FEDERAL RESERVE BANK OF MINNEAPOLIS
Research Division

STAFF REPORT
No. 631

Constructing Pure-Exchange Economies with Many Equilibria

Revised December 2021

Pascal Gauthier
Nordic Investment Bank

Timothy J. Kehoe
University of Minnesota, Federal Reserve Bank of Minneapolis, and National Bureau of Economic Research

Erwan Quintin
University of Wisconsin-Madison

DOI: https://doi.org/10.21034/sr.631
Keywords: Uniqueness of equilibrium; Multiplicity of equilibrium; Computation of equilibrium
JEL classification: C60, C62, C63, D51

Federal Reserve Bank of Minneapolis
Research Department Staff Report 631

December 2021

Constructing Pure-Exchange Economies with Many Equilibria*

Pascal Gauthier
Nordic Investment Bank

Timothy J. Kehoe
University of Minnesota,
Federal Reserve Bank of Minneapolis,
and National Bureau of Economic Research

Erwan Quintin
University of Wisconsin School of Business

ABSTRACT

We develop a restart algorithm based on Scarf's (1973) algorithm for computing approximate Brouwer fixed points. We use the algorithm to compute all of the equilibria of a general equilibrium pure-exchange model with four consumers, four goods, and 15 equilibria. The mathematical result that motivates the algorithm is a fixed-point index theorem that provides a sufficient condition for uniqueness of equilibrium and a necessary condition for multiplicity of equilibria. Examining the structure of the model with 15 equilibria provides us with a method for constructing higher dimensional models with even more equilibria. For example, using our method, we can construct a pure-exchange economy with eight consumers and eight goods that has (at least) 255 equilibria.

JEL Codes: C60, C62, C63, D51

Keywords: uniqueness of equilibrium; multiplicity of equilibrium; computation of equilibrium

*This paper is forthcoming in *Economic Theory*. The authors developed the numerical example as part of a course on advanced topics in microeconomic theory at the University of Minnesota. Kehoe was the professor, and Gauthier and Quintin were students. Kehoe proposed the example, and Gauthier and Quintin implemented the restart algorithm and mapped out all of the equilibria. We thank the referee for pointing out an error in our original figure 1 and for making several valuable suggestions. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System. Contact information: pascal.gauthier@nib.int, tkehoe@umn.edu, equintin@bus.wisc.edu.

1. Introduction

This paper has been written in honor of Nicholas Yannelis on the occasion of his 65th birthday. In 1991, Nicholas was one of the founders of the Society for the Advancement of Economic Theory, and he followed Charalambos Aliprantis as the editor of the society's journal, *Economic Theory*, after the tragic death of Aliprantis in 2009. Nicholas, the SAET, and *Economic Theory* have played major roles in keeping alive the study of general equilibrium theory in economic theory, even as much of the focus in economic theory has shifted to game theory and decision theory. Keeping alive the study of general equilibrium theory is essential for economics as a whole because applications of general equilibrium in macroeconomics, international trade, industrial organization, and development have become increasingly important. Both Timothy Kehoe and Erwan Quintin are members of *Economic Theory*'s editorial board and regular participants in the SAET annual summer conferences. Together with coauthor Pascal Gauthier, Kehoe and Quintin are happy to be able to honor Yannelis with a paper that provides new insights into the fundamentals of general equilibrium theory.

This paper takes an approach to calculating the equilibria of general equilibrium pure-exchange economies and developing sufficient conditions for uniqueness of equilibrium and necessary conditions for multiplicity of equilibria that was developed during the 1960s, '70s, and '80s by Herbert Scarf, his coauthors, and his students at Yale University. Kehoe remembers discussing with Ludo Van der Hayden the sort of restart computational algorithm employed in this paper and discussing the fixed-point index theorem developed in this paper with Scarf, B. Curtis Eaves, and David Backus at Yale in the second half of the 1970s.

In this paper, we employ Scarf's algorithm to compute equilibria in pure-exchange economies. (See Scarf, 1967a and Scarf with Hansen, 1973.) Scarf's algorithm calculates equilibria by finding approximate fixed points of mappings from the unit simplex of price vectors into itself. The algorithm finds an approximate fixed point by moving from one subsimplex — a small subset of a grid of points — to another on the unit simplex. The algorithm starts at a corner of the simplex and cannot cycle. Since the algorithm has to terminate, it necessarily ends at an approximate fixed point. Scarf's algorithm can be viewed as a constructive proof of Brouwer's fixed point theorem.

We show that keeping track of the orientation of vertices on the subsimplices along the path traveled by Scarf's algorithm leads to a simple proof of an alternative version of Eaves and Scarf's

(1976) index theorem. This index theorem also suggests a restart algorithm that builds on the same principles as Scarf's original algorithm. To illustrate the value of this restart algorithm, we describe an economy that has a large number of equilibria and provide an example with 15 equilibria, in which all of the equilibria are connected via the computational algorithm we propose.

2. A simple pure-exchange economy

Consider a Walrasian pure-exchange economy with m consumers. Each of these consumers has a strictly concave and monotonically increasing utility function $u_i:\mathbb{R}^n_+\to\mathbb{R}$ that specifies his or her preferences over nonnegative vectors of consumption of n goods $\hat{x}^i=(x^i_1,\ldots,x^i_n)$. Each consumer is also endowed with a vector of these goods $w^i=(w^1_i,\ldots,w^i_n)$ that is strictly positive. An equilibrium of this economy is a vector of nonnegative prices, not all zero, $\hat{p}=(\hat{p}_1,\ldots,\hat{p}_n)$ and an allocation of consumption vectors to the consumers $(\hat{x}^1,\ldots,\hat{x}^m)$ such that

1. for each consumer i, $\hat{x}^i$ solves

$$\max\ u_i(x^i_1,\ldots,x^i_n)$$
$$\text{s.t. } \sum_{j=1}^n \hat{p}_j x^i_j \le \sum_{j=1}^n \hat{p}_j w^i_j$$
$$x^i_j \ge 0\,;$$

2. for each good j,

$$\sum_{i=1}^m \hat{x}^i_j \le \sum_{i=1}^m w^i_j\ .$$

Given our assumptions on utility functions and endowments, the utility maximizing response of consumer i to a price vector p can be expressed as a function $x^i(p)=(x^i_1(p),\ldots,x^i_n(p))$ that is continuous, at least at strictly positive price vectors. Since the budget constraint does not change if all prices are multiplied by a positive constant, the demand function is homogeneous of degree zero:

$$x^i(\theta p)=x^i(p)\ \text{ for all }\ \theta>0\,.$$

Furthermore, since utility is monotonically increasing, the demand function satisfies the budget identity

$$p'x^i(p) = p'w^i \, .$$

(Here, $p'x^i(p)$ is the inner product $\sum_{j=1}^n p_j x_j^i(p)$.)

There is a minor technical problem that the demand function may not be continuous at price vectors in which some $p_i = 0$. The easiest way to solve this problem is to impose the additional constraints

$$x_j^i \leq 2\sum_{k=1}^m w_j^k$$

in the maximization problem of consumer i, $i = 1,...,m$. Since the demand function of each consumer is continuous, none of these constraints can bind in some open neighborhood of any equilibrium.

The aggregate of all the consumers' responses to a price vector p is summarized in the excess demand function $f : \mathbb{R}_+^n \setminus \{0\} \rightarrow \mathbb{R}^n$, $f(p) = (f_1(p),\ldots,f_n(p))$, where

$$f_j(p) = \sum_{k=1}^m (x_j^i(p) - w_j^i) \, .$$

The properties of the individual demand functions $x^i(p)$ imply that $f(p)$ is continuous and is homogeneous of degree zero,

$$f(\theta p) = f(p) \text{ for all } \theta > 0,$$

and obeys Walras's law,

$$p' f(p) = \sum_{j=1}^n p_j f_j(p) = 0 \, .$$

Specifying consumer responses in terms of excess demand functions, we can simplify the definition of an equilibrium to a vector of prices $\hat{p} \in \mathbb{R}_+^n \setminus \{0\}$ such that

$$f_j(\hat{p}) \leq 0 \, , \; j = 1,...n \, .$$

Walras's law ensures that excess demand is actually equal to zero if the corresponding price is positive. In terms of our previous definition, the equilibrium is $\hat{p}, x^1(\hat{p}),\ldots,x^m(\hat{p})$. Since $f(p)$ and the individual $x^i(p)$ are homogeneous of degree zero, we need to impose some sort of normalization on prices: two price vectors p and q are essentially the same if $p = \theta q$ for some $\theta > 0$.

Given the properties of the excess demand function, we can restrict our search for equilibrium price vectors to the unit simplex

$$S = \left\{ p \in \mathbb{R}^n \,\middle|\, \sum_{j=1}^{n} p_j = 1,\ p_j \geq 0 \right\}.$$

Let us define

$$g_j(p) = \frac{\max[p_j + f_j(p), 0]}{\max[p_1 + f_1(p), 0] + \ldots + \max[p_n + f_n(p), 0]},\ j = 1, \ldots, n,$$

for all $p \in S$. Then it is easy to verify that g is continuous and maps S into itself, $g : S \to S$. Furthermore, any fixed point $\hat{p} = g(\hat{p})$ is a competitive equilibrium.

