refer to this assumption as the assumption that the banker has deep pockets.

There are four cutoff levels of debt: $\overline{b}(a)$, $\overline{B}(a)$, $a = 0,1$:

1. If $B \leq \overline{b}(0)$, the government does not default when the private sector is in a recession even if international bankers do not lend, and, if $B > \overline{b}(0)$, the government defaults when the private sector is in a recession if international bankers do not lend.

2. If $B \leq \overline{b}(1)$, the government does not default when the private sector is in normal conditions even if international bankers do not lend, and, if $B > \overline{b}(1)$, the government defaults when the private sector is in normal conditions if international bankers do not lend.

3. If $B \leq \overline{B}(0)$, the government does not default when the private sector is in a recession if international bankers lend, and, if $B > \overline{B}(0)$, the government defaults when the private sector is in a recession even if international bankers lend.

4. If $B \leq \bar{B}(1)$, the government does not default when the private sector is in normal conditions if international bankers lend, and, if $B > \bar{B}(1)$, the government defaults when the private sector is in normal conditions even if international bankers lend.

The assumption that once $z = 0$, it stays equal to 0 forever says that a country that defaults is permanently excluded from international borrowing or lending. This assumption can be modified at the cost of complicating the analysis. The assumption has two consequences for the relation of the bond price q to the current state s. First, once default has occurred, international bankers do not lend:

$$q(B',(B,a,0,\zeta)) = 0\,. \tag{7}$$

Second, during a crisis, international bankers do not lend:

$$q(B',(B,a,1,\zeta)) = 0 \tag{8}$$

whenever $B > \bar{b}(a)$ and $\zeta > 1-\pi$. Otherwise, the bond price q only depends on the amount of bonds B' that the government offers for sale.

We focus on the case where

$$\bar{b}(0) < \bar{b}(1) < \bar{B}(0) < \bar{B}(1)\,. \tag{9}$$

The first-order condition for the international bankers' utility maximization problem implies that

$$q(B',s) = \beta E z(B'(s'),s',q(B'(s'),s'))\,, \tag{10}$$

which implies that in recessions

$$q(B',(B,0,1,\zeta)) = \begin{cases} \beta & \text{if } B' \leq \bar{b}(0) \\ \beta\big(p+(1-p)(1-\pi)\big) & \text{if } \bar{b}(0) < B' \leq \bar{b}(1) \\ \beta(1-\pi) & \text{if } \bar{b}(1) < B' \leq \bar{B}(0) \\ \beta p(1-\pi) & \text{if } \bar{B}(0) < B' \leq \bar{B}(1) \\ 0 & \text{if } \bar{B}(1) < B' \end{cases} \tag{11}$$

and in normal times

$$q(B',(B,1,1,\zeta))=\begin{cases}\beta & \text{if } B'\le \bar{b}(1)\\ \beta(1-\pi) & \text{if } \bar{b}(1)<B'\le \bar{B}(1)\\ 0 & \text{if } \bar{B}(1)<B'\end{cases}. \tag{12}$$

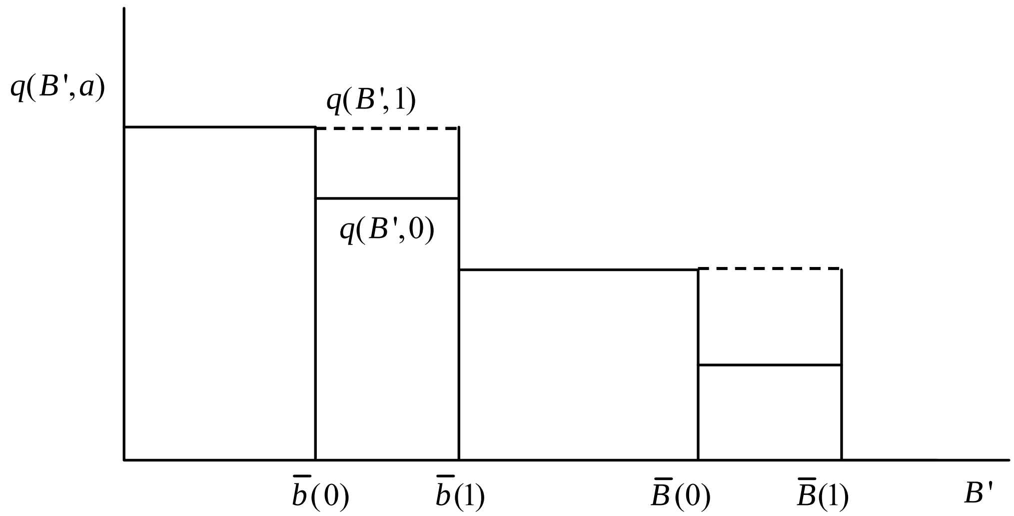

Figure 7: Bond prices as a function of new bonds offered and state of private sector

There are other possibilities. Suppose, for example, that $\bar{B}(0)<\bar{b}(1)$, that is,

$$\bar{b}(0)<\bar{B}(0)<\bar{b}(1)<\bar{B}(1). \tag{13}$$

Here the solution to the international bankers' utility maximization problem implies that

$$q(B',(B,0,1,\zeta))=\begin{cases}\beta & \text{if } B'\le \bar{b}(0)\\ \beta\left(p+(1-p)(1-\pi)\right) & \text{if } \bar{b}(0)<B'\le \bar{B}(0)\\ \beta p & \text{if } \bar{B}(0)<B'\le \bar{b}(1)\\ \beta p(1-\pi) & \text{if } \bar{b}(1)<B'\le \bar{B}(1)\\ 0 & \text{if } \bar{B}(1)<B'\end{cases}. \tag{14}$$

In the case where

$$\bar{b}(0)<\bar{b}(1)=\bar{B}(0)<\bar{B}(1), \tag{15}$$

the region where $\bar{B}(0)<B'\le \bar{b}(1)$ in the bond schedule (14) disappears.

The second and third cases, where $\bar{B}(0) \leq \bar{b}(1)$, are only possible for catastrophic recessions, where A is low. In the rest of our analysis, we focus on the first case, where $\bar{B}(0) > \bar{b}(1)$. The analysis is easily modified to cover the two cases where $\bar{B}(0) \leq \bar{b}(1)$.

2.2. Definition of equilibrium

An equilibrium is a value function for government $V(s)$ and policy functions $B'(s)$ and $z(B',s,q)$ and $g(B',s,q)$, a value function for bankers $W(b,B',s)$ and policy correspondence $b'(b,B',s)$, and a bond price function $q(B',s)$ such that

1. Given the policy functions $z(B',s,q)$ and $g(B',s,q)$ and the price function $q(B',s)$, $V(s)$ and $B'(s)$ solve the government's problem at the beginning of the period:

$$\begin{gathered} V(B,a,z_{-1},\zeta) = \max \;\; u(c,g) + \beta EV(B',a',z,\zeta') \\ \text{s.t. } c = (1-\theta)y(a, z(B', s, q(B', s))) \\ g(B',s,q(B',s)) + z(B',s,q(B',s))B = \theta y(a,z) + q(B',s)B'. \end{gathered} \tag{16}$$

2. $b'(b,B',s)$ solves the banker's problem and $q(B',s)$ is consistent with market clearing and rational expectations:

$$B'(s) \in b'(b,B',s) \tag{17}$$

$$q(B',s) = \beta E z(B',s,q(B',s)) . \tag{18}$$

3. Given the value function $V(s)$, $z(B',s,q)$ and $g(B',s,q)$ solve the government's problem at the end of the period:

$$\begin{gathered} \max \;\; u(c,g) + \beta EV(B',a',z,\zeta') \\ \text{s.t. } c = (1-\theta)y(a,z) \\ g + zB = \theta y(a,z) + qB' \\ z = 0 \text{ or } z = 1, \text{ but } z = 0 \text{ if } z_{-1} = 0 . \end{gathered} \tag{19}$$

Notice that, when the government solves its problem at the beginning of the period, it takes as given the optimal responses of both international bankers and itself later in the period. In particular, the government cannot commit to repaying its debt and not defaulting later in the period. Furthermore, since the occurrence of a crisis depends on the amount of debt to be repaid, B, not the amount of debt offered for sale, B', once a sunspot has occurred that signals that a self-fulfilling crisis will take place during that period, there is nothing that the government can do to avoid it.

This model has many equilibria. Our definition of equilibrium restricts our attention to equilibria with a simple Markov structure. Many other possibilities exist. If we include the date in the state $s=(B,a,z_{-1},\zeta,t)$, for example, we could allow crises to occur only in even periods t or in periods that are prime numbers, or we could allow the probability of a crisis π to be time varying in other ways. We could, for example, have π itself follow a Markov process so as to mimic the sort of time-varying spreads seen in the data in figure 1. The advantage of our simple Markov structure is that it makes it easy to characterize and compute equilibria.

3. Self-fulfilling debt crises

In the general model, we need to resort to numerical examples to illustrate the possibilities and do comparative statics analysis. Before turning to the results for the general model, we study two special cases, where we can provide analytical characterizations of the equilibria in which we are interested. The first is the case where $a=1$, that is, where the private sector has recovered and where there is no incentive for the government to gamble for redemption. This is a simplified version of the Cole-Kehoe model (1996, 2000) without private capital. To keep our discussion simple, we omit the details of proofs that can be found in Cole and Kehoe (2000).

Notice that we can easily modify the analysis of this case to study the limiting case where $a=0$ and $p=0$, that is, where there is a recession but no possibility for recovery, simply by replacing $\bar{y}$ with $A\bar{y}$ in what follows. In this case, where self-fulfilling crises are possible, but where there is no incentive for the government to gamble for redemption, the optimal strategies of the government involve either leaving debt constant or running it down to eliminate the possibility of a crisis. In the next section, we consider the other extreme case, where recovery is possible but self-fulfilling crises are not.

