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A MODEL OF COMMODITY MONEY

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ABSTRACT

Commodity money is modeled as one or two of the capital goods in a one-consumption good and one or two capital-good, overlapping generations model. Among the topics addressed using versions of the model are (i) the nature of the inefficiency of commodity money; (ii) the validity of quantity-theory predictions for commodity money systems; (iii) the circumstances under which one commodity emerges naturally as the commodity money; (iv) the role of inside money (money backed by private debt) in commodity money systems; and (v) the circumstances under which a government can choose the commodity to serve as the commodity money.

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A Model of Commodity Money

This paper describes and analyzes a set of models designed to answer the following questions about commodity money systems. Are commodity money systems generally inefficient? Does the degree of inefficiency depend on the commodity that becomes the commodity money? Under what circumstances do observations of a quantity-theory-of-money nature turn up under commodity systems? Are interest rates determined by the return on the commodity money, or can commodity money be driven out because it cannot compete with interest-bearing inside money? Does one commodity emerge as a natural commodity money because of its depreciation rate and technological conditions of production? Can a government choose the commodity or combination of commodities that is to serve as the commodity money? Does a commodity money system limit seignorage, thereby disciplining fiscal policy? Finally, what happens if restrictions are placed on privately issued inside money, as has often been the case historically?

The models we use to approach these questions are growth models with a single consumption good (at each date) and one or several capital goods (at each date) and with technological features of production designed to make the capital goods resemble commodity monies. People in our model are members of overlapping generations, each living for two periods, who care only about their own consumption. Finally, aside from explicit legal restrictions, nothing in the models distinguishes a priori the demands for assets that one may wish to identify as monies from the demands for other assets.

In section 1, we describe the physical environment of our model. In section 2, we present conditions for a perfect foresight, competitive equilibrium--henceforth called, simply, an equilibrium--under laissez-faire; we also describe the equilibrium in some simple cases. We use those descriptions in section 3 to answer some of the questions listed above. In section 4, we consider governmental choice of the commodity standard. We examine in section 5 the role of legal restrictions that limit the degree to which private debts can compete with the commodity money and the potential role of a central bank as intermediary. Finally, in the concluding section, we comment on other issues, including differences between our modeling approach and that of others.

1. The Model

In this section, we describe a version of the model based on one consumption good and one capital good. We call the consumption good "bread"--bread at t, bread at t + 1, and so on--and we call the capital good "gold"--gold at t, gold at t + 1, and so on. The model uses discrete time, and the current or initial date is labeled $t = 1$. We now describe, in turn, total resources, technology, and people and their preferences.

Total Resources

At each date, $Y(t)$ units of time t bread are exogenously available to the economy. We also assume that the economy starts at $t = 1$ with $Z(1)$ units of time 1 gold, which is the only exogenously available gold. Thus, although the economy gets a new endowment of bread each period, it does not get a new endowment of gold.

Technology

The technology includes two aspects: a static one with possibilities for converting time t bread into time t gold and vice versa; and an intertemporal one with possibilities for converting time t gold into time t + 1 gold. We assume that no direct technology is available for converting time t bread into time t + 1 bread.

[Insert Figure 1]

In Figure 1, which describes the static features of the technology, Z(t) represents the amount of gold the economy starts with at date t. This amount is determined by actions taken at t - 1 and earlier in a way that we will describe below. The segment of the frontier to the right of B represents the possibility of devoting some of the bread resource, Y(t), to the production of additional gold. The parameter λ_{bg} (the subscripts connote "bread into gold") has the units ounces of time t gold (output) per ounce of time t bread (input). The segment of the frontier to the left of B represents the possibility of devoting some of the gold resource, Z(t), to the production of additional bread. This is meant to represent an "industrial use" for gold. Given that we assume that only bread appears as an argument of utility functions (gold is not consumed directly), the existence of such a use for gold is critical and justifies calling this object a commodity or capital good. The parameter λ_{gb} ("gold into bread") has the units ounces of time t bread (output) per ounce of time t gold (input). We assume that $0 < \lambda_{gb} < 1/\lambda_{bg}$.

We assume a very simple intertemporal technology. As already noted, we assume that time t bread cannot be directly converted into time t + 1 bread. As for gold, for any $k > 0$, k units of time t gold can be converted into $(1-\delta)k$ units of time t + 1 gold. In other words, gold depreciates geometrically at the rate δ per period. Thus, letting ZP(t) be net gold production at t (the amount of gold produced at t minus the amount used as input into the production of bread at t), the starting stock at t and net gold production at t are related to the starting stock at t + 1 according to $Z(t+1) = (1-\delta)[Z(t) + ZP(t)]$.

Population and Preferences

The population consists of overlapping generations, each living for two periods. Each member h of generation t (young at t, old at t + 1), $t \geq 1$, has a utility function $u_t^h[c_t^h(t), c_t^h(t+1)]$, where $c_t^h(s)$ is the consumption of time s bread by h in generation t. We assume that u_t^h is twice continuously differentiable, increasing, and strictly quasi-concave. Later on, we consider additional assumptions imposed implicitly by way of assumptions on aggregate saving. Each member of generation 0, the old at the first date, wants to maximize his or her own consumption of time 1 bread.

2. Equilibrium under Laissez-Faire

We distinguish between the individual endowments of members of generation 0 and everyone else. We assume that the members of generation 0 own among them $Z(1)$, the starting stock of gold. They also have an endowment of time 1 bread. Everyone else has endowments of bread only; member h of generation t, $t \geq 1$, has an endowment consisting of some time t bread, $w_t^h(t) > 0$, and some time $t + 1$ bread, $w_t^h(t+1) > 0$. We assume that the individual endowments exhaust $Y(t)$.

Roughly speaking, an equilibrium consists of sequences of quantities and prices such that the quantities are utility maximizing and clear markets when individuals take the relevant (current and future) prices parametrically. We now proceed to derive equilibrium conditions.

First, letting $p(t)$ be the price of time t gold in terms of time t bread, we have the following (see Figure 1):

<u>Proposition 1</u>: In any equilibrium, $\lambda_{gb} \leq p(t) \leq 1/\lambda_{bg}$. Moreover, if $ZP(t) > 0$, then $p(t) = 1/\lambda_{bg}$; while if $ZP(t) < 0$, then $p(t) = \lambda_{gb}$. (We omit the proofs of this and other obvious propositions. Other proofs, as noted, appear in the Appendix.)

We now describe the choice problem of a member of generation t, $t \geq 1$, but only for prices that satisfy Proposition 1. This allows us to ignore, for the moment, static production possibilities because at such prices they do not enhance consumption opportunities. We do, however, explicitly introduce the possibility of storing gold and two markets in loans: loans stated in terms of bread at t, $\ell^h(t)$, with gross payoff in bread at t + 1, denoted $r(t)\ell^h(t)$; and loans stated in terms of time t gold, $\ell_g^h(t)$, with gross payoff in gold at t + 1, denoted $i(t)\ell_g^h(t)$. [Think of r(t) as the gross real rate of interest and of i(t) as the gross nominal rate of interest.] Consumption of bread at t and t + 1 for h is then related to loans granted and gold stored by

$$(1) \qquad c_t^h(t) \leq w_t^h(t) - \ell^h(t) - p(t)\ell_g^h(t) - p(t)z^h(t+1)/(1-\delta)$$

$$(2) \qquad c_t^h(t+1) \leq w_t^h(t+1) + r(t)\ell^h(t) + p(t+1)i(t)\ell_g^h(t) + p(t+1)z^h(t+1).$$

Here, $z^h(t+1)$ is the output from the gold h stores at t. For each such unit, the individual must have stored $1/(1-\delta)$ units at t, where δ is the physical depreciation rate. The aggregate Z(t+1) is simply the sum of every $z^h(t+1)$, the sum being taken over the members of generation t.

Since $\ell^h(t)$ can be positive (lending by h) or negative (borrowing by h), equations (1) and (2) are equivalent in terms of consumption bundles to the following single constraint obtained by solving equation (1) for $\ell^h(t)$ and substituting the result into equation (2):

$$(3) \qquad c_t^h(t) + c_t^h(t+1)/r(t) \leq w_t^h(t) + w_t^h(t+1)/r(t) + \ell_g^h(t)[p(t+1)i(t)/r(t)-p(t)] + z^h(t+1)[p(t+1)/r(t)-p(t)/(1-\delta)].$$

Since utility is increasing in consumption and consumption is bounded by resources in equilibrium, it follows that in equilibrium the right-hand side of equation (3) must be bounded. Therefore, in any equilibrium, $i(t) = r(t)p(t)/p(t+1)$, $r(t) \geq (1-\delta)p(t+1)/p(t)$ and with equality if $z^h(t+1) > 0$; and

$$(4) \qquad c_t^h(t) + c_t^h(t+1)/r(t) \leq w_t^h(t) + w_t^h(t+1)/r(t).$$

In other words, utility maximizing consumption choices in equilibrium are constrained by equation (4).

Let $s_t^h[r(t)]$ (s for saving) be the solution for $w_t^h(t) - c_t^h(t)$ to the problem, maximize u_t^h subject to equation (4). It follows that s is a bounded (above), continuous function that has a unique zero. We will describe equilibrium conditions in terms of the aggregate saving function for generation t, $S_t[r(t)] = \sum s_t^h[r(t)]$, the sum being taken over the members of generation t. From now on, we make the convenient but strong assumption that S_t is strictly increasing.

These results allow us to work with the following definition of an equilibrium.

Definition: An equilibrium consists of nonnegative sequences for $r(t)$, $p(t)$, and $Z(t+1)$ and a sequence for $ZP(t)$ that for all $t \geq 1$ satisfy

(i) $\quad S_t[r(t)] = p(t)Z(t+1)/(1-\delta)$;

(ii) $\quad r(t) \geq p(t+1)(1-\delta)/p(t)$ and with equality if $Z(t+1) > 0$;

(iii) $\quad Z(t+1) = (1-\delta)[Z(t) + ZP(t)]$;

(iv) $\lambda_{gb} \leq p(t) \leq 1/\lambda_{bg}$ with

$$p(t) = \begin{cases} \lambda_{gb} & \text{if } ZP(t) < 0 \\ 1/\lambda_{bg} & \text{if } ZP(t) > 0 \end{cases}$$

We now describe the equilibrium for some special cases.