3. Computation of equilibria

Brouwer's (1912) fixed-point theorem says that there exists a price vector $\hat{p} \in S$ such that $\hat{p} = g(\hat{p})$, which implies that there exists a competitive equilibrium. Scarf (1967a) and Scarf with Hansen (1973) provide an algorithm for computing a fixed point of any continuous mapping of S into itself. Hence, Scarf's algorithm can be viewed as a constructive proof of the existence of competitive equilibrium.

To make Scarf's algorithm concrete, consider the two-dimensional unit simplex depicted in Figure 1. In general, a k-dimensional simplex is the convex hull of $k+1$ vectors, called vertices, $v^1, \ldots, v^{k+1}$, that have the property that the k vectors $v^1 - v^{k+1}, \ldots, v^k - v^{k+1}$ are linearly independent. The three vertices of the simplex S depicted in Figure 1, for example, are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. A face of a simplex is a simplex whose vertices are vertices of the larger simplex. In Figure 1, for example, the point $(1,0,0)$ is a zero-dimensional face of S, and the line segment that is the convex hull of $(1,0,0)$ and $(0,1,0)$ is a one-dimensional face. A simplicial subdivision (sometimes referred to as a *triangularization*) of S divides S into smaller simplices so that every point in S is an element of some subsimplex and the intersection of any two subsimplices is either empty or a face of both. The particular simplicial subdivision of the $(n-1)$-dimensional unit simplex that we employ has as vertices points of the form $(a_1 / D,...,a_n / D)$, where $a_1,...,a_n$ are nonnegative integers that sum to D. We refer to D as the grid size. Figure 1 illustrates this simplicial subdivision, the Freudenthal (1942) subdivision first used for computation of approximate fixed points by Kuhn (1968), for the case $n=3$ and $D=10$.[1]

Scarf's algorithm can be viewed as a constructive proof of a version of Sperner's (1928) lemma:

Sperner's Lemma: Assign to every vertex of a simplicial subdivision of S a label, an integer from the set $1,\dots,n$, with the property that every vertex v on the boundary of S receives a label i for which $v^i = 0$. Then, there exists a subsimplex $\hat{S}$ whose vertices have all of the labels $1,\dots,n$.

It is easy to show that if we label interior vertices in such a way that vertex v receives a label i for which $g_i(v) \geq v_i$, then any price vector in a completely labeled subsimplex, $\hat{p} \in \hat{S}$, is an approximate fixed point, $\|\hat{p} - g(\hat{p})\| < \varepsilon$, and hence an approximate equilibrium.

Proving the existence of a fixed point requires a non-constructive step in which we take a sequence of simplicial subdivisions generated by a sequence of D_i with $D_i \to \infty$, so that the mesh of the simplicial subdivision — the maximum distance between any two points in a subsimplex — in Figure 1 tends to 0. The sequence of points, each of which lies in a completely labeled subsimplex, generated by this sequence of D_i has a convergent subsequence (and perhaps more than one). The continuity of f implies that a point to which this subsequence converges is a fixed point of g and therefore an equilibrium of the economy.

Notice that an approximate equilibrium satisfies a condition of the form $\|f(\hat{p})\| < \varepsilon$, where the continuity of f implies that we can reduce ε by increasing D and thereby reducing the mesh of the simplicial subdivision. The distinction between approximate equilibria and exact equilibria is important for some economic problems: see, for example, Kubler and Schmedders, 2005. This distinction is not as relevant here. Although we could invent examples in which an approximate equilibrium is far from an exact equilibrium for a simplicial subdivision with a course mesh, that is not the case in the example with 15 equilibria that we study. For each of the 15 equilibria, we

[1] Scarf (1967a) and Scarf with Hansen (1973) do not use simplicial subdivisions but rather a concept called *primitive sets*. See the Appendix for a discussion.

use Newton's method to calculate the equilibrium, using the midpoint of the completely labeled subsimplex as the starting point. Newton's method is convergent as long as we start close enough to an equilibrium, which is a solution to the system of equations

$$f_i(p_1,...,p_{n-1},\overline{p}_n)=0,\ i=1,...,n-1,$$

where we normalize $\overline{p}_n$ as fixed at its initial value at the approxiate fixed point. (When and if Newton's method converges, we can renormalize prices so that $p_1+...+p_{n-1}+\overline{p}_n=1$.) Sure enough, in each of the 15 cases, Newton's method converges very rapidly to a vector of prices extremely close to the starting guess, and at this vector of prices, all n excess demand functions are equal to 0 to a specified tolerance determined by the machine accuracy of the computer.

The algorithm for finding a completely labeled subsimplex starts in the corner of S where there is a subsimplex with boundary vertices with all of the labels $2,\ldots,n$. Notice that the algorithm starts at the bottom-left corner in Figure 1. If the additional vertex of this subsimplex has the label 1, then the algorithm stops. Otherwise, it proceeds to a new subsimplex with all of the labels $2,\ldots,n$. The original subsimplex has two faces with all of these labels. One of them contains the interior vertex. The algorithm moves to the unique other subsimplex that shares this face. If the new vertex of this subsimplex has the label 1, the algorithm stops. Otherwise, it proceeds, moving to the unique subsimplex that shares the new face and has the labels $2,\ldots,n$. The algorithm cannot cycle. Cycling requires that some subsimplex be the first that the algorithm encounters for the second time, but the algorithm must have previously encountered both of the subsimplices that share the two faces of the subsimplex with the labels $2,\ldots,n$. Nor can the algorithm try to exit through a boundary face: the only boundary face that has the labels $2,\ldots,n$ is the corner in which the algorithm started, and we have just argued that the algorithm cannot cycle.

The crucial replacement step in the algorithm is easy to carry out. The first subsimplex encountered by the algorithm in Figure 1, for example, has as vertices the vectors

$$\begin{bmatrix} .9 \\ 0 \\ .1 \end{bmatrix},\ \begin{bmatrix} .9 \\ .1 \\ 0 \end{bmatrix},\ \begin{bmatrix} .8 \\ .1 \\ .1 \end{bmatrix}.$$

Since these vertices have the labels 2, 3, and 2, respectively, we move to the new subsimplex by keeping the second and third vertices and dropping the first vertex. Given the regular structure of the Freudenthal simplicial subdivision, elementary Cartesian geometry tells us that we can do this by completing a parallelogram, adding the second and third vertices and subtracting the first:

$$\begin{bmatrix} .9 \\ .1 \\ 0 \end{bmatrix} + \begin{bmatrix} .8 \\ .1 \\ .1 \end{bmatrix} - \begin{bmatrix} .9 \\ 0 \\ .1 \end{bmatrix} = \begin{bmatrix} .8 \\ .2 \\ 0 \end{bmatrix}.$$

This procedure can be readily extended to higher dimensions. A minor complication is that we need to know which two of the $n-1$ vertices we have to add together before subtracting the vertex being dropped. Scarf with Hansen (1973) provide a simple Fortran program for carrying out the replacement step.

Some readers will notice that to keep our discussion simple, we have focused our analysis on the original algorithm of Scarf with Hansen (1973). In doing so, we have ignored the rich literature on computation of fixed points — and the associated algorithms — that followed it. Zangwill and Garcia (1981) provide an early survey of this literature. One point worth noting is that there are algorithms that make the mesh of the simplicial subdivision smaller and smaller during computation, thus avoiding the need to use Newton's method at the end, as we do.[2] Merrill (1971) developed the first such algorithm. This sort of algorithm finds an approximate fixed point for a fixed simplicial subdivision, then refines the grid, in effect increasing D. It then finds a new, presumably more accurate approximate fixed point for the finer grid. Because this sort of algorithm starts with a solution for the previous simplicial subdivision to generate an almost complexly labeled subsimplex to start the algorithm for the subdivision with the smaller mesh, it is sometimes called a restart algorithm. The algorithm that we use also restarts at a previous solution, but it keeps the simplicial subdivision fixed and restarts in search of additional approximate fixed points rather than a more accurate calculation of the same fixed point.

4. The index theorem

We can show that there are an odd number of solutions to Scarf's algorithm for approximating fixed points — that is, an odd number of completely labeled subsimplices: the path followed from

[2] Scarf with Hansen (1973) do something similar to using Newton's method starting at an approximate fixed point.

the corner subsimplex with labels $2,\dots,n$ leads to a unique completely labeled subsimplex. Suppose that there is an additional completely labeled subsimplex. It then shares its face with labels $2,\dots,n$ with a unique other subsimplex. Restart Scarf's algorithm at this subsimplex. Either the additional vertex to this subsimplex had the label 1, in which case it is completely labeled, or it did not, in which case it has another face with all of the labels $2,\dots,n$. Move to the unique other subsimplex that shares this face and continue as before. The restarted algorithm cannot encounter any subsimplex in the path from the corner to the original completely labeled subsimplex. To do so, there must be some subsimplex in the path that is the first that it encounters for the second time, but it then must have previously encountered both of the two subsimplices that share the two faces of this subsimplex with the labels $2,\dots,n$. The algorithm must therefore terminate in yet another completely labeled subsimplex. Consequently, all completely labeled subsimplices come in pairs, except the original one located by the algorithm starting in the corner.

