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A TEST OF THE INTERTEMPORAL ASSET PRICING MODEL

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ABSTRACT

Restrictions that general equilibrium theory place upon average returns are found to be strongly violated by the U.S. data in the 1889-1978 period. This result is robust to model specification and measurement problems. We conclude that equilibrium models which are not Arrow-Debreu economies are needed to rationalize the large average equity premium that prevailed during the last 90 years.

1. Introduction

During the 90-year period, 1889-1978, the average annual real return on equity was 6.95 percent, while the average real return on short-term, virtually risk-free bills was only 0.76 percent. This paper examines whether this large equity premium can be accounted for by the curvature on the utility function alone without recourse to incomplete intertemporal trading opportunities. Our conclusion is that it cannot.[1/]

Our testing procedure is nonstandard, in that it does not utilize the usual statistical techniques and is of independent methodological interest. In the Lucasian research tradition, we formulate "fully articulated artificial economic systems,"[2/] that is specifications of preferences and technology for which a competitive equilibria exist and can be characterized, and we examine whether they quantitatively mimic selected features of the historical data. Here we searched for an economy for which the averages of both equity returns and short-term interest rates match those observed for the U.S. economy over the 90-year period (1889-1978). What we tested, therefore, was a central prediction of a complete, internally consistent model as opposed to a subset of equilibrium conditions from some only partially specified model.

It is true that any of the structures employed in the test can be rejected on other grounds. For example, our structure has but two possible growth rates of per capita consumption, where as in fact there were 89 different growth rates in the sample period. What we contend is that if no economy in the restricted class of Arrow-Debreu economies considered can produce mean equity and risk-free returns consistent with the historically observed average values, then no Arrow-Debreu economy with full intertemporal state-contingent trading opportunities can resolve the puzzle.

We do not claim that no Arrow-Debreu competitive equilibrium model is consistent with the observation on equity premium cited above. If state-dependent utility functions are admitted, any observation can be explained. It is also true that with extreme curvature on the utility function, an Arrow-Debreu economy can be constructed which produces these observations.[3/] However, the tenet of cross-model verification rules out extreme curvature on at least three grounds: First, the assumption is inconsistent with a wealth of microeconomic studies on individual consumption behavior. Responses to temporary changes in income are not close to zero. Secondly, it is inconsistent with the risk aversion parameter values that Hansen and Singleton (1982) found. Finally, it would imply much larger variability in real returns than those historically observed if there were low frequency movements in the consumption growth rate as in fact there were.

Intuitively, the reason why increased risk aversion does not give us the desired result is that while increased curvature on the utility function does increase the equity premium, it simultaneously increases the average real risk-free rate. This is due to the fact that real per capita consumption has grown at an average of nearly 2 percent per year. In the case of a growing economy, agents with high risk aversion effectively discount the future to a greater extent than agents with low risk aversion (relative to a nongrowing economy). Due to growth, future consumption will probably exceed present consumption and since the marginal utility of future consumption is less than that of present consumption, real interest rates will be higher on average.

In this paper, we employ a variation of Lucas' pure exchange model. Since per capita consumption has grown over time, we assume that the growth rate of the endowment follows a Markov process. This is in contrast to the assumption in Lucas' model that the endowment level follows a Markov process. Our assumption, which required an extension of competitive equilibrium theory, enables us to capture the nonstationarity in the consumption series associated with the large increase in per capita consumption that occured in the 1889-1978 period.

With our structure, the process on the endowment is exogenous and there is neither capital accumulation nor production. Modifying the technology to admit these opportunities cannot overturn our conclusion because expanding the set of technologies in this way does not increase the set of joint equilibrium processes on consumption and asset prices (see Donaldson and Mehra (1983)). As opposed to standard testing techniques, the failure of the model hinges not on the acceptance/rejection of a statistical hypothesis but on its inability to generate average returns even close to those observed. If we had been successful in finding an economy which passed our not very demanding test, as we expected, we planned to add capital accumulation and production to the model using a variant of Brock's (1982) or Prescott and Mehra's (1980) general equilibrium stationary structures and to perform additional tests.