An Unchanging Monotone Saving Function and No Depreciation

Here we assume $S_t[r(t)] = S[r(t)]$ for all t and $\delta = 0$. Let r^* be such that $S(r^*) = 0$.

Proposition 2. If $r^* \geq 1$, then there exists an equilibrium with $r(t) = r^*$, $p(t) = \lambda_{gb}$, and $Z(t+1) = 0$ for all $t \geq 1$. If $r^* < 1$, then there exists an equilibrium with $r(t) = 1$, $p(t) = p$, and $Z(t+1) = S(1)/p$ for all $t \geq 1$ where

$$p = \begin{cases} \lambda_{gb} & \text{if } Z(1) \geq S(1)/\lambda_{gb} \\ S(1)/Z(1) & \text{if } S(1)/\lambda_{gb} > Z(1) > S(1)\lambda_{bg} \\ 1/\lambda_{bg} & \text{if } Z(1) \leq S(1)\lambda_{bg} \end{cases}$$

That this is an equilibrium is immediate from the definition given above. In this equilibrium, all the endogenous variables are constant over time and, in the case $r^* < 1$, the size of the starting stock of gold, $Z(1)$, determines $p(t)$.

With monotonicity of S, the Proposition 2 equilibrium is unique in all significant respects. We state this as

Proposition 3: Under the assumptions of Proposition 2, there is a unique equilibrium $r(t)$ sequence. (For a proof, see the Appendix.)

Only as regards $p(t)$ is the Proposition 2 equilibrium not unique, and then only in the case $r^* > 1$. In that case, if $Z(1) > 0$, then

$p(1) = \lambda_{gb}$ in order that $ZP(1) = -Z(1)$. However, other terms of the $p(t)$ sequence are somewhat free. Although (ii) and (iv) must be satisfied, they do not determine a unique $p(t)$ sequence when $r^* > 1$.

An Unchanging Saving Function and Depreciation

Depreciation changes matters in several respects. First, if the amount of gold held is not to approach zero, then gold must be produced. That, in turn, requires that its price be at the upper bound, $1/\lambda_{bg}$. The rate of return on gold at this price is $1 - \delta$. Hence, with depreciation, the crucial value of r^* is $1 - \delta$. Finally, if $r^* < 1 - \delta$, then although $p(t)$ must eventually be such as to permit gold to be produced, it need not start out that high. If $Z(1)$ is sufficiently large, then the price starts out below $1/\lambda_{bg}$. The description of the equilibrium in this case is somewhat complicated, and we leave the details to the Appendix. There we give a more or less constructive proof of the existence of equilibrium, a proof of the following proposition.

Proposition 4. If $r^* \geq 1-\delta$, then there exists an equilibrium with $r(t) = r^*$, $p(t) = \lambda_{gb}$, and $Z(t+1) = 0$ for all $t \geq 1$. If $r^* < 1-\delta$, then there exists an equilibrium with $r(t) = 1-\delta$, $p(t) = 1/\lambda_{bg}$, and $Z(t+1)/(1-\delta) = \lambda_{bg}S(1-\delta)$ for all $t \geq T$; where $1 \leq T \leq \overline{T}$, $\overline{T}$ is the smallest positive integer j that satisfies

$$(5) \qquad (1-\delta)^{j-1}Z(1) \leq \lambda_{bg}S(1-\delta),$$

$1 > r(1) > r(2) > \ldots > r(T-1) > r(T)$, and $p(1) < p(2) < \ldots < p(T-1) < p(T)$.

If $r^* < 1 - \delta$, then desired saving at $r(t) = 1 - \delta$ and $p(t) = 1/\lambda_{bg}$ is $\lambda_{bg}S(1-\delta)$ in units of gold. If $Z(1)$ does not exceed this quantity, then $\overline{T} = 1$ [see the definition of $\overline{T}$ and equation (5)]; hence, $T = 1$. If, however, $Z(1)$ exceeds this quantity, then the price starts out lower and increases in such a way that $r(t)$ decreases. The monotonicity of S plays a role in our proof that there exists an equilibrium with these properties. It also provides uniqueness subject only to the kind of provision about $p(t)$ that is noted above.

Proposition 5: Under the assumptions of Proposition 4, there is a unique equilibrium $r(t)$ sequence. (For a proof, see the Appendix.)

Note, of course, that uniqueness of the $r(t)$ sequence implies uniqueness of the $Z(t+1)$ sequence and of the consumption allocation.

Growth and Two Limiting Cases

Here we discuss briefly simple proportional growth and two special static production cases: $\lambda_{bg} = 0$ (gold cannot be produced) and $\lambda_{gb} = 1/\lambda_{bg}$ (constant costs).

As regards growth, suppose that for all $t \geq 1$ the aggregate saving function for generation $t + 1$ is a positive constant, n, times the aggregate saving function of generation t. (This happens if the number of type i members of generation $t + 1$ is n times the number of type i members of generation t for all types i

and for all $t \geq 1$, where a type is a description of preferences and endowments.) With all other assumptions intact, there are obvious analogues of propositions 2 and 4.

If $r^* \geq 1 - \delta$, then the claims made in Proposition 4 hold. If $r^* < 1 - \delta$, then a comparison between n and $1 - \delta$ is relevant. If $n = 1 - \delta$, then the description of Proposition 2 applies; if $n > 1 - \delta$, then that of Proposition 4 applies; finally, if $n < 1 - \delta$, then the price of gold must eventually be at its lower bound.

If gold cannot be produced, $\lambda_{bg} = 0$, then propositions 2 and 4 do not apply because the price can rise indefinitely. This permits a very different equilibrium to occur.

Proposition 6: If $n > 1 - \delta$, $S(n) > 0$, and $\lambda_{bg} = 0$; then there exists a unique equilibrium with $r(t) = n$ for all t.

In such an equilibrium, the depreciation rate plays no role.

The special case $\lambda_{gb} = 1/\lambda_{bg}$ is covered by propositions 2 and 4. It merits comment only as being particularly simple. It is the case discussed by many writers, some of whom assert that it prevails only in the "long run." One way to capture what is meant by "long run" is to assume a production technology with adjustment costs in a model with uncertainty. We do not study such setups here.

3. Features of the Laissez-Faire Equilibrium

We will now apply these results to some of the questions listed at the beginning of this paper.

Inefficiency of Commodity Money

Since bread is the only consumption good in our model, the standard definition of inefficiency applied to our model takes the following simple form: a sequence $C(t)$, $t \geq 1$, where $C(t)$ is total consumption of time t bread, is inefficient if there exists a feasible alternative, $\hat{C}(t)$, with $\hat{C}(t) \geq C(t)$ for all $t \geq 1$ and with strict inequality for some $t \geq 1$. It is immediate from this definition that many of the equilibria described above are inefficient; the inefficient ones are those in which a commodity money is held and $n \geq 1 - \delta$.

Thus, for example, consider the Proposition 2 equilibrium in the case $r^* < 1$. This equilibrium satisfies $C(1) < Y(1) + \lambda_{gb}Z(1)$, $C(t) = Y(t)$ for all $t \geq 2$. It can be dominated by the feasible path, $\hat{C}(1) = Y(1) + \lambda_{gb} Z(1)$, $\hat{C}(t) = Y(t)$, $t \geq 2$.

Not surprisingly, the inefficiency is "worse" with depreciation. Thus, a Proposition 4 equilibrium with $r^* < 1 - \delta$ satisfies $C(1) < Y(1) + \lambda_{gb} Z(1)$; $C(t) \leq Y(t)$; $t = 2, 3, \ldots, T$; and $C(t) < Y(t)$, $t > T$. This can be dominated by the same $\hat{C}(t)$ path used to dominate the Proposition 2 equilibrium.

In addition to these inefficiency results, a paradox relates the degree of inefficiency to the productivity of the bread-into-gold production technology. In general, the more productive the technology--the larger λ_{bg}--the "worse" the equilibrium. Thus, consider two economies, A and B, which are identical in all respects except for λ_{bg}; $\lambda^A_{bg} > \lambda^B_{bg}$. It follows that the corresponding laissez-faire equilibria satisfy $C^A(t) \leq C^B(t)$ for all $t \geq 1$. Moreover, it is easy to construct examples, with or without depreciation, in which some of the inequalities are strict. Of course, this ceases to be paradoxical when one recognizes that any production of the commodity money is inefficient and that less tends to be produced as it becomes more costly, in terms of the consumption good, to produce it.

These inefficiencies are "capital theoretic"; they are instances of capital overaccumulation. It is well-known that the no-last-period feature of our model is necessary for producing such outcomes. If there were a last period, a last generation, then under our assumptions any competitive equilibrium is both efficient and Pareto optimal. Thus, although our model provides an interpretation of the usual claim that commodity money is inefficient--why use resources to mine gold just to bury it under Fort Knox?--one wonders whether there are not more plausible interpretations.

An alternative interpretation must rely, somehow, on the notion that the commodity money is a relatively low return asset--relative, that is, to other <u>available</u> assets. To use this notion within a laissez-faire equilibrium, however, one must explain why one of the higher return assets does not take over as the commodity money. We do not think that it is easy to provide such an explanation.

The Quantity Theory of Money

Does this model imply the quantity theory, in the sense of proportionality between the price level and the quantity of money? One way to consider this is to compare the equilibria for two economies that are physically identical except for starting stocks of gold, Z(1).

We get a quantity theory outcome from this comparison in Proposition 2 (or, more generally, when $n = 1 - \delta$) if $r^* < 1$ and if Z(1) for both economies is in the interval $[S(1)\lambda_{bg}, S(1)/\lambda_{gb}]$. The model is then compatible with the following remarks of Samuelson:

> Given physical amounts of tobacco, food, ballet, etc., have significance in terms of the want pattern of the consumer, but it is not possible to attach similar significance to a given number of physical units of money, say to a number of ounces of gold. It would be otherwise in the case of gold which was to be used to fill teeth, but such uses of gold in the industrial arts we purposely neglect. The amount of money which is needed depends upon the work that is to be done, which in turn depends upon the prices of all goods in terms of gold (Foundations of Economic Analysis, [1947], pp. 118-19).

