PACS File #: 2660

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Game Theory and Its Relevance
for Economic Theorizing

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January 1980

Staff Report #: 54

According to the folklore of economics, game theory has failed. This paper argues that that is an incorrect interpretation of the game theory literature. When faced with a well-posed problem, game theory provides a solution. When faced with an ill-posed problem, game theory fails to provide a solution. This is, indeed, the best one can hope for from a method of analysis! Further, some suggestions are made for facing game theory with well-posed economic problems.

A General Method of Solution for Game Theory and Its Relevance for Economic Theorizing

by John Bryant

The determination of the mechanism for ordering strategies in a game theoretic conflict is the keystone of economic science, at least insofar as economics is to remain an outgrowth of that (otherwise relatively minor) school of English philosophy, Utilitarianism. A method for the solution of the general game is presented in this paper, and the implications for economic theorizing discussed.

The Structure of a Game

First, let us turn to the problem of game theory. Let A be the set of states of the world, and let T be the set of individuals. For each $t \in T$, let X_t be the subset of A for which individual t has his unique complete preordering (preference ordering) $\geq_t$ on points in X_t. In game theory it is usually assumed that $X_t = X \subset A$ for all t. Now we get to the conflict part of game theory. For each $t \in T$, there exists a collection of subsets of A, W_t, such that individual t can restrict the states of the world to belong to one of the sets in W_t, $S_t \in W_t$. The strategy of the individual t is choice of a member of W_t. In game theory it is typically assumed that for any collection of sets

$$\{S_t\}_{t \in T}$$

such that $S_t \in W_t$ for all t, $\bigcap_{t \in T} S_t$ is a point in X, and that for all t, $S_t \in W_t$ implies $S_t \subset X$. The problem in game theory is to determine a preference ordering on the sets $S_t \in W_t$ which, in an appropriate sense, is consistent with the preference ordering $\geq_t$ on points in X.

There is, of course, an obvious extension of the preference ordering on points to preference orderings on sets, dominance. For $S', S'' \subset X$, $S' \geq_t S''$ if $s' \in S'$, $s'' \in S''$ implies $s' \geq_t s''$. In practice this extension is not very useful, as dominance is a very strong condition. Moreover, dominance is the only noncontroversial extension of the preference ordering on points.

The resolution of this game theoretic conflict is the keystone of economics. Once one goes beyond the "Robinson Crusoe economy," one is in a game theoretic conflict situation. Whether one treats this problem explicitly or not, one's model must somehow resolve the conflict.

One obvious approach to the conflict situation is simply to start with preferences over sets. As we do not observe choices on points in X but on sets in

X anyway, preferences on sets can be our primitive. One first has to determine desirable properties for such preferences. However, one immediately confronts a disadvantage to this approach. Independence of preference orderings is not a desirable property. In games that are solvable by the traditional game theoretic approach we know that strategies chosen by an individual depend upon other individuals' evaluations of their own strategies. Moreover, the desirable properties of the preference orderings over sets may be exactly those produced by assuming preference orderings over points in X, and an appropriate mechanism of resolving the conflict. In any case, we proceed with the traditional game theoretic approach and start with preferences over points in X.

Resolution of the Conflict: Consistency

Game theory provides one basic approach to solving the conflict situation, consistency. Any "mechanism" for resolving the conflict situation should not be, roughly speaking, self-contradictory. It should be consistent with the preference orderings. Knowledge that the conflict is to be resolved by the mechanism should not give the individual reason to diverge from the mechanism's implication for the individual's own decision. Rather, the individual's best choice, given the mechanism, is to take part in the mechanism. Any mechanism should obey a fixed-point property.

For noncooperative games, this consistency property takes a simple form, equilibrium. $s^e = \bigcap_{t\varepsilon T} S_t^e$ for $S_t^e \varepsilon W_t$ for all t is an equilibrium if and only if for all t

$$s^e \geq_t \Big(\bigcap_{\substack{v\varepsilon T \\ v\neq t}} S_v^e \Big) \cap S_t$$

for all $S_t \varepsilon A_t$.

Application of consistency to cooperative games is not so obvious. Indeed, there is no noncontroversial definition of consistency for the cooperative game. As a result, the cooperative game has proven intractable. There is always one possibility worth considering when a problem proves intractable. The problem is intractable because it is not a well-posed problem. We now argue that this is true for the cooperative game.