This paper consists of six sections. Section 2 describes the economy, while the existence and optimality of equlibrium are considered in Section 3. In Section 4, we derive the basic relationships for asset prices and returns. The tests of the model are discussed in Section 5, and Section 6 concludes the paper.

2. The Economy

The economy we consider was judiciously selected so that the joint process governing the growth rates in aggregate per capita consumption and asset prices would be stationary and easily determined. The economy has a single representative "stand-in" household. This unit orders its preferences over random consumption paths by

$$(1) \qquad E_0\{\textstyle\sum_{t=0}^{\infty}\beta^t U(c_t)\}$$

where c_t is per capita consumption, β is the subjective time discount factor, $E_0\{\cdot\}$ is the expectation operator conditional upon information available at time zero (which denotes the present time) and $U\colon\ R_+ \to R$ is the increasing concave utility function. To insure that the equilibrium process will be stationary, the utility function is further restricted to be of the constant relative risk aversion class,

$$(2) \qquad U(c,\alpha) = \frac{c^{1-\alpha}-1}{1-\alpha};\ 0 < \alpha < \infty.$$

The parameter α measures the curvature of the utility function. When $\alpha = 1$, the utility function is defined to be the logarithmic function, which is the limit of the above function as α approaches one.

We assume that there is one productive unit producing the perishable consumption good and there is one equity share that is competitively traded. Since only one productive unit is considered, the return on this share of equity is also the return on the market. The firm's output is constrained to be less than or equal to y_t. It is the firm's dividend payment in period t as well.

The growth rate in y_t is subject to an ergodic Markov chain; that is

(3) $$y_{t+1} = x_{t+1}\, y_t$$

where $x_{t+1} \epsilon \{\lambda_1, \dots, \lambda_n\}$ and

(4) $$\Pr \{x_{t+1} = \lambda_j ; x_t = \lambda_i\} = \phi_{ij}.$$

The λ_i are all nonnegative and $y_0 > 0$. The random variable y_t is observed at the beginning of the period, at which time dividend payments are made. All securities are traded ex-dividend. We also assume that the matrix A with elements $a_{ij} \equiv \beta\, \phi_{ij}\, \lambda_j^{1-\alpha}$ for i, j = 1, ..., n is stable; that is, lim A^m as $m \to \infty$ is zero. In the Appendix, it is shown that this is necessary and sufficient for the expected utility to exist if the stand-in households consumes y_t every period.

3. Equilibrium

In order for this to be a Debreu competitive equilibrium model, it is necessary to map our model into his structure. This requires, among other things, a specification of a linear space, say L, to serve as the commodity space. Given that, in our economy, economic activity takes place over an infinite number of periods, the space is necessarily infinite dimensional, which results in a few mathematical subtleties. Our commodity space is the normed linear space of infinite sequences of vectors with the t^{th} vector indexed by the event, $e_t \equiv (x_1, \ldots, x_t)$. The set of possible period t events, E_t, is finite having cardinality n^t. The norm for $z \varepsilon L$ is

$$\|z\| = \sup_t \max_{e_t \varepsilon E_t} \left| \frac{z_t(e_t)}{y_t} \right|$$

where $y_t = y_0 \, x_1 \, \ldots \, x_t$ is the event contingent maximium output of the firm. The element $z_t(e_t)$ is the quantity of the good delivered in period t conditional upon e_t occurring.

The households consumption possibility set is

$$C = \{c \varepsilon L: \; y_t/2 \leq c_t(e_t) \text{ all } e_t \varepsilon E_t \text{ all } t\},$$

which is stronger than the requirement that consumption be nonnegative. The endowment of the stand-in household is the zero element of L and the firm's production possibility set is

$$W = \{w \varepsilon L: \; w_t(e_t) \leq y_t \text{ all } e_t \varepsilon E_t \text{ all } t\}.$$

This completes the representation of our economy in the Debreu framework.