Evidently, the quantity theory result is not robust in models like ours. It disappears, for example, if our static production assumptions are replaced by those that would produce a smooth production possibilities curve in Figure 1. That being so, either our model is defective or one should not be concerned if one's model of commodity money does not give results resembling those of the quantity theory. The attitude one adopts depends, in part, on the pervasiveness of the quantity theory observations.

Those most often cited concern large discoveries of the precious metals. However, it is not clear to us that these discoveries have been accompanied by anything other than the outcomes that would proceed from ordinary price theory, according to which a large discovery of "x" is likely to be accompanied by a large price decrease in x, whether x is gold, oil, cotton, or Picassos. To argue that a special theory is required to explain the observations, one would have to show that they regularly come "close" to satisfying proportionality; we agree that outcomes satisfying proportionality almost exactly should be "rare" according to ordinary price theory models.

Interest Rates and Outside and Inside Money

Our models give rise to two possibilities in relation to interest rates and portfolios in equilibrium. Before describing them, we want to note that our equilibria generally contain private securities, which are loans issued by members of generation t who are sufficiently (compared with other members) heavily endowed with time t + 1 bread relative to time t bread or who have a sufficiently strong preference for time t bread. At time t, these loans are safe titles to bread or gold at t + 1 and are "as good" as gold. If, therefore, we regard gold holdings as commodity money, and, hence, as outside money, then we ought to regard the loans as inside money, if only because they and gold are perfect substitutes from the point of view of potential holders when both are held.

One possible pattern of equilibrium portfolios is that only private loans are held; that is, there is only inside money. This occurs when $r^* > 1 - \delta$. The other general possibility is that both loans and gold are held. In the first case, the gross real interest rate, $r(t)$, can be higher than any possible equilibrium return on the commodity money. In the second case, it is equal to that on the commodity money. These possibilities match almost completely two possibilities Samuelson described in the following passage:

> It is true that in a world involving no transaction friction and no uncertainty, there would be no reason for a spread between the yield on any two assets, and hence there would be no difference in the yield on money and on securities. Hicks concludes, therefore, that securities will not bear interest but will accommodate themselves to the yield on money. It is equally possible and more illuminating to suppose that under these conditions money adjusts itself to the yield of securities. In fact, in such a world securities themselves would circulate as money and be acceptable in transactions; demand bank deposits would bear interest, just as they often did in this country in the period of the twenties. And if money could not make the adjustment, as in the case of metal counters which Aristotle tells us are barren, it would pass out of use, wither away and die, become a free good (Foundations of Economic Analysis, [1947], p. 123).

Our gold does not become a free good in the case $r^* > 1 - \delta$, but it does pass out of use.

From this and a subsequent passage, it seems clear that Samuelson thought the relevant case empirically to be $r^* > 1 - \delta$, or that of a pure credit economy. Even so, it is not clear why he goes on to say:

> Of course, the above [disappearance of outside money] does not happen in real life, precisely because uncertainty, contingency needs, non-synchronization of revenues and outlay, transaction frictions, etc., etc., all are with us. But the abstract special case [of disappearance] analyzed above should warn us against the facile assumption that the average levels of the structure of interest rates are determined solely or primarily by these differential factors (p. 124).

If interest rates in this context mean the spread between the yield on securities and the yield on currency, and if eliminating "these differential factors" eliminates the spread (by eliminating either outside money or the spread), then it seems that an observed spread should be attributed to these factors.

We can also use the outside-inside money interpretation of gold and loans to comment on Friedman's estimate of the resource costs of a commodity standard. He writes:

> The amount is by no means negligible---for example, under a pure commodity standard, the United States would at present be devoting about 2 1/2% of its national product or about $8 billion a year to produce directly or indirectly through foreign trade additional amounts of the monetary commodity to add to the amounts already in circulation or in warehouses (A Program for Monetary Stability, [1960], p. 5).

As described in the footnote accompanying Friedman's text, this estimate is the amount required to provide for growth in currency plus demand and time deposits at commercial banks (M_2), assuming that this total is replaced by holdings of a commodity (that is, by holdings of outside money). Such a replacement is what Friedman means by a "pure" commodity standard. But "pure" must be quite different from laissez-faire, under which the total money supply would probably include at least as much inside money as under the current standard. This would greatly reduce the cost of providing for growth under a commodity standard.

Multiple Commodity Monies

We now describe a class of economies that are identical to the previous ones, except that an additional storable good called silver is also potentially available. The economy starts out with a stock $Z_s(1) > 0$ of silver at $t = 1$, all in the hands of those who are old at $t = 1$. Silver is characterized by a technology analogous to that governing gold. The two parameters λ_{bs} and λ_{sb}, respectively, give the terms on which bread can be converted into silver and on which silver can be converted into bread. In particular, bread can be converted at constant returns to scale into silver at the rate of λ_{bs} units of silver per unit of bread; silver can be converted into bread at constant returns to scale at the rate of λ_{sb} units of bread per unit of silver. We assume that $\lambda_{sb} < 1/\lambda_{bs}$. Silver and gold cannot be converted directly into one another. Silver depreciates geometrically at the rate δ_s per period, $0 < \delta_s < 1$, while gold depreciates at the rate δ_g per period, $0 < \delta_g < 1$. We now let $Z_g(t)$ be the stock of gold held at the beginning of period t; $Z_s(t)$ the stock of silver at the beginning of period t; $ZP_g(t)$ and $ZP_s(t)$ the production of gold and silver, respectively, during period t; and $p_g(t)$ and $p_s(t)$ the prices of gold and silver, respectively, in terms of bread.

We define an equilibrium for this economy as follows.

Definition: An equilibrium consists of nonnegative sequences for $r(t)$, $p_g(t)$, $p_s(t)$, $Z_g(t+1)$, and $Z_s(t+1)$ and sequences for $ZP_s(t)$ and $ZP_g(t)$ for $t \geq 1$ that satisfy:

(i) $\quad S_t[r(t)] = p_g(t)Z_g(t+1)/(1-\delta_g) + p_s(t)Z_s(t+1)/(1-\delta_s);$

(ii) $\quad r(t) \geq p_g(t+1)(1-\delta_g)/p_g(t)$ with equality if $Z_g(t+1) > 0;$

(iii) $\quad r(t) \geq p_s(t+1)(1-\delta_s)/p_s(t)$ with equality if $Z_s(t+1) > 0;$

(iv) $\quad Z_g(t+1) = (1-\delta_g)[Z_g(t)+ZP_g(t)];$

(v) $\quad Z_s(t+1) = (1-\delta_s)[Z_s(t)+ZP_s(t)];$

(vi) $\quad \lambda_{gb} \leq p_g(t) \leq 1/\lambda_{bg}$ with

$$p_g(t) = \begin{cases} \lambda_{gb} & \text{if } ZP_g(t) < 0 \\ 1/\lambda_{bg} & \text{if } ZP_g(t) > 0; \end{cases}$$

(vii) $\quad \lambda_{sb} \leq p_s(t) \leq 1/\lambda_{bs}$ with

$$p_s(t) = \begin{cases} \lambda_{sb} & \text{if } ZP_s(t) < 0 \\ 1/\lambda_{bs} & \text{if } ZP_s(t) > 0. \end{cases}$$

We shall consider the case of proportional growth in which the saving function satisfies $S_t(r) = n^{t-1}S_1(r)$, $t \geq 1$, and $n > 1$. As above, we let r^* be the value that satisfies $S_1(r^*) = 0$.

Proposition 7 (Unequal Depreciation Rates): Assume that $\delta_g < \delta_s$. There exists a unique equilibrium. If $r^* \geq 1 - \delta_g$, then $Z_g(t+1) = Z_s(t+1) = 0$ for all $t \geq 1$. If $r^* < 1 - \delta_g$, then $r(t) = 1 - \delta g$, pg(t) = 1/λbg, and Zg(t+1)/(1-δg) = λbgS1(1-δg)nt-1 for all $t \geq T$, where T is a finite integer (determined in a manner analogous to that described in Proposition 4); and $p_s(t) = \lambda_{sb}$, $Z_s(t+1) = 0$ for all $t \geq 1$. For $t \leq T$, returns and prices of gold obey $n > r(1) > r(2) > \ldots > r(T-1) > r(T)$ and $p_g(1) < p_g(2) < \ldots < p_g(T-1) < p_g(T)$.

Proposition 7 states that the commodity with the lower rate of depreciation is stored. Thus, it provides an example in which "the market" selects one commodity to act as outside money. Proposition 10 will provide another example, one not based on different depreciation rates.

In preparation for discussing government interventions that influence the choice of standard, we will begin to assume equal depreciation rates. To obtain the following proposition, we assume $0 < \lambda_{gb} < 1/\lambda_{bg}$ and $0 < \lambda_{sb} < 1/\lambda_{bs}$. We also assume that $Z_s(1)/\lambda_{bs} = Z_g(1)/\lambda_{bg}$ and $\lambda_{gb}\lambda_{bg} = \lambda_{sb}\lambda_{bs}$, although we will see below that these conditions can be weakened.

<u>Proposition 8 (Equal Depreciation Rates)</u>: Assume that $\delta_g = \delta_s = \delta$. If $r^* \geq 1 - \delta$, then there exists a unique equilibrium with $r(t) = r^*$, $p_g(t) = \lambda_{gb}$, $p_s(t) = \lambda_{sb}$, $Z_g(t+1) = Z_s(t+1) = 0$ for all $t \geq 1$. If $r^* < 1 - \delta$, then there exist multiple equilibria, each having $r(t) = 1 - \delta$ for $t \geq T$, where T is a finite number that is determined in a manner analogous to that described in Proposition 4. We can categorize these equilibria as follows:

(a) An equilibrium with $p_g(t) = 1/\lambda_{bg}$ and $Z_g(t+1)/(1-\delta) = \lambda_{bg}S_1(1-\delta)n^{t-1}$ for $t \geq T$; $p_s(t) = \lambda_{sb}$ and $Z_s(t+1) = 0$ for all $t \geq 1$; and $n > r(1) > \ldots > r(T)$ and $p_g(1) < p_g(2) < \ldots < p_g(T)$.