We start by assuming that $\pi=0$. Notice that, since a recovery has already occurred in the private sector, p is irrelevant. To derive the optimal government policy, we solve

$$
\begin{aligned}
&\max \ \sum_{t'=t}^{\infty} \beta^{t'} u(c_{t'}, g_{t'}) \\
&\text{s.t. } c_{t'} = (1-\theta)\bar{y} \qquad (20) \\
&g_{t'} + B_{t'} = \theta\bar{y} + \beta B_{t'+1} \\
&B_t = B \\
&B_{t'} \le \bar{B}(1).
\end{aligned}
$$

The first-order conditions are

$$\beta^{t'} u_g((1-\theta)\bar{y}, g_{t'}) = \lambda_{t'} \qquad (21)$$

$$\lambda_{t'+1} = \beta\lambda_{t'}, \qquad (22)$$

and the transversality condition is

$$\lim_{t'\to\infty} \lambda_{t'} B_{t'+1} \ge 0. \qquad (23)$$

The first-order conditions imply that

$$u_g((1-\theta)\bar{y}, g_{t'}) = u_g((1-\theta)\bar{y}, g_{t'-1}), \qquad (24)$$

in particular, that $g_{t'} = \hat{g}$ is constant. Since $\hat{g}$ is constant, the budget constraint in the government's problem is

$$B_{t'+1} = \frac{1}{\beta}\left(\hat{g} + B_{t'} - \theta\bar{y}\right). \qquad (25)$$

Suppose that

$$\hat{g} = \theta\bar{y} - (1-\beta)B. \qquad (26)$$

Then $B_s = B$. Otherwise, since $\beta < 1$, $B_{t'}$ is explosive. Too low a $\hat{g}$ results in a path for $B_{t'}$ that violates the transversality condition. Too high a $\hat{g}$ results in a path for $B_{t'}$ that hits $\bar{B}(1)$. Neither can be optimal.

We can calculate the value of being in state $s = (B, a, z_{-1}, \zeta) = (B,1,1,\zeta)$ as

$$V(B,1,1,\zeta) = \frac{u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)B)}{1-\beta}. \qquad (27)$$

The calculation of utility when default has occurred, when $z=0$, is mechanical. In that case $B=0$ and

$$c=(1-\theta)Z\overline{y} \tag{28}$$

$$g=\theta Z\overline{y}\,. \tag{29}$$

Notice that, once a default has occurred, ζ and π are irrelevant. Consequently, when $s=(B,a,z_{-1},\zeta)=(B,1,0,\zeta)$,

$$V(B,1,0,\zeta)=V_d(1)=\frac{u((1-\theta)Z\overline{y},\theta Z\overline{y})}{1-\beta}\,. \tag{30}$$

Let us calculate $\overline{b}(1)$. Let $V_n(B,a,q)$ be the value of not defaulting when the price of new debt is q. The utility of repaying B even if the international bankers do not lend is

$$V_n(B,1,0)=u((1-\theta)\overline{y},\theta\overline{y}-B)+\frac{\beta u((1-\theta)\overline{y},\theta\overline{y})}{1-\beta}, \tag{31}$$

whereas the utility of defaulting $V_d(a)$ is

$$V_d(1)=\frac{u((1-\theta)Z\overline{y},\theta Z\overline{y})}{1-\beta}\,. \tag{32}$$

Consequently, $\overline{b}(1)$ is determined by the equation

$$V_n(\overline{b}(1),1,0)=V_d(1) \tag{33}$$

$$u((1-\theta)\overline{y},\theta\overline{y}-\overline{b}(1))+\frac{\beta u((1-\theta)\overline{y},\theta\overline{y})}{1-\beta}=\frac{u((1-\theta)Z\overline{y},\theta Z\overline{y})}{1-\beta}\,. \tag{34}$$

Determining $\overline{B}(1)$ is more complicated because it depends on the optimal debt policy in the crisis zone. Suppose that $B_0>\overline{b}(1)$ and the government decides to reduce B to $\overline{b}(1)$ in T periods, $T=1,2,...,\infty$. The first-order conditions for the government's problem imply that

$$g_t=g^T(B_0), \tag{35}$$

that is, government spending is constant while the government is reducing its debt. The government's budget constraints are

$$
\begin{aligned}
g^T(B_0) + B_0 &= \theta\bar{y} + \beta(1-\pi)B_1 \\
g^T(B_0) + B_1 &= \theta\bar{y} + \beta(1-\pi)B_2 \\
&\vdots \\
g^T(B_0) + B_{T-2} &= \theta\bar{y} + \beta(1-\pi)B_{T-1} \\
g^T(B_0) + B_{T-1} &= \theta\bar{y} + \beta\bar{b}(1)\,.
\end{aligned} \tag{36}
$$

Multiplying each equation by $(\beta(1-\pi))^t$ and adding, we obtain

$$
\sum\nolimits_{t=0}^{T-1}(\beta(1-\pi))^t g^T(B_0) + B_0 = \sum\nolimits_{t=0}^{T-1}(\beta(1-\pi))^t \theta\bar{y} + (\beta(1-\pi))^{T-1}\beta\bar{b}(1) \tag{37}
$$

$$
g^T(B_0) = \theta\bar{y} - \frac{1-\beta(1-\pi)}{1-(\beta(1-\pi))^T}\left(B_0 - (\beta(1-\pi))^{T-1}\beta\bar{b}(1)\right). \tag{38}
$$

Notice that

$$
g^\infty(B_0) = \lim\nolimits_{T\to\infty} g^T(B_0) = \theta\bar{y} - (1-\beta(1-\pi))B_0\,. \tag{39}
$$

We can compute the value $V^T(B_0)$ of each of the policies of running down the debt in T periods, $T = 1,...,\infty$. Letting $V_t^T(B_0)$ be the value of the policy where there are still t periods to go in running down debt, we can write

$$
\begin{aligned}
V_T^T(B_0) &= u((1-\theta)\bar{y}, g^T(B_0)) + \beta(1-\pi)V_{T-1}^T(B_0) + \frac{\beta\pi u((1-\theta)\overline{Zy}, \theta\overline{Zy})}{1-\beta} \\
V_{T-1}^T(B_0) &= u((1-\theta)\bar{y}, g^T(B_0)) + \beta(1-\pi)V_{T-2}^T(B_0) + \frac{\beta\pi u((1-\theta)\overline{Zy}, \theta\overline{Zy})}{1-\beta} \\
&\vdots \\
V_2^T(B_0) &= u((1-\theta)\bar{y}, g^T(B_0)) + \beta(1-\pi)V_1^T(B_0) + \frac{\beta\pi u((1-\theta)\overline{Zy}, \theta\overline{Zy})}{1-\beta} \\
V_1^T(B_0) &= u((1-\theta)\bar{y}, g^T(B_0)) + \frac{\beta u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)\bar{b}(1))}{1-\beta}\,.
\end{aligned} \tag{40}
$$

Notice that g increases from $g^T(B_0)$ to $\theta\bar{y} - (1-\beta)\bar{b}(1)$ in period T. To calculate $V^T(B_0)$, we use backward induction:

$$
\begin{aligned}
V_2^T(B_0) = & (1+\beta(1-\pi))u((1-\theta)\overline{y}, g^T(B_0)) \\
& + \frac{\beta\pi u((1-\theta)Z\overline{y}, \theta Z\overline{y})}{1-\beta} \\
& + \beta(1-\pi)\frac{\beta u((1-\theta)\overline{y}, \theta\overline{y}-(1-\beta)\overline{b}(1))}{1-\beta}
\end{aligned}
$$

$$
\begin{aligned}
V_3^T(B_0) = & (1+\beta(1-\pi)+(\beta(1-\pi))^2)u((1-\theta)\overline{y}, g^T(B_0)) \\
& + (1+\beta(1-\pi))\frac{\beta\pi u((1-\theta)Z\overline{y}, \theta Z\overline{y})}{1-\beta} \\
& + (\beta(1-\pi))^2\frac{\beta u((1-\theta)\overline{y}, \theta\overline{y}-(1-\beta)\overline{b}(1))}{1-\beta} \\
& \vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
V_T^T(B_0) = & (1+\beta(1-\pi)+(\beta(1-\pi))^2+...+(\beta(1-\pi))^{T-1})u((1-\theta)\overline{y}, g^T(B_0)) \\
& + (1+\beta(1-\pi)+(\beta(1-\pi))^2+...+(\beta(1-\pi))^{T-2})\frac{\beta\pi u((1-\theta)Z\overline{y}, \theta Z\overline{y})}{1-\beta} \\
& + (\beta(1-\pi))^{T-2}\frac{\beta u((1-\theta)\overline{y}, \theta\overline{y}-(1-\beta)\overline{b}(1))}{1-\beta},
\end{aligned}
$$

and, of course, $V^T(B_0) = V_T^T(B_0)$:

$$
\begin{aligned}
V^T(B_0) = & \frac{1-(\beta(1-\pi))^T}{1+\beta(1-\pi)}u((1-\theta)\overline{y}, g^T(B_0)) + \frac{1-(\beta(1-\pi))^{T-1}}{1+\beta(1-\pi)}\frac{\beta\pi u((1-\theta)Z\overline{y}, \theta Z\overline{y})}{1-\beta} \\
& + (\beta(1-\pi))^{T-2}\frac{\beta u((1-\theta)\overline{y}, \theta\overline{y}-(1-\beta)\overline{b}(1))}{1-\beta}.
\end{aligned} \tag{42}
$$

Notice that

$$
V^\infty(B_0) = \frac{u((1-\theta)\overline{y}, \theta\overline{y}-(1-\beta(1-\pi))B_0)}{1+\beta(1-\pi)} + \frac{\beta\pi u((1-\theta)Z\overline{y}, \theta Z\overline{y})}{(1-\beta)(1+\beta(1-\pi))}. \tag{43}
$$

To find $\overline{B}(1)$, we solve

$$
\begin{aligned}
& \max\left[V^1(\overline{B}(1)), V^2(\overline{B}(1)), ..., V^\infty(\overline{B}(1))\right] \\
& = u((1-\theta)Z\overline{y}, \theta Z\overline{y}+\beta(1-\pi)\overline{B}(1))) + \frac{\beta u((1-\theta)Z\overline{y}, \theta Z\overline{y})}{1-\beta} \cdot
\end{aligned} \tag{44}
$$

Our arguments have produced the following analytical characterization of $V(B,1,1,\zeta)$:

$$V(B,1,1,\zeta)=\begin{cases}\dfrac{u((1-\theta)\bar{y},\theta\bar{y}-(1-\beta)\bar{b}(1))}{1-\beta} & \text{if } B\le\bar{b}(1)\\ \max\left[V^1(B),V^2(B),\ldots,V^\infty(B)\right] & \text{if } \bar{b}(1)<B\le\bar{B}(1),\ \zeta\le 1-\pi\\ \dfrac{u((1-\theta)Z\bar{y},\theta Z\bar{y})}{1-\beta} & \text{if } \bar{b}(1)<B\le\bar{B}(1),\ 1-\pi<\zeta\\ \dfrac{u((1-\theta)Z\bar{y},\theta Z\bar{y})}{1-\beta} & \text{if } \bar{B}(1)<B\end{cases}. \quad (45)$$

Some of the different possibilities for optimal government strategies — which vary with the initial debt — are illustrated in figure 8.