We can say more. Notice that if we order the vertices of the completely labeled subsimplex located from the bottom left by the algorithm in Figure 1 by their labels 1, 2, 3, then these vertices are oriented counterclockwise. Our index theorem says that if we assign a completely labeled subsimplex the index +1 if its labels are oriented counterclockwise and the index −1 if the labels are oriented clockwise, then the sum of the indices of all the completely labeled subsimplices is $+1$. Consequently, if we can ensure that any completely labeled subsimplex must have the index $+1$ — that is, its vertices must be oriented counterclockwise — we know that there is a unique completely labeled subsimplex.

Index Theorem: Let $\hat{S}_i$ be a completely labeled subsimplex, and let $\left\{\hat{S}_1,\hat{S}_2,\dots,\hat{S}_\ell\right\}$ be the set of all completely labeled subsimplices. Then,

$$\sum_{\hat{S}_i\in\left\{\hat{S}_1,\hat{S}_2,\dots,\hat{S}_\ell\right\}}\operatorname{index}\left(\hat{S}_i\right)=+1\,.$$

A simple proof of this index theorem follows the same lines as the previous argument that there are an odd number of completely labeled subsimplices; we need just to keep track of the orientation of vertices as we move along a path. Notice that in the path from the bottom-left corner in the first subsimplex, the vertices are oriented counterclockwise if we order them $v^{new},v^2,\dots,v^n$, where v^j

is the boundary vertex with label j and v^{new} is the interior vertex. It is easy to verify that counterclockwise orientation of the vertices corresponds to the determinant of the 3×3 matrix

$$[v^2 \quad v^3 \quad v^{new}] = \begin{bmatrix} .9 & .9 & .8 \\ 0 & .1 & .1 \\ .1 & 0 & .1 \end{bmatrix},$$

which is .01, being positive. Since v^{new} has the label 2 in this example, the algorithm does not stop. It moves to another almost completely labeled subsimplex with vertices $\tilde{v}^2$, $\tilde{v}^3$, $\tilde{v}^{new}$, where $\tilde{v}^3 = v^3$ but $\tilde{v}^{new}$ becomes $v^3 + v^{new} - v^2$ and v^2 becomes v^{new}. These steps — multiplying v^2 by -1 and adding v^3 and v^{new}, then interchanging the resulting vector with v^{new} in the above matrix — do not change the algorithm's determinant:

$$\det[\tilde{v}^2 \quad \tilde{v}^3 \quad \tilde{v}^{new}] = \det\begin{bmatrix} .8 & .9 & .8 \\ .1 & .1 & .2 \\ .1 & 0 & 0 \end{bmatrix} = 0.01 .$$

Continuing this process, we see that the oriented path index defined by the sign of the determinant of the matrix of ordered vertices $[v^2 \quad v^3 \quad v^{new}]$ remains equal to +1 along the path. When the path reaches the completely labeled subsimplex, we define the index of the subsimplex to be

$$\operatorname{sgn}(\det[[v^2 \quad v^3 \quad v^{new}]) = +1 .$$

Along paths that connect two other completely labeled subsimplices, we need only note that the index changes sign at the first step but then remains constant.

In Figure 1 there are seven completely labeled subsimplices. There are three along the path that starts at the lower left-hand corner and the two restarts from the first completely labeled subsimplex. Notice that the first completely labeled subsimplex has index +1 because its vertices are oriented counterclockwise:

$$\det[v^2 \quad v^3 \quad v^1] = \det\begin{bmatrix} .6 & .5 & .5 \\ .2 & .3 & .2 \\ .2 & .2 & .3 \end{bmatrix} = 0.01 .$$

In contrast, the completely labeled subsimplex near the lower right-hand corner has the index −1 because its vertices are oriented clockwise:

$$\det[v^2 \quad v^3 \quad v^1] = \det\begin{bmatrix} .2 & .2 & .3 \\ .6 & .7 & .6 \\ .2 & .1 & .1 \end{bmatrix} = -0.01.$$

If we start at this subsimplex, at the first step, we replace the vector v^1 with the vector $\widetilde{v}^{new} = v^2 + v^3 - v^1$, which changes the sign of the determinant. Along the subsequent path to another completely labeled subsimplex, the index remains equal to +1, finally arriving at a completely labeled subsimplex whose index is +1 — that is, one whose vertices are oriented counterclockwise.

The above index theorem is general in three respects. First, the index of a completely labeled subsimplex depends only on its local properties and not on the algorithm used to locate it. It does not even depend on the shape of the subsimplex, as long as the mesh is fine enough. For example, we could have started the algorithm in the almost completely labeled subsimplex with boundary vertices with the labels 1 and 2, as shown in the top corner of Figure 1, and although we would have generated a completely new set of paths, the indices of the completely labeled subsimplices would have remained the same. Second, the argument is easily extended to n dimensions. Third, the argument can be applied to any simplicial subdivision.

By developing conditions that guarantee that completely labeled simplices have positive indices, we could develop conditions that are necessary and sufficient for uniqueness of solutions to the fixed point algorithm. This approach is rather awkward, however, since the integer label contains little information about the underlying function $g(p)$. Eaves and Scarf (1976) take another approach and prove an index theorem for a version of Scarf's algorithm in which a vertex v receives a label that is a vector that includes all of the information about the value of $g(v)$. Taking limits as the mesh of the subdivision gets smaller — as the grid size D increases — we can show that the index of the Eaves-Scarf theorem for vector labels (almost always) agrees with the fixed point index introduced into economics by Dierker (1972):

$$\text{index}(\hat{p}) = \text{sgn}\left(\det[I - Dg(\hat{p})]\right).$$

Once again,

$$\sum_{\{\hat{p}|\hat{p}=g(\hat{p})\}} \text{index}(\hat{p}) = +1,$$

and the condition that $\text{index}(\hat{p}) = +1$ at all equilibrium prices $\hat{p}$ is necessary and sufficient for uniqueness of equilibrium.[3] See the Appendix for a discussion of other index theorems found in the economic literature.

5. The restart algorithm

The proof of the index theorem that we provided in the previous section suggests a restart algorithm to compute approximate equilibria other than those found by the Scarf approach. Our restart algorithm builds on the same principles as Scarf's algorithm and requires adding only a few lines of code to the Fortran routine provided by Scarf with Hansen (1973).

Consider a completely labeled subsimplex reached by the Scarf algorithm. Adjacent to this simplex are $n-1$ almost completely labeled subsimplices, including the penultimate subsimplex visited by the algorithm. Assuming for concreteness that the algorithm began from a subsimplex with all labels but the label 1, dropping label 1 from the final subsimplex would take us back to this penultimate subsimplex. Furthermore, following the Scarf procedure after this step would lead back to the corner in which the algorithm started. On the other hand, dropping any vertex other than the vertex with label 1 would lead to an almost completely labeled subsimplex not yet visited by the algorithm. Restarting the algorithm from this new subsimplex — making sure not to reverse the original step in the second step, which is trivially guaranteed by imposing a first-in, last-out rule — starts a new path through the subset of almost completely labeled subsimplices, which yields a natural way to search for completely labeled subsimplices other than those reachable by the standard Scarf search algorithm.

This search method has several virtues beyond the simplicity inherent in Scarf's algorithm. First, by the same argument we used in the previous section to establish that the number of completely

[3] The typical statement of the index theorem has the index defined as $\text{index}(\hat{p}) = \text{sgn}\left(\det[I - Dg(\hat{p})]\right)$ and the sum of the indices as $(-1)^n$.

labeled subsimplices must be odd, a restart cannot return to the subsimplex from which it started. Likewise, the algorithm cannot exit through a boundary, non-corner subsimplex.

The restarted algorithm can, however, lead to a corner subsimplex. Consider, for example, the case in which a unique completely labeled subsimplex exists. Then we know that the path followed by Scarf algorithm from each of the n corners leads to that unique subsimplex. It follows immediately that any restart from that subsimplex must lead to a corner.

If the restarted algorithm does not lead to a corner, then it must lead to a completely labeled subsimplex. Each new subsimplex, in turn, gives $n-1$ new ways to restart the algorithm and navigate different parts of the subset of almost completely labeled subsimplices. Note that implementing this method requires only a rule to restart the algorithm once a completely labeled subsimplex or a corner is encountered. Once a corner is encountered, returning to the previous completely labeled subsimplex and dropping a different vertex is a natural way to proceed.

Implementing this procedure thus requires recording restarts already executed from completely labeled subsimplices. Doing so also prevents the algorithm from becoming trapped in long cycles. While individual restarts cannot return to the subsimplex from where they started, they can display longer cycles involving two or more completely labeled simplices.

Figure 1 illustrates the possible outcomes of restarts. The solid path that starts in the lower-left corner finds its first completely labeled simplex in eight steps. The two possible restarts from that simplex lead to the same completely labeled simplex from which one new restart possibility opens up, which leads to yet another completely labeled simplex and two more restart possibilities.

To see how the restart algorithm can produce long cycles, consider the dashed path from the top corner. A first restart from the resulting completely labeled simplex leads to an adjacent completely labeled simplex. A restart from that new subsimplex brings us back to the first subsimplex encountered from the top.