(b) An equilibrium symmetric to equilibrium (a) with the roles of gold and silver reversed.

(c) A class of equilibria with $p_s(t) = 1/\lambda_{bs}$ and $p_g(t) = 1/\lambda_{bg}$, with $Z_s(t+1) > 0$ and $Z_g(t+1) > 0$ satisfying $S_1(1-\delta)n^{t-1} = Z_g(t+1)/(1-\delta)\lambda_{bg} + Z_s(t+1)/(1-\delta)\lambda_{bs}$ for $t \geq T$, with $n > r(1) > \ldots > r(T)$, $p_g(1) < p_g(2) < \ldots < p_g(T)$, and $p_s(t)/p_g(t) = \lambda_{bg}/\lambda_{bs}$ for all $t \geq 1$. Notice that $Z_s(t+1)$ and $Z_g(t+1)$ are not uniquely determined for $t \geq T$.

(d) A class of equilibria with $p_s(t) = 1/\lambda_{bs}$ and $p_g(t) = p_g$, $t \geq T$, where p_g is a constant that satisfies $\lambda_{gb} < p_g < 1/\lambda_{bg}$ with $Z_g(t+1) = (1-\delta_g)\ Z_g(t)$ for all $t > 1$, $n > r(1) > \ldots > r(T)$, $p_s(1) < p_s(2) < \ldots < p_s(T)$, and $p_s(t)/p_g(t) = 1/p_g\lambda_{bs}$ for all $t \geq 1$, with $Z_g(t+1)$ satisfying (a) with the above values of $Z_g(t+1)$, $p_g(t)$, and $p_s(t)$ for $t \geq 1$.

(e) A class of equilibria symmetric to class (d) with the roles of silver and gold reversed.

If $r^* < 1 - \delta$, equilibria of types (a) and (b) exist even if we abandon the conditions $Z_s(1)/\lambda_{bs} = Z_g(1)/\lambda_{bg}$ and $\lambda_{gb}\lambda_{bg} = \lambda_{sb}\lambda_{bs}$. These conditions are sufficient, although not necessary, to permit equilibria of types (c), (d), and (e) to exist. The role of the conditions is to permit the bread prices of both metals to rise in proportion in each period to T, so that each metal can bear the same rate of return exceeding $1 - \delta$ prior to T. To illustrate how such conditions are needed, suppose that

$Z_g(1) > (1-\delta)S_1(1-\delta)$, that $\lambda_{gb}\lambda_{bg} < 1$, and that $\lambda_{sb}\lambda_{bs} = 1$. In this case, equilibria of types (c), (d), and (e) do not exist because the technology does not permit the price of silver to move over time, as would be required for an initial period.

Any two equilibria from among types (a)-(e), with $T > 1$ for one of them, differ in prices of gold and silver, rates of return, and consumption allocations assigned to generations born before dates when the steady state commenced for each of the equilibria. However, the consumption allocations are identical for generations born after steady states are attained.

The range of indeterminacies in the Proposition 8 economy gets much smaller when we make the technologies for gold and silver perfectly reversible. We state this in the following proposition, which describes an economy that is a special limiting case of a Proposition 8 economy.

<u>Proposition 9 (Reversible Technologies)</u>: Suppose that $\lambda_{bs}\lambda_{sb} = \lambda_{bg}\lambda_{gb} = 1$. If $r^* < 1 - \delta$, then there exists a continuum of equilibria, each having $r(t) = 1 - \delta$, $p_g(t) = \lambda_{gb}$, and $p_s(t) = \lambda_{sb}$ for $t \geq 1$; and with any nonnegative sequences $Z_g(t+1)$, $Z_s(t+1)$ that solve $S_1(1-\delta)n^{t-1} = \lambda_{gb}Z_g(t+1)/(1-\delta) + \lambda_{sb}Z_s(t+1)/(1-\delta)$ for $t \geq 1$.

In the Proposition 9 equilibrium, prices and rates of return are determinate, only the quantities of gold and silver stored are indeterminate when $r^* < 1 - \delta$. The consumption allocations are identical in all Proposition 9 equilibria with $r^* < 1 - \delta$, including the consumption of the current old at $t = 1$. The Proposition 9 economy thus fits Friedman's description:

> The commodity in question might be gold or silver or copper or bricks or some combination of these or of other goods in fixed proportions, as under any of the variety of symmetallic or commodity reserve standards that have been proposed . . . . The maintenance of a commodity standard requires the use of real resources to produce additional amounts of the monetary commodity--of men and other resources to dig gold or silver or copper out of the ground or to produce whatever other commodities constitute the standard. In a stationary economy, production is needed solely to make good losses through wear and tear; in a growing economy, also to provide for an increase in the stock of money. Interestingly enough, the amount of resources required to provide for growth does not depend on the commodity or commodities used as the standard, but only on the cash balance preferences of the public and on the rate of growth of the economy (A Program for Monetary Stability, [1960], pp. 4-5).

We now describe a variation on the above economy in which one of the metals is getting cheaper to produce over time. We assume that $\lambda_{gb} = 1/\lambda_{bg}$ for all $t \geq 1$ but that the technology for silver is changing in the following way. At time t, bread can be converted into silver at constant returns to scale at a rate of $\lambda_{bs}(t)$ units of silver per unit of bread, and the technology is perfectly reversible so that $\lambda_{sb}(t) = 1/\lambda_{bs}(t)$. We also assume that silver gets cheaper to produce over time so that $\lambda_{bs}(t+1) > \lambda_{bs}(t)$ for all $t \geq 1$. For this economy, we have:

Proposition 10 (A Time-Varying Technology for Silver): Suppose that $r^* < 1 - \delta$. There then exists a unique equilibrium with $p_g(t) = 1/\lambda_{bg}$, $p_s(t) = 1/\lambda_{bs}(t)$, $Z_s(t+1) = 0$, and $Z_g(t+1)/(1-\delta) = \lambda_{bg}S_1(1-\delta)n^{t-1}$ for $t \geq 1$.

In the Proposition 10 economy, the metal that is getting cheaper to produce over time (silver) is not produced or stored because it is dominated in rate of return by the metal that is not getting cheaper to produce (gold).

We now return to the case in which the technology is constant through time. The next proposition considers an economy that is another limiting case of a Proposition 8 economy.

<u>Proposition 11 (Gold Cannot Be Produced)</u>: Suppose that $0 < \lambda_{bs} = 1/\lambda_{sb}$ but that $0 = \lambda_{bg} < 1/\lambda_{gb}$. For convenience, we also assume that $Z_g(1) < S_1(1-\delta)/\lambda_{gb}$ so that the initial gold stock is not too big. Suppose that $r^* < 1 - \delta$. There then exist the following three classes of equilibria:

(a) Gold and silver: A continuum of equilibria with $r(t) = 1 - \delta$, $p_s(t) = 1/\lambda_{bs}$, and $p_g(t) = p_g$ where p_g is a constant that obeys $\lambda_{gb} < p_g < S_1(1-\delta)/Z_g(1)$ for $t \geq 1$; $Z_g(t+1) = (1-\delta)^t Z_g(1)$ for $t \geq 1$; $Z_s(t+1) = \lambda_{bs}[(1-\delta)n^{t-1}S_1\ (1-\delta) - p_g Z_g(1)(1-\delta)^t]$.

(b) Silver only: A unique equilibrium with $p_g(t) = \lambda_{gb}$, $p_s(t) = 1/\lambda_{bs}$, $Z_g(t+1) = 0$, and $Z_s(t+1) = \lambda_{bs}S_1(1-\delta)n^{t-1}$ for all $t \geq 1$.

(c) Gold only: A unique equilibrium with $r(t) = n$, $p_s(t) = 1/\lambda_{bs}$, $Z_s(t+1) = 0$, $Z_g(t+1) = (1-\delta)^t Z_g(1)$ for $t \geq 1$, and $p_g(t) = S_1(n)\ n^{t-1}/Z_g(1)(1-\delta)^{t-1}$.

Equilibria in (a) and (b) differ with respect to the consumption allocated to the old at t = 1 since this can be affected by the initial prices of gold and silver. Otherwise, equilibria of these classes lead to identical consumption allocations for all generations born from time 1 onward. Equilibrium (c) results in different allocations for all generations, as compared with the other equilibria. In it, the rate of return for the holding of gold equals n.

4. Government Choice of a Commodity Standard

The class of models that we are using is shot through with propositions of a Modigliani-Miller kind, in which agents see through various institutions and, having the technologies to do so, costlessly undo their effects. This property of the models poses problems in determining whether the government can be said to choose or to influence the choice of a commodity money. Consider two government measures that might be thought to influence the choice of commodity: administering a government mint and choosing the commodity in which to denominate government debt. Setting up a government mint to stamp coins costlessly, even assigning the government mint a monopoly, has in itself no effect in our models because private agents can costlessly assay and melt down coins. Private agents can easily be imagined to render innocuous government decisions to stamp differing amounts of gold, silver, and other metals as "one dollar." For example, if the government freely coins gold and silver dollars at a ratio of 16:1 and requires that all stored metals be stamped "dollars," the equilibria of the models of propositions 7-11 are unaffected because there is nothing to prevent gold dollars from exchanging for silver at a discount or a premium. Similarly, if fiscal policy is held constant in a natural sense, the equilibria of the propositions 8-11 economies (suitably modified to include a role for government finance) would be unaffected by a decision to denominate the government debt in silver as opposed to gold.

These properties of our model reveal the need to impose supplementary legal restrictions to render the government capable of influencing the choice of commodity standard. Without imposing such legal restrictions, it is difficult either to imagine a role for the government in choosing a standard commodity or to make sense of Gresham's law, which seems to hinge on imputing to the government a significant role in influencing the choice of standard.

Partly to illustrate how stringent such restrictions must be in our economies, this section describes two hypothetical systems of government intervention that are directed at influencing the choice of the commodity to be used as the standard. The systems of intervention are particular abstract versions of bimetallism and symmetallism. We will describe each of these systems of intervention in turn, together with its effects on some of the different economies that were described in propositions 8-11. Throughout the section, we deal with identical depreciation rates; furthermore, for simplicity, depreciation rates are equal to zero.