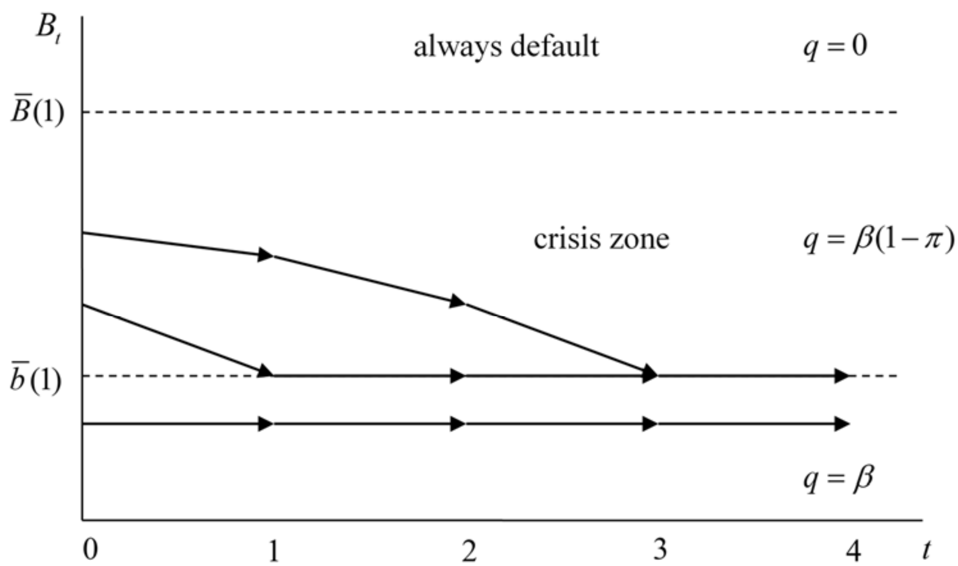

Figure 8: Optimal debt policy with self-fulfilling crises

4. Consumption smoothing without self-fulfilling crises

Suppose now that $a=0$ and $\pi=0$. That is, no self-fulfilling crises are possible, but the private sector is in a recession and faces the probability p, $1>p>0$, of recovering in every period, as depicted in figure 9. We can also interpret this as the limiting case in which crises can occur, but the government and the international bankers assign probability $\pi=0$ to them.

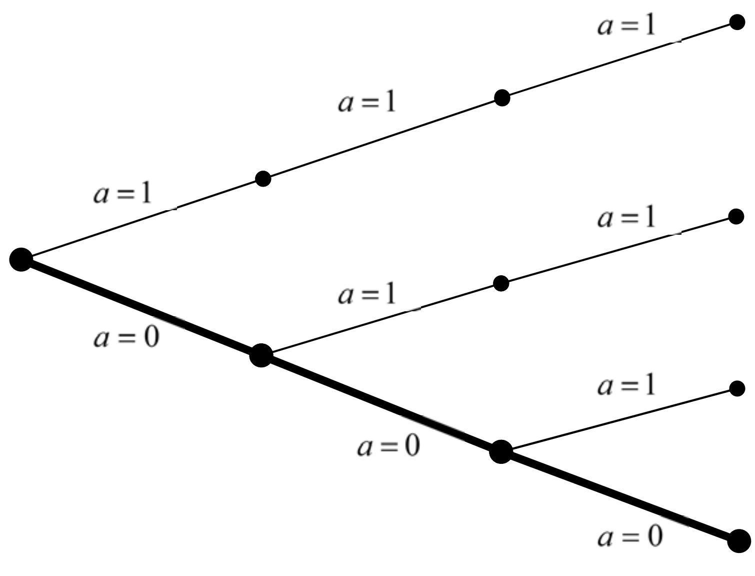

Figure 9: Uncertainty tree with recession path highlighted

In this section, we argue that the optimal government policy is to increase its debt as long as $a=0$. In fact, if the country is unlucky in the sense that $a=0$ long enough, the government may choose to eventually default. Consequently, the upper limits on the debt, $\bar{B}(0)$ and $\bar{B}(1)$, are crucial for our analysis. Because $\pi=0$, the optimal policy for debt after a recovery has occurred is to keep debt constant. Consequently, the condition that determines $\bar{B}(1)$ is similar to condition (34) for determining $\bar{b}(1)$ in the previous section:

$$\begin{aligned} &u((1-\theta)Z\bar{y},\theta Z\bar{y}+\beta\bar{B}(1))-u((1-\theta)\bar{y},\theta\bar{y}-(1-\beta)\bar{B}(1)) \\ &\quad =\frac{\beta}{1-\beta}\Big(u((1-\theta)\bar{y},\theta\bar{y}-(1-\beta)\bar{B}(1))-u((1-\theta)Z\bar{y},\theta Z\bar{y})\Big). \end{aligned} \tag{46}$$

To determine $\bar{B}(0)$, we suppose that, when the government has debt $B\le\bar{B}(0)$, it borrows B', where $\bar{B}(0)<B'\le\bar{B}(1)$, at price βp, then repays the next period if the private sector recovers and defaults otherwise. The value of borrowing $\bar{B}(1)$ at price βp, repaying the current debt, and then repaying in the next period if the private sector recovers and defaulting otherwise is

$$
\begin{aligned}
V_n(B) = {} & u((1-\theta)A\bar{y}, \theta A\bar{y} + \beta p\bar{B}(1) - B) \\
& + \beta(1-p)\left(\frac{u((1-\theta)AZ\bar{y}, \theta AZ\bar{y})}{1-\beta(1-p)} + \frac{\beta p u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{(1-\beta)(1-\beta(1-p))}\right) \\
& + \frac{\beta p}{1-\beta} u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)\bar{B}(1)).
\end{aligned} \tag{47}
$$

The value of borrowing $\bar{B}(1)$ at price βp and then defaulting is

$$
\begin{aligned}
V_d(B) = {} & u((1-\theta)AZ\bar{y}, \theta AZ\bar{y} + \beta p\bar{B}(1)) \\
& + \beta(1-p)\left(\frac{u((1-\theta)AZ\bar{y}, \theta AZ\bar{y})}{1-\beta(1-p)} + \frac{\beta p u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{(1-\beta)(1-\beta(1-p))}\right) \\
& + \frac{\beta p}{1-\beta} u((1-\theta)Z\bar{y}, \theta Z\bar{y}).
\end{aligned} \tag{48}
$$

The equation that determines $\bar{B}(0)$ is, therefore,

$$
V_n(\bar{B}(0)) = V_d(\bar{B}(0)) \tag{49}
$$

$$
\begin{aligned}
& u((1-\theta)A\bar{y}, \theta A\bar{y} + \beta p\bar{B}(1) - \bar{B}(0)) + \frac{\beta p}{1-\beta} u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)\bar{B}(1)) \\
& = u((1-\theta)AZ\bar{y}, \theta AZ\bar{y} + \beta p\bar{B}(1)) + \frac{\beta p}{1-\beta} u((1-\theta)Z\bar{y}, \theta Z\bar{y}).
\end{aligned} \tag{50}
$$

The government may, in fact, choose a lower level of the debt than $\bar{B}(1)$, but $\bar{B}(0)$ is the highest level of the debt B' at which the government can borrow at price $q(B', s) = \beta$ where there is no possibility of a self-fulfilling crisis, $\pi = 0$, and the economy is in a recession, $a = 0$. If the constraint $B' \leq \bar{B}(1)$ does not bind, we can calculate the optimal B' by solving

$$
\begin{aligned}
\max\, & u((1-\theta)A\bar{y}, \theta A\bar{y} + \beta pB' - B) \\
& + \beta(1-p)\left(\frac{u((1-\theta)AZ\bar{y}, \theta AZ\bar{y})}{1-\beta(1-p)} + \frac{\beta p u((1-\theta)Z\bar{y}, \theta Z\bar{y})}{(1-\beta)(1-\beta(1-p))}\right) \\
& + \frac{\beta p}{1-\beta} u((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)B').
\end{aligned} \tag{51}
$$

The first-order condition is

$$
u_g((1-\theta)A\bar{y}, \theta A\bar{y} + \beta pB' - B) = u_g((1-\theta)\bar{y}, \theta\bar{y} - (1-\beta)B'). \tag{52}
$$

Letting $\hat{B}'(B)$ be the solution to this problem,

$$B'(B) = \min[\hat{B}'(B), \overline{B}(1)]. \tag{53}$$

There are two cases:

1. The government chooses to never violate the constraint $B \le \overline{B}(0)$, and the optimal debt converges to $\overline{B}(0)$ if $a = 0$ for a sufficiently large number of periods.

2. The government chooses to default in T periods if $a = 0$ for a sufficiently large number of periods.

For case 1, where the government chooses to never violate the constraint $B \le \overline{B}(0)$, to be an equilibrium, the expected discounted value of steady state utility at $B = \overline{B}(0)$ must be higher than that of case 2, running up the debt one more time at price βp, repaying if the private sector recovers, and defaulting otherwise,

$$\begin{aligned}
&u((1-\theta)A\overline{y}, \theta A\overline{y} - (1-\beta)\overline{B}(0)) \\
&\quad + \beta(1-p)\left(\frac{u((1-\theta)A\overline{y}, \theta A\overline{y} - (1-\beta)\overline{B}(0))}{1-\beta(1-p)} + \frac{\beta p u((1-\theta)\overline{y}, \theta\overline{y} - (1-\beta)\overline{B}(0))}{(1-\beta)(1-\beta(1-p))}\right) \\
&\quad + \frac{\beta p}{1-\beta} u((1-\theta)\overline{y}, \theta\overline{y} - (1-\beta)\overline{B}(0)) \\
&\quad \ge u((1-\theta)A\overline{y}, \theta A\overline{y} + \beta p B'(\overline{B}(0)) - \overline{B}(0)) \\
&\quad + \beta(1-p)\left(\frac{u((1-\theta)AZ\overline{y}, \theta AZ\overline{y})}{1-\beta(1-p)} + \frac{\beta p u((1-\theta)Z\overline{y}, \theta Z\overline{y})}{(1-\beta)(1-\beta(1-p))}\right) \\
&\quad + \frac{\beta p}{1-\beta} u((1-\theta)\overline{y}, \theta\overline{y} - (1-\beta)B'(B'(\overline{B}(0)))).
\end{aligned} \tag{54}$$

Examining condition (54), we see that the crucial parameters in determining which of these two cases holds for a particular economy are the severity of the recession $1-A$ and the probability of recovery p. The government never chooses to sell debt $B'(\overline{B}(0)) > \overline{B}(0)$, gambling on recovery the next period unless the probability of recovery is high — which also implies that the price it obtains for this debt, βp, is high — and unless the recession is severe. If the default penalty $1-Z$ is sufficiently large, the government never chooses to sell debt $B'(\overline{B}(0)) > \overline{B}(0)$, but the range of parameters for which case 2 occurs is not sensitive to Z. The major impact of decreasing the default penalty $1-Z$ is to shrink the crisis zone by lowering $\overline{B}(0)$.