A simple rule that prevents the repetition of a restart performed from an equilibrium simplex already found prevents the algorithm from cycling. With this rule in place, the restart search algorithm must end after a finite number of steps. Each newly found approximate equilibrium offers $n-1$ restart possibilities. Navigating each of the corresponding paths takes but a few seconds in applications of typical sizes. The algorithm continues only as long as it keeps yielding new equilibria. Once all possible restarts lead either to already discovered completely labeled subsimplices or to a corner, the procedure ends. Naturally, there is no guarantee that this method will reveal all completely labeled subsimplices. It does, however, yield all completely labeled subsimplices that are connected by a path of almost completely labeled subsimplices to the equilibria reached from a corner by Scarf's algorithm.

6. Constructing exchange economies with many equilibria

To illustrate the search algorithm described in the previous section, we consider a family of pure-exchange economies with 2^k goods and 2^k consumers, where $k = 1, 2, 3, \ldots$, and select parameters such that each of these economies has a large number of equilibria.

We first consider the case in which $k = 1$ — that is, the case with two goods and two consumers. Consumer i, $i = 1, 2$, chooses $(\hat{x}_1^i, \hat{x}_2^i)$ to solve

$$\max \ \frac{a_1^i (x_1^i)^b - 1}{b} + \frac{a_2^i (x_2^i)^b - 1}{b}$$

$$\text{s.t. } \hat{p}_1 x_1^i + \hat{p}_2 x_2^i \le \hat{p}_1 w_1^i + \hat{p}_2 w_2^i$$

$$x_j^i \ge 0 .$$

To make things simple, we choose the parameters a_j^i and w_j^i to be symmetric in the sense that $a_1^1 = a_2^2$, $a_2^1 = a_1^2$, $w_1^1 = w_2^2$, and $w_2^1 = w_1^2$. This symmetry implies that

$$f_1(0.5, 0.5) = f_2(0.5, 0.5) .$$

Walras's law then implies that $(0.5, 0.5)$ is an equilibrium. A sufficient condition for the economy to have multiple equilibria is that $\text{index}(0.5, 0.5) = -1$. Symmetry implies that this condition is also necessary because if $(\hat{p}_1, \hat{p}_2)$ is an equilibrium, then $(\hat{p}_2, \hat{p}_1)$ and $\text{index}(\hat{p}_1, \hat{p}_2) = \text{index}(\hat{p}_2, \hat{p}_1)$.

In the case in which $n = 2^k = 2$, we can calculate

$$\text{index}(\hat{p}_1, \hat{p}_2) = \text{sgn}\left(\det[-\bar{J}(\hat{p})]\right) = \text{sgn}\left(-\frac{\partial f_1(\hat{p}_1, \hat{p}_2)}{\partial p_1}\right);$$

that is, the index of an equilibrium in a two-good pure-exchange economy is –1 if the graph of f_1 crosses 0 from below.[4]

There are two classic results from general equilibrium theory that provide sufficient conditions for uniqueness of equilibrium: that the demand function f satisfies the weak axiom of revealed preference, or that it exhibits gross substitutability (see, for example, Kehoe, 1998).

Before discussing conditions that imply the weak axiom of revealed preference, we note that the utility functions are homothetic because there is a monotonically increasing transformation of u_i that is homogenous of degree one,

$$g(u_i) = \left(bu_i + 2\right)^{\frac{1}{b}},\ g'(u_i) = \left(bu_i + 2\right)^{\frac{1-b}{b}} > 0 .$$

Antonelli (1886), Gorman (1953), and Nataf (1953) show that if utility functions are homothetic and identical but endowments are different, then aggregate excess demand behaves as if it were excess demand of a single consumer. Consequently, it satisfies the weak axiom of revealed preference. Small perturbations to an economy with identical homothetic utility functions do not necessarily result in an economy whose excess demand function satisfies the weak axiom, but they do result in an economy with a unique equilibrium. The reasoning is simple: the equilibrium price set varies continuously at a regular economy. If $(a_1^1, a_2^1) = (a_1^2, a_2^2)$, then the economy has a unique equilibrium in which

$$\frac{\partial f_1(\hat{p}_1, \hat{p}_2)}{\partial p_1} < 0 ;$$

therefore, it is a regular economy. Consequently, to construct an example with multiple equilibria, we need to have consumers with sufficiently different utility functions; that is, (a_1^1, a_2^1) should be very different from (a_1^2, a_2^2). In the Appendix, we show that we need a large deviation from $(a_1^1, a_2^1) = (a_1^2, a_2^2)$ to obtain multiple equilibria.

[4] We could also calculate $\text{index}(\hat{p}_1, \hat{p}_2) = \text{sgn}\left(-\frac{\partial f_2(\hat{p}_1, \hat{p}_2)}{\partial p_2}\right)$. The homogeneity of degree zero of f and the assumption that $f(\hat{p}_1, \hat{p}_2) = 0$ imply that the two partial derivatives have the same sign.

Eisenberg (1961) and Chipman (1974) show that if utility functions are homothetic but possibility different and the endowment vectors (w_1^1, w_2^1) and (w_1^2, w_2^2) are proportional, then aggregate excess demand again behaves as if it were excess demand of a single consumer. As in the case of identical homothetic utility functions, we can argue that to construct an example with multiple equilibria, we need to have consumers whose endowments are sufficiently non-proportional. In the Appendix, we show that we need a large deviation from proportional endowments to obtain multiple equilibria.

Even if (a_1^1, a_2^1) is very different from (a_1^2, a_2^2) and (w_1^1, w_2^1) is far from proportional to (w_1^2, w_2^2), the economy has a unique equilibrium if the individual demand functions exhibit gross substitutability. The reasoning is simple in the two-good economy. To have

$$\frac{\partial f_1(\hat{p}_1, \hat{p}_2)}{\partial p_1} > 0,$$

we need to have the income effect in the Slutsky (1915) decomposition of demand for at least one consumer dominate the income effect, which is always negative. As is well known, if the CES curvature parameter b satisfies $b \geq 0$ so the elasticity of substitution, $\eta = 1/(1-b)$, is greater than or equal to 1, then

$$\frac{\partial x_j^i(p_1, p_2)}{\partial p_j} < 0.$$

Consequently, $b \geq 0$ is sufficient for uniqueness of equilibrium. Mas-Colell (1991) points out that a weaker, more general condition on individual demand functions,

$$\frac{x' D^2 u_i(x) x}{D u_i(x) x} \leq 4,$$

implies a monotonicity property that implies that demand is strictly deceasing in its own price. He credits this result to Mityushin and Polterovich (1978), although it turns out that Milleron (1974) has the same condition. In the case of CES utility functions, the Milleron-Mityushin-Polterovich condition becomes

$$\frac{x' D^2 u_i(x) x}{D u_i(x) x} = 1 - b \leq 4.$$

Consequently, for arbitrary utility parameters (a_1^i, a_2^i) and endowments (w_1^i, w_2^i), the condition that $b \geq -3$ — or equivalently, $1/(1-b) \geq 0.25$ — is sufficient for uniqueness. Therefore, we need $b < -3$ — that is, $\eta = 1/(1-b) < 0.25$ — for non-uniqueness. In the Appendix, we show how much lower than -3 b has to be for there to be multiple equilibria.

Consider the example of a two-good pure-exchange economy of non-uniqueness developed by Kehoe (1991, 1998):

$$\begin{bmatrix} a_1^1 & a_2^1 \\ a_1^1 & a_2^1 \end{bmatrix} = \begin{bmatrix} 1024 & 1 \\ 1 & 1024 \end{bmatrix},$$

$$\begin{bmatrix} w_1^1 & w_2^1 \\ w_1^1 & w_2^1 \end{bmatrix} = \begin{bmatrix} 12 & 1 \\ 1 & 12 \end{bmatrix},$$

and $b = -4$. Kehoe (1998) draws the Edgeworth box for this economy, which depicts its three equilibria: $\hat{p}^1 = (0.88708, 0.11292)$ with $\text{index}(\hat{p}^1) = +1$, $\hat{p}^2 = (0.50000, 0.50000)$ with $\text{index}(\hat{p}^2) = -1$, and $\hat{p}^3 = (0.11292, 0.88708)$ with $\text{index}(\hat{p}^3) = +1$.

The referee has suggested two interpretations of this example. The first is a world with two symmetric countries that open to trade. In the non-symmetric equilibria, the countries are treated very differently. It is worth noting that since all three equilibria are in the core, each country benefits from opening to trade, but one benefits far more. The second interpretation is an economy with uncertainty, in which two groups of consumers have very different subjective probabilities and are very optimistic in the sense that they put large probabilities $\pi_j^i = a_j^i / (a_1^i + a_2^i)$ on the states where they are rich. In this interpretation, we see that competitive markets can generate highly unequal final allocation even if there is no aggregate risk.

We now consider families of economies with 2^k goods and 2^k consumers. The economy in which $k = 1$ is the previously mentioned two-good economy with two consumers. The economy in which $k = 2$ has four goods and four consumers. Consumers 1 and 2 have preferences and endowments of goods 1 and 2 very close to those in two-good economy, but they also put small utility weights on, and have small endowments of, goods 3 and 4. We can think of consumers 1 and 2 and goods 1 and 2 as forming a sub-economy. Similarly, consumers 3 and 4 and goods 3 and 4 form another sub-economy that is symmetric. We impose parameters so that the relationship between sub-economy 1 and sub-economy 2 is the same as the one between consumer 1 and consumer 2 in our original two-good economy.