Bimetallism

We suppose that the government imposes a legal restriction allowing evidences of private indebtedness (consumption loans) to be held but prohibiting the direct storage of gold and silver. Instead, the government requires that only (intrinsically useless) paper certificates called dollars that it alone issues and that are "backed" in a particular way by gold or silver can be stored. The government sets up a printing press, which operates costlessly, and issues dollars according to the following rules. It offers freely to issue dollars for silver that is brought to it at the price of d_s dollars per unit of silver and freely to issue dollars for gold at the price of d_g dollars per unit of gold. The government stores the gold and silver that are brought to it. Furthermore, the government offers to redeem dollar certificates for a metal of the government's own choosing, giving $1/d_s$ units of silver per dollar or $1/d_g$ units of gold per dollar. The government's choice is to pay in gold if $p_g(t)/p_s(t) < d_g/d_s$ and in silver otherwise.

This set of government regulations is economically equivalent to one in which the government simultaneously sets up a mint, offers freely to coin or stamp silver dollars at a price of d_s and gold dollars at a price of d_g, and costlessly enforces a legal restriction that in all private trades a "dollar is a dollar"; consequently, gold and silver dollars are always required to circulate at par.[1/] Despite their economic equivalence, we prefer to think of the scheme in the terms described in the preceding paragraph. Under both versions it will be seen that in some of the economies of propositions 8-11, but not in others, there are

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[1/] Milton Friedman and Anna Schwartz [1963, fn, p. 27] point to the need for some device to fix the market rate of exchange between two monies if Gresham's law is to apply.

strong incentives for avoiding the restriction that "a dollar is a dollar."

Bimetallism With Equal Depreciation Rates (Proposition 8)

We consider the Proposition 8 economy with $\delta_g = \delta_s = 0$ and $r^* < 1$. As we have seen, this economy has a continuum of equilibria, one that can be indexed by price ratios p_g/p_s in the closed interval $[\lambda_{gb}\lambda_{bs}, 1/\lambda_{bg}\lambda_{sb}]$. Any constant ratio in this interval is an equilibrium price ratio corresponding to one of the equilibria described in Proposition 8. In the context of this economy, the institution of bimetallism works as follows. If the government sets the "mint ratio" d_g/d_s in the interval $[\lambda_{gb}\lambda_{bs}, 1/\lambda_{bg}\lambda_{sb}]$, the effect is to select one equilibrium from within the continuum, namely the one that satisfies $d_g/d_s = p_g/p_s$. The particular choice of d_g/d_s affects consumption allocations of those born before T, and in some versions T itself, in the manner described above. In equilibria in which p_g/p_s is set in the open interval $(\lambda_{gb}\lambda_{bs}, 1/\lambda_{bg}\lambda_{sb})$, both gold and silver are used as "backing"; in this sense, bimetallism works.

If the government sets $d_g/d_s > 1/\lambda_{bg}\lambda_{sb}$, the effect is to select equilibrium (a); if the government sets $d_g/d_s < \lambda_{gb}\lambda_{bs}$, the effect is to select equilibrium (b). There is a sense in which these cases exhibit a version of Gresham's law. In the former case, gold is "overvalued at the mint," causing it alone to be taken there. In the latter case, silver is overvalued at the mint, and it alone is taken.

In this economy, the government has some latitude for setting the price of gold relative to silver, p_g/p_s. There is no uniquely determined "market price ratio" from which the government can depart only at the cost of driving one metal or the other out of use as dollars. However, a special case of the Proposition 8 economy evidently does have this property, namely the Proposition 9 economy in which both gold and silver technologies are perfectly reversible. In this economy, if $d_g/d_s > \lambda_{gb}\lambda_{bs}$, then only gold is brought to the mint; if $d_g/d_s < \lambda_{gb}\lambda_{bs}$, then only silver is brought to the mint. In this way, the choice of d_g/d_s resolves the indeterminacy about which metals are stored in the Proposition 9 economy. However, the institution of bimetallism as we have specified it leaves all of the real aspects of the economy unaffected by the choice of d_g/d_s. The choice of d_g/d_s only affects whether the "dollar" represents gold or silver "certificates."

<u>Bimetallism When Silver Gets Cheaper to Produce (Proposition 10)</u>

In the Proposition 10 economy, we assume that d_g and d_s are constant over time and set in the following way. We assume that $d_g/d_s > p_g(1)/p_s(1) = \lambda_{gb}\lambda_{bs}(1)$ so that silver is initially undervalued at the mint; as a result, only gold is taken to the mint and stored. We assume that there is a finite $T > 1$ such that $d_g/d_s > \lambda_{gb}\lambda_{bs}(t)$ for $t < T$ and $d_g/d_s < \lambda_{gb}\lambda_{bs}(t)$ for $t \geq T$. For $t < T$, only gold is taken to the mint and stored. For such t, the institution of bimetallism is innocuous because agents would freely choose to store gold, which dominates silver in rate of return. For $t \geq T$, only silver is brought to the mint, since from T on it becomes overvalued at the mint. For $t > T$, the institution of bimetallism makes a difference because some lenders would like to store gold but are prevented from doing so by the legal restriction requiring that only minted dollar certificates be stored. The gross rate of return on loans and dollars for $t > T$ becomes $r(t) = \lambda_{bs}(t)/\lambda_{bs}(t+1)$. The equilibrium condition determining the amount of silver minted and stored for $t > T$ is $n^{t-1}S_1[\lambda_{bs}(t)/\lambda_{bs}(t+1)] = \lambda_{bs}(t)[Z_s(t+1)+ Z_g(T)]$, where $Z_g(T)$ is the amount of gold stored by the government from $(T-1)$ to T.[2/]

We note that borrowers are better off after silver displaces gold, since both loans and dollars bear a lower rate of return than when gold is the standard. There is an inflation after silver drives out gold. Thus, after $\lambda_{bs}(t)$ has risen enough by date T to make silver cheap enough to function as dollars, there is a new force encouraging lenders and holders of dollars to evade or agitate against the institutions that operate to depress the rate of return on assets.

Bimetallism When Gold Cannot Be Produced (Proposition 11)

In the economy of Proposition 11, the government is free to set p_g/p_s at any value in the interval $[\lambda_{gb}\lambda_{bs}, S_1(1)/Z_g(1)]$. This will determine a unique equilibrium from among the continuum described in Proposition 11. In those equilibria for which $\lambda_{gb}\lambda_{bs} < d_g/d_s < \lambda_{bs}S_1(1)/Z_g(1)$, all of the initial gold stock is used as dollars; additional silver is taken to the mint at a rate determined by the equation $S_t(1) = p_gZ_g(1) + \lambda_{sb}Z_s(t+1)$. If the government sets $d_g/d_s > \lambda_{bs}\ S_1(1)/Z_g(1)$, then the effect is to select the gold-only (c) equilibrium of Proposition 11.

[2/] This assumes that $n^{t-1}S_1[\lambda_{bs}(T)/\lambda_{bs}(T+1)] > \lambda_{bs}(T) Z_g(T)$ and that $n^{t-1}S_1[\lambda_{bs}(t)/\lambda_{bs}(t+1)] > n^{t-2}S_1[\lambda_{bs}(t-1)/\lambda_{bs}(t)]$ for $t > T + 1$, so that there is a motive to coin more dollars from T onward.

In this economy, as in the Proposition 8 one, the government is free to preserve bimetallism while setting d_g/d_s within some interval governed by the range of indeterminacy inherited from the technology. Also, a version of Gresham's law obtains in the sense that an attempt to set d_g/d_s too high moves the economy to a gold-only equilibrium. In this economy, however, the gold-only equilibrium differs from the others in allocations, rates of return, and economic efficiency.

Symmetallism

Under symmetallism, the government again has a monopoly on issuing dollars. Besides evidences of private indebtedness, only dollars can be stored. The government defines a dollar as consisting of γ units of gold and σ units of silver. It stands ready to issue freely a dollar certificate to anyone who brings it γ units of gold and σ units of silver, and it costlessly stores the gold and silver. The government is willing to redeem dollar certificates for $1/\gamma$ units of gold plus $1/\sigma$ units of silver.

We let $p_d(t)$ be the bread price of a dollar at time t, measured in units of bread per dollar. In equilibrium, we must have

$$p_d(t) = \gamma p_g(t) + \sigma p_s(t). \tag{6}$$

We let D(t) be the total stock of dollars at the beginning of t, which obeys

$$D(t) = Z_g(t)/\gamma = Z_s(t)/\sigma. \tag{7}$$

An equilibrium under symmetallism with dollars being stored satisfies the following version of (i) (in the definition of equilibrium),

$$S_t[p_d(t+1)/p_d(t)] = p_d(t)D(t+1). \tag{8}$$

Symmetallism With Equal Depreciation Rates (Proposition 8)

For the Proposition 8 economy, with $n > 1$, there is a unique equilibrium of type (c). There is a unique price ratio $p_s(t)/p_g(t) = \lambda_{bg}/\lambda_{bs}$ that is determined by the technology and the need eventually to accommodate growth.

Symmetallism When Silver Gets Cheaper to Produce (Proposition 10)

In the Proposition 10 economy, there is a unique equilibrium in which the bread price of dollars obeys $p_d(t) = \gamma/\lambda_{bg} + \sigma/\lambda_{bs}(t)$ for all $t \geq 1$. The equilibrium quantity of dollars is determined by $S_t\{[\gamma/\lambda_{bg}+\sigma/\lambda_{bs}(t+1)]/[\gamma/\lambda_{bg}+\sigma/\lambda_{bs}(t)]\} = D(t+1)/[\gamma/\lambda_{bg}+\sigma/\lambda_{bs}(t)]$, with $Z_g(t) = \gamma D(t)$ and $Z_s(t) = \sigma D(t)$. The equilibrium rate of return in this economy is between that which emerges in the gold-only equilibrium under laissez-faire and that which emerges eventually under bimetallism in the silver-only equilibrium.

