Portfolio Autarky: A Welfare Analysis

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ABSTRACT

Portfolio autarky obtains when residents of every country
are prohibited from owning real assets located in other countries.
Such a regime and a laissez-faire regime, both characterized by free
trade in goods, are studied in a model whose resource and technology
assumptions are those of the standard two-country, two-(nonreproducible)
factor, two-(nonstorable) good model. But to ensure a market for assets
(land), the model is peopled by overlapping generations; each two-period
lived individual supplies one unit of labor only in the first period of
his life. Unique equilibria are described and shown to exist, and, in
terms of a "growth model" version of the Pareto criterion, laissez-
faire is shown to be optimal and portfolio autarky to be nonoptimal.

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I. INTRODUCTION

A.   Is free trade beneficial? That question has been asked many

times. The articles and books on the issue of free trade versus autarky,

if stacked up, would reach quite a way to heaven. But if the exchange

of goods by residents of different countries may be prohibited, so may

the exchange of assets. There is then another type of autarky, the

portfolio or asset analogue of goods autarky. And in this paper we

present a welfare analysis of that less familiar type: as we refer to

it, portfolio autarky, which for us obtains when the residents of every

country are prohibited from owning real assets, by assumption physically

immobile, that are located in other countries.[1]

As might be expected, what we do in the pages that follow is prove several welfare propositions. We must confess, however, that not one of them says anything very surprising. Thus, we prove that our portfolio autarky equilibrium solution is not in general Pareto-optimal. But who would not have guessed that?

Yet we do more than prove what any reasonable economist, if pressed about portfolio autarky, would conjecture. We provide a new representation (or model) of a barter-trade world, a representation that is, we believe, very much in the Heckscher-Ohlin tradition but more interesting than that which has come to dominate in the trade theory literature and in international economics textbooks.[2/]

B. Most contributors to barter-trade theory have assumed that there are two factors of production, labor and capital. And of those who have, virtually all have assumed, a few explicitly and the great majority implicitly, that capital is not marketable. The individuals who populate the world economy of traditional barter-trade theory do not therefore make portfolio decisions. If capital is physically mobile, they may decide where to employ their respective endowments. But they do not decide how much of what kinds of capital to hold.

Obviously, then, it would not have done for us to be entirely traditional in our choice of assumptions. We do assume, as many contributors to barter-trade theory have, that there are only two known production processes. Each requires labor and, as we assume, land (non-reproducible capital). And each yields one non-storable output.

Land is then the only asset. But departing from tradition, we take land as being marketable.

To ensure a non-trivial portfolio demand for land, a demand for land as an asset, we assume, as Samuelson did [6], that each individual lives for two periods, but supplies labor (one unit) only in the first. Nor does anyone come into the world with so much as an acre of land, or in the second period of life receive any kind of transfer payment. To consume in the second period, an individual must therefore have acquired land in the first.

Our world economy is thus populated in each period of time by members of two different (overlapping) generations. There are the young, those who are, as we say, of age one; and there are the old, those of age two. Further, the young do make portfolio decisions. But what their range of portfolio choices is depends on which international economic policy regime they are living under: our financial autarky regime, which is distinguished by a prohibition, applicable world-wide, on the ownership of land located in other countries; or our laissez-faire regime, which is distinguished by complete freedom of portfolio choice.

Whichever regime obtains, though, the young and the old are quite unrestricted in their choice of goods. Free trade in goods is a characteristic of both regimes.

C. We may then describe our task as being first to show that there do indeed exist equilibrium solutions for our policy regimes and, second, to evaluate those equilibrium solutions. In evaluating them, we use a

"growth model" version of the traditional welfare criterion of welfare economics, a version first used, we believe, by Malinvaud [5].

In section II we set out our assumptions and show that they imply a kind of dichotomy: the equilibrium goods price is independent of regime (and time as well). Then in section III we establish the existence of unique equilibrium solutions for our policy regimes and, in passing, show that trade balance is a necessary characteristic of the portfolio autarky equilibrium solution but not of the laissez-faire solution (see [1] and [2]). And in section IV, where we evaluate the two equilibrium solutions, we show the following: (a) that except in the odd instance the portfolio autarky equilibrium solution is not Pareto-optimal; (b) that the laissez-faire equilibrium solution is Pareto-optimal; and (c) that although the laissez-faire equilibrium solution is not in general Pareto-superior to that of the portfolio autarky regime, there are labor and land tax rates that yield a Pareto-superior laissez-faire equilibrium solution.

We suggested above that our representation of a barter-trade world, what might be described as a Heckscher-Ohlin-Samuelson representation, may be more interesting than the traditional representation. Some readers may agree and wonder whether, if there are overlapping generations, the usual "gains from trade" argument goes through. In an appendix to this paper, we show that it does not. If tastes of all individuals are the same and all the individuals of each country are similarly "endowed," the portfolio autarky equilibrium solution may still not be Pareto-superior to the solution of a regime of complete

autarky, a regime distinguished by prohibitions, applicable world-wide, on international exchanges of assets and goods.

II. SOME PRELIMINARIES

A.   For our purposes, it suffices that there be only two national economies or countries. (They are indexed by the variable k.) Each has a population that is constant through time. At the beginning of period t, N_k individuals, who will live for two periods, are born in country k. Consequently, in period t the population of country k is made up of N_k age-one individuals, the members of generation t, and a like number of age-two individuals, the members of generation t-1.

One of our more restrictive assumptions is that tastes are the same. Members of different generations have the same tastes or utility function. Nor does it matter where an individual is born. The life-time utility of individual h, a member of generation t, is given by

$$U[c^h(t)] = a_1 f[g(c_{11}^h(t), c_{21}^h(t))] + a_2 f[g(c_{12}^h(t), c_{22}^h(t))]$$

where $c_{ij}^h(t)$ is consumption of good i (i = 1,2) at age j (j = 1,2) by individual h of generation t and

$$c^h(t) = (c_{11}^h(t), c_{21}^h(t), c_{12}^h(t), c_{22}^h(t)).$$

We take the a_j to be positive and the function f to be increasing, strictly concave and twice differentiable. Further, $f'(y) \to \infty$ as $y \to 0$ and

$$f'(y) + yf''(y) \geq 0.$$

As for the function g, it is homogeneous of degree one in its two arguments; and for $y_i > 0$, the function

$$V(y_1/y_2) = [\partial g(y_1,y_2)/\partial y_2]/[\partial g(y_1,y_2)/\partial y_1]$$

is such that $V > 0$, $V' > 0$, $V(y_1/y_2) \to 0$ as $(y_1/y_2) \to 0$ and $V(y_1/y_2) \to \infty$ as $(y_1/y_2) \to \infty$.

In both countries, output of good i is constrained by the production function $F_i(n,\ell)$, where n and ℓ are respectively the amounts of labor and land used in the production of that good. The F_i are also homogeneous of degree one in their two arguments; and for n, $\ell > 0$, the functions

$$G_i(n/\ell) = [\partial F_i(n,\ell)/\partial \ell]/\partial F_i(n,\ell)/\partial n]$$

satisfy the conditions imposed on V. Moreover, the F_i are different in a strong sense that rules out factor intensity reversals; for $n/\ell > 0$, $G_1(n/\ell) > G_2(n/\ell)$.