Figure 10 illustrates some optimal government strategies in case 1 as functions of the initial debt.

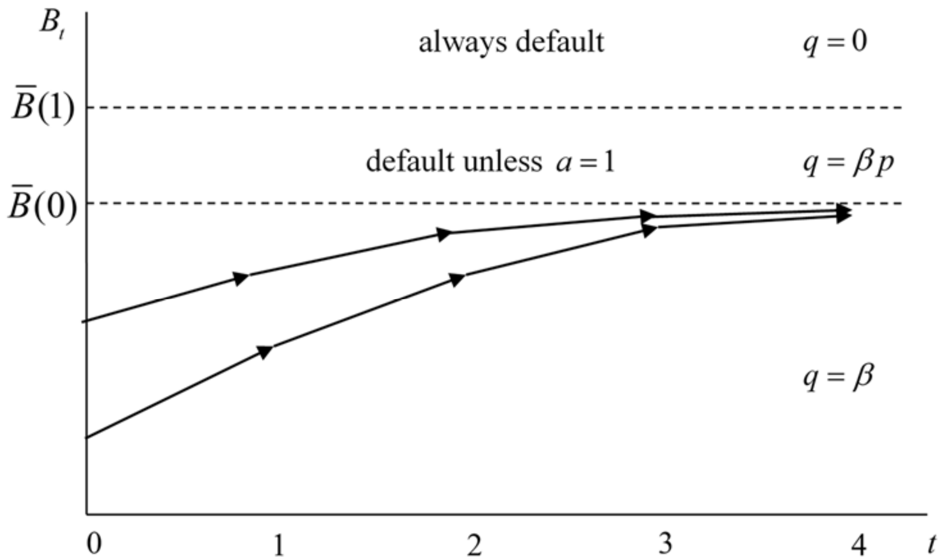

Figure 10: Optimal debt policy gambling for redemption when $B' \leq \overline{B}(0)$ binds

In case 2, the government chooses to violate the constraint $B \leq \overline{B}(0)$ with its sale of debt in period T, defaulting in period $T+1$ unless the private sector recovers. In figure 11, we illustrate two possibilities, which depend on B_0. In one $T=1$, and in the other $T=2$.

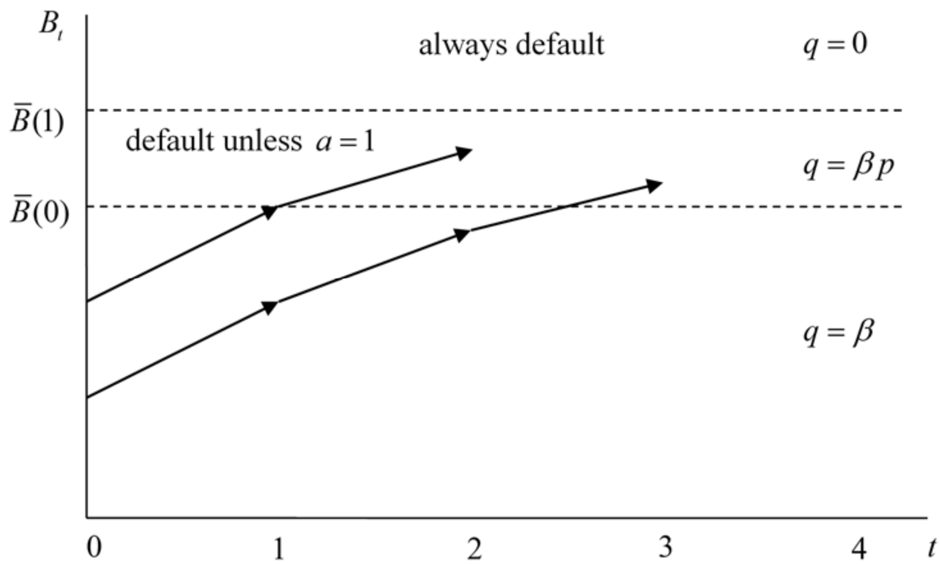

Figure 11: Optimal debt policy gambling for redemption when $B \leq \bar{B}(0)$ does not bind

5. Numerical results for the general model

We now solve the full model numerically to evaluate when gambling for redemption might happen in equilibrium. To solve for the equilibrium, we need to choose a functional form for the utility function,

$$u(c, g) = \log(c) + \gamma \log(g - \bar{g}). \tag{55}$$

The period length is one year, and the parameters we choose for our benchmark scenario are displayed in table 1.

Parameter	Value	Target
A	0.90	average government revenue loss (see figure 3)
Z	0.95	default penalty in Cole and Kehoe (1996)
p	0.20	expected recovery in five years
β	0.98	yield on safe bonds 2 percent annual
π	0.04	real interest rate in crisis zone 6 percent annual
γ	0.20	arbitrary value
θ	0.36	government revenues as a share of output
$\bar{g}$	25.0	minimum government expenditure

Table 1: Parameter values in the benchmark scenario

Although we do not derive our parameter values from a careful calibration of the model, we have chosen them to illustrate the possibilities that exist for reasonable parameter values. We choose a default penalty $1-Z$ of 5 percent, as do Cole and Kehoe (1996). Alonso-Ortiz *et al.* (2017) provide empirical evidence for a default penalty in this range. Mendoza and Yue (2012) and Sosa-Padilla (2014) build models in which sovereign default endogenously generates an output loss. Their quantitative exercises suggest larger output losses (between 6 and 12 percent). Our benchmark cost of default is smaller, but we assume that the default cost is permanent, whereas theirs is transitory. Sensitivity analysis shows that increasing the default penalty, that is, decreasing Z, increases the upper debt limits $\bar{B}(0)$ and $\bar{B}(1)$.

We have chosen the probability of a recovery p of 20 percent, which implies that the expected waiting time for a recovery is $1/p$ or five years. This parameter is crucial to generating gambling for redemption. Higher probabilities of recovery generate more gambling; lower probabilities of recovery generate less.

The probability of a panic π is arbitrary, and we fix it at 4 percent, generating spreads of approximately 4 percent over the safe rate in the crisis zone. This magnitude is consistent with the average risk premia across countries we observe in the data in figure 1. As we have explained, we could have π itself follow a Markov process generating the sorts of time-varying spreads observed in the data. Higher probabilities of self-fulfilling crises generate less gambling; lower probabilities of self-fulfilling crises generate more.

We have chosen A so that a recession results in a drop in GDP of 10 percent for the benchmark scenario. More severe recessions — we also report results for 20 percent drops in GDP — generate more gambling; less severe recessions generate less.

We have set a miminum government expenditure level $\bar{g}$ at 25 percent of GDP in normal times. In normal times the government collects tax revenue of 36, but sees this number fall to 32 in the recession. The minimum expenditure, which can be interpreted as entitlements that leave less room for discretionary spending, implies that the curvature of utility is high once the recession hits.

In the description of the model, we have assumed one-period bonds for simplicity. When we set the period length equal to one year, this assumption is restrictive for the economies in the Eurozone that we consider, although it makes more sense for emerging market economies such as Mexico in the 1990s (see Cole and Kehoe, 1996). Table 2 shows the average maturity of debt in the PIIGS and Germany.

	2008	2009	2010
Germany	6.3	5.9	5.9
Greece	8.4	7.9	7.1
Ireland	4.3	5.6	5.9
Italy	6.8	7.1	7.2
Portugal	6.2	6.1	5.8
Spain	6.6	6.4	6.6

Table 2: Weighted average term to maturity for total debt (years)

It is important to note that the average maturity data in table 2 are only suggestive of the maturity structure of debt that is relevant for our model. What matters for us is how much debt becomes due every period. Consider, for example, a government that has half of its debt evenly distributed across maturity dates in 10-year bonds and half in 1-year bonds. The simple average maturity of debt is 3.25 (= (1/2)×(1 + (10 + 1)/2)) years, but 55 percent of the debt becomes due every year, more than if the debt were evenly distributed across maturities with 2-year bonds. In data like those in table 2, future payments are discounted so that, with a 5 percent annual interest rate, the weighted average maturity of the debt in our example becomes 2.84 years, but 60.7 percent of the present discounted value of the debt becomes due every year.

Our theory is silent about the maturity structure of debt, which we take as exogenous. In our model, longer maturities are always better, since it is the need to roll over debt that generates the risk of default. There is a growing literature devoted to understanding maturity decisions; see, for example, Aguiar *et al.* (2016), Arellano and Ramanarayanan (2012), Broner *et al.* (2013), Hatchondo *et al*. (2016), Mihalache (2017), and Sanchez *et al.* (2016).

To understand the crucial role of debt maturity for our results, we extend the model to introduce the feature that a given fraction δ of the existing stock of debt becomes due every period. As in Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012), this feature renders the maturing of debt "memoryless," so that it is not necessary to keep track of the entire distribution of maturities of the debt.

Now the government's problem is to choose c, g, B', z to solve

$$\begin{aligned} V(s) &= \max \;\; u(c,g) + \beta E V(s') \\ &\text{s.t. } c = (1-\theta) y(a,z) \\ g + z\delta B &= \theta y(a,z) + q(B',s)(B' - (1-\delta)B)\,. \end{aligned} \tag{56}$$

In general, the entire distribution of maturities of debt would determine the amount of debt due at any given period, δB, and this is not necessarily related to average maturity. Notice that problem (56) reduces to the one-period debt case when $\delta = 1$, or to infinitely lived debt, consols, as δ tends to 0.

With multiperiod debt, we need to change the price functions (11) and (12). In the benchmark scenario, where $\bar{b}(0) < \bar{b}(1) < \bar{B}(0) < \bar{B}(1)$, we can define prices recursively when the sunspot ζ is such that there is no self-fulfilling crisis, for normal and recession times, respectively:

$$q(B',1) = \begin{cases} \beta\left[\delta + (1-\delta) q'(\cdot)\right] & \text{if } B' \le \bar{b}(1) \\ \beta(1-\pi)\left[\delta + (1-\delta) q'(\cdot)\right] & \text{if } \bar{b}(1) < B' \le \bar{B}(1) \\ 0 & \text{if } \bar{B}(1) < B' \end{cases} \tag{57}$$

$$q(B',0) = \begin{cases} \beta\left[\delta + (1-\delta) q'(\cdot)\right] & \text{if } B' \le \bar{b}(0) \\ \beta\left(p + (1-p)(1-\pi)\right)\left[\delta + (1-\delta) q'(\cdot)\right] & \text{if } \bar{b}(0) < B' \le \bar{b}(1) \\ \beta(1-\pi)\left[\delta + (1-\delta) q'(\cdot)\right] & \text{if } \bar{b}(1) < B' \le \bar{B}(0) \\ \beta p(1-\pi)\left[\delta + (1-\delta) q'(\cdot)\right] & \text{if } \bar{B}(0) < B' \le \bar{B}(1) \\ 0 & \text{if } \bar{B}(1) < B' \end{cases}\,. \tag{58}$$

Our benchmark scenario is a model where

$$\delta = 1/6\,, \tag{59}$$

which is the value implied by debt being evenly distributed across maturities with 6-year bonds, consistent with the empirical evidence in table 2, subject to the reservations that we have noted.