Symmetallism When Gold Cannot Be Produced (Proposition 11)

An equilibrium of the Proposition 11 economy solves equations (6), (7), and (8) with $Z_g(t) = Z_g(2)$ for all $t \geq 2$. We will first seek a solution with $Z_g(t+1) = Z_g(1)$ for $t \geq 1$, which amounts to assuming that the initial stock $Z_g(1)$ is sufficiently small. Under this condition, equation (8) becomes

(9) $$S_t[p_d(t+1)/p_d(t)] = p_d(t)Z_g(1)/\gamma.$$

In the special case $S_t(r) = n^{t-1}S_1(r)$ with $n > 1$, equation (9) has a solution

(10) $$n^{t-1}S_1(n) = p_d(t)Z_g(1)/\gamma.$$

This gives the unique solution of the model, provided that the implied value of $p_g(1) > \lambda_{gb}$. The implied value of $p_g(t)$ is $p_g(t) = p_d(t)/\gamma - \sigma/\gamma\lambda_{bs}$. If $Z_g(1) < S_1(n)/(\lambda_{gb}+\sigma/\gamma\lambda_{bs})$, then $p_g(1) > \lambda_{gb}$ and the solution is, indeed, given by equation (10). If not, then the solution is $p_d(t)n^{t-1}S_1(n)\gamma/Z_g(2)$, where $Z_g(2) = S_1(n)/ (\lambda_{gb}+\sigma/\gamma\lambda_{bs})$. The old at $t = 1$ convert $Z_g(1) - Z_g(2)$ into bread.

We note that in the Proposition 11 model, symmetallism has the effect of selecting an equilibrium with a rate-of-return sequence that is identical to that associated with the gold-only equilibrium. Symmetallism thus eliminates the $r(t) = 1$ equilibria in which gold and silver coexist as backing for dollars. Therefore, in these economies, borrowers generally prefer bimetallism to symmetallism.

Seignorage

In steady states where the per capita real government debt held by the private sector is constant, we regard the government as earning seignorage if government revenue accrues from maintenance of the stock of its indebtedness. This happens if the real rate of return on its debt is less than the growth rate n. Under laissez-faire, the return on government debt in a steady state cannot be less than the maximum of $1 - \delta$ and r^*. The kind of legal restriction discussed above, under which outside money must be in the form of government-issued "dollars," removes $1 - \delta$ as a lower bound on the return on government debt. Hence, if $r^* < 1 - \delta < n$, the existence of such a restriction enhances seignorage possibilities. In the next section, we discuss legal restrictions on private borrowing and lending that can remove r^* as a lower bound on the return on government debt. Here, in order to discuss the sense in which adopting a commodity standard limits seignorage, we will briefly describe seignorage through fiat-money issue under laissez-faire.

Under laissez-faire, if the growth rate n exceeds $1 - \delta$ and exceeds r^*, then there is "room" in the economy for seignorage earning government debt. One form that this debt can take is a stock of fiat money that the market prices. If the stock of fiat money is kept constant over time, an equilibrium rate of return on fiat money and private securities is n. Now imagine that the government lets the stock of fiat money evolve according to $M(t+1) = \beta M(t)$, where $M(t)$ is the stock of fiat money at time t and β is a constant greater than one. Imagine, also, that the government spends the newly created money on "bread," which it consumes. Provided that n/β exceeds $1 - \delta$ and r^*, there is an equilibrium in which fiat money is valued and has a rate of return equal to n/β.

Although adoption of a commodity standard would seem to rule out the issue of fiat money, it is misleading to conclude that it limits seignorage. The seignorage possibility just described can be achieved without having the government issue fiat money. It can be achieved by having the government properly choose and keep outstanding a stock of government debt denominated, say, in gold or, better yet, in bread. This conclusion, which resembles one of those Modigliani-Miller results referred to above, illustrates why it is misleading to say that the adoption of a commodity standard limits seignorage and imposes fiscal discipline.

There is, however, a sense in which limitation of seignorage does take place. Given a rule under which outside money is limited to government-issued dollars and given $r^* < 1 - \delta$, adoption of a commodity standard in any of the variants described above makes $1 - \delta$ a lower bound on the return on government debt. In these circumstances, however, other standards could be adopted that would permit a lower return on government debt and would, perhaps, enhance seignorage. In this sense, adoption of a commodity standard limits seignorage.

5. Restrictions on the Issue of Private Inside Money and the Role of an Intermediating Central Bank

By an intermediating central bank, we mean one that exchanges assets through open market operations or some sort of discount window. These exchanges have no effect unless the central bank has effective monopoly power--for example, over note issue. If it does not have such power, then a Modigliani-Miller type of irrelevance result applies to its intermediation activities. If it has effective monopoly power, then its choice of an intermediation strategy matters by affecting interest rates, the price level, and the equilibrium allocation.

In our view, whether a central bank has effective monopoly power does not depend on the commodity standard or on whether there is a commodity or fiat standard in effect; rather, it depends on whether there are restrictions on privately issued liabilities and assets--for example, on issues of notes by private banks, reserve requirements against bank deposits, and so on.

Because the laissez-faire regime examined above has, by definition, no such restrictions, an intermediating central bank has no role under that regime. In describing laissez-faire, we have assumed that private borrowers can costlessly issue loans in a form that competes perfectly with any potential commodity money. However, the irrelevance of central bank intermediation holds in more general setups. If the costs of intermediating private loans are the same for private intermediaries and the central bank, then the existence of such costs does not overturn the irrelevance result. Indeed, intermediation by a central bank is relevant only if there are potential profits to such intermediation. When these exist, we attribute them to legal restrictions on private intermediation. Such profits, of course, constitute one of the forms of seignorage mentioned in the last section.

Since this point of view and many of its implications have been spelled out elsewhere (see Bryant and Wallace 1980, Wallace 1981, and Sargent and Wallace 1982), we will use it here to interpret some of the well-known debates concerning the Bank Charter Act of 1845 (Peel's Act), which limited bank note issue in England.

This Act required a 100 percent gold backing of all bank note issues beyond a certain inherited amount. Some proponents of the Act wanted the stock of "currency" to behave exactly as would a currency that consisted entirely of gold. Bank notes consisting of warehouse receipts to gold were, on this view, the only acceptable kind of "paper currency." The Act accomplished this; it prevented private borrowers--directly or through banks--from issuing liabilities in the form of small denomination bearer notes.

Peel's Act did not give the Bank of England an effective monopoly because it was subject to the 100 percent marginal reserve requirement on note issue. We regard the Bank of England as having had and as having exercised monopoly power only to the extent that the terms of the Act limiting its note issue were subject to suspension. In fact, they were suspended on several occasions during the second half of the 19th century.

Jevons, a defender of the Act, described the views of some of its opponents:

> The objectors to the Bank Charter Act urge that we want more currency, but they cannot really mean more metallic currency. We must not look to changes in the law to increase the amount of specie in the country, and, as I have remarked, any one can get sovereigns if he has the needful gold. . . . What the currency theorists want, then, is not more gold, but more promises to pay gold. The Free-Banking School especially argue that it is among the elementary rights of an individual to make promises, and that each banker should be allowed to issue as many notes as he can get his customers to take, keeping such a reserve of metallic money, as he thinks in his own private discretion, sufficient to enable him to redeem his promises.

> But this free issue of paper representative money does not at all meet the difficulty of the money market, which is a want of gold, not of paper; on the contrary, an unlimited issue of paper would tend to reduce the already narrow margin of gold upon which we erect an enormous system of trade (Money and the Mechanism of Exchange, [1918], pp. 307-8).

Jevons was dubious of the notion that "more promises to pay gold" could meet the needs of the money market. However, that notion is understandable if we interpret the Act as limiting the form that private debts can take and, therefore, as making it unnecessarily expensive to conclude private agreements to borrow and lend. By doing that, the Act made effective interest rates to borrowers higher than they would otherwise be. The money market could have used more promises to pay gold in forms convenient for most lenders in the sense that relieving the restrictions would have led to a different outcome, one in which borrowers' demands would have been met at lower interest rates.

6. Concluding Remarks

The propositions and interpretations that we have advanced in this paper are products of the models that we have chosen to use. They stem particularly from the finance-theory approach to the pricing of all assets that is inherent in our growth models. The principle of finance theory to which we refer is the notion that assets are priced according to the consumption streams that they support. Some features of our results are also sensitive to particular features of the technology that we have specified and are not robust. For example, results of a quantity-theory kind are very special. Other results, such as those regarding differential depreciation rates for different metals and the inefficiency of commodity money systems, are more robust within this class of growth theories.

Many researchers agree that growth models, with their associated "finance-theory like" property, are promising tools for building theories of asset pricing for privately issued securities. Such models are much more controversial, however, as models of "money" and maybe of government debts in general. Thus, our model and our selection of issues differ from those in recent writings on commodity standards, including those of Barro (1979), Whitaker (1979), and Flood and Garber (1981). These differences partly reflect the fact that we have chosen to explore the consequences of a different set of first principles for pricing assets.

These differences having been noted, we would like to assert the virtues of models like ours for helping to understand and clarify a number of ideas and issues. We find these models particularly useful for distinguishing between inside and outside money and for bringing into focus such distinctions as that between a "pure" gold standard and a "gold reserve standard." In our models, these systems differ either because more private borrowing and lending emerge in the latter--say because of differing endowment patterns and preferences across agents--or because laissez-faire prevails in the latter and legal restrictions are imagined to inhibit intermediation in the former. Because our models are rigged to handle heterogeneous agents in a tractable way, they are manageable vehicles for studying these and other questions involving private borrowing and lending. (Indeed, it is this manageability in the two-period lived overlapping generations model, not an attraction to its demographics, which prompts our use of this model.) Our models also shed useful insights on issues of economic inefficiency of commodity standards, the delicacy of quantity theory predictions, and the nature of the restrictions that must be imposed to get a version of Gresham's law to operate.

Finally, we would like to add a question to the list in our introduction, one that is in the same spirit but not fully answered. How does adopting a commodity standard, as opposed to a fiat standard, affect the "Phillips curve?" Lucas's (1972) analysis of the Phillips curve was conducted using a growth model similar to ours but assuming a fiat money standard that was administered with a special structure of monetary transfers. Since the nature of the Phillips curve correlation in Lucas's setup depends on how the money is handed out, there is a presumption that moving to a commodity standard would alter the Phillips curve in such a model in interesting ways. A complete analysis would be of substantial interest because interesting observations on the Phillips curve are drawn from periods and countries using a commodity standard.