Labor and land are both perfectly immobile internationally, so country k has a land endowment, denoted by L_k, and a labor endowment, denoted by N_k.[3/] And since the two factors are perfectly mobile domestically, the input or factor constraints for country k are

$$N_{1k} + N_{2k} \leqq N_k$$

$$L_{1k} + L_{2k} \leqq L_k$$

where N_{ik} and L_{ik} are respectively the amounts of the labor and land of

country k that are used in the production of good i.

As is well known, it follows from our assumptions that the upper boundary of the country k production set is given by a differentiable and concave function, which we denote by

$$X_{2k} = s_k(X_{1k})$$

where X_{ik} is the country k output of good i. The function s_k is defined on the interval $[0, F_1(N_k, L_k)]$; and $s_k(0) = F_2(N_k, L_k)$ and $s_k[F_1(N_k, L_k)] = 0$. Also, $s_k' < 0$ is bounded from below and away from zero; and $s_k'' > 0$. Finally, if $X_i = \sum_k X_{ik}$ is world output of good i, then there is a world production set the boundary of which is

$$X_2 = s(X_1)$$

where the function s has properties analogous to those of the s_k.

B.   We turn now to an explanation of our notion of equilibrium. If there are alternative policy regimes, then government (the governments of the two countries, acting as one) has a choice. Obviously, though, that choice is made at some point in time, which for convenience we assume to be the current or first period (t = 1). For us then the relevant indi- viduals, those who must be taken into account in any welfare reckoning, are all the members of all the generations from the first on and, in addition, since they are alive in the current or first period, all the members of generation zero. It follows that in order to evaluate a regime we have to determine the evolution of the world economy under that

regime from the first period on. That we do in the usual way, by requiring that the prices and quantities of every date be market-clearing solutions to the choice or maximization problems of individuals and firms. Any sequence of such prices and quantities is an equilibrium.

We can, though, give a more detailed definition of an equilibrium. But we first have to describe two choice problems: that faced by the current young and by the young of the second and succeeding generations; and that faced by the current old.

The objective of member h of generation $t \geq 1$, who like all other individuals takes prices as given, is to maximize his life-time utility, $U[c^h(t)]$.[4/] To do that, he chooses a non-negative consumption vector $c^h(t)$. In addition, he chooses some amount of the land of his own country to be purchased in period t and, if unconstrained, an amount of the land of the other country, also to be purchased in period t. We let $q_k^h(t)$ denote the amount of land of country k purchased in period t by member h of generation t. And we require that $q_k^h(t) \geq 0$.

As was indicated in the introduction, every member of generation t supplies one unit of labor in period t (and none in period t+1). For his one unit, member h of generation $t \geq 1$ receives a before-tax labor income $w_h(t)$, which like all other prices (and tax rates) is in units of the first good. In period t, though, he pays a head tax $\mu_h(t)$, so his after-tax income is $w_h^*(t) = w_h(t) - \mu_h(t)$. And it is $w_h^*(t)$ that limits his period t expenditure on goods and land:

(1) $\quad c_{11}^h(t) + p(t)c_{21}^h(t) + \sum_k T_k(t)q_k^h(t) \leqq w_h^*(t)$

where $p(t)$ is the period t price of the second good and $T_k(t)$ is the period t price of land located in country k.

In period $t+1$, member h of generation t rents his country k land holdings to the firms (industries) of country k. The period t rental rate in that country is $r_k(t)$. (Competition among firms ensures that there is only one.) And member h sells all his land in period $t+1$. Since at the time of sale he pays a tax $v_k(t+1)$ on each unit of his country k land, his period $t+1$ expenditure on goods is constrained as follows:

(2) $\quad c_{12}^h(t) + p(t+1)c_{22}^h(t) \leqq \sum_k q_k^h(t)[T_k(t+1) + r_k^*(t+1)]$

where $r_k^*(t) = r_k(t) - v_k(t)$.

To state it in the customary way, the problem of member h of generation $t \geqq 1$ is then to maximize $U[c^h(t)]$ by the choice of $c^h(t) \geqq 0$ and the $q_k^h(t) \geqq 0$, subject to given period t and period $t+1$ prices and constraints (1) and (2). If, however, he is living under the portfolio autarky regime, then there is an additional constraint: if he resides in the second country, then $q_1^h(t) = 0$; and if he resides in the first, then $q_2^h(t) = 0$. So member h is more or less constrained, depending on which regime obtains. There are, as it were, two versions of his choice problem, a laissez-faire version and a portfolio autarky version.

With what has already been said, the choice problem faced in the first period by members of generation zero can be briefly stated. For member h, it is to maximize $U[c^h(0)]$, but by the choice of only the $c_{12}^h(0) \geqq 0$ and subject to given first period prices and the $t = 0$ version

of constraint (2).

We assume that the members of generation zero come into the first period owning among them all of the land of the world: $\sum_h q_1^h(0) = L_1$ and $\sum_h q_2^h(0) = L_2$. But we impose no particular distribution of land ownership, so in a sense there are arbitrary initial conditions.

For our purposes, it is enough that all members of any generation $t \geq 1$ who reside in country k pay the same head tax. That tax we denote by μ_k. We also take the tax on the land of country k as being independent of time and so write ν_k for the per unit tax. And by assumption government chooses the μ_k and ν_k subject to a world economy budget-balance constraint

$$(3) \quad \sum_k \mu_k N_k + \sum_k \nu_k L_k = 0$$

and the additional constraints

$$(4) \quad \begin{aligned} w_k^*(t) &> 0 \\ r_k^*(t) &> 0. \end{aligned}$$

It remains only for us to describe the choice problem of firms and to set out our market-clearing conditions. For any firm located in country k, the problem is to choose non-negative quantities of country k land and labor so as to maximize period t profit, subject to given $r_k(t)$ and $w_k(t)$. That, it should be noted, is precisely the problem of the firm of the traditional literature (see 4, chapter 2),

so our supply-side maximization conditions, which there is no need to write down, are those of that literature.

Among the market-clearing conditions, which hold for all i, k and $t \geq 1$, are

$$(5) \quad C_i(t) \equiv \sum_h c_{i1}^h(t) + \sum_h c_{i2}^h(t-1) = \sum_k X_{ik}(t)$$

where $h = 1,2,\ldots,N_1,N_1+1,\ldots,N_1+N_2(= N)$ and $X_{ik}(t)$ is the output of good i in period t by the firms of country k, and

$$(6) \quad \sum_h q_k^h(t) = L_k.$$

Condition (5) requires for good i that aggregate demand in period t, denoted by $C_i(t)$ and defined as the sum of the demands of the members of generation t and t-1, equal aggregate supply, the sum of the supplies of all the firms of the two countries. And condition (6) requires that period t aggregate asset demand for country k land, the sum of the demands of the members of generation t, equal the exogenous supply. The two conditions hold independent of regime.

There are two other market-clearing conditions: a labor market condition which requires that the period t aggregate demand for the labor of country k, the sum of the demands of all the firms of country k, equal the total supplied by the country k members of generation t (the exogenous quantity N_k); and a land market condition, a second one, which requires that the period t rental demand for country k land, the sum of the demands of the firms of country k, equal the exogenous supply L_k. And although we do not bother to state our factor market conditions formally, it is

important that they are exactly those of the traditional literature.