The cooperative game is, ultimately, nonsensical. Remember that the game as a complete model is taken to describe the entire relevant environment. In a game at a point in time all players decide which $S_t \varepsilon W_t$ to restrict the world to. At this point in time all previous conversations, agreements, and so on, are irrelevant. Therefore cooperation is impossible. The fact of having to make independent choices at a point in time, the basic structure of the game, itself rules out cooperation in that choice. The cooperative game violates a basic assumption of Utilitarianism, individual choice.

Does this ultimate nonsensicalness of the cooperative game imply that game theory cannot confront the existence of coalitions? Not at all. One can have a coalition if individuals can bind themselves to strategies prior to the decision point of a game. But that decision to bind oneself must come at a particular previous point, a point of time in which the decision to bind can be analyzed as another noncooperative game. Coalitions appear in a sequence of noncooperative games in which decisions in early games determine the W_t's of later games. The whole sequence of games should be analyzed at the initial point as a noncooperative super game. The fact that this procedure is conceptually justified does not, of course, imply that in practice it is tractable.

However, that there may be many possible sequences of noncooperative games corresponding to a single cooperative game does not invalidate this procedure. Quite the contrary. Rather, it implies that a search for a general solution to a cooperative game is misguided, the cooperative game is not a well-posed problem. The cooperative game provides only a partial description of the relevant environment. Loosely speaking, in removing the key assumption of independent choice by allowing binding contracts, the cooperative game renders the model incomplete. Additional structure must be provided to replace the deleted assumption.

There is another possible structure for the game that should be considered. Decisions are not made at a point in time, but on an open set, before a point in time. However, it is a basic fact of existence that events always occur, there is no empty set. Therefore, in such a game there must exist a null strategy of not making a decision. Then the game can be viewed as an uncountable "sequence" of games in which strategies, or the null strategy, must be announced at each point in time.

As all games reduce to the simple noncooperative game, this is what we consider from now on. We restrict our attention to equilibria.

Nonuniqueness of Equilibria

Any "mechanism" for resolution of the conflict situation should be consistent. Therefore, if there is a unique equilibrium in a game, it must be the solution generated by any mechanism. Our search for the resolution of the conflict is, then, over.

For a wide class of games there is at least one equilibrium, which is not surprising given the Brouwer fixed-point theorem of analysis. However, uniqueness of the equilibrium is much more special. Therefore, we concentrate on the problem of multiple equilibria.

The interpretation of multiple equilibria has not been resolved. Yet, the crucial problem in game theory, and therefore in economics, is the resolution of the problem of multiple equilibria. We now turn to a discussion of several approaches to the resolution of this problem.

One approach is to reject traditional theory entirely. It simply cannot address a host of interesting problems.

A second approach is to restrict ourselves to models with a unique equilibrium, as being the only possible models that describe reality. This approach may be more positive than it sounds. The need to produce a unique equilibrium may yield useful restrictions on technologies, for example, which in turn generate strategy sets of the required form.

A third, and common, approach is to assume that any of the equilibria is possible. This has the advantage that at least one will describe the outcomes under all possible "mechanisms."

This approach is, of course, incomplete, as it does not show how any mechanism is generated, nor determine which equilibrium obtains. In particular, this approach leaves open the important question of the stability of the solution. What, if anything, might cause a shift from one equilibrium to another?

Nash has extended the notion of equilibrium to provide a broader solution concept (see [5], p. 106). If there are many equilibria, but with interchangeable strategies, then the game is solvable. Suppose $v \in V$ indexes the equilibria $s^v = \bigcap_{t \in T} S_t^v$, $S_t^v \in W_t$, is an equilibrium for $v \in V$. The game is solvable if for any

$$\{v_t\}_{t \in T}$$

where $v_t \in V$ for all t, $\bigcap_{t \in T} S_t^{v_t}$ also is an equilibrium. However, unless $s^u \geq_t s^v$ and $s^u \leq_t s^v$ for all u, $v \in V$, and all t, this extension is not appealing. If this indifference does not hold, there still is a substantive issue of which equilibrium obtains. The Nash solution concept may, however, provide a criterion for deciding that there is no mechanism for resolving the conflict situation, and that theory provides only a partial description of behavior.

More generally, a fourth approach is to take seriously the result that there is no solution. The imposition of rationality is not, by itself, enough to determine economic agents' decisions. We must look elsewhere for a completion of the model. As behavior does occur, this suggests that completion is a necessary part of an economic model. Completion cannot be ducked on the grounds of being outside the purview of economics, as the conflict solution is the essence of economics. The implication is that a coherent economic model must include "noneconomic" elements.