Appendix

This appendix contains proofs of propositions 3, 4, and 5. The proofs of our other propositions are either straightforward or closely follow one of the proofs given below.

Proof of Proposition 3.

We prove this by contradiction, letting $^{-}$'s denote an alternative equilibrium and $\hat{}$'s denote the Proposition 2 equilibrium. We suppose that for some date, t, $\bar{r}(t) \neq \hat{r}(t)$, and show that this gives rise to a contradiction.

Suppose $\bar{r}(t) < \hat{r}(t)$. This is possible only if $\hat{r}(t) = 1$ and $S(1) > 0$. Therefore, $\bar{r}(t) < 1$. But, then, since $\bar{p}(t+1) < \bar{p}(t)\bar{r}(t)$, $ZP(t+1) < 0$ and hence $S[\bar{r}(t+1)] < S[\bar{r}(t)]$. This implies $\bar{r}(t+1) < \bar{r}(t)$; hence, $\bar{p}(t+2) < \bar{p}(t+1)\bar{r}(t) < \bar{p}(t)[\bar{r}(t)]^2$. Repeating these steps, we conclude that $\bar{p}(t+k) < \bar{p}(t)[\bar{r}(t)]^k$ for all $k > 0$. Since $\bar{r}(t) < 1$, this violates Proposition 1.

Suppose $\bar{r}(t) > \hat{r}(t)$. It follows that $\bar{r}(t) > 1$ and that $S[\bar{r}(t)] > 0$, which implies $\bar{p}(t+1) = \bar{r}(t)\bar{p}(t)$ and $ZP(t+1) > 0$. Hence, we have $S[\bar{r}(t+1)] > S[\bar{r}(t)]\bar{r}(t)$. It follows that $\bar{r}(t+1) > \bar{r}(t)$ and that $\bar{p}(t+2) = \bar{p}(t+1)\bar{r}(t+1)$ $\bar{p}(t)[\bar{r}(t)]^2$. Proceeding in this way, we get $S[\bar{r}(t+k)] > S[\bar{r}(t)][\bar{r}(t)]^k$ for all $k > 0$. This violates the upper bound on $S(r)$.

Proof of Proposition 4.

What has to be proved are the assertions about T and $r(t)$ and $p(t)$ for $t < T$ in the case $r^* < 1 - \delta$.

We first consider the possibility that $T = \overline{T}$ and that $p(t) > \lambda_{gb}$ for all t. We denote the implied rates of return and prices by $\overline{r}(t)$ and $\overline{p}(t)$, respectively.

If $p(t) > \lambda_{gb}$ for all t, then no gold is used to produce bread. It follows that the return on saving at $\overline{T} - 1$ and the price of gold at $\overline{T} - 1$ satisfy

$$\text{(A1)} \qquad S[\overline{r}(\overline{T}-1)] = \overline{p}(\overline{T}-1)Z(1)(1-\delta)^{\overline{T}-2} .$$

Using $\overline{r}(\overline{T}-1) = (1/\lambda_{gb})(1-\delta)/\overline{p}(\overline{T}-1)$ to eliminate $\overline{p}(\overline{T}-1)$ from (A1), we get

$$\text{(A2)} \qquad \overline{r}(\overline{T}-1)S[\overline{r}(\overline{T}-1)] = (1/\lambda_{bg})Z(1)(1-\delta)^{\overline{T}-1}.$$

The monotonicity of S implies that (A2) has a unique solution for $\overline{r}(\overline{T}-1)$. We now show that this solution is the interval $(1-\delta, 1)$.

From the definition of $\overline{T}$ as the smallest integer j that satisfies equation (4), it follows that $(1-\delta)S(1-\delta) < (1/\lambda_{bg}) Z(1)(1-\delta)^{\overline{T}-1}$. This implies that $\overline{r}(\overline{T}-1) > 1 - \delta$.

If $r \geq 1$, then $rS(r) \geq S(r) > S(1-\delta) \geq (1/\lambda_{bg})Z(1) (1-\delta)^{\overline{T}-1}$, where the last inequality follows from the definition of $\overline{T}$. This implies that the solution to (A2) is less than unity.

Now we proceed, working backwards, to find $\overline{r}(\overline{T}-2)$, then $\overline{r}(\overline{T}-3)$, and finally $\overline{r}(1)$.

For any k such that $1 < k \leq \overline{T} - 1$,

$$\text{(A3)} \qquad S[\overline{r}(\overline{T}-k)] = \overline{p}(\overline{T}-k)Z(1)(1-\delta)^{\overline{T}-k-1} .$$

Using $\overline{r}(\overline{T}-k) = \overline{p}(\overline{T}-k+1)(1-\delta)/\overline{p}(\overline{T}-k)$ to eliminate $\overline{p}(\overline{T}-k)$ from (A3) and using (A3) for $S[\overline{r}(\overline{T}-k+1)]$, we see that (A3) is equivalent to

$$\text{(A4)} \qquad \bar{r}(\bar{T}-k)S[\bar{r}(\bar{T}-k)] = S[r(\bar{T}-k+1)].$$

With $r(\bar{T}-k+1)$ given, this is an equation of the form $rS(r) = S(x)$ with x given. From the properties of S it follows that if $x \in (1-\delta, 1)$, then this equation has a unique solution for r that is in the interval $(x, 1)$. Since we have established that $\bar{r}(\bar{T}-1) \in (1-\delta, 1)$, it follows that solving (A4) repeatedly--for $k = 2$, then $k = 3$, and finally $k = \bar{T} - 1$--gives us a unique $\bar{r}(1)$, $\bar{r}(2)$, ..., $\bar{r}(\bar{T}-1)$ that satisfies $1 > \bar{r}(1) > \bar{r}(2) > \ldots > \bar{r}(\bar{T}-1) > (1-\delta)$. This, in turn, implies a unique $\bar{p}(1)$, $\bar{p}(2)$, ..., $\bar{p}(\bar{T}-1)$ using condition (ii) from the definition of equilibrium at equality and $\bar{p}(\bar{T}) = 1/\lambda_{bg}$. This is our Proposition 4 equilibrium if $\bar{p}(t) \geq \lambda_{gb}$ for all t.

We now use the above sequences to describe the Proposition 4 equilibrium if it happens that $\bar{p}(t) < \lambda_{gb}$ for some t. Let $\bar{p}(\bar{T}-K)$ be the largest of the $\bar{p}(t)$ that is less than λ_{gb}. [Since $\bar{p}(\bar{T}-1) > \bar{p}(\bar{T}-2) > \ldots > \bar{p}(1)$, if we work backwards from $\bar{p}(\bar{T}-1)$, then $\bar{p}(\bar{T}-K)$ is the first $\bar{p}(t)$ that is less than λ_{gb}.] In this case we let $T = K + 1$; we let $p(1) = \lambda_{gb}$; and we let $p(2)$, $p(3)$, ..., $p(K)$, and $Z(2)$ be the solution to the following K equations:

$$S[p(2)(1-\delta)/\lambda_{gb}] = \lambda_{gb}Z(2)/(1-\delta)$$

$$S[p(3)(1-\delta)/p(2)] = p(2)Z(2)$$

$$\vdots$$

$$\text{(A5)} \qquad S[p(t+1)(1-\delta)/p(t)] = p(t)Z(2)(1-\delta)^{t-2}$$

$$\vdots$$

$$S[(1/\lambda_{bg})(1-\delta)/p(K)] = p(K)Z(2)(1-\delta)^{K-2}.$$

We now show that equations (A5) have a unique solution and that that solution satisfies the claims made in Proposition 4.

We deal only with the case $K > 2$. If $K = 1$, the existence and uniqueness is trivial. We begin by rewriting equations (A5) in terms of rates of return; $r(t) = p(t+1)(1-\delta)/p(t)$. In particular, we consider the $K + 1$ equations

$$\text{(A6)}\qquad \begin{aligned} S[r(1)] &= \lambda_{gb}Z(2)/(1-\delta) \\ S[r(2)] &= r(1)S[r(1)] \\ &\vdots \\ S[r(K)] &= r(K-1)S[r(K-1)] \\ r(K)S[r(K)] &= (1/\lambda_{bg})Z(2)(1-\delta)^{K-1}. \end{aligned}$$

The last two equations of (A6) come from the last equation of (A5). It is easy to see that there is a one-to-one correspondence between solutions to (A6) and solutions to (A5). [Notice that any $r(t)$; $t = 1, 2, \ldots, K$ that solves (A6) satisfies the condition that the product of these $r(t)$'s equals $(1-\delta)^K(1/\lambda_{bg})/\lambda_{gb}$.]

We define a mapping from values of $Z(2)$ on the right-hand side (RHS) of the last equation of (A6) to values of $Z(2)$ on the RHS of the first equation of (A6). We will establish enough facts about this mapping to imply the existence of a fixed point (and certain facts about the fixed point).

We let the domain for this mapping be the interval $D = [d_1, d_2]$ where

$$d_1 = S(1-\delta)\lambda_{bg}/(1-\delta)^{K-2}$$

$$d_2 = S(1)\lambda_{bg}/(1-\delta)^{K-1}.$$

The mapping is defined by proceeding equation by equation through (A6), from the last to the first. That is, for any element of D inserted in place of Z(2) on the RHS of the last equation of (A6), one solves that equation for r(K). Then, using that solution for r(K) in the LHS of the next-to-last equation of (A6), one solves for r(K-1). One proceeds backward in this fashion, finally solving the first equation of (A6) for Z(2), which is the value of the mapping. We denote the mapping by Γ.

It is obvious from this definition that Γ is a continuous, increasing function on D. We now show that $\Gamma(d_1) > d_1$ and that $\Gamma(d_2) < d_2$.