An equilibrium is then a non-negative solution of conditions (5) and (6), the factor market conditions and the first-order conditions of the choice or maximization problems described above, including that of firms.[5/] The solution is made up of a second-period consumption allocation for the members of generation zero, $c_2(0) = (c_{12}(0), c_{22}(0))$, where

$$c_{12}(0) = (c_{12}^1(0), c_{12}^2(0), \ldots, c_{12}^N(0))$$

and what we refer to as equilibrium sequences or time paths defined over $t = 1, 2, \ldots$ for the remaining endogenous variables, the $w_k(t)$, $r_k(t)$, $T_k(t)$, $p(t)$ and

$$c(t) = (c^1(t), c^2(t), \ldots, c^N(t)).$$

Above, we suggested that members of generation $t \geq 1$ are confronted by given period t and period $t+1$ prices. And it might be asked how that can be. Member h of generation $t \geq 1$ can be regarded as maximizing the expectation of $U[c^h(t)]$ over some subjective distribution of period $t+1$ prices. But our world economy is non-stochastic. So if the subjective distribution is equated to the actual distribution, the distribution determined by the economic structure, then it collapses to a point (the coordinates of which are actual period $t+1$ prices). And one way of equating the distributions is to endow individuals with complete knowledge of the economic structure. That is what we do. We assume that every individual knows the choice problems faced by firms

and members of his and succeeding generations, what the market-clearing conditions are and what regime and tax rates will prevail in the current and all future periods. Individuals can then and do accurately forecast period t+1 prices and, as they must to determine those prices, the prices for all future periods. Our equilibrium can therefore be thought of as consisting of first-period equilibrium values and, for all future periods, forecasts of all the endogenous variables.

C.   Having got through the necessary preliminaries, we can now state our

> Proposition 1: On our assumptions, there exists a
> unique equilibrium relative price sequence $\{\bar{p}(t)\}$,
> where $\bar{p}(t) = \bar{p}$ for all t, that is determined only
> by factor endowments and the utility and production
> functions. In particular, $\bar{p}$ is independent of the
> choice of regime and the choice of tax rates.

To prove the proposition, we first characterize our aggregate supply functions, the $S_{ik}[p(t)]$. Since the F_i are classical, or satisfy the assumptions of the traditional literature, and since our supply-side maximization and factor market conditions are those of that literature, our supply functions are too. So it would seem enough that we simply state certain properties of those functions.

The functions $S_{ik}[p(t)]$, which give the period t outputs of the first and second goods by the firms of country k, are continuous.

And for $p(t) \geq 0$, $S_{1k}[p(t)]$ is non-increasing; it is decreasing for $p(t)$ on the interval $[p_{1k}^*, p_{2k}^*]$, where p_{1k}^* and p_{2k}^* are numbers determined by the F_i and N_k and L_k and $p_{2k}^* > p_{1k}^* > 0$; and $S_{1k}(p_{2k}^*) = 0$. Further $S_{2k}[p(t)]$ is non-decreasing; for $p(t)$ on the interval $[p_{1k}^*, p_{2k}^*]$, it is increasing; and $S_{2k}(p_{1k}^*) = 0$.

The period t aggregate or world supply of good i, the RHS of equation (5), is given by $S_i[p(t)] = \sum_k S_{ik}[p(t)]$. Letting $p_1^* = \min(p_{11}^*, p_{12}^*)$ and $p_2^* = \max(p_{21}^*, p_{22}^*)$, we thus have that $S_1[p(t)]$ is non-increasing for all $p(t)$; for $p(t)$ on the interval $[p_1^*, p_2^*]$, it is decreasing; and $S_1(p_2^*) = 0$. And $S_2[p(t)]$ is non-decreasing; for $p(t)$ on the interval $[p_1^*, p_2^*]$ it is increasing; and $S_2(p_1^*) = 0$.

Whichever regime obtains, the aggregate supply of good i in period $t \geq 1$ is given by $S_i[p(t)]$; that must be, since for firms there is but one choice problem. And whichever regime obtains

(7) $\quad \sum_k [w_k(t)N_k + r_k(t)L_k] = S_1[p(t)] + p(t)S_2[p(t)]$;

independent of regime, total factor payments and the value of world output are the same. For the F_i, which constrain firms no matter which regime obtains, are linearly homogeneous.

And now, as a preliminary to getting an equilibrium restriction on $p(t)$, a restriction that holds independent of regime, we derive a relationship between aggregate demand quantities, the $C_i(t)$. Since $U[c^h(t)]$ is strictly concave and constraints (1) and (2) are linear in $c^h(t)$, there is a unique maximizing $c^h(t)$, denoted by $\tilde{c}^h(t)$, associated with every positive price vector. Further, $\tilde{c}^h(t) > 0$. That follows from the conditions

imposed on the function V and the conditions $f'(0) = \infty$ and $w_k^*(t), r_k^*(t) > 0$. In consequence, the elements of $\tilde{c}^h(t)$, the consumption quantities desired or demanded by member h of generation t, are constrained thusly:

$$(8) \quad a_j f'\{g[\tilde{c}_{1j}^h(t), \tilde{c}_{2j}^h(t)]\}(\partial g/\partial c_{1j}) = \tilde{\lambda}_j^h(t)$$

$$(9) \quad a_j f'\{g[\tilde{c}_{1j}^h(t), \tilde{c}_{2j}^h(t)]\}(\partial g/\partial c_{2j}) = \tilde{\lambda}_j^h(t)p(t-1+j)$$

where $\tilde{\lambda}_j^h(t) > 0$ is the optimal $\lambda_j^h(t)$, the Lagrangean multiplier associated with constraint j (=1,2).[6]/ For $t = 0$, the first order conditions (8) and (9) hold for $j = 2$ and all h; and for $t \geq 1$, they hold for all j and h. But then for $t \geq 1$ and all h

$$(10) \quad V[\tilde{c}_{11}^h(t)/\tilde{c}_{21}^h(t)] = p(t);$$

and for $t \geq 0$ and all h

$$(11) \quad V[\tilde{c}_{12}^h(t)/\tilde{c}_{22}^h(t)] = p(t+1).[7]/$$

And since V^{-1} exists, it follows that for $t \geq 1$

$$(12) \quad C_1(t) = C_2(t)V^{-1}[p(t)]$$

where $C_i(t) = \sum_h \tilde{c}_{i1}^h(t) + \sum_h \tilde{c}_{i2}^h(t-1)$. Equation (12), the sought-after restriction on the total desired (maximizing) quantities of the two goods, can also be described as an equilibrium condition. And, as may be obvious, it holds independent of regime. That is to say, the first-order conditions (8) and (9) hold whichever regime obtains. For no matter which regime they are living under, the members of generation $t \geq 1$ are subject to

constraints (1) and (2); and the members of generation zero are subject to constraint (2).