To calculate the thresholds for the safe zone $\bar{b}(0)$ and $\bar{b}(1)$, we need to make an assumption about what it means for international bankers to panic even if the government repays. To keep things simple, we assume that the panic lasts one period and then, if the country repays, international bankers resume lending. This is a somewhat arbitrary assumption about out-of-equilibrium behavior. The calculation (34) of $\bar{b}(1)$, for example, becomes

$$\begin{aligned} &u((1-\theta)\bar{y},\theta\bar{y}-\delta\bar{b}(1))+\frac{\beta u\left((1-\theta)\bar{y},\theta\bar{y}-\dfrac{1-\beta}{1-\beta(1-\delta)}\delta(1-\delta)\bar{b}(1)\right)}{1-\beta} \\ &=\frac{u((1-\theta)Z\bar{y},\theta Z\bar{y})}{1-\beta}. \end{aligned} \tag{60}$$

In figure 12 we plot the policy functions, together with the debt thresholds.

First, notice the value of the thresholds. The lower threshold is about 66 percent of GDP. Below this level, the government would not find it optimal to default even if it could not roll over its stock of debt. Any debt above this level makes the economy vulnerable to a self-fulfilling crisis. The upper threshold is about 126 percent of GDP. Above this level, interest payments are so large that the government would choose to default even if investors were willing to refinance the stock of debt.

If the economy is in normal times, then the optimal policy is to keep debt constant when the economy is in the safe region and decrease debt step-by-step when the economy is in the crisis zone. This is exactly the policy prescription of Cole and Kehoe (2000).

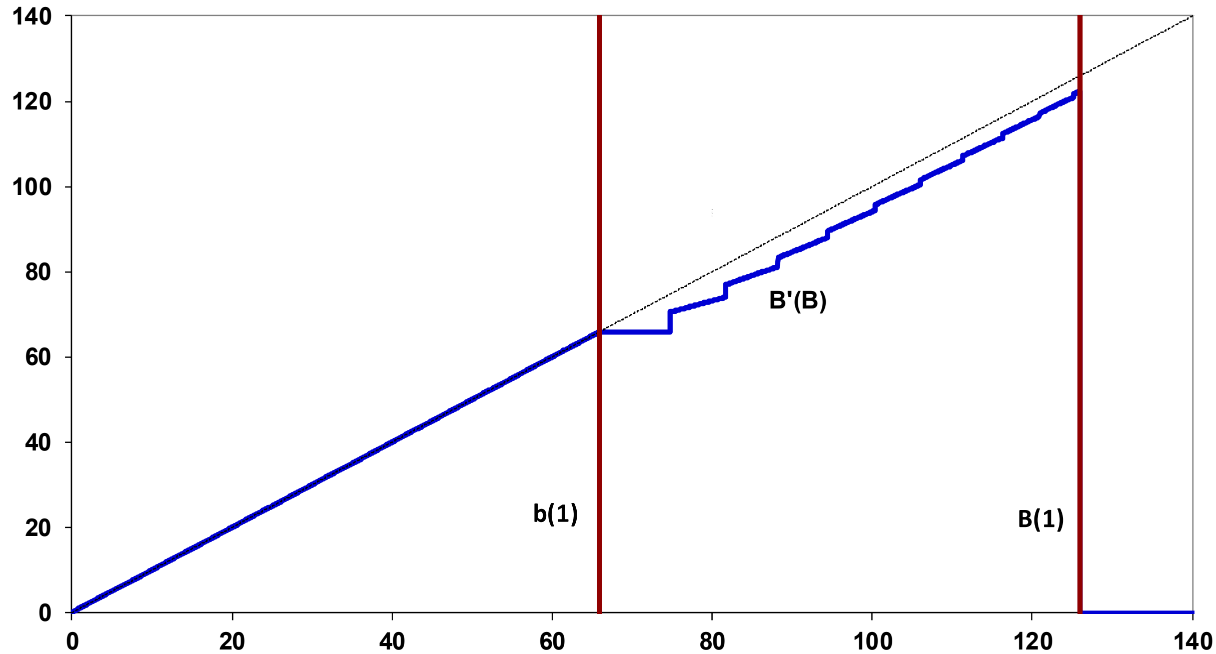

Figure 12: Policy function in normal times

Consider now that the economy unexpectedly falls into a recession, and GDP and fiscal revenues fall by 10 percent. Figure 13 plots the impact of such a change. Notice that, as soon as the economy enters into the recession, both the lower and upper thresholds decrease. In our numerical example, governments of countries with a low level of debt — below about 44 percent of the original GDP, about 49 percent of GDP when the recession hits — are still safe, and, as a result, they smooth consumption, increasing the level of debt until it reaches $\bar{b}(0)$.

Debt in the region between $\bar{b}(0)$ and $\bar{b}(1)$ displays interesting dynamics. These are debt levels where the economy was not vulnerable to a panic before the recession. When the recession hits, the economy becomes vulnerable and interest rates jump from 2 to 6 percent. In terms of debt dynamics, this region is split. Governments in countries with initial debt that is close to the safe threshold in the recession, $\bar{b}(0)$, choose to lower debt to avoid paying the spread and to avoid the probability of a self-fulfilling crisis. The government in a country with a larger initial level of debt gambles for redemption until debt reaches the safe threshold in normal times, $\bar{b}(1)$, and stays there waiting for a recovery. Recall that above $\bar{b}(1)$ the government would have to pay an even higher spread and would face an even higher probability of a self-fulfilling crisis than it would if it were at or below $\bar{b}(1)$.

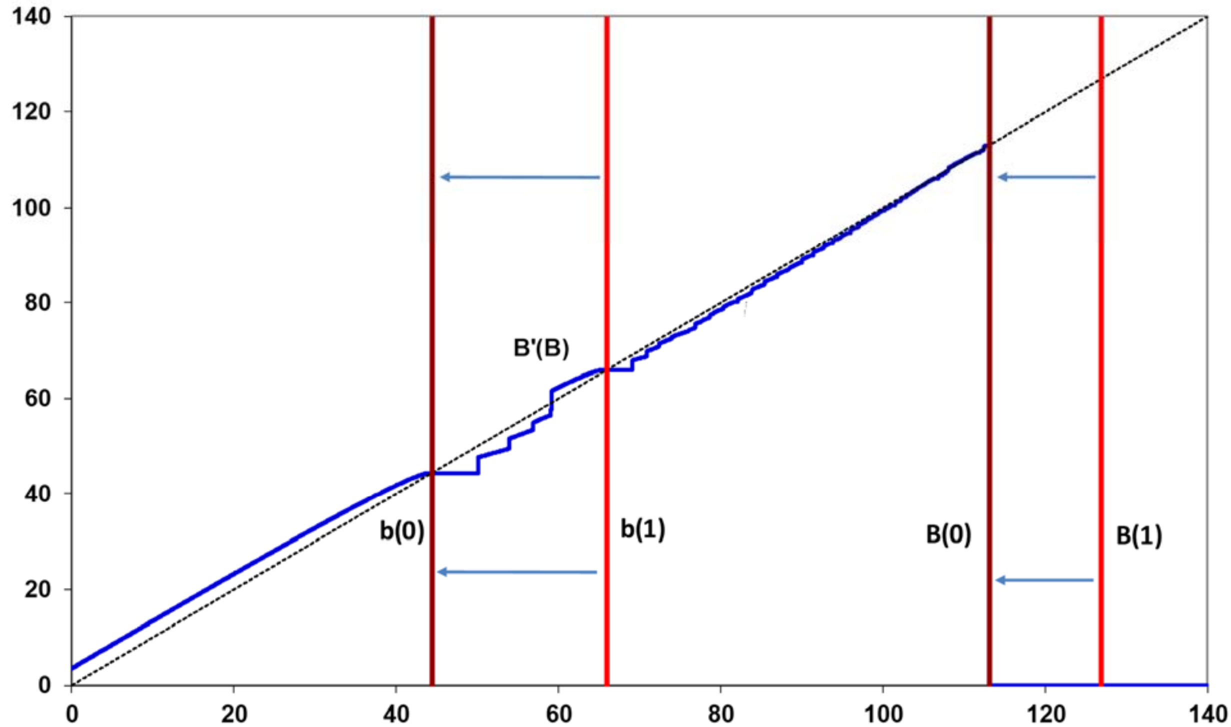

Figure 13: Policy function in recession

Initial debt in the region between $\bar{b}(1)$ and $\bar{B}(0)$ represents debt levels for economies that were vulnerable before the crisis. The interest rate stays high as before. Some economies with levels of debt above but close to 66 find it optimal to reduce their level of debt slowly and wait for a recovery. A close examination of the policy function at debt levels above 108 shows that it is optimal to increase debt slightly for some values of debt. In general, however, it is optimal to keep debt levels close to constant, waiting for recovery or a crisis to occur. Notice that a default cost is large enough, and the probability of a recovery small enough, to deter the governments in these countries from gambling beyond $\bar{B}(0)$ and risking default if a recovery does not happen.

Finally, in the region between $\bar{B}(0)$ and $\bar{B}(1)$, we find economies that in good times could refinance their debt by paying a high enough interest rate as long as investors were willing to roll over their debt, but that, as soon as the economy hits a recession, become insolvent and default immediately. This would be the only fundamental default we would observe in this model. All other defaults are generated by sunspots. Bocola and Dovis (2016) use a model with a choice of debt maturity to quantify the importance of sunspots shocks in the data.

Notice that the model accommodates all types of behavior: some governments increase their levels of debt at high, intermediate, and zero spreads, whereas other governments choose to reduce their levels of debt to reduce their interest payments, as in Cole and Kehoe (2000).