As regards $\Gamma(d_1)$, note that when $Z(2) = d_1$, then $r(K) = 1 - \delta$. Thus our task is to find the corresponding $S[r(1)]$. But $S[r(K)] = S(1-\delta)$, and equations (A6) are exactly those equations we solve when $\overline{T} = K$ and Z(1) is such that no gold gets produced at $\overline{T}$. Thus, by (A3), the corresponding $S[r(1)]$ satisfies

$$\text{(A7)} \qquad S[r(1)] = \overline{p}(\overline{T}-K+1)Z(1) = \overline{p}(\overline{T}-K+1)S(1-\delta)\lambda_{bg}/(1-\delta)^{K-1}$$

where the second equality uses the condition that no gold is produced at $t = K$. Upon inserting the second equality of (A7) on the LHS of the first equation of (A6), we find [as a solution for Z(2)]

$$\Gamma(d_1) = \overline{p}(\overline{T}-K+1)S(1-\delta)\lambda_{bg}/(1-\delta)^{K-2}\lambda_{gb}.$$

Since $\overline{p}(\overline{T}-K+1) > \lambda_{gb}$, we conclude that $\Gamma(d_1) > d_1$.

As regards $\Gamma(d_2)$, note that when d_2 is inserted for $Z(2)$ on the RHS of the last equation of (A6), we get $r(K) = 1$. It follows immediately that the corresponding $S[r(1)]$ is $S(1)$. Thus

$$\Gamma(d_2) = (1-\delta)S(1)/\lambda_{gb}.$$

Therefore, $\Gamma(d_2) < d_2$ if $(1-\delta)/\lambda_{gb} < \lambda_{bg}/(1-\delta)^{K-1}$ or if $\lambda_{gb} > (1-\delta)^K/\lambda_{bg}$. This follows from the fact that $\lambda_{gb} > \overline{p}(\overline{T}-K)$ and that $\overline{r}(t) < 1$ for all t. In other words, $\overline{p}(\overline{T}-K) > (1-\delta)^K/\lambda_{bg}$ follows from the fact that the $\overline{p}(t)$ prices increase at a rate less than $1/(1-\delta)$.

The above facts about Γ imply that there is a d^* in the interval (d_1, d_2) such that $\Gamma(d^*) = d^*$. This d^* and the associated $r(1)$, $r(2)$, ..., $r(K)$ satisfy (A6). Moreover, because d^* is not an endpoint of D, the associated r's satisfy the claim made in Proposition 4.

We now show that d^* is unique. Suppose instead that there are two fixed points of Γ, $d^{(1)}$ and $d^{(2)}$, with $d^{(1)} < d^{(2)}$. It follows from (A6) that $r(t)^{(1)} < r(t)^{(2)}$ for t = 1, 2, ..., K. But this violates the condition on the product of the r's that any solution satisfies. Hence, d^* is unique.

<u>Proof of Proposition 5</u>

We proceed by contradiction, letting $\overline{\ }$'s denote an alternative equilibrium and $\hat{\ }$'s the Proposition 4 equilibrium.

First, suppose that $\overline{r}(t) < \hat{r}(t)$ for some t. This implies that the $\hat{\ }$ equilibrium satisfies $S[\hat{r}(t)] > 0$ for all t, and hence $\hat{r}(t) = 1 - \delta$ for all $t > T$. Since the proof of Proposition

4 shows that positive saving equilibria with $r(t) = 1 - \delta$ for $t \geq T$ have a unique T and unique $r(t)$ for $t < T$, we can, without loss of generality, take $\bar{r}(t) < 1 - \delta$. This possibility is ruled out by proceeding exactly as in the first part of the proof of Proposition 3; essentially, by repeatedly using $\bar{p}(t+k) \leq \bar{p}(t+k-1)\bar{r}(t+k-1)/(1-\delta)$ to get $\bar{p}(t+k) \leq \bar{p}(t)[\bar{r}(t)/(1-\delta)]^k$, which violates the lower bound on the price of gold.

Now suppose that $\bar{r}(t) > \hat{r}(t)$ for some t. It follows that $\bar{r}(t) > 1 - \delta$ and that $S[\bar{r}(t)] > 0$; therefore, $\bar{p}(t+1) = \bar{r}(t)\bar{p}(t)/(1-\delta)$. This and $\bar{r}(t) > (1-\delta)$ imply $\bar{p}(t+1) > \bar{p}(t)$ and, hence, $\overline{ZP}(t+1) \geq 0$ and $\bar{Z}(t+2) \geq (1-\delta)\ \bar{Z}(t+1)$. Multiplying both sides of this last inequality by $\bar{p}(t+1)/(1-\delta)$, we get

$$\text{(A8)} \qquad S[\bar{r}(t+1)] \geq \bar{p}(t+1)\bar{Z}(t+1) = \bar{r}(t)\bar{p}(t)\bar{Z}(t+1)/(1-\delta) = \bar{r}(t)S[\bar{r}(t)].$$

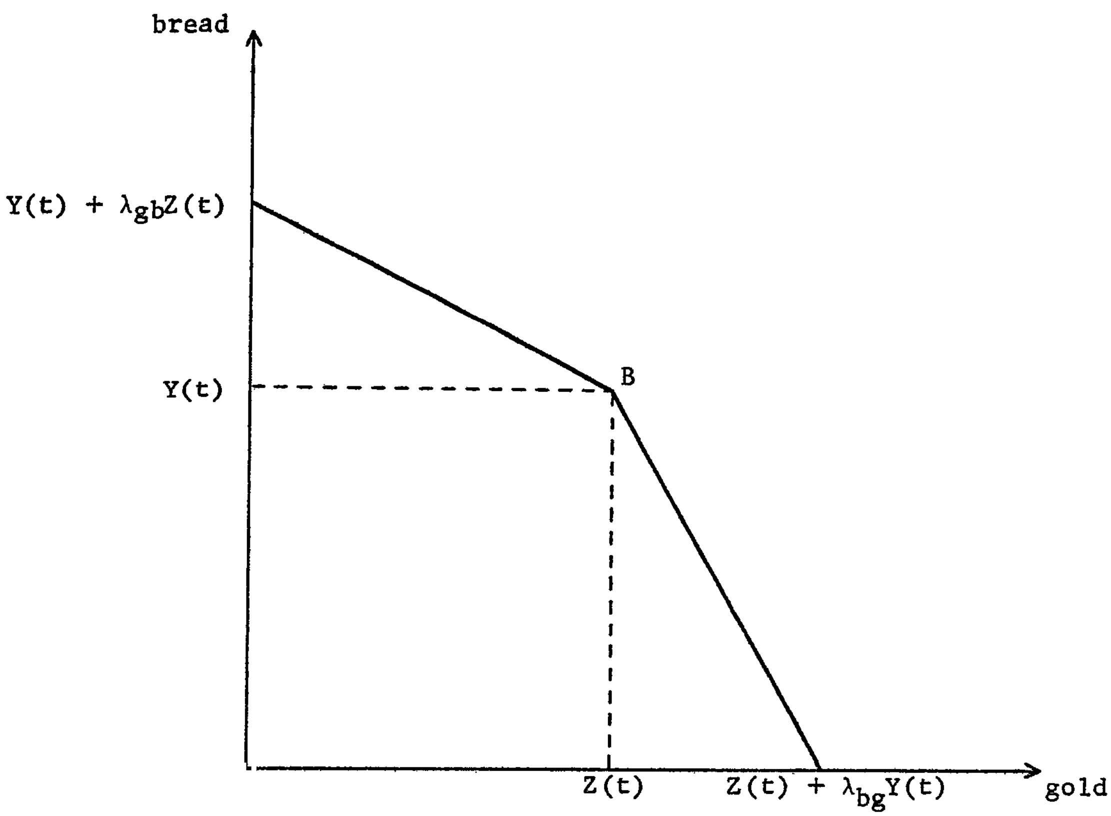
We now consider separately the possibilities $\bar{p}(t+1) = 1/\lambda_{bg}$ and $\bar{p}(t+1) < 1/\lambda_{bg}$. In the former case, it follows that $\bar{p}(t+2) \leq \bar{p}(t+1)$, so $\bar{r}(t+1) \leq 1 - \delta$. The strict inequality was ruled out above. The equality, $\bar{r}(t+1) = 1 - \delta$, implies $\bar{r}(t+k) = 1 - \delta$ for all $k \geq 1$, which, as noted above, cannot be other than a Proposition 4 equilibrium.

If $\bar{p}(t+1) < 1/\lambda_{bg}$, then we get equality in (A8) and, hence, $\bar{r}(t+1) = \bar{r}(t) = 1$ and $\bar{p}(t+2) = \bar{p}(t+1)/(1-\delta)$.

But the latter implies that saving at t + 2 is at least as large as saving at t + 1. Thus, $\bar{r}(t+2) \geq 1$, which implies $\bar{p}(t+3) \geq \bar{p}(t+2)/(1-\delta)$. Proceeding in this manner, we violate the upper bound on the price of gold.

Figure 1

The Static Production Possibility Frontier



References

Barro, Robert J. 1979. Money and the price level under the gold standard. Economic Journal 89: (March) 13-33.

Bryant, John, and Wallace, Neil. 1980. A suggestion for further simplifying the theory of money. Research Department Staff Report 62. Federal Reserve Bank of Minneapolis.

Flood, Robert, and Garber, Peter. 1981. Gold monetization and gold discipline. Federal Reserve International Finance Paper 189. Washington, D.C.

Friedman, Milton. 1960. A program for monetary stability. New York: Fordham University Press.

Friedman, Milton, and Schwartz, Anna. 1963. A monetary history of the United States 1867-1960. Princeton: Princeton University Press.

Jevons, W. Stanley. 1918. Money and the mechanism of exchange. New York and London: D. Appleton and Company.

Lucas, Robert E., Jr. 1972. "Expectations and the neutrality of money. Journal of Economic Theory 4: (April) 102-24.

Samuelson, Paul A. 1947. Foundations of economic analysis. Cambridge: Harvard University Press.

Sargent, Thomas and Wallace, Neil. 1982. The real bills doctrine vs. the quantity theory: A reconsideration. Journal of Political Economy 90 (December) 1212-36.

Wallace, Neil. 1981. A Modigliani-Miller theorem for open-market operations. American Economic Review 71: (June) 267-74.

Whitaker, John K. 1979. An essay on the pure theory of commodity money. Oxford Economic Papers 31: (November) 339-57.