In the 2^k-good economy, consumer i, $i = 1,...2^k$ orders consumption vectors $x^i \in \mathbb{R}_+^{2^k}$ according to the utility function

$$u_i(x^i) = \sum_{i=1}^{2^k} a_j^i \frac{(x_j^i)^b - 1}{b},$$

where $a_j^i > 0$ for all i and j and $b < 1$. Letting $\theta_j^i = (a_j^i)^\eta$ for all i and j, consumer i's demand function is

$$x_j^i(p) = \begin{cases} \dfrac{\theta_j^i \sum_{\ell=1}^{2^k} p_\ell w_\ell^i}{p_j^\eta \sum_{\ell=1}^{2^k} \theta_\ell^i p_\ell^{1-\eta}} & \text{if } p_j > 0 \text{ and } \dfrac{\theta_j^i \sum_{\ell=1}^{2^k} p_\ell w_\ell^i}{p_j^\eta \sum_{\ell=1}^{2^k} \theta_\ell^i p_\ell^{1-\eta}} \le 2\sum_{i=1}^{2^k} w_\ell^i \\ 2\sum_{i=1}^{2^k} w_\ell^i & \text{if } p_j = 0 \text{ or if } \dfrac{\theta_j^i \sum_{\ell=1}^{2^k} p_\ell w_\ell^i}{p_j^\eta \sum_{\ell=1}^{2^k} \theta_\ell^i p_\ell^{1-\eta}} > 2\sum_{i=1}^{2^k} w_\ell^i \end{cases}$$

for all $p \in \mathbb{R}_+^{2^k}$, where w_j^i denotes consumer i's endowment of commodity j. (Remember that bounding the individual demand functions by twice the aggregate endowment of the corresponding good has no effect on the aggregate excess demand function in an open neighborhood of any equilibrium, nor does it have any effect on our index theorems.) The excess demand function for commodity j, $j = 1,...2^k$, is

$$f_j(p) = \sum_{i=1}^{2^k} \left(x_j^i(p) - w_j^i \right).$$

An equilibrium price for this economy is a price $\hat{p} \in \mathbb{R}_+^{2^n} \setminus \{0\}$ such that $f_j(\hat{p}) \le 0$ for all j. In fact, given our utility functions, any equilibrium price must be strictly positive, which, by Walras's law, implies that $f_j(\hat{p}) = 0$ for all j.

The economy in which $k = 1$ is the two-good economy with three equilibria that we have studied. Notice that in this economy,

$$\begin{bmatrix} \theta_1^1 & \theta_2^1 \\ \theta_1^1 & \theta_2^1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

To specify utility functions, we use the parameters θ_j^i, rather than a_j^i, because they are less messy. With each increase in k, we double the size of the economy by building on two-good, two-consumer sub-economies, as shown in tables 1 and 2:

Table 1: Utility parameters θ_j^i

$i\downarrow, j\rightarrow$	1	2	3	4	$\ldots$	2^k-1	2^k
1	4^k	4^{k-1}	$4^{\max[k-2,0]}$	$4^{\max[k-2,0]}$	$\ldots$	1	1
2	4^{k-1}	4^k	$4^{\max[k-2,0]}$	$4^{\max[k-2,0]}$	$\ldots$	1	1
3	$4^{\max[k-2,0]}$	$4^{\max[k-2,0]}$	4^k	4^{k-1}	$\ldots$	1	1
4	$4^{\max[k-2,0]}$	$4^{\max[k-2,0]}$	4^{k-1}	4^k	$\ldots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
2^k-1	1	1	1	1	$\ldots$	4^k	4^{k-1}
2^k	1	1	1	1	$\ldots$	4^{k-1}	4^k

Table 2: Endowments w_j^i

$i\downarrow, j\rightarrow$	1	2	3	4	$\ldots$	2^k-1	2^k
1	12^k	12^{k-1}	$12^{\max[k-2,0]}$	$12^{\max[k-2,0]}$	$\ldots$	1	1
2	12^{k-1}	12^k	$12^{\max[k-2,0]}$	$12^{\max[k-2,0]}$	$\ldots$	1	1
3	$12^{\max[k-2,0]}$	$12^{\max[k-2,0]}$	12^k	12^{k-1}	$\ldots$	1	1
4	$12^{\max[k-2,0]}$	$12^{\max[k-6,0]}$	12^{k-1}	12^k	$\ldots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
2^k-1	1	1	1	1	$\ldots$	12^k	12^{k-1}
2^k	1	1	1	1	$\ldots$	12^{k-1}	12^k

We have already seen the parameters for the economy in which $n = 2^k = 2$. For the economy in which $n = 2^k = 4$,

$$\begin{bmatrix} \theta_1^1 & \theta_2^1 & \theta_3^1 & \theta_4^1 \\ \theta_1^2 & \theta_2^2 & \theta_3^2 & \theta_4^2 \\ \theta_1^3 & \theta_2^3 & \theta_3^3 & \theta_4^3 \\ \theta_1^4 & \theta_2^4 & \theta_3^4 & \theta_4^4 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 1 & 1 \\ 4 & 16 & 1 & 1 \\ 1 & 1 & 16 & 4 \\ 1 & 1 & 4 & 16 \end{bmatrix},$$

and

$$\begin{bmatrix} w_1^1 & w_2^1 & w_3^1 & \mathrm{w}_4^1 \\ w_1^2 & w_2^2 & w_3^2 & \mathrm{w}_4^2 \\ w_1^3 & w_2^3 & w_3^3 & w_4^3 \\ w_1^4 & w_2^4 & w_3^4 & w_4^4 \end{bmatrix} = \begin{bmatrix} 144 & 12 & 1 & 1 \\ 12 & 144 & 1 & 1 \\ 1 & 1 & 144 & 12 \\ 1 & 1 & 12 & 144 \end{bmatrix}.$$

To construct the parameters for the economy in which $n = 2^k = 8$, we take the 4×4 utility parameter matrix for the economy in which $k = 2$, multiply every element by 4, then put this matrix as the upper-left diagonal submatrix and as the lower-right diagonal submatrix in the 8×8 utility parameter matrix for the economy in which $k = 3$. We set every element in the two 4×4 off-diagonal submatrices equal to 1. We construct the endowment matrices in the same way, but we multiply by 12 at every step. Two of the crucial features of replicating the economy by increasing k are that we maintain symmetry, which implies that $(\hat{p}_1, ..., \hat{p}_{2^k}) = (2^{-k}, ..., 2^{-k})$ is an equilibrium, and that $\det[-J(\hat{p})] < 0$, which implies that there are multiple equilibria.

We now demonstrate that we can use the restart algorithm to find all of the equilibria of the economy in which $n = 2^k = 4$. We use the Freudenthal simplicial subdivision of the unit simplex of prices in which $D = 1,000,000$. We start Scarf's algorithm in the corner $(1,0,0,0)$. Once the algorithm locates the first approximate equilibrium, there are three almost completely labeled subsimplices adjacent to the equilibrium simplex from which the algorithm can be restarted, corresponding to each column of the equilibrium matrix other than the one last added by Scarf's algorithm. As Figure 2 shows, each of these three restart possibilities leads to a new equilibrium. Applying our restart algorithm to these new equilibria then leads to additional equilibria. In fact, exhausting every possible restart produces 15 equilibria, displayed in Table 3. Figure 2 displays one of several possible road maps through this set of 15 approximate equilibria based on our restart algorithm.

Table 3: Equilibria

equilibrium	p_1	p_2	p_3	p_4	index
1	0.25000	0.25000	0.25000	0.25000	−1
2	0.43187	0.06813	0.43187	0.06813	−1
3	0.43187	0.06813	0.06813	0.43187	−1
4	0.06813	0.43187	0.06813	0.43187	−1
5	0.06813	0.43187	0.43187	0.06813	−1
6	0.25640	0.25640	0.42036	0.06684	+1
7	0.25640	0.25640	0.06684	0.42036	+1
8	0.42036	0.06684	0.25640	0.25640	+1
9	0.06684	0.42036	0.25640	0.25640	+1
10	0.49969	0.49969	0.00031	0.00031	−1
11	0.12828	0.87110	0.00031	0.00031	+1
12	0.87110	0.12828	0.00031	0.00031	+1
13	0.00031	0.00031	0.49969	0.49969	−1
14	0.00031	0.00031	0.87110	0.12828	+1
15	0.00031	0.00031	0.12828	0.87110	+1

The fixed point indices in the final column can be calculated by calculating either the orientation of the completely labeled subsimplex in the grid with $D = 1,000,000$ that approximates $\hat{p}$ or the sign of $\det[-\bar{J}(\hat{p})]$. The two calculations agree.

A natural question to ask is whether there are equilibria other than those connected to equilibria encountered from one corner of the simplex by our restart algorithm. Notice that the indices of our 15 equilibria sum to +1, so that our list of 15 equilibria satisfies a necessary condition to be exhaustive. A further search that starts Newton's method at every point in a fine grid on the unit simplex strongly suggests that there are no other equilibria in our example. See the Appendix for details about this exhaustive search.