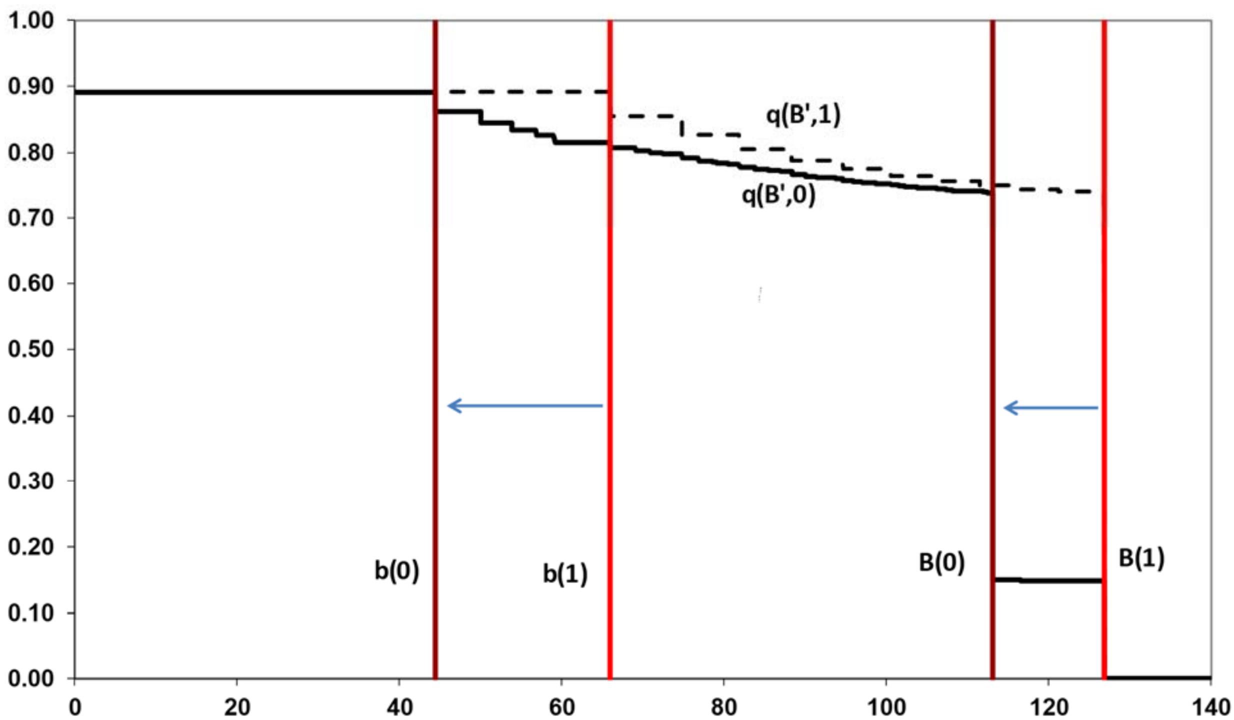

Figure 14: Bond prices

Figure 14 depicts the prices for bonds in normal times, $q(B',1)$, and in the recession, $q(B',0)$. Notice that the bond prices in the recursive schedules (57) and (58) generate more complex price determination than do the schedules of one-period bonds in the schedules (11) and (12) depicted in figure 7.

In our example, the regions where gambling for redemption occurs are relatively small, for levels of debt below but close to $\bar{b}(1)$ and $\bar{B}(0)$. As we have discussed, a key parameter in our exercise is the severity of the recession. More severe recessions generate more incentives to gamble for redemption. Figure 15 displays the policy function in a recession of 20 percent, $A = 0.80$, which is the magnitude of the most severe recessions experienced in southern European economies in 2008. In this case, the lower debt threshold falls from 66 to 23, and the upper threshold falls from 126 to 93. The impact of the recession is so large that immediate default occurs for levels of debt far below the upper threshold in normal times. In addition, the values of debt for which there will be an increase in financing costs is large. Finally, notice that, virtually for all levels of debt, the optimal policy is to gamble for redemption and to hope for a recovery. The only exception is a small range of debt above 23, for which it would be optimal to reduce debt in order to exit the crisis zone and access financing at the risk-free rate.

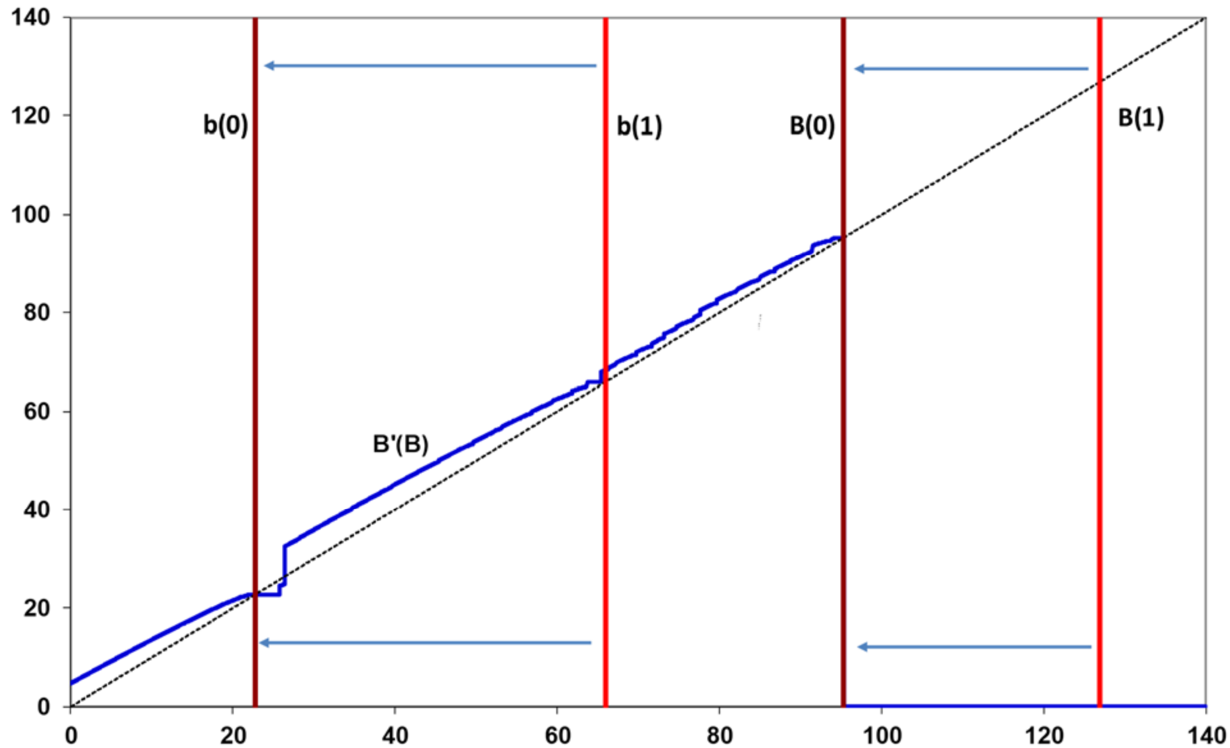

Figure 15: Policy function in a severe recession, A = 0.80

If we stay with the case of a severe recession where $A = 0.80$, but increase the probability of recovery to $p = 0.5$, we find even more gambling for redemption. Notice that, in figure 16, the optimal government policy is to increase debt for all bond levels except the points $b(0)$, $b(1)$, and $B(0)$.

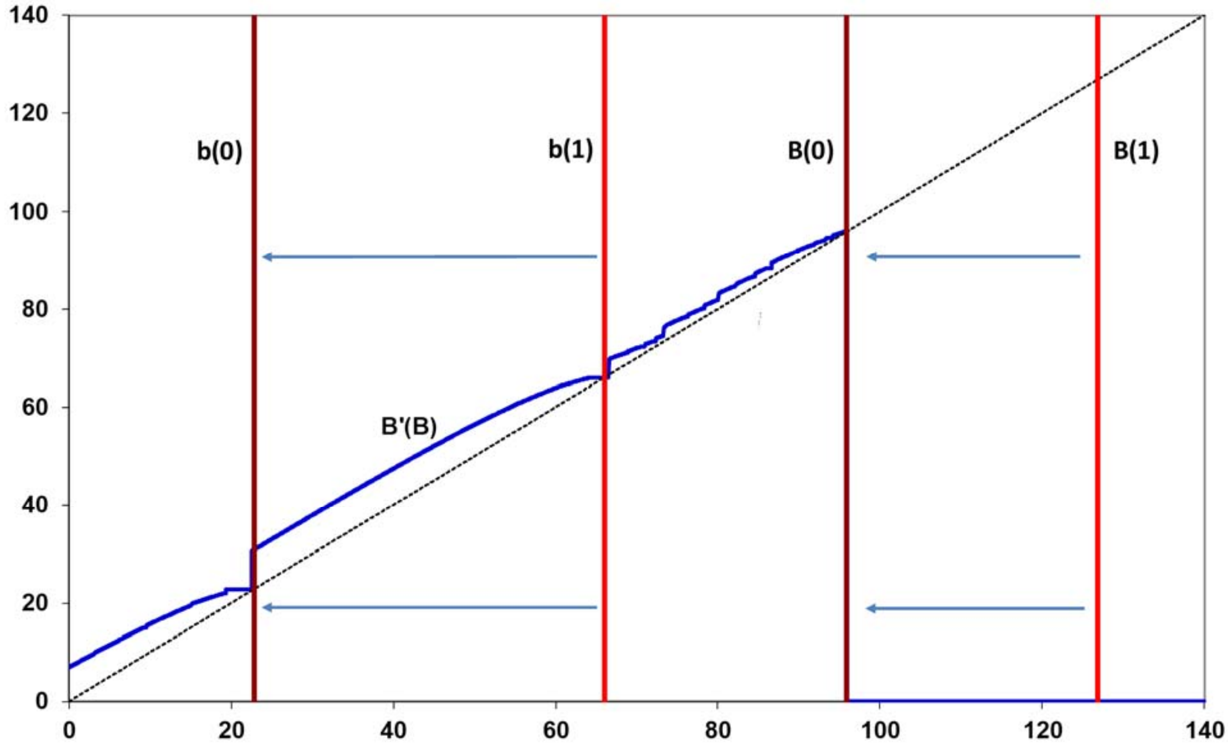

Figure 16: Policy function in a severe recession, $A = 0.80$, with a high probability of recovery, $p = 0.5$

To understand the role of debt maturity, we can solve the model for different values of δ. As δ is made smaller, the lower and upper thresholds converge because the government does not need to roll over debt, only pay debt service, as discussed in Cole and Kehoe (1996). In figure 17 we present the results for $\delta = 1.0$.

As δ becomes progressively larger, that is, the maturity of debt decreases, the thresholds decrease. In fact, the levels of the thresholds become more in line with the experience of emerging economies that borrow short term, where only low levels of debt are sustainable and rollover crises can emerge for low levels of debt (above 11 percent of normal times output). In contrast, as δ becomes smaller, the lower and upper thresholds converge as rollover risk disappears and large levels of debt can be sustained.