The allocation $c_t^*(e_t) = w_t^*(e_t) = y_t$ is a Pareto optimum as more is preferred to less. In the Appendix, the expected utility of plan c^* is shown to exist. As plan $2c^*$ belongs to C, the element c^* is not a saturation point for the stand-in household.

The consumption possibility set C is convex; the expected utility functional u: $C \to R$ is concave and continuous; the production possibility set W is convex and has an interior point; c^* is not a saturation point for the stand-in household. By Theorem 2 of Debreu (1954, page 590) this optimum can be supported by a valuation equilibrium subject to the conditions of the Remark (page 591). The conditions of the Remark are satisfied, for a point exists in C having valuation less than c^*. The point with $c_0 = y_0/2$ and $c_t(e_t) = c^*_t(e_t)$ is such a point.

This theorem of Debreu does not guarantee that the equilibrium valuation function v: $L \to R$ has the dot product representation, which is required in the subsequent analysis. The needed result is now established.[4]

Let L^n be the linear subspace of L for which $z_t = 0$ for $t > n$. Let $\pi_n(z)$ denote the projection of z on L^n. The following valuation function p, which does have dot product representation, will be shown to also support the optimum allocation:

$$p(z) = \lim_{n\to\infty} v[\pi_n(z)] = \sum_t \sum_{e_t} p_t(e_t)\, z_t(e_t).$$

If $z \varepsilon C$ implied $\pi_n(z) \varepsilon C$, the result would be an application of Theorem 1 in Prescott and Lucas (1972, page 418). Their theorem holds under the following slightly weaker conditions. Letting c^n denote the element with $c^n_t(e_t) = c(e_t)$ for $t < n$ and $c^n_t(e_t) = y_t$ for $t > n$, the Prescott-Lucas assumptions that $c \varepsilon C$ implies $\pi_n(c) \varepsilon C$ and that $c, c' \varepsilon C$ and $u(c) > u(c')$ implies $u[\pi_n(c)] > u(c')$ for sufficiently large n are modified by replacing $\pi_n(c)$ by c^n. This slightly more general version of their theorem is established by substituting $p(\cdot) + \lim v(0^n)$ for $p(\cdot)$ wherever it appears in their proof, where 0 denotes the zero element of L.

4. Asset Prices and Returns

The state of the economy at the beginning of the period t is the pair (x_t, y_t). These two variables are sufficient for determining the period t equilibrium decisions and prices and the equilibrium predictive probability distributions of future prices. Our economy is recursive in that the equilibrium prices and the decision rules which specify their actions are time invariant functions of the state. Our principal concern in this study is with recursive securities; that is, with securities whose payments s periods hence, that is in period t + s for s = 0,1,2, ..., is a function of the state in period t + s and s only. An example of such a security is the equity. Its dividend in any period is a time invariant function of the state in that period. Bonds with fixed payment schedules are recursive securities as well, since their payments in any period is constant independent of the state (that is, they are a degenerate function of the state).

For any security, with process $\{d_t\}$ on payments, its price in period t is

$$p_t = E_t \{\sum_{s=t+1}^{\infty} \beta^{s-t} U'(c_s) d_s / U'(c_t)\}. \tag{5}$$

Prices in period t are in terms of units of the period t consumption and all prices are ex-dividend or interest payments. For recursive securities with payment $\{d_{s+t} = d_s(x_{t+s}, y_{t+s})\}$

$$p_t = p(x_t, y_t) = E\{\sum_{s=t+1}^{\infty} \beta^{s-t} U'(c_s) d_{s-t}(x_s, y_s) / U'(c_t) \mid x_t, y_t\} \tag{6}$$

The above expectation is well defined given that $c_s = y_s$ for $s = t, t+1, \ldots.$ The important property of recursive securities is that their prices are time-invariant functions of the state.