A fifth approach is to impose an extra-model procedure for picking a particular equilibrium. This approach has appeal from the point of view of positive economics. However, it is ultimately unsatisfying, as it leaves unanswered the question of how market participants actually get to this result. One can advocate a particular equilibrium on the grounds of good properties which it exhibits. This suggests that another agent, or the same agents at a previous time, set in place restrictions on strategies that make the desired equilibrium the unique equilibrium. Such restrictions should, however, be included in the model. Moreover, the super game in which those restrictions are imposed should be explicitly analyzed. Perhaps this is expressed in saying that economics is innately political economy.

The most common tack taken in economics is the <u>assumption</u> of competitive equilibrium. As competitive equilibrium preceded game theory, it is not clear how this assumption is intended. Perhaps its advocates intend that it be used only in games in which it is inherently a unique Nash equilibrium, but that seems to contradict its widespread, casual use. Rather, it seems to be an extra-model restriction. In some unexplained manner strategies are restricted to make competitive equilibrium a Nash equilibrium, and the only Nash equilibrium.

The last approach to resolving the conflict situation which we consider is additional assumptions on the individual's psyche beyond just a preference ordering on points in X. There is a common theme running through the preceding discussion of multiple equilibria. Multiple equilibria exist because the model is incomplete, it is underspecified. One obvious possible incompleteness is that preference orderings on states of the world are just not a complete description of the individual as economic animal. Perhaps it is not at all surprising that a preference ordering is not a complete description of the individual in a conflict situation, and that as a result equilibria are not, in general, unique. We should not expect to predict behavior solely on the basis of orderings on states of the world. There simply is more to an individual in a social situation than a preference ordering. Utilitarianism must be supplemented.

This last approach suggests two possible procedures. One could impose additional attributes on the individual psyche. This is not something which economists have much experience with, there is little guidance in the economics literature as to what such attributes should be. Secondly, one could start with an observed equilibrium, and assuming conventional utility functions try to characterize the set of additional attributes that could produce that equilibrium.

Minimax-Nash

There is, however, some guidance in the game theory literature to useful restrictions on the individual's psyche. Indeed, the second major contribution of game theory, after consistency, is the recommendation of an additional attribute of the individual psyche. When all else fails, the individual maximizes security level. This is the concept of minimax.

It is worth stressing that minimax is a further additional description of the individual. It is, for example, totally consistent for the individual's preference ordering to exhibit risk preference, and for the individual to be a "maximiner." We now consider the use of the minimax concept to solve the conflict situation.

In an earlier paper, [4], the author has suggested a particular solution concept for games with multiple equilibria. Each individual views herself as in a two-person game in which the other "person" is the rest of the economy playing one of the equilibria. The individual acts as a "maximiner" in this contracted game. If the outcome from all individuals behaving in this manner is an equilibrium, that is the solution. Of course, not all games are solvable in this sense.

What does happen if "minimax-Nash" does not produce a solution? There are two choices one can make. First, assume that the individual has additional attributes of her psyche which do result in an equilibrium being chosen. The individual is a "maximiner" only when that behavior generates an equilibrium. Second, one can take maximizing security level as more nearly absolute. For example, the individual's psyche is such that she maximizes security level in the contracted game if that yields equilibrium, but if that procedure does not yield equilibrium, the individual maximizes security level in the original game. This is, of course, a much stronger imposition of "maximin" behavior as it overrides the consistency criterion. It is worth noting that such a strong imposition of the minimax concept also could be used to solve games when there is no equilibrium.

Summary and Concluding Comments

Every conflict situation should be modeled as a noncooperative game. When the noncooperative game does not yield a unique equilibrium, this should be treated as an incomplete specification of the game. To complete an incomplete model one may add more to the assumptions on the individual psyche than just a preference ordering, or alter the model in some other way.

The keystone of economics, the ordering of strategies in a game theoretic conflict situation, is conceptually feasible using the above method, although in practice it may be difficult. This method of solution may in many applications imply that "economics" is better termed "political economy." Purely "economic" concepts do not make a complete economic model. This further suggests that use of this method may put us well on the way to the goal of coherent "micro-micro" modeling of political economy: models in which criteria like Pareto optimality are irrelevant. That is to say, Pareto optimality holds vacuously. What is, is, and that is all there is to it.

References

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