Evidently, though, for $t \geq 1$, $\tilde{c}^h(t)$ and the $\tilde{q}_k^h(t)$, the optimal $q_k^h(t)$, satisfy the equality versions of constraints (1) and (2); and for $t = 1$, the $\tilde{c}_{12}^h(0)$ and the exogenous $q_k^h(0)$ satisfy the equality version of constraint (2). We have then, by summing the equality version of constraint (2) over the members of generation $t-1$ and the equality version of constraint (1) over the members of generation t and adding, that

$$(13) \quad C_1(t) + p(t)C_2(t) = \sum_h w_h^*(t) + \sum_h \sum_k q_k^h(t-1)r_k^*(t)$$
$$+ \sum_h \sum_k T_k(t)[q_k^h(t-1) - q_k^h(t)].$$

And making use of our assumptions about tax rates and equations (6), the land market equilibrium conditions, we get a second restriction on the equilibrium $C_i(t)$, or a second equilibrium condition, namely

$$(14) \quad C_1(t) + p(t)C_2(t) = \sum_k [w_k(t)N_k + r_k(t)L_k]$$

which, like equation (12), holds independent of regime.

It follows from equations (14), (12) and (7) that

$$C_2(t)\{p(t) + V^{-1}[p(t)]\} = S_1[p(t)] + p(t)S_2[p(t)].$$

And since for an equilibrium $C_2(t) = S_2[p(t)]$, we thus have that any equilibrium price sequence $\{\bar{p}(t)\}$ satisfies the conditions

$$(15) \quad S_2[p(t)]V^{-1}[p(t)] = S_1[p(t)]$$

where $t \geq 1$. And so, as alleged, any equilibrium period t goods price,

$\bar{p}(t)$, depends only on the utility and production functions and factor endowments. It is independent of regime, for condition (15) holds no matter which regime obtains. Also, $\bar{p}(t) = \bar{p}$ for $t \geq 1$, since the functions S_i and V are independent of t.

What remains to be shown then is that equation (15) determines a unique and positive p. That is immediate, though, since $V^{-1}(p) > 0$ is continuous and increasing for $p > 0$. So $S_2(p)V^{-1}(p) \geq 0$ is continuous, non-decreasing and, for $p \geq p_1^*$, increasing. And therefore, by the properties of the function $S_1(p)$, outlined above, there must be a unique p, say $\bar{p} > 0$, that satisfies equation (15).[8/]

Having established that the equilibrium price sequence, $\{\bar{p}(t)\}$, is unique and stationary (independent of t), we may take it that there exist unique equilibrium factor return sequences, $\{\bar{w}_k(t)\}$ and $\{\bar{r}_k(t)\}$, also stationary. For the equilibrium wage of country k, we write $\bar{w}_k$; and for the equilibrium land rental, $\bar{r}_k$. And we have that independent of regime the two goods are consumed in the same proportions by all members of all generations; that is, whichever regime obtains

$$(16) \quad \tilde{c}_{1j}^h(t)/\tilde{c}_{2j}^h(t) = V^{-1}(\bar{p})$$

for all h and j if $t \geq 1$ and for all h and j = 2 if $t = 0$.

III. EQUILIBRIUM: EXISTENCE AND SOME PROPERTIES

A.   Our immediate task is to prove

Proposition 2: Whichever regime obtains, if it is
assumed to obtain for $t \geq 1$, then there exists a
unique and stationary equilibrium.

For us, an equilibrium is stationary if all of the price sequences
are stationary. So what we have to show is the following: if the
laissez-faire (or portfolio autarky) regime obtains and, by common
assumption, will into the indefinite future, then there exist unique,
positive and stationary land price sequences $\{T_k(t)\}$. Because it is
convenient to do so, we first establish the existence of the laissez-
faire equilibrium.

If living under the laissez-faire regime, the members of
generation $t \geq 1$ may purchase the land of either or both countries.
Therefore, if the laissez-faire regime obtains, there are first-order
conditions

$$(17) \quad -\tilde{\lambda}_1^h(t)T_k(t) + \tilde{\lambda}_2^h(t)[T_k(t+1) + \bar{r}_k^*] \leq 0$$

which hold for all h and k and with equality if $\tilde{q}_k^h(t) > 0$. So under
the laissez-faire regime, $\beta_1(t) = \beta_2(t) = \beta(t)$ for $t \geq 1$, where

$$\beta_k(t) = T_k(t)/[T_k(t+1) + \bar{r}_k^*] = \tilde{\lambda}_2^h(t)/\tilde{\lambda}_1^h(t)$$

is the reciprocal of the one-period yield on the land of country k.
That is obviously so if for any member of generation $t \geq 1$ the $k = 1$
and $k = 2$ versions of constraints (17) both hold with equality. And
assuming the two versions hold with equality for no member yields a

contradiction.[9/]

Now, then, appealing to the homogeneity of the function g, Proposition 1 and equation (16), we may rewrite equations (9) as follows:

$$(18) \quad a_j f'[\tilde{c}^h_{2j}(t)\bar{g}]\bar{g}_2 = \tilde{\lambda}^h_j(t)\bar{p}$$

where $\bar{g} = g[V^{-1}(\bar{p}),1]$ and $\bar{g}_2 = \partial g/\partial c^h_{2j}$, which is the same function of $\bar{p}$ for $j = 1,2$. And therefore any possible laissez-faire equilibrium $c^h(t)$, which we denote by $\hat{c}^h(t)$, satisfies

$$(19) \quad a_2 f'[c^h_{22}(t)\bar{g}]/a_1 f'[c^h_{21}(t)\bar{g}] = \beta(t)$$

and

$$(20) \quad c^h_{21}(t) + \beta(t)c^h_{22}(t) = \overline{w^*_K}/\bar{P}$$

where $\bar{P} = V^{-1}(\bar{p}) + \bar{p}$.[10/] But $\hat{c}^h(t)$, which maximizes $U[c^h(t)]$ subject to constraints (1) and (2), is unique. So we have

$$(21) \quad \hat{c}^h_{2j}(t) = \phi^h_j[\beta(t)]$$

as the solutions to equations (19) and (20). For any h, the $\hat{c}^h_{2j}(t)$ also depend on either $\overrightarrow{w^*_1}$ or $\overrightarrow{w^*_2}$. That is why the functions ϕ_j are indexed by h. But for each j there are only two functions, one for the residents of the first country and one for the residents of the second.

From equations (19) and (20),

$$(22) \quad A((\phi^h_1)',(\phi^h_2)')' = b$$

where, denoting $f[\hat{c}_{2j}^h(t)\bar{g}]$ by $f(j)$,

$$A = \begin{pmatrix} -f'(2)f''(1) & f'(1)f''(2) \\ 1 & \beta \end{pmatrix}$$

and

$$b = \begin{pmatrix} a_1[f'(1)]^2/a_2\bar{g} \\ -\hat{c}_{22}^h \end{pmatrix}.$$

And so, by the conditions imposed on the function f, which imply among other things that det A > 0, we have the first of the properties of the functions $\phi_2^h(\beta)$ that subsequently we will need:

(i) $(\phi_2^h)' < 0$

(ii) $\phi_2^h(\beta) \to \infty$ as $\beta \to 0$

(iii) $\phi_2^h(\beta) \to 0$ as $\beta \to \infty$

(iv) $\beta(\phi_2^h)'/\phi_2^h \le -1$.