As the concern in this study is only with recursive securities, the subscript t can be dropped. This is accomplished by redefining the state to be the pair (c,i) if $y_t = c$ and $x_t = \lambda_i$. With this convention, the price of the equity share (there is one infinitely divisible share in the firm) from (6) is

$$p^e(c,i) = \beta \sum_{j=1}^{n} \phi_{ij} (\lambda_j c)^{-\alpha} [p^e(\lambda_j c, j) + c\lambda_j] \, c^{\alpha}. \tag{7}$$

This holds because $d(c,i) = c$ for all i for this recursive security, $\lambda_j c_j$ is next period's consumption (and dividends), $c^{-\alpha}$ is the marginal utility of current consumption and the agents have the option (which is not exercised in equilibrium) to sell the security. Given our assumption, there is a unique positive function satisfying (7) and it is the equilibrium price of the equity.

We conjecture that this function has the form

$$p^e(c,i) = w_i c, \tag{8}$$

where w_i is a constant. Making this substitution in (7) and dividing by c yield

$$w_i = \beta \sum_{j=1}^{n} \phi_{ij} \lambda_j^{(1-\alpha)} (w_j + 1) \tag{9}$$

for $i = 1, \ldots, n$. This is a system of n linear equations in n unknowns. The assumption that guaranteed existence of equilibrium, guarantees the existence of unique positive solution to this system. This verifies the conjecture.

The period return if the current state is (c,i) and next period state $(\lambda_j c, j)$ is

$$(10) \qquad r_{ij}^e = \frac{p^e(\lambda_j c, j) + \lambda_j c - p^e(c,i)}{p^e(c,i)} = \frac{\lambda_j(w_j+1)}{w_i} - 1$$

using (8).

The equity's expected period-return if the current state is i is

$$(11) \qquad R_i^e = \sum_{j=1}^{n} \phi_{ij} r_{ij}^e \; .$$

Capital letters are used to denote expected return. With the subscript i, it is the expected return conditional upon the current state being (c,i). Without this subscript it is the expected return with respect to the stationary distribution. The superscript indicates the type of security.

The second recursive security that we consider is the one period real bill or riskless asset. It pays one unit of the consumption period next period with certainty. From (6)

$$(12) \qquad p_i^f = p^f(c,i) = \beta \sum_{j=1}^{n} \phi_{ij} U'(\lambda_j c)/U'(c)$$

$$= \beta \sum_{j=1}^{n} \phi_{ij} \lambda_j^{-\alpha}$$

The certain return on this riskless security is

(13) $$R_i^f = 1/p_i^f - 1,$$

when the current state is (c,i).

The statistics that are probably most robust to the modelling specification are the means over time. Let $\pi \varepsilon R^n$ be the vector of stationary probabilities on i. This exists because the process on i has been assumed to be stationary. The vector π is the solution to the system of equations $\pi = \phi^T \pi$ with $\sum_{i=1}^{n} \pi_i = 1$ and $\phi^T = \{\phi_{ji}\}$. The expected returns on the equity and the risk-free security are, respectively,

(14) $$R^e = \sum_{i=1}^{n} \pi_i R_i^e \quad \text{and} \quad R^f = \sum_{i=1}^{n} \pi_i R_i^f.$$

Robust estimates of these parameters of the model are time averages. The risk premium for the equity security is $R^e - R^f$, a parameter that is used in the test.

5. The Tests

The parameters defining preferences are α and β while the parameters defining technology are the ϕ_{ij} and λ_i. Our approach is to assume two states for the Markov chain and to restrict the process as follows

$$\lambda_1 = 1+\mu+\delta \qquad \lambda_2 = 1+\mu-\delta$$

$$\phi_{11} = \phi_{22} = \phi \qquad \phi_{12} = \phi_{21} = (1-\phi).$$

The parameters μ, ϕ, and δ now define the technology. We require $\delta > 0$ and $0 < \phi < 1$. This particular parameterization was selected because it permitted us to independently vary the average growth rate of output by changing μ, the variability of consumption by altering δ, and the serial correlation of the growth rates by adjusting ϕ.