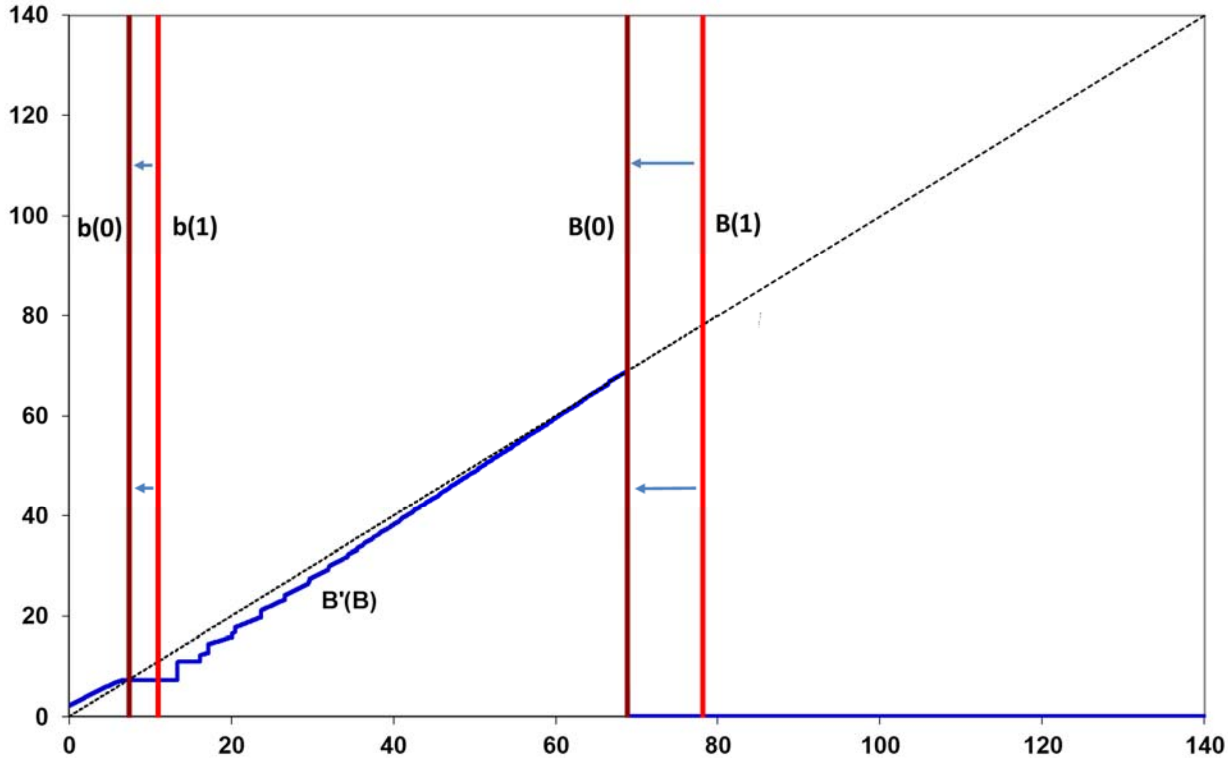

Figure 17: Policy function in recession for one-period bonds, $\delta = 1.0$

In this case, with a recession of 10 percent, there is no gambling for redemption, although the government does increase the levels of debt $B < b(0)$.

6. Conclusions

We provide a theory that accounts for governments optimally increasing their levels of debt even in situations in which increasing debt levels makes them more vulnerable to a sovereign debt crisis. The key trade-off is between, on the one hand, decreasing debt to avoid the risk premium and eliminate the possibility of a self-fulfilling crisis and, on the other hand, increasing debt to smooth consumption when faced with a recession. If the consumption smoothing effect dominates, we call this optimal policy gambling for redemption, and our model can relate this feature of the equilibrium to the fundamentals of the economy and the maturity structure of debt. We show that gambling for redemption would be limited for moderate recessions but would be widespread for more severe recessions.

We have restricted our analysis to a game between a government and atomistic international investors given the set of fundamentals. Two obvious extensions are left out of the paper. First, we could study the role of a third party with deep pockets that is able to bail out the government if a panic occurs. Roch and Uhlig (2016) and Conesa and Kehoe (2014) analyze the optimal debt policy in a model with the possibility of bailouts. Second, we could extend our analysis to study a game in which the government can endogenously choose to engage in costly reforms to increase fiscal revenues as done by Conesa *et al.* (2017).

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Appendix A: Characterization of equilibria of the model without self-fulfilling crisis

Our analysis in section 4 distinguishes between two cases. We consider first case 1, where the government chooses to never violate the constraint $B \leq \bar{B}(0)$. The optimal government policy is the solution to the dynamic programming problem:

$$V(B,a) = \max u((1-\theta)A^{1-a}\bar{y}, \theta A^{1-a}\bar{y} + \beta B' - B) + \beta EV(B',a') \tag{A.1}$$

$$\text{s.t. } B \leq \bar{B}(0).$$

We write the Bellman's equation explicitly as

$$V(B,0) = \max u((1-\theta)A\bar{y}, \theta A\bar{y} + \beta B' - B) + \beta(1-p)V(B',0) + \beta pV(B',1) \tag{A.2}$$

$$V(B,1) = \max u((1-\theta)\bar{y}, \theta\bar{y} + \beta B' - B) + \beta V(B',1). \tag{A.3}$$

The first-order condition is

$$\beta u_g((1-\theta)A^{1-a}\bar{y}, \theta A^{1-a}\bar{y} + \beta B' - B) = \beta EV_B(B',a'), \tag{A.4}$$

and the envelope condition is

$$V_B(B,a) = -u_g((1-\theta)A^{1-a}\bar{y}, \theta A^{1-a}\bar{y} + \beta B' - B). \tag{A.5}$$

The envelope condition implies that $V(B,a)$ is decreasing in B. A standard argument — that the operator on the space of functions defined by Bellman's equation maps concave value functions into concave value functions — implies that $V(B,a)$ is concave in B.

The first-order condition (A.4) implies that the policy function for debt $B'(B,a)$ is increasing in B while the policy function for government spending $g(B,a)$ is decreasing in B. Our assumption that

$$u_g((1-\theta)A\bar{y}, \theta A\bar{y} - B) > u_g((1-\theta)\bar{y}, \theta\bar{y} - B) \tag{A.6}$$

implies that $B'(0,0) > 0$ and that it is impossible for $B'(B,0) = B$ unless the constraint $B' \leq \bar{B}(0)$ binds, which implies that $B'(B,0) > B$.

We now consider case 2, where the government chooses to violate the constraint $B \leq \bar{B}(0)$ with its sale of debt in period T, defaulting in period $T+1$ unless the private sector recovers. The optimal government policy is the solution to the finite horizon dynamic programming problem:

$$
\begin{aligned}
V_t(B_t) = \max u((1-\theta)A\overline{y}, \theta A\overline{y} + \beta B_{t+1} - B_t) \\
+\beta(1-p)V_{t+1}(B_{t+1}) + \beta p \frac{u((1-\theta)\overline{y}, \theta\overline{y} + (1-\beta)B_{t+1}))}{1-\beta}
\end{aligned} \tag{A.7}
$$

$$
\text{s.t. } B_t \leq \overline{B}(0)\,.
$$

We solve this problem by backward induction with the terminal value function:

$$
\begin{aligned}
V_T(B_T) = \max u((1-\theta)A\overline{y}, \theta A\overline{y} + \beta p B_{T+1} - B_T) \\
+\beta(1-p)\frac{u((1-\theta)Z\overline{y}, \theta Z\overline{y}))}{1-\beta} + \beta p \frac{u((1-\theta)\overline{y}, \theta\overline{y} + (1-\beta)B_{T+1}))}{1-\beta}
\end{aligned} \tag{A.8}
$$

$$
\text{s.t. } \overline{B}(1) \geq B_{T+1} \geq \overline{B}(0)\,.
$$

We then choose the value of T for which $V_0(B_0)$ is maximal. As long as the constraint $B_{T+1} \geq \overline{B}(0)$ binds, we can increase the value of $V_0(B_0)$ by increasing T.

The algorithm for calculating the optimal policy function is a straightforward application of backward induction. We work backward from the period in which the government borrows at price βp and defaults in the next period unless a recovery of the private sector occurs. Define

$$
\begin{aligned}
V_T(B) = \max u((1-\theta)A\overline{y}, \theta A\overline{y} + \beta p B' - B) \\
+\beta(1-p)\frac{u((1-\theta)Z\overline{y}, \theta Z\overline{y}))}{1-\beta} + \beta p \frac{u((1-\theta)\overline{y}, \theta\overline{y} + (1-\beta)B'))}{1-\beta}
\end{aligned} \tag{A.9}
$$

$$
\text{s.t. } \overline{B}(0) \leq B' \leq \overline{B}(1)\,.
$$

The steps of the algorithm are as follows:

1. Solve for the value function $V_0(B)$ and the policy function $B_0'(B)$ on a grid of bonds B on the interval $[\underline{B}, \overline{B}(0)]$. We can set the lower limit $\underline{B}$ equal to any value, including a negative value. In an application with a given initial stock of debt, we could set $\underline{B} = B_0$. We have already solved this problem analytically. The solution is $B'(B) = \min[\hat{B}'(B), \overline{B}(1)]$ unless $B'(B) < \overline{B}(0)$, in which case $B_0'(B) = \overline{B}(0)$. Consequently,

$$
B_0'(B) = \max\left[\overline{B}(0), \min[\hat{B}'(B), \overline{B}(1)]\right]. \tag{A.10}
$$

The values of B for which $B'(B) < \overline{B}(0)$ are those for which it is not optimal to set $T = 0$.

2. Let $t=0$, and set $\tilde{B}_0 = \overline{B}(0)$.

3. Solve for the value function $V_{t+1}(B,0)$ and the policy function $B_{t+1}'(B)$ in the Bellman's equation

$$\begin{aligned} V_{t+1}(B,0) = \max u((1-\theta)A\overline{y}, \theta A\overline{y} + \beta B' - B) \\ + \beta(1-p)V_t(B',0) + \beta p \frac{u((1-\theta)\overline{y}, \theta\overline{y} + (1-\beta)B))}{1-\beta}. \end{aligned} \quad (A.11)$$

Let $\tilde{B}_t$ be the largest value of B for which $V_{t+1}(B,0) \geq V_t(B,0)$.