Furthermore, as can be verified by rather considerable algebraic manipulation of the expression for $(\phi_2^h)'$ [see equation (22)], property (iv) holds if and only if

$$f'(y) + yf''(y) \ge 0$$

which by assumption is a condition satisfied by the function f.[11]/

Property (iii) is immediate from equation (20). And property (ii) follows from the equality version of constraint (1) and equation (19). For by the equality version of constraint (1), $\hat{c}_{21}^{h}$ is bounded from above. And therefore $f'(\hat{c}_{21}^{h}\bar{g})$ is bounded from below. So by equation (19), $f'(\hat{c}_{22}^{h}\bar{g}) \to 0$ as $\beta \to 0$; and by the conditions imposed on the function f, $\hat{c}_{22}^{h} \to \infty$.

And now, having established properties (i) - (iv), we derive equilibrium restrictions on the $\{T_k(t)\}$. From the equality version of constraint (2) and equations (16) and (21), we get for all h and $t \geq 1$

$$\bar{P}\phi_2^h[\beta(t)] = \sum_k \hat{q}_k^h(t)[T_k(t+1) + \bar{r}_k^*].$$

And summing over the members of generation t and making use of equations (6) yields

$$(23) \quad \bar{P}\phi_2[\beta(t)] = \sum_k L_k[T_k(t+1) + \bar{r}_k^*]$$

where, as is easily shown, $\phi_2[\beta(t)] = \sum_h \phi_2^h[\beta(t)]$ satisfies properties (i) - (iv). But any laissez-faire equilibrium land price sequences, denoted by $\{T_k(t)\}$, satisfy not only equation (23); they also satisfy

$$(24) \quad T_1(t)/T_2(t) = \bar{r}_1^*/\bar{r}_2^*$$

for all t.

To establish equation (24), we first show that $M > \hat{T}_k(t) > m > 0$ for all t and k. Since the $\hat{c}_{22}^h(t)$ satisfy equation (5), they are bounded. (At $p = \bar{p}$, output of the second good is certainly finite.) But then, by property (ii), $\beta(t) > 0$ for all t; and by the definition

of $\beta(t)$, $\hat{T}_k(t) > m = \min(m_1, m_2) > 0$ for all t. Also, by equation (6), $\hat{q}_k^h(t) > 0$ for all k and some h; and therefore, by constraint (1), the $\hat{T}_k$ must be bounded from above by some number M.

If we denote $\hat{T}_1(t)/\hat{T}_2(t)$ by $z_1(t)$ and $\hat{T}_2(t)/[\hat{T}_2(t) + \bar{r}_2^*]$ by $z_2(t)$, then $\beta_1(t) = \beta_2(t)$ implies that $z_1(t)$ satisfies the difference equation

$$(25) \quad z_1(t) = [z_1(t+1) - \bar{r}_1^*/\bar{r}_2^*]z_2(t+1) + \bar{r}_1^*/\bar{r}_2^*.$$

And consequently for any $K \geq 1$

$$(26) \quad z_1(t) = [z_1(t+K) - \bar{r}_1^*/\bar{r}_2^*] \prod_{k=1}^{K} z_2(t+k) + \bar{r}_1^*/\bar{r}_2^*.$$

But by the boundedness of the $\hat{T}_k(t+k)$, $z_1(t+k)$ is bounded and $z_2(t+k)$ is positive and bounded away from one. So equation (24) follows on taking the limit of both sides of equation (26) as $K \to \infty$.

We have then from equations (23) and (24) and the definition of $\beta(t)$ that any stationary equilibrium land price sequence $\{\hat{T}_2(t)\}$ must satisfy

$$(27) \quad \bar{P}\phi_2[T_2/(T_2 + \bar{r}_2^*)] = L[1 + T_2/\bar{r}_2^*]$$

where $L = \sum_k L_k \bar{r}_k^*$. And there clearly is a unique and positive solution to equation (27). For the RHS is a positive, increasing and linear function of $T_2 \geq 0$. And since $\beta = T_2/(T_2 + \bar{r}_2^*)$ increases as T_2 increases and $\beta = 0$ for $T_2 = 0$, it follows from properties (i) and (ii) that the LHS of equation (27) is a continuous decreasing function of T_2 that tends to infinity as $T_2 \to 0$.

So there do exist unique laissez-faire equilibrium land price

sequences that are stationary and what must now be demonstrated is that there do not exist any non-stationary equilibrium sequences. Any laissez-faire equilibrium sequence is (a) bounded by m and M and (b) satisfies the first-order difference equation

$$(28) \quad \bar{P}\phi_2\{T_2(t)/[T_2(t+1) + \bar{r}_2^*]\} = L[1 + T_2(t+1)/\bar{r}_2^*]$$

which follows from equations (23) and (24) and the definition of $\beta(t)$. But any non-stationary equilibrium sequence, denoted $\{T_2^*(t)\}$, being other than constant, must also satisfy condition (c): there exists a value of t, say $\bar{t}$, such that $|T_2^*(\bar{t}) - \hat{T}_2| = \delta$ where δ is some positive number. And as we show, any sequence satisfying conditions (a) and (b) cannot satisfy condition (c).[12]

By properties (i) - (iii) of the functions ϕ_2^h, there is for any $T_2(t+1) \geq 0$ one and only one value of $T_2(t)$ that satisfies equation (28). Consequently, we may write

$$T_2^*(t) = H[T_2^*(t+1)]$$

where the function H has a unique fixed point, $\hat{T}_2$, and where, as can be verified by differentiating equation (28)

$$H'[T_2(t+1)] = \beta(t) + \phi_2[\beta(t)]/\phi_2'[\beta(t)].$$

And as a first step in our proof, we establish that any sequence $H(y) = H^1(y)$, $H[H(y)] = H^2(y),\ldots$, denoted by $\{H^k(y)\}$, converges to the fixed point $\hat{T}_2$. In doing that, we refer to Figure I, wherein we show a possible H function.