The parameters were selected so that the average growth rate of per capita consumption, the variance of the growth rate of per capita consumption and first order correlation of this growth rate all with respect to the model's stationary distribution matched the sample values for the U.S. economy between 1889-1978.[5/] The resulting parameter's values were $\mu = .018$, $\delta = .035$ and $\phi = .43$. Given these values, the nature of the test is to search for parameters α and β for which the model's averaged risk free rate and equity risk premium match with those observed for the U.S. economy over this 90-year period.

The average real return on a relatively riskless short-term security over the 1889-1978 period was 0.76 percent. The securities used were 90 day government treasury bills in the 1931-1978 period, treasury certificates for the 1920-1930 period and 60 to 90-day prime commercial paper rate prior to 1920.[6/] These securities do not correspond perfectly with the real bill, but insofar as unanticipated inflation is negligible and/or uncorrelated with the growth rate x_{t+1} conditional upon information at time t, the expected real return for the nominal bill will equal R_i^f. Litterman (1980) found using vector autoregressive analysis that the innovation in the inflation rate in the postwar period (quarterly data) has standard deviation of only one-half of 1 percent and that this innovation is nearly orthogonal to the subsequent path of the real GNP growth rate. Consequently, the average realized real return on a nominally denoted short-term bill should be close to that which would have prevailed for a real bill if such a security were traded.[7/] The average real return on the Standard and Poor's 500 Composite Stock Index over the 90 years considered was 6.95 percent per annum. This leads to an average equity premium of 6.19 percent (standard error 1.7 percent).

One set of possible problems are associated with errors in measuring the inflation rate. Such errors do not affect the computed risk premium as they bias both the real risk-free rate and the equity rate by the same amount. A potentially more serious problem is that these errors bias our estimates of the growth rate of consumption and the risk-free real rate. Therefore, only if the tests are insensitive to biases in measuring the inflation rate should the tests be taken seriously. A second measurement problem arise because of tax considerations. The theory implicitly is considering effective after-tax returns which vary over income classes. In the earlier part of the period, tax rates were low. In the latter period, the low real rate and sizable equity risk premium hold for after-tax returns for all income classes (see Fisher and Lorie (1978)).

Given the estimated process on consumption, Figure 1 depicts the set of values of the average risk-free rate and equity risk premium which are both consistent with the model and result in average real risk free rates between 0 and 3 percent.[8/] These are values that can be obtained by varying preference parameters α and β. The observed real return of 0.76 percent and equity premium of 6 percent is clearly inconsistent with the predictions of the model. The largest premium obtainable with the model is 0.2 percent, which is not close to the observed value.

Robustnes of Results

In an attempt to reconcile the large discrepancy between the theory and the observation, we tested the sensitivity of our results to model misspecification. We found that the conclusions were not sensitive to changes in the parameters μ, which is the average growth rate of consumption, and not very sensitive to δ, the variability of the consumption growth rate. As the persistence parameter ϕ increased ($\phi = 0.5$ corresponds to independence over time), the premium decreased. Reducing ϕ (introducing stronger negative correlation in the consumption growth rate) had only small effects. We also modified the process on consumption by introducing additional states that permitted us to increase higher moments of the stationary distribution of the growth rate without varying the first or second moments. The maximal equity premium increased by .02 to .22 only. These exercises lead us to the conclusion that the result of the test is not sensitive to the specification of the process generating consumption.

That the results were not sensitive to increased persistence in the growth rate, that is increases in ϕ, implies low frequency movements or nonstationarities in the growth rate do not increase the equity premium. Indeed, by assuming stationarity, we biased the test towards acceptance.