4. Repeat step 3 until $\tilde{B}_t = \underline{B}$.

Let T be such that $\tilde{B}_T = \underline{B}$. We can prove that $\underline{B} < \tilde{B}_{T-1} < \tilde{B}_{T-2} < \cdots < \tilde{B}_1 < \overline{B}(0)$. Our algorithm divides the interval $[\underline{B}, \overline{B}(0)]$ into subintervals $[\underline{B}, \tilde{B}_{T-1})$, $[\tilde{B}_{T-1}, \tilde{B}_{T-2})$,…, $[\tilde{B}_1, \overline{B}(0; p, 0)]$. If the initial capital stock B_0 is in the subinterval $[\tilde{B}_t, \tilde{B}_{t-1}]$, then the optimal government policy is to increase B, selling debt B, $\overline{B}(0) < B \leq \overline{B}(1)$, in period $t-1$, and defaulting in period t unless the private sector recovers. The optimal sequence of debt is B_0, $B_{t-1}'(B_0)$, $B_{t-2}'(B_{t-1}'(B_0))$ …, $B_0(B_1(\cdots(B_{t-1}'(B_0))))$.

Appendix B: The algorithm for computing an equilibrium in the general model

The algorithm computes the four debt thresholds, the value functions, and the policy functions.

1. Compute the value function $V\left(B,a,z_{-1},\zeta\right)$ of being in the default state, where $B=0$ and $z_{-1}=0$. To simplify notation, we denote it $V_d\left(a\right)$. Notice that these values are independent of the sunspot ζ , which becomes irrelevant after a default has occurred. The value function of defaulting in normal times, where $a=1$, is

$$V_d(1)=u(\left(1-\theta\right)Zy,\theta Zy)+\beta V_d(1) \tag{B.1}$$

which is just a constant:

$$V_d(1)=\frac{1}{1-\beta}u((1-\theta)Zy,\theta Zy)\,. \tag{B.2}$$

Similarly, in a recession, where $a=0$,

$$V_d(0)=u((1-\theta)AZy,\theta AZy)+\beta pV_d(1)+\beta(1-p)V_d(0) \tag{B.3}$$

which is also a constant:

$$\begin{aligned}V_d(0)=&\frac{1}{1-\beta\left(1-p\right)}u((1-\theta)AZy,\theta AZy)\\&+\frac{\beta p}{\left(1-\beta\left(1-p\right)\right)\left(1-\beta\right)}u((1-\theta)Zy,\theta Zy)\end{aligned}\,. \tag{B.4}$$

Notice that the value functions become those obtained above whenever the state of the economy determines that a self-fulfilling debt crisis happens or has happened in the past. To simplify notation, from this point on, we describe how to compute the value functions in the case of no default and drop the variable z_{-1} that determines whether a government has defaulted in the past and the sunspot ζ as arguments of the value functions. That is, from this point on, $V(B,a)$ is the value function if default has not happened today or anytime in the past.

2. Guess initial values for the thresholds $\bar{b}(0), \bar{b}(1), \bar{B}(0), \bar{B}(1)$, where $\bar{b}(0) < \bar{b}(1) < \bar{B}(0) < \bar{B}(1)$, and the associated prices. (We could also modify the algorithm to calculate an equilibrium in the case where $\bar{b}(0) < \bar{B}(0) < \bar{b}(1) < \bar{B}(1)$.)

3. Perform value function iteration on a finite grid of values of debt to compute the value function in normal times, $a = 1$. Guess an initial value function in normal times if default has not happened in the past, $\tilde{V}(B,1)$, and an optimal debt policy, which is needed to recursively compute the prices, $\tilde{q}(B',1)$, defined as in equation (57), in the case with multiperiod debt. Then:

3.1. For values of initial debt $B \le \bar{B}(1)$, the value function is

$$V(B,1) = \max[V_1(B,1), V_2(B,1)], \tag{B.5}$$

where V_1, V_2 are the value functions if next period bonds are in the regions $B' \le \bar{b}(1)$ or $\bar{b}(1) < B' \le \bar{B}(1)$, respectively:

$$\begin{aligned} V_1(B,1) = \max\, & u((1-\theta)y, \theta y + \tilde{q}(B',1)(B' - (1-\delta)B) - \delta B) + \beta \tilde{V}(B',1) \\ \text{s.t. } & B' \le \bar{b}(1) \end{aligned} \tag{B.6}$$

and

$$\begin{aligned} V_2(B,1) = \max\, & u((1-\theta)y, \theta y + \tilde{q}(B',1)(B' - (1-\delta)B) - \delta B) + \beta(1-\pi)\tilde{V}(B',1) + \beta\pi V_d(1) \\ \text{s.t. } & \bar{b}(1) < B' \le \bar{B}(1). \end{aligned} \tag{B.7}$$

3.2. For high values of initial debt, $B > \bar{B}(1)$, set $V(B,1) = V_d(1)$.

3.3. If there is multiperiod debt, compute the pricing function, $q(B,1)$, recursively, as in equation (57), using the optimal policy function.

3.4. If $\max_B \left| V(B,1) - \tilde{V}(B,1) \right| > \varepsilon$ and $\max_B \left| q(B,1) - \tilde{q}(B,1) \right| > \varepsilon$, where ε is a preset convergence criterion, then $\tilde{V}(B,1) = V(B,1)$ and $\tilde{q}(B,1) = q(B,1)$ and go to 3.1. Else, go to 4.

4. Perform value function iteration on a finite grid of values of debt to compute the value function in a recession, $a = 0$. Guess an initial value function if we are in a recession and the government

has not defaulted in the past: $\tilde{V}\left(B,0\right)$, and an optimal debt policy, which is needed to recursively compute the prices, $\tilde{q}(B',0)$, defined as in equation (58). Remember the value function in normal times, $V(B,1)$, is already known from step 3. Then:

4.1. For values of initial debt where $B \leq \bar{B}(0)$, the value function is

$$V(B,0) = \max[V_1(B,0), V_2(B,0), V_3(B,0), V_4(B,0)], \tag{B.8}$$

where V_1, V_2, V_3, V_4 are the associated value functions if next period bonds are in the regions $B' \leq \bar{b}(0)$, $\bar{b}(0) < B' \leq \bar{b}(1)$, $\bar{b}(1) < B' \leq \bar{B}(0)$, $\bar{B}(0) < B' \leq \bar{B}(1)$, respectively:

$$\begin{aligned} V_1(B,0) = \max u(&(1-\theta)Ay, \theta Ay + \tilde{q}(B',0)(B'-(1-\delta)B) - \delta B) \\ &+ \beta p V(B',1) + \beta(1-p)\tilde{V}(B',0) \\ \text{s.t. } & B' \leq \bar{b}(0) \end{aligned} \tag{B.9}$$

$$\begin{aligned} V_2(B,0) = \max u(&(1-\theta)Ay, \theta Ay + \tilde{q}(B',0)(B'-(1-\delta)B) - \delta B) \\ &+ \beta p V(B',1) + \beta(1-p)\pi V_d(0) + \beta(1-p)(1-\pi)\tilde{V}(B',0) \\ \text{s.t. } & \bar{b}(0) < B' \leq \bar{b}(1) \end{aligned} \tag{B.10}$$

$$\begin{aligned} V_3(B,0) = \max u(&(1-\theta)Ay, \theta Ay + \tilde{q}(B',0)(B'-(1-\delta)B) - \delta B) \\ &+ \beta p \pi V_d(1) + \beta p(1-\pi)V(B',1) \\ &+ \beta(1-p)\pi V_d(0) + \beta(1-p)(1-\pi)\tilde{V}(B',0) \\ \text{s.t. } & \bar{b}(1) < B' \leq \bar{B}(0) \end{aligned} \tag{B.11}$$

$$\begin{aligned} V_4(B,0) = \max u(&(1-\theta)Ay, \theta Ay + \tilde{q}(B',0)(B'-(1-\delta)B) - \delta B) \\ &+ \beta p \pi V_d(1) + \beta p(1-\pi)V(B',1) + \beta(1-p)V_d(0) \\ \text{s.t. } & \bar{B}(0) < B' \leq \bar{B}(1). \end{aligned} \tag{B.12}$$

4.2. For high values of initial debt, $B > \bar{B}(0)$, set $V(B,0) = V_d(0)$.

4.3. If there is multiperiod debt, compute the pricing function, $q(B,0)$, recursively, as in equation (58), using the optimal policy function.

4.4. If $\max_B \left|V\left(B,0\right) - \tilde{V}\left(B,0\right)\right| > \varepsilon$ and $\max_B \left|q\left(B,0\right) - \tilde{q}\left(B,0\right)\right| > \varepsilon$, then $\tilde{V}\left(B,0\right) = V\left(B,0\right)$ and $\tilde{q}\left(B,0\right) = q\left(B,0\right)$ and go to 4.1. Else, go to 5.

5. Update the threshold values:

5.1. Choose $\overline{b}_{new}(0)$ to be the highest point in the grid for B for which

$$u((1-\theta)Ay, \theta Ay - \delta B) + \beta p V((1-\delta)B,1) + \beta(1-p)V((1-\delta)B,0) \geq V_d(0) \text{ .} \quad \text{(B.13)}$$

5.2. Choose $\overline{b}_{new}(1)$ to be the highest point in the grid for which

$$u((1-\theta)y, \theta y - \delta B) + \beta V((1-\delta)B,1) \geq V_d(1) \text{ .} \quad \text{(B.14)}$$

5.3. Choose $\overline{B}_{new}(0)$ to be the highest point in the grid for which

$$\begin{aligned} V\left(B,0\right) \geq u((1-\theta)ZAy, \theta ZAy + q(B',0)(B'(B,0)-(1-\delta)B)) \\ + \beta p V_d(1) + \beta(1,p)V_d(0), \end{aligned} \quad \text{(B.15)}$$

where $q(B',0)$ are the prices computed in step 5.

5.4. Choose $\overline{B}_{new}(1)$ to be the highest point in the grid for which

$$V(B,1) \geq u((1-\theta)Zy, \theta Zy + q(B',1)(B'(B,1)-(1-\delta)B)) + \beta V_d(1) \text{ .} \quad \text{(B.16)}$$

5.5. If $\left|\overline{b}_{new}(0)-\overline{b}(0)\right| > \varepsilon$ or $\left|\overline{b}_{new}(1)-\overline{b}(1)\right| > \varepsilon$ or $\left|\overline{B}_{new}(0)-\overline{B}(0)\right| > \varepsilon$ or $\left|\overline{B}_{new}(1)-\overline{B}(1)\right| > \varepsilon$, then $\overline{b}(0) = \overline{b}_{new}(0), \overline{b}(1) = \overline{b}_{new}(1), \overline{B}(0) = \overline{B}_{new}(0), \overline{B}(1) = \overline{B}_{new}(1)$ and go to 3. Else, exit.

Notice that the lower threshold in normal times, $\overline{b}(1)$, can be computed directly since no information about policy functions is required. Hence, the iterative procedure would not be necessary, but we choose to do it this way to be consistent with the computation of the upper thresholds and the lower threshold in recession, which do depend on the policy function and hence require an iterative procedure.