Mas-Colell (1977) proves that for any non-empty, compact set of prices on the unit simplex, there is an economy with utility maximizing consumers for which the prices in this set are the set of equilibria. For many of the sets of prices, however, the equilibria are not regular. Mas-Colell therefore restricts himself to regular economies and proves that for any set of an odd number of prices, there is an economy with utility maximizing consumers for which the prices in this set are the set of equilibria and that they satisfy the index theorem. Applying Mas-Colell's result to our example implies that we could choose two additional price vectors and make them into equilibrium price vectors, one with index +1 and the other with index −1. The economy that would have these 17 equilibria, however, would presumably have a very different aggregate excess demand function, at least near the two new equilibria. The strong assumptions imposed by our choice of symmetric CES utility functions limit the relevance of Mas-Colell's (1977) results for our family of examples.

In fact, intuition strongly suggests that this economy generates exactly 15 equilibria. As tables 3 and 4 make clear, the economy juxtaposes two sets of two consumers — consumers 1 and 2 on one hand, and consumers 3 and 4 on the other — that care primarily about two distinct sets of goods. We know that each of these two sub-economies would independently yield three equilibria. Juxtaposing these two sets of consumers thus gives us nine possible permutations in which the consumers in each sub-economy trade primarily with one another. These are the first nine equilibria listed in Table 3.

What explains the presence of the other six equilibria? Equilibria 10, 11, and 12 feature price vectors that assign little value to the endowments of consumers 3 and 4. Consumers 1 and 2 therefore have most of the income in equilibrium and, as a result, exert a dominant influence on relative prices. Since they symmetrically value goods 3 and 4, all equilibria must be such that $p_3 = p_4$. On the other hand, the three possible relative prices between goods 1 and 2 that obtained when consumers 1 and 2 traded only with one another remain possible. In effect, therefore, shutting down the influence of consumers 3 and 4 produces three new equilibria. Symmetrically, shutting down the influence of consumers 1 and 2 by putting low values on their endowments — goods 1 and 2 — gives us another 3 equilibria: equilibria 13, 14 and 15 in Table 3.

Notice that a similar phenomenon occurs with the two equilibria besides $\hat{p}^2 = (0.5, 0.5)$ in the two-good economy. Since $\text{index}(p^2) = -1$, we know that there have to be at least two more equilibria. In one of them, in which $\hat{p}^1 = (0.88708, 0.11292)$, the influence of consumer 2 is shut down by the low value of his endowment. In the other, in which $\hat{p}^3 = (0.11292, 0.88708)$, the influence of consumer 1 is shut down. Notice that the sum of the indices of the first nine equilibria in Table 3 is −1, which implies that there have to be at least two more equilibria. When we shut down the influence of one of the sub-economies, however, a two-consumer sub-economy with three equilibria still determines prices. In the two-good economy, however, shutting down the influence of one of the consumers leaves us with a one-consumer sub-economy that necessarily has a unique equilibrium.

This reasoning suggests that as k grows large, the number of equilibria becomes large as well. We can quite precisely predict the progression of that number. Let N_k be the number of equilibria in the *k*-fold replica described in tables 1 and 2. When the size of the economy doubles — as we go from k to k+1 — each of the $(N_k)^2$ possible permutations of the equilibria that existed at stage k produces an equilibrium at stage $k+1$. It also becomes possible to shut down the influence of each of the two sub-economies that compose replica $k+1$. This yields another $2N_k$ equilibria. Our logic prompts us to conjecture:

Conjecture: For all $k \geq 0$, $N_{k+1} = (N_k)^2 + 2N_k$.

Given the compelling economic intuition that underlies this conjecture, it should come as no surprise that the expression above, as the sum of an odd and an even number, is always odd, as it must be if the conjecture is correct. In addition, while we leave a formal proof of the conjecture for future work, it is clear that the expression above provides at least a lower bound on the number of equilibria the construction outlined in this section can generate. That lower bound quickly becomes large, as Table 4 illustrates.

Table 4: Number of equilibria in *k*-fold replica

k	2^k	N_k
1	2	3
2	4	15
3	8	255
4	16	65,535
5	32	4,294,967,295

References

Antonelli, G. B. (1886) *Sulla Teoria Matematica Della Economia Politica*, Tipografia del Folchetto. (Translated as "On the Mathematical Theory of Political Economy," translated by J. S. Chipman in J. S. Chipman, L. Hurwicz, M. K. Richter, and H. F. Sonnenschein, editors (1971), *Preferences, Utility, and Demand: A Minnesota Symposium*. Harcourt Brace Jovanovich, 333–360.)

Arrow, K. J. and T. J. Kehoe (1994) "Distinguished Fellow: Herbert Scarf's Contributions to Economics," *Journal of Economic Perspectives*, 8: 161–181.

Brouwer, L. E. J. (1912) "Über Abbildung von Mannigfaltigkeiten**,"** *Mathematische Annalen*, 71: 97–115.

Chipman, J. S. (1974) "Homothetic Preferences and Aggregation," *Journal of Economic Theory*, 8: 26–38.

Debreu, G. (1970) "Economies with a Finite Set of Equilibria," *Econometrica*, 38: 387–392.

Dierker, E. (1972) "Two Remarks on the Number of Equilibria of an Economy," *Econometrica*, 40: 951–953.

Eaves, B. C. and H. E. Scarf (1976) "The Solution of Systems of Piecewise Linear Equations," *Mathematics of Operations Research*, 1: 1–27.

Eisenberg, E. (1961), "Aggregation of Utility Functions," *Management Science*, 7: 337–350.

Freudenthal, H. (1942) "Simplizialzerlegungen von Beschränkter Flachheit," *Annals of Mathematics*, 43: 580–582.

Gorman, W. M. (1953) "Community Preference Fields," *Econometrica*, 21: 63–80.

Guillemin, V. and A. Pollack (1974) *Differential Topology*, Prentice Hall.

Kehoe, T. J. (1980) "An Index Theorem for General Equilibrium Models with Production," *Econometrica*, 48: 1211–1232.

Kehoe, T. J. (1985) "Multiplicity of Equilibria and Comparative Statics," *Quarterly Journal of Economics*, 100: 119–147.

Kehoe, T. J. (1991) "Computation and Multiplicity of Equilibria," in W. Hildenbrand and H. Sonnenschein, editors, *Handbook of Mathematical Economics*, volume IV, North-Holland, 2049–2143.

Kehoe, T. J. (1998) "Uniqueness and Stability," in A. P. Kirman, editor, *Elements of General Equilibrium Analysis*, Basil Blackwell, 38–87.

Kubler, F. and K. Schmedders (2005) “Approximate versus Exact Equilibria in Dynamic Economies,” *Econometrica*, 73: 1205–1235.

Kuhn, H. W. (1968) “Simplicial Approximation of Fixed Points,” *Proceedings of the National Academy of Sciences of the United States of America*, 61: 1238–1242.

Kuhn, H. W. (1969) “Approximate Search for Fixed Points,” *Computing Methods in Optimization Problems*, 2: 199–211.

Lefschetz, S. (1926) “Intersections and Transformations of Complexes and Manifolds,” *Transactions of the American Mathematical Society*, 28: 1–49.

Mas-Colell, A. (1977) “On the Equilibrium Price Set of an Exchange Economy,” *Journal of Mathematical Economics*, 4: 117–126.

Mas-Colell, A. (1991) “On the Uniqueness of Equilibrium Once Again,” in W. A. Barnett, B. Cornet, C. d’Aspremont, J. Gabszewicz, and A. Mas-Colell, editors, *Equilibrium Theory and Applications*, Cambridge University Press, 275–296.

Merrill, O. H. (1971) “Applications and Extensions of an Algorithm That Computes Fixed Points of Certain Non-empty Convex Upper Semi-continuous Point to Set Mappings,” Department of Industrial Engineering, University of Michigan, Technical Report 71-7.

Milleron, J. C. (1974) “Unicité et Stabilité d l’Equilibre en Economie de Distribution,” Séminaire de Econométrie Roy-Malinvaud, unpublished manuscript.

Milnor, J. (1965) *Topology from the Differentiable Point of View*. University Press of Virginia.

Mityushin, L. G. and W. M. Polterovich (1978) “Kriterij Monotonnosti Funktsij Sprosa,” *Ekonomika i Matematicheskie Metody*, 14: 122–128. (Translated as “Criteria for Monotonicity of Demand Functions,” in Munich Personal RePEc Archive (2010) https://mpra.ub.uni-muenchen.de/20097/.)

Nataf, A. (1953) “Possibilité d’Agrégation dans le Cadre de la Théorie des Choix,” *Metroeconomica*, 5 :22–30.

Scarf, H. E. (1967a) “On the Approximation of Fixed Points of a Continuous Mapping,” *SIAM Journal of Applied Mathematics*, 15: 1328–1343.

Scarf, H. E. (1967b) “The Core of an N Person Game,” *Econometrica*, 35: 50–69.

Scarf, H. E. (1986) “Neighborhood Systems for Production Sets with Indivisibilities,” *Econometrica*, 54: 507–532.