We also examined whether aggregation affects the results for the case that the growth rates were independent between periods, which they approximately were given the estimated ϕ was near one-half. Varying the underlying time period from one one-hundredths of a year to two years had a neglible affect upon the admissable region. Consequently, the test appears robust to the use of annual data in estimating the process on consumption.

Effects of Firm Leverage

The security priced in our model does not correspond to the common stocks traded in the U.S. economy. In our model there is only one type of capital while in an actual economy there is virtually a continuum of capital types with widely varying risk characteristics. The stock of a typical firm traded in the stock market entitles its owner to the residual claim on output after all other claims including wages have been paid. The share of output accruing to stockholders is much more variable than that accruing to holders of other claims against the firm. Labor contracts, for instance, may incorporate an insurance feature as labor claims on output are in part fixed having been negotiated prior to the realization of output. Hence, a disproportionate part of the uncertainty in output is probably borne by equity owners.

The firm in our model corresponds to one producing the entire output of the economy. Clearly, the riskiness of the stock of this firm is not the same as that of the Standard and Poor's 500 Composite Stock Price Index. In an attempt to match the two securities we price and calculate the risk premium of a security whose dividend next period is actual output less a fraction of expected output. Let θ be the fraction of expected date t+1 output committed at date t by the firm. Then equation (7) becomes

$$(15) \qquad p^e(c,i) = \beta \sum_{j=1}^{n} \phi_{ij}(\lambda_j c)^{-\alpha} \, [p^e(\lambda_j c,j)+c\lambda_j-\theta \sum_{k=1}^{n} \phi_{ik} c\lambda_k]c^{\alpha}$$

As before, it is conjectured and verified that $p^e(c,i)$ has the functional form $w_i c$. Substituting $w_i c$ for $p^e(c,i)$ in (15) yields the set of linear equations

$$(16) \qquad w_i = \beta \sum_{j=1}^{n} \phi_{ij} \, \lambda_j^{-\alpha} \, [\lambda_j w_j + \lambda_j - \sum_{k=1}^{n} \phi_{ik}\lambda_k],$$

for $i = 1, \ldots, n$. This system was solved for the equilibrium w_i and equations (10), (11), and (14) used to determine the average equity premium.

As the corporate profit share of output is about 10 percent, we set $\theta = 0.9$. Thus, 90 percent of expected output is committed and all the risk is born by equity owners who receive 10 percent of output on average. To our surprise, this increases the equity risk premium by less than 0.1 percent. This is the case because financial arrangements have no effect upon resource allocation and, therefore, the underlying Arrow-Debreu prices. Large fixed payment commitments on the part of the firm do not reverse the test's outcome.

6. Conclusion

The equity premium puzzle we think is not why are average equity returns so high, but why are average risk-free rates so low. This is not the only example of some asset receiving a lower return than that implied by Arrow-Debreu general equilibrium theory. Currency, for example, is dominated by Treasury bills with positive nominal yields yet sizable amounts of currency are held. The equity premium we conjecture is an important one whose resolution may lead to an understanding of the mechanism by which open market operations affect real output and employment.

We doubt whether heterogeniety, per se, of the agents will alter the conclusion. Within the Debreu (1954) competitive framework, Constandinides (1981) has shown heterogenous agent economies also impose the set of restrictions tested here (as well as others). Some features must be introduced into the environment that make certain types of intertemporal trades among agents infeasible. In the absence of such markets, there can be variability in consumption of groups, yet little variability in aggregate consumption. The fact that certain types of contracts may be nonenforceable is one reason for the nonexistence of markets that would otherwise arise to share risk. It would be surprising if a government enforced a contract that offset the effect of a change it made in the social security system. Similarly, entering into contracts with as yet unborn generations is not feasible.[9] Such non-Arrow-Debreu competitive equilibrium models may rationalize the large equity risk premium that has characterized the behavior of the U.S. economy over the last 90 years. To test such theories it probably would be necessary to have consumption data by income or age groups.