Note that to the left of $\hat{T}_2$, the function, as drawn, lies above the positively sloped 45° line; and to the right of $\hat{T}_2$, it lies below that line. That is as must be. Any admissible H function is so bounded. For at $T_2(t+1) = \hat{T}_2$, $\beta(t) < 1$; and therefore, by property (i) above, $H' < 1$ at $T_2(t+1) = \hat{T}_2$. Then, in some neighborhood about the
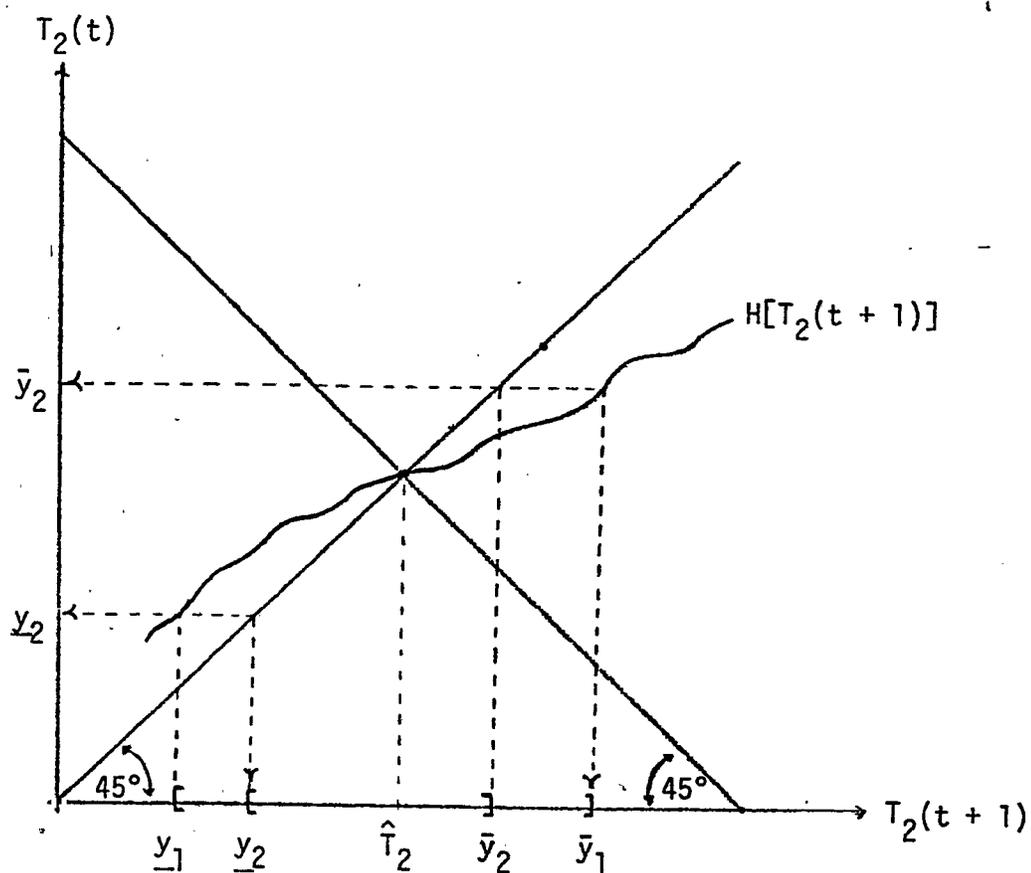

FIGURE I

fixed point, $H[T_2(t+1)] \gtrless T_2(t+1)$ as $T_2(t+1) \lessgtr \hat{T}_2$. But since H, a continuous function, has a unique fixed point, that is true for $T_2(t+1) \geq 0$.

And note that the H function of Figure I lies below the negatively sloped 45° line passing through the point $(\hat{T}_2, \hat{T}_2)$, to the left of $\hat{T}_2$; and to the right of $\hat{T}_2$, it lies above that line. For any admissible H function, that too must be. By the above property (iv), $H' \geq 0$, from which it follows that $H[T_2(t+1)] \leq \hat{T}_2$ for $T_2(t+1) < \hat{T}_2$; and $H_2[T_2(t+1)] \geq \hat{T}_2$ for $T_2(t+1) > \hat{T}_2$.$\underline{13/}$

Now, then, repeated applications of the H function of Figure I, bounded as it is, generates a sequence of nested closed intervals; and if the first contains $\hat{T}_2$, all do. Two such intervals, $[\underline{y}_1, \bar{y}_1]$ and $[\underline{y}_2, \bar{y}_2]$, are depicted in the figure. And that the second is necessarily a proper subset of the first is easily verified. Project the first onto the $T_2(t)$-axis using the H function; and using the positively sloped 45° line, project the resulting interval back onto the $T_2(t+1)$-axis. But for any admissible H function, repeated application starting from $I_1 = [\underline{y}_1, \bar{y}_1]$, which contains $\hat{T}_2$, yields the sequence I_k of nested closed intervals, all containing $\hat{T}_2$. And, more particularly, repeated application generates number sequences $\{\underline{y}_k\}$ and $\{\bar{y}_k\}$. Moreover, since $\{\underline{y}_k\}$, the sequence of lower end-points, is increasing and bounded from above, it converges to $\hat{T}_2$; and the sequence of upper end-points, $\{\bar{y}_k\}$, which is decreasing and bounded from below, also converges to $\hat{T}_2$.$\underline{14/}$ Since we may define $I_1 = [m, M]$, we have then that for any element of a laissez-faire equilibrium land price sequence and any $\varepsilon > 0$, there exists a $K(\varepsilon)$ such that $|H^k[T_2^*(t)] - \hat{T}_2| < \varepsilon$ for $k \geq K$. We may, however, choose $\varepsilon = \delta/2$ and consider the element $T_2^*(\bar{t}+k)$. Then

$$|H^k[T_2^*(\bar{t}+k)] - \hat{T}_2| = |T_2^*(\bar{t}) - \hat{T}_2| < \delta/2$$

from which it may be concluded that the sequence $\{T_2^*(t)\}$ does not satisfy condition (c). There are then no non-stationary laissez-faire equilibrium land price sequences.

B.   We have been awhile in proving that a unique and stationary laissez-faire equilibrium exists. But having gone on at such length, we can be brief in establishing the existence of a unique and stationary portfolio autarky equilibrium.

If the portfolio autarky regime obtains, then for member h of generation $t \geq 1$ one of the $q_k^h(t)$ is not a choice variable; depending on where he resides, either $q_1^h(t) = 0$ or $q_2^h(t) = 0$. Thus, $\bar{c}^h(t)$, the equilibrium consumption choice of member h of generation t, satisfies equations (18), although with different optimal $\lambda_j^h(t)$, and equation (17) either for $k = 1$ or $k = 2$ (but not both). To put the point another way, $\bar{c}^h(t)$ satisfies equations (19) and (20) with $\beta(t)$ replaced by $\beta_k(t)$. And we therefore have

$$(29) \quad \bar{c}_{2j}^h(t) = \phi_j^k[\beta_k(t)]$$

where, again, either $k = 1$ or $k = 2$. The $\bar{c}_{2j}^h(t)$ do depend on $\bar{w}_k^*$. But since the after-tax equilibrium wage is the same for all residents of country k, the functions ϕ_j are also the same. That explains why we write ϕ_j^k rather than ϕ_j^h.

Since under the portfolio autarky regime the $\bar{c}^h(t)$ are the same for all country k residents of generation $t \geq 1$, the equilibrium

$q_k^h(t)$, denoted $\bar{q}_k(t)$, are too [see the equality version of constraint (1)]. And by equations (6), if the portfolio autarky regime obtains, then $\bar{q}_k(t) = L_k/N_k$. So we have from constraint (2) in its equality version (and without summing over the country k members of generation $t \geq 1$) that

$$(30) \quad \bar{P}\phi_2^k\{T_k(t)/[T_k(t+1) + \bar{r}_k^*]\} = \bar{r}_k^*(L_k/N_k) + (L_k/N_k)T_k(t+1)$$

where $k = 1,2$. But equation (30) is very much like equation (28)--in relevant respects, just like it. And consequently the argument used above to establish the existence of a unique and stationary laissez-faire equilibrium land price, $\hat{T}_2$, establishes that for country k there exists a unique portfolio autarky equilibrium land price sequence $\{\bar{T}_k(t)\}$, where $\bar{T}_k(t) = \bar{T}_k$.