Scarf, H. E. (1991), “The Origins of Fixed Point Methods,” in J. K. Lenstra, A. H. G. Rinnooy Kan, and A. Schrijver, editors, *History of Mathematical Programming, A Collection of Personal Reminiscences*, North-Holland, 126–134.

Scarf, H. E. with the collaboration of T. Hansen (1973) *The Computation of Economic Equilibria*, Yale University Press.

Slutsky, E. (1915) "Sulla Teoria del Bilancio del Consumatore," *Giornale degli Economisti e Rivista di Statistica*, 51: 1–26. (Translated as "On the Theory of the Budget of the Consumer," translated by O. Ragusa in K. E. Boulding and G. J. Stigler, editors (1953) *Readings in Price Theory*, Allen & Unwin, 26–56.)

Sperner, E. (1928) "Ein Satz über Untermengen einer endlichen Menge," *Mathematische Zeitschrift*, 27: 544–548.

Varian, H. R. (1975), "A Third Remark on the Number of Equilibria of an Economy," *Econometrica*, 43: 985–986.

Zangwill, W. I. and C. B. Garcia (1981) *Pathways to Solutions, Fixed Points, and Equilibria*. Prentice-Hall.

Appendix

Simplicial subdivisions and primitive sets

The original version of Scarf's algorithm (Scarf, 1967a) did not use simplicial subdivisions of the unit simplex. Rather, Scarf used subsets of neighboring points that he called *primitive sets*, a concept that he had found useful in proving the existence of an allocation in the core of cooperative games (Scarf, 1967b) and that he later used in studying profit maximization by firms with increasing returns to scale and indivisibilities in production (Scarf, 1986). Kuhn (1968, 1969) modified Scarf's algorithm to use the simplicial subdivision depicted in Figure 1, which is the Freudenthal (1942) subdivision. Kuhn sent Scarf an early version of Kuhn (1968), and Scarf realized that the simplicial subdivision used by Kuhn resulted in an algorithm that was almost identical to the Scarf (1967a) algorithm in which the primitive sets had the regular form developed by his student Terje Hansen. Scarf with Hansen (1973) and Scarf (1991) explain the history and the connections between simplicial subdivisions and primitive sets. There is a minor difference between the two algorithms in that Scarf's algorithm starts at a subsimplex at a corner, a zero-dimensional face, of the unit simplex, and Kuhn's algorithm starts at a subsimplex on the side, an ($n-2$)-dimensional face. What is identical is the simple formula for moving from one subsimplex to another, which we present in Section 3. Our analysis uses Scarf's version of his algorithm and is based on the description of the algorithm by Arrow and Kehoe (1994).

Alternative versions of the index theorem

The index theorem used by Dierker (1972) and that developed by Eaves and Scarf (1976) are special cases of the Lefschetz fixed-point theorem (Lefschetz, 1926).[5] Other economists, starting with Varian (1975), have used indices for equilibria from the Poincaré-Hopf theorem for vector fields on a manifold (Milnor, 1965 and Guillemin and Pollack, 1974). They normalize prices to lie on the intersection of the positive orthant with the unit sphere,

$$\bar{S} = \left\{ p \in \mathbb{R}^n \,\middle|\, \sum\nolimits_{j=1}^{n} p_j^2 = 1,\ p_j > 0 \right\},$$

[5] It was Solomon Lefschetz who submitted Harold Kuhn's 1968 paper to the National Academy of Sciences for publication in its *Proceedings*.

and observe that Walras's law implies that f generates a tangent vector field on $\bar{S}$. If we define the index of a 0 of f as

$$\text{index}(\hat{p}) = \text{sgn}\left(\det[-\bar{J}(\hat{p})]\right),$$

where $\bar{J}(\hat{p})$ is the $(n-1)\times(n-1)$ matrix formed by deleting one row and the same column for the Jacobian matrix $Df(\hat{p})$, then the Poincaré-Hopf theorem says that

$$\sum_{\{\hat{p}|f(\hat{p})=0\}} \text{index}(\hat{p}) = +1.$$

It is easy to perform elementary row and column operations on $I - Dg(\hat{p})$ that do not change its determinant to prove that its determinant has the same sign as that of $-\bar{J}$ (see, for example, Kehoe, 1980).[6] Furthermore, Kehoe (1980) shows that the index theorem can be applied to economies in which some goods can have a price equal to zero in equilibrium and to economies in which excess demand is non-differentiable at some prices as long as the non-differentiability does not occur at an equilibrium. Guillemin and Pollack (1974, Chapter 3) prove the equivalence of the differentiable version of the Lefschetz fixed-point theorem and the Poincaré-Hopf theorem. They even provide some intuition for the connection between the definition of the index based on derivatives and that based on orientation of subsimplex in a simplicial subdivision, as in Lefschetz (1926) and in our index theorem in section 4.

Critical economies

We show how large the deviations from the three conditions $b \geq -3$, $(a_1^1, a_2^1) = (a_1^2, a_2^2)$, and $(w_1^1, w_2^1) = (w_1^2, w_2^2)$, each of which is sufficient for uniqueness on equilibrium in the economy with $n = 2$, need to be. To do this, we calculate the parameters of the critical economies where we relax these conditions one by one until the unique symmetric equilibrium bifurcates into three equilibria.

[6] It is worth noting also that the Lefschetz fixed point index, as well as the Poincaré-Hopf index of a 0 of a vector field, can take on any positive or negative integer value, but indices that are not +1 or −1 occur only when the relevant determinant is 0. Debreu (1970) proves that almost all economies, in a very precise sense, have only regular equilibria — equilibria in which the relevant determinant is not 0. Critical economies, where the relevant determinant is 0 at some equilibrium, are interesting. They are points in parameter space where the set of equilibria can change discontinuously and the number of equilibria can change. Critical economies are examples of mathematical catastrophes.

We know that if we choose $b=-3$, keeping a_j^i and w_j^i fixed, then the economy has a unique equilibrium because of the Milleron-Mitiuschin-Polterovich result. We can calculate that the critical economy where

$$\frac{\partial f_1(0.5,0.5)}{\partial p_1}=0$$

occurs at $b=-3.33333$. For $b\geq -3.3333$, the economy has a unique equilibrium, and for $b<-3.3333$, it has three equilibria. We also know that if we choose $(a_1^1,a_2^1)=(a_1^2,a_2^2)$, keeping b and w_j^i fixed, then the economy has a unique equilibrium because of the Antonelli-Gorman-Nataf result. Setting $a_2^1=a_1^2=1$, we can calculate that the critical economy that divides the parameters for which there is a unique equilibrium from those parameters for which there are three equilibria occurs at $a_1^1=a_2^2=41.65972$. We also know that if we choose $(w_1^1,w_2^1)=(w_1^2,w_2^2)$, keeping b and a_j^i fixed, then the economy has a unique equilibrium because of the Eisenberg-Chipman result. Setting $w_2^1=w_1^2=1$, we can calculate that the critical economy occurs at $w_1^1=w_2^2=9.71429$.

Exhaustive search in the economy with $n=4$

To search for equilibria in the example with $n=4$, we start Newton's method at every point in the simplex of the grid of prices of the form

$$(p_1^i,p_2^i,p_3^i,p_4^i)=(a_1^i/D,a_2^i/D,a_3^i/D,a_4^i/D),$$

where a_j^i are integers such that $a_j^i>0$, $a_1^i+a_2^i+a_3^i+a_4^i=D$, and $D=100$. This is a grid of 156,849 points. To get some intuition for the number of points in the grid, we observe that in $\mathbb{R}^3$, the unit cube of points (p_1^i,p_2^i,p_3^i) where $1\geq p_j^i\geq 0$ contains $100^3=1{,}000{,}000$ grid points of the form $(a_1^i/D,a_2^i/D,a_3^i/D)$, where a_j^i are integers such that $D\geq a_j^i\geq 0$ and $D=100$. It easy to verify that, while the unit cube in $\mathbb{R}^3$ has volume 1, its subset, the tetrahedron in $\mathbb{R}^3$ of points of the form (p_1^i,p_2^i,p_3^i), where $p_j^i\geq 0$ and $p_1^i+p_2^i+p_3^i\leq 1$, has volume 1/6. Since we are ignoring grid points on the boundary of this tetrahedron, we should expect there to be fewer than $100^3/6 \approx 166,667$ grid points of this form.