FOOTNOTFS

1/There are other interesting features of these time series and procedures for testing them. The variance bound tests of LeRoy and Porter (1981) and Shiller (1980), are particularly innovative and constructive. They did indicate that consumption risk was important (see Grossman and Shiller (1981) and LeRoy and LaCavita (1981)).

2/See Lucas (1980) and Lucas (1983, especially pp. 16-18).

3/In a private communication, Fischer Black using the Merton (1973) continuous time model with investment opportunities constructed the example with a curvature parameter (α) of 55. We thank him for the example.

4/This result could also be established by verifying Mackey continuity of preferences and then applying a theorem of Bewley (1970) or Brown and Lewis (1981).

5/ We thank Sanford Grossman and Robert Shiller for providing us with the data they used in their study (1981). Consumption is per capita consumption of nondurables and services. Services of consumer durables are not included.

6/The data was obtained from Homer (1963) and Ibbotson and Singuefield (1979).

7/See LeRoy (1982) for a further discussion of the theoretical relation.

8/There are also admissible points with real riskless returns less than 3 percent that are associated with an extreme equity premium of 14 percent or more. We do not focus on these values because they require extreme values for the risk aversion parameter of between 50 and 150. This is possible because the probability of negative growth in consumption is positive.

[9] See Wallace (1980) for an exposition on the use of the overlapping generations model and the importance of legal constraints in explaining rate of return anamolies.

APPENDIX

We first establish that the expected utility of the element $c^* \varepsilon C$ with the $c_t^*(e_t) = y_t$ exists. Let $v_t(y,i)$ be the expected utility for the first t + 1 periods of the plan if $y_0 = y$ and $x_0 = \lambda_i$. It satisfies the recursion

$$\text{(A1)} \qquad v_{t+1}(y,i) = \frac{y^{1-\alpha}-1}{1-\alpha} + \beta \sum_j \phi_{ij} \; v_t(\lambda_j y, j).$$

for t = 0, 1, ... The initializing function is

$$\text{(A2)} \qquad v_0(y,i) = \frac{y^{(1-\alpha)}-1}{1-\alpha} .$$

By definition, the expected utility of c^* is the limit of $v_t(y,i)$ as t goes to infinity.

It is easily verified by mathematical induction that

$$\text{(A3)} \qquad v_t(y,i) = \frac{\gamma_{it}}{(1-\alpha)} \, y^{(1-\alpha)} - \frac{1-\beta^{t+1}}{(1-\alpha)(1-\beta)}$$

by noting it is true for t = 0 and using (A1) to conclude that if it is true for t then it is true for t + 1. The substitution of (A3) into (A1) yields

$$\text{(A4)} \qquad \gamma_{i,t+1} = 1 + \beta \sum_j \phi_{ij} \; \lambda_j^{(1-\alpha)} \; \gamma_{jt} \text{ for } i = 1, \ldots, n.$$

The requirement for the expected utility to exist is that the difference equation (A4) converge given $\gamma_{i0} = 1$ all i. It will occur if and only if the n x n matrix $A = [\beta \phi_{ij} \lambda_j^{(1-\alpha)}]$ has eigenvalues which all lie within the unit circle in the complex plan or equivalently that $\lim A^n = 0$. This is true by assumption.

The expected utility exists for all $c \varepsilon C$ and is continuous because $c \varepsilon C$ constraints event contingent consumption $c_t(e_t)$ to be at least half $c^*_t(e_t)$ and not more than $\|c\|$ times $c^*_t(e_t)$. This uniformly bounds the percentage difference between the $c_t(e_t)$ insuring the expected utility of c exists given the expected utility of c* exists. Continuity follows because the sequence $c_n \varepsilon L$ converging to $c \varepsilon L$ requires the percentage difference between $c_n(e_t)$ and $c(e_t)$ go to zero uniformly in t and e_t. This implies the limit of $u(c_n)$ is $u(\lim c_n)$.

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