C.   As may be obvious, it is not in general true, though, that the portfolio autarky equilibrium land prices, $\bar{T}_1$ and $\bar{T}_2$, satisfy equation (24). To show that, we prove our

Proposition 3: If factor rental equalization obtains and $\mu_k = \nu_k = 0$ for $k = 1,2$, then $L_2/N_2 \gtreqless L_1/N_1$ implies $\bar{\beta}_2 \lesseqgtr \bar{\beta}_1$, where $\bar{\beta}_k = \bar{T}_k/(\bar{T}_k + \bar{r}_k^*)$ is the portfolio autarky equilibrium value of the discount rate, β_k.

We have by the conditions of the proposition that $\bar{r}_k^* = \bar{r}_k = \bar{r}$ and $\bar{w}_k^* = \bar{w}_k = \bar{w}$ and, by equations (30), that

$$(31) \quad \bar{P}\phi_2^k[T_k/(T_k + \bar{r})] = \bar{r}(L_k/N_k) + (L_k/N_k)T_k$$

are the conditions determining the $\overline{T}_k$. But if $w_k^* = \overline{w}$, then the LHS's

of the two equations (31), the $k = 1$ version and the $k = 2$ version, are

the same function of their respective arguments. [See

equations (19) and (20)]. The RHS's are different linear functions for

the two countries if and only if the endowment ratios differ; both

intercept and slope are larger for the country with the larger value

of L_k/N_k. It follows that an ordering of the L_k/N_k implies an inverse

ordering of the equilibrium T_k, and, hence, of the β_k.

D.   Later on, we show (in effect) that $\overline{\beta}_1 = \overline{\beta}_2$ is among the necessary

and sufficient conditions for the Pareto-optimality of a stationary con-

sumption allocation. And since the portfolio autarky equilibrium consump-

tion allocation is a stationary allocation, we will subsequently be

appealing to our Proposition 3. We will also find it helpful to have our

> Proposition 4: If the laissez-faire and portfolio
>
> autarky regime tax rates are the same, then $\overline{\beta}_1 = \overline{\beta}_2 = \overline{\beta}$
>
> implies $\hat{\beta} = \overline{\beta}$, where $\hat{\beta}$ is the laissez-faire equilibrium
>
> value of β; and $\overline{\beta}_1 \neq \overline{\beta}_2$ implies
>
> $$\min(\overline{\beta}_1, \overline{\beta}_2) < \hat{\beta} < \max(\overline{\beta}_1, \overline{\beta}_2).$$

Because there is a common function ϕ_2 for all members of

country k, we may write [see equation (27)]

$$(32) \quad \overline{P}\sum_k N_k \phi_2^k(\hat{\beta}) = [1/(1 - \hat{\beta})]\sum_k L_k \overline{r}_k^*$$

for by the definition β, $T_k/\overline{r}_k^* = \beta_k/(1 - \beta_k)$. But by the stationarity

of the $\{\overline{T}_k(t)\}$ we also have [see equations (30)] that

$$(33) \quad \overline{P}N_k \phi_2^k(\overline{\beta}_k) = L_k \overline{r}_k^* + \overline{T}_k L_k$$

and, by addition of the $k = 1$ and $k = 2$ versions of equation (33), that

$$(34) \quad \overline{P} \sum_k N_k \phi_2^k(\overline{\beta}_k) = \sum_k [1/(1 - \overline{\beta}_k)] L_k \overline{r}_k^*.$$

And subtracting equation (34) from equation (32), we get

$$(35) \quad \overline{P} \sum_k \{N_k [\phi_2^k(\hat{\beta}) - \phi_2^k(\overline{\beta}_k)]\}$$

$$= \sum_k L_k \overline{r}_k^* \{[1/(1 - \hat{\beta})] - [1/(1 - \overline{\beta}_k)]\}$$

from which it follows that if $\overline{\beta}_1 = \overline{\beta}_2 = \overline{\beta}$, then $\hat{\beta} = \overline{\beta}$. For if the $\overline{\beta}_k$ are equal, then $\hat{\beta} > \overline{\beta}$ implies that the RHS of equation (35) is positive and, by property (i) of the functions ϕ_2^k, that the LHS is negative; and $\hat{\beta} < \overline{\beta}$ implies that the RHS and LHS of equation (35) are, respectively, negative and positive.

And if $\overline{\beta}_1 \neq \overline{\beta}_2$? Then $\hat{\beta} \geq \max(\overline{\beta}_1, \overline{\beta}_2)$ implies that the two sides of equation (35) are of opposite sign; and $\hat{\beta} \leq \min(\overline{\beta}_1, \overline{\beta}_2)$ does too. So we have our Proposition 4.

E.   In this subsection, we show that trade balance is not a necessary characteristic of the steady-state equilibrium for a world of trading countries. We do so because it could be, as Gale observed a few years ago [1, p. 141], that many if not most economists have one way or another persuaded themselves that in the steady-state trade balance must obtain. We are a little doubtful. But there may well be some who believe that.

To be more precise, what we prove in this subsection is our

Proposition 5: If $\mu_k = \nu_k = 0$ for $k = 1,2$, then in

equilibrium under portfolio autarky there is trade balance. But if in addition $\overline{\beta}_1 \neq \overline{\beta}_2$, then in equilibrium under laissez-faire there is trade imbalance.[15/]

By the definition of τ_k, the trade balance of country k, and equations (16) and (29)

$$(36) \quad \tau_k = S_{1k}(\overline{p}) + \overline{p}S_{2k}(\overline{p}) - \overline{p}N_k[\phi_1^k(\beta_k) + \phi_2^k(\beta_k)]$$

for $t \geq 2$. (Recall that for all residents of country k the functions ϕ_j^h are the same.) But if the portfolio autarky regime obtains and $\mu_k = \nu_k = 0$, then by the equality versions of constraints (1) and (2)

$$\overline{p}\phi_1^k(\overline{\beta}_k) + \overline{T}_k\overline{q}_k = \overline{w}_k$$

and

$$\overline{p}\phi_2^k(\overline{\beta}_k) - (\overline{T}_k + \overline{r}_k)\overline{q}_k = 0.$$

And therefore, since $\overline{q}_k = L_k/N_k$

$$\overline{p}N_k[\phi_1^k(\overline{\beta}_k) + \phi_2^k(\overline{\beta}_k)] = \overline{w}_kN_k + \overline{r}_kL_k$$

or, with the F_i being linearly homogeneous

$$(37) \quad \overline{p}N_k[\phi_1^k(\overline{\beta}_k) + \phi_2^k(\overline{\beta}_k)] = S_{1k}(\overline{p}) + \overline{p}S_{2k}(\overline{p}).$$

So we have by equations (37) and (36) that $\overline{\tau}_k = 0$ for $k = 1,2$, where $\overline{\tau}_k$, the portfolio autarky equilibrium τ_k, is the value of τ_k at $\beta_k = \overline{\beta}_k$.

To complete our proof, we note that

$$(1/\overline{P}N_k)(\partial\tau_k/\partial\beta) = -\partial[\sum_j \phi_j^k(\beta)]/\partial\beta > 0.$$

The inequality is implied by equation (22) and $\beta < 1$. It follows from Proposition 4 that if $\overline{\beta}_1 \neq \overline{\beta}_2$, then $\hat{\tau}_k \neq 0$ for $k = 1,2$, where $\hat{\tau}_k$, the laissez-faire equilibrium τ_k, is the value of τ_k at $\beta_k = \hat{\beta}$. And since $\sum_k \hat{\tau}_k = 0$, the country with the greater portfolio autarky equilibrium β has a laissez-faire equilibrium trade deficit.

More particularly, if factor rental equalization obtains, then the "land poor" country, the country with the smaller land/labor endowment, has a laissez-faire equilibrium trade deficit. That follows from Proposition 3. Even under the laissez-faire regime, though, the current account balance of country k is zero, at least for $t \geq 2$; that is, $\hat{\tau}_k$ equals rental payments on the foreign-owned land of country k less rental income on the foreign land owned by residents of country k. So if there is factor rental equality, then the residents of the land-poor country own land the value of which exceeds that of the land of their country. And, therefore, one possible pattern of land ownership under the laissez-faire regime is the following: the residents of the land-poor country own all the land of their country and, in addition, some of the land of the other country.

IV. A WELFARE ANALYSIS

A.　　　　In this section, we evaluate the laissez-faire and portfolio autarky equilibria. We start off, though, by indicating what we mean by a stationary consumption allocation and then, in something of a digression,

explain (defend) our use of what earlier on we described as a growth theory version of the traditional welfare criterion.

A consumption allocation (or perhaps better, a complete consumption allocation) is an infinite-dimensional vector

$$(c_2(0), c(1), c(2), \ldots)$$

where, as above

$$c(t) = (c^1(t), c^2(t), \ldots, c^N(t)).$$

And we say that a (complete) consumption allocation is stationary if given that member h of generation $\bar{t} \geq 1$ has the allocation $c^h(\bar{t})$, then there is at least one member of every other generation $t \geq 1$ who has the allocation $c^h(\bar{t})$. Thus, if an allocation is stationary, then the members of the various generations can be ordered or indexed so that $c(t) = c(t+1)$ for all $t \geq 1$. But to say that an allocation is stationary is to leave the first element of the allocation vector, namely $c_2(0)$, which gives the age-two consumption bundles of the members of generation zero, quite unrestricted.

And now to explain our choice of a welfare criterion, in which task we find it convenient to use a diagram (Figure II). We note first that any consumption vector $c^h(t)$ that satisfies equation (16) is of the form

$$(c_{21}^h(t)(V^{-1}(\bar{p}), 1), c_{22}^h(t)(V^{-1}(\bar{p}), 1))$$

and can therefore be thought of as a point in the (c_{22}, c_{21}) plane.

Further, since the utility of any such allocation is

$$\sum_j a_j f[c_{2j}^h(t)\overline{g}]$$

it follows that for consumption allocations $c^h(t)$ satisfying equation (16), we can represent any individual's utility function by a family of strictly convex indifference curves in the (c_{22}, c_{21}) plane. And the laissez-faire equilibrium consumption allocation, the allocations of members h of generation $t = 1, 2, \ldots$, is a point like $(\hat{c}_{22}^h, \hat{c}_{21}^h)$ of Figure II. (We say members h, since the complete laissez-faire equi-
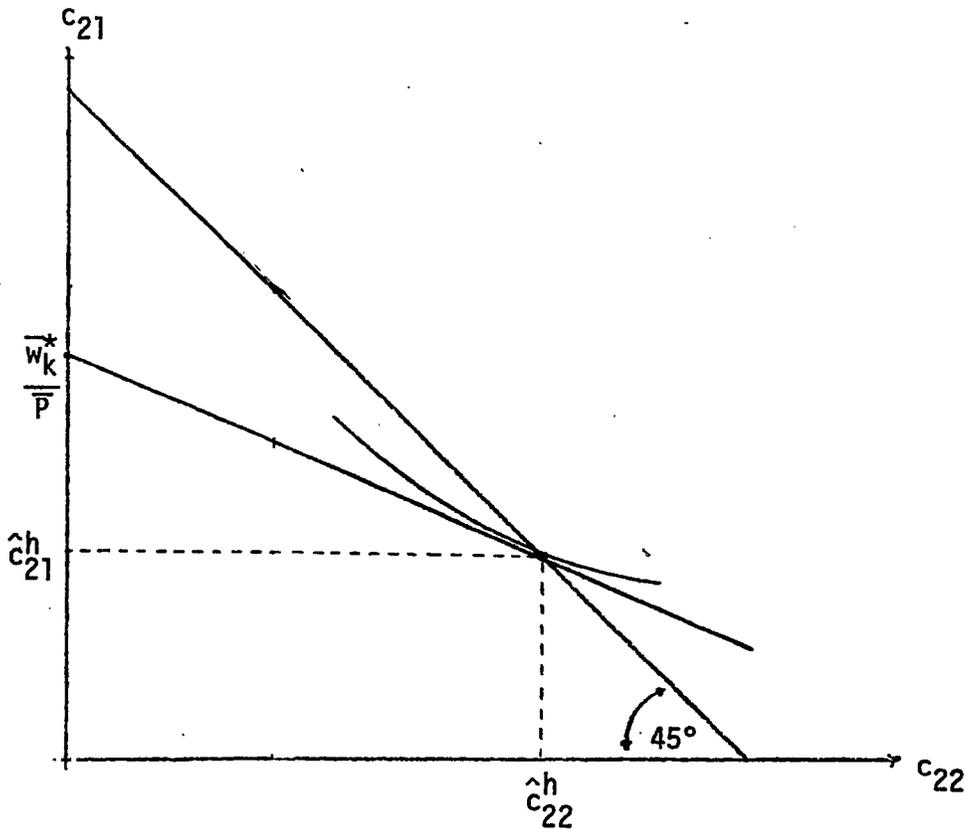

FIGURE II

librium consumption allocation, like the complete portfolio autarky allocation, is stationary.) Because a laissez-faire equilibrium allocation satisfies equations (19) and (20) for some value of $\beta < 1$, it is a point of tangency of an indifference curve and a line of slope greater than -1 that intersects the c_{21} axis at $\overline{w}_k^*/\overline{P} > 0$.[16] And as may be evident, an equilibrium allocation is not the "golden-rule" allocation. For there necessarily are preferred allocations that lie northwest of $(\hat{c}_{22}^h, \hat{c}_{21}^h)$ along the negatively sloped 45° line passing through $(\hat{c}_{22}^h, \hat{c}_{21}^h)$. Moreover, those preferred allocations, since they satisfy the output constraint for the second good, are feasible.[17]

Yet, having established that neither the laissez-faire nor the portfolio autarky equilibrium consumption allocation yields maximum utility for the members of generation t = 1,2,..., we cannot claim to have quickly finished the task of evaluating the equilibria of our two regimes. Evidently, the purpose in evaluating any policy regime is to determine whether its equilibrium ought to be altered, presumably by government. But for a world populated, as ours is, by overlapping generations, an equilibrium consumption allocation may not yield the maximum sustainable utility and still any reallocation may decrease the utility of some individual(s).

Consider a "preferred" allocation of Figure II. It gives more first period consumption to the members of generation $t \geq 1$ than does either of the equilibrium allocations; and it gives less second period consumption. If, however, that