We start Newton's method at each point in the tetrahedron $(p_1^{i,0}, p_2^{i,0}, p_3^{i,0}) = (p_1^i, p_2^i, p_3^i)$:

$$\begin{bmatrix} p_1^{i,k+1} \\ p_2^{i,k+1} \\ p_3^{i,k+1} \end{bmatrix} = \begin{bmatrix} p_1^{i,k} \\ p_2^{i,k} \\ p_3^{i,k} \end{bmatrix} - \lambda_k \begin{bmatrix} \frac{f_1(p^{i,k})}{\partial p_1} & \frac{f_1(p^{i,k})}{\partial p_2} & \frac{f_1(p^{i,k})}{\partial p_3} \\ \frac{f_2(p^{i,k})}{\partial p_1} & \frac{f_2(p^{i,k})}{\partial p_2} & \frac{f_2(p^{i,k})}{\partial p_3} \\ \frac{f_3(p^{i,k})}{\partial p_1} & \frac{f_3(p^{i,k})}{\partial p_2} & \frac{f_3(p^{i,k})}{\partial p_3} \end{bmatrix}^{-1} \begin{bmatrix} f_1(p^{i,k}) \\ f_2(p^{i,k}) \\ f_3(p^{i,k}) \end{bmatrix},$$

once again leaving $p_4^{i,k}$ fixed at $p_4^{i,0} = 1 - p_1^i - p_2^i - p_3^i$ and renormalizing so that $\hat{p}_1^i + \hat{p}_2^i + \hat{p}_3^i + \hat{p}_4^i = 1$ when and if Newton's method converges. Every regular equilibrium has an open neighborhood in which Newton's method converges to it. For each of equilibria 1–9 in Table 3, this open neighborhood is large, with thousands of grid points providing initial values of prices for Newton's method to converge to it. For equilibria close to the boundary of the simplex, equilibria 10–15, there are fewer such grid points, but each equilibrium has hundreds of grid points that provide initial values of prices for Newton's method to converge to it. Our exhaustive search locates all of the equilibria in Table 3, each of them many times, and it locates no other equilibrium. Convergence of the algorithm is enhanced, especially starting at grid points near the boundary, by using a small step size for the first two iterations, $\lambda_1 = \lambda_2 = 0.2$, then using $\lambda_k = 1$ for $k = 3, 4, \ldots$. We abandon Newton's method if it leads to a price vector with a nonpositive element or if it has not converged after eight iterations. Lack of convergence after eight iterations happens seldom. For most grid points as starting values, Newton's method either converges rapidly or explodes and leads to nonpositive prices.

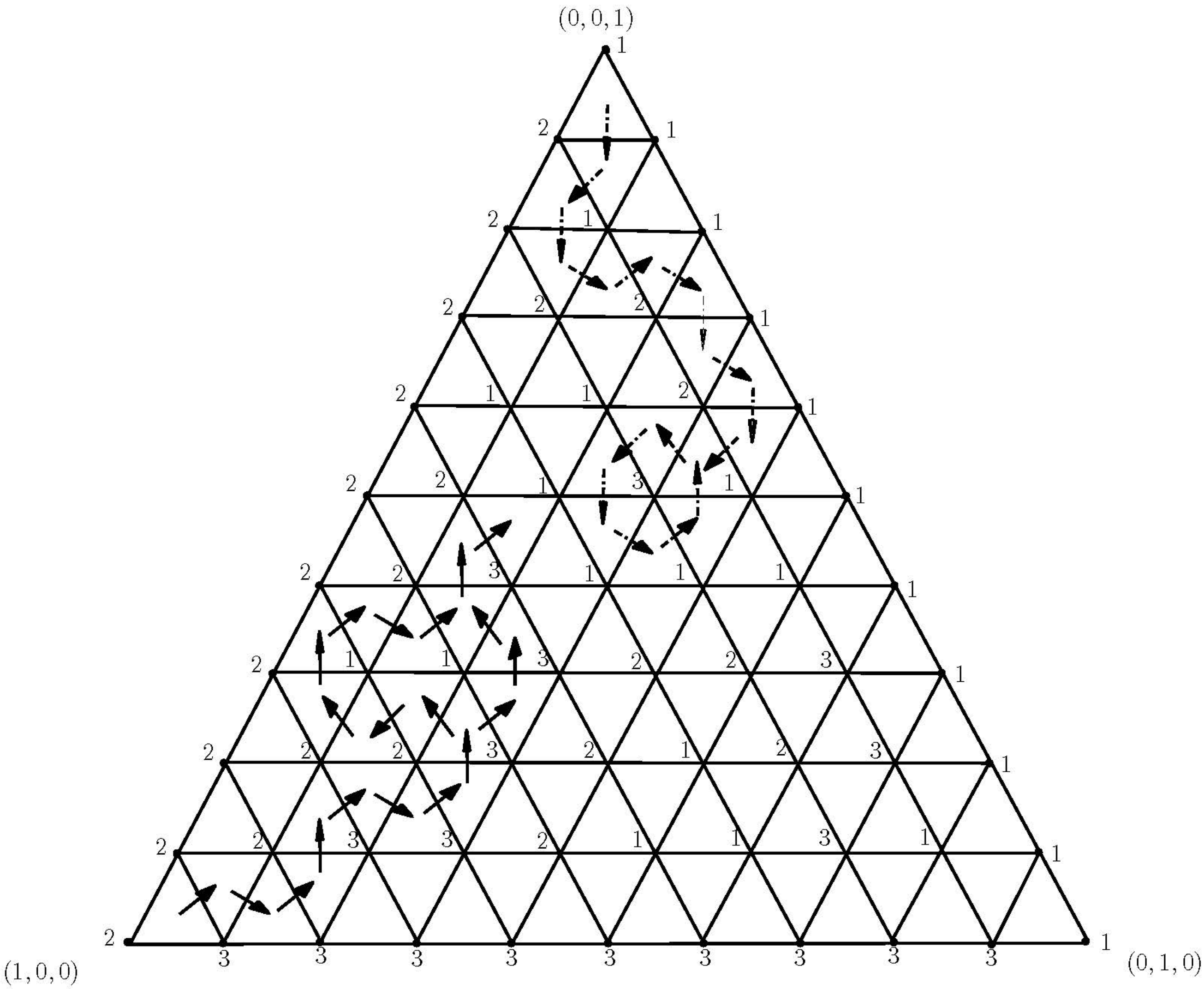

Figure 1: Scarf's algorithm with restarts

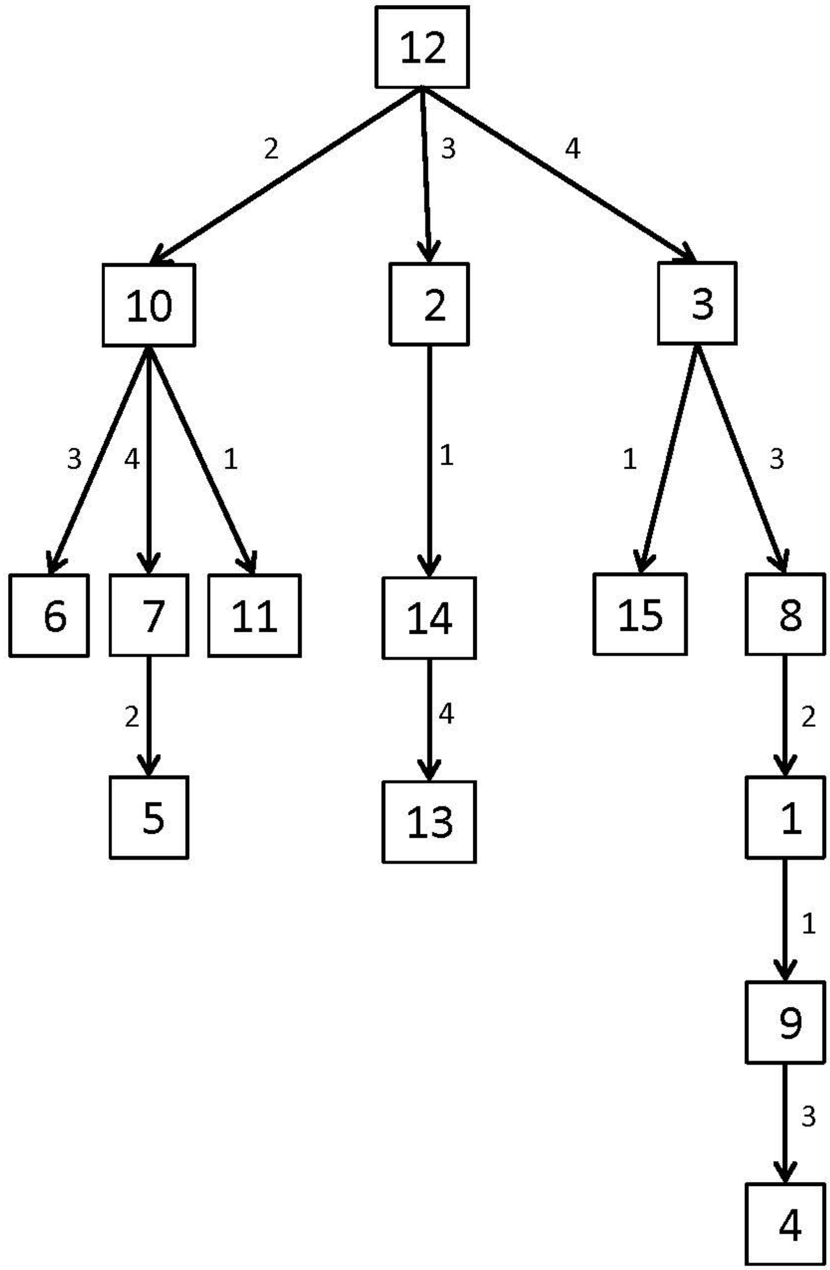

Figure 2: A road map to all equilibria

Notes: The boxed numbers are the equilibria as numbered in Table 3. The numbers alongside the arrows show the label of the column to drop to go from one completely labeled subsimplex to another. The road map depicted here is not unique. There are many ways to navigate the set of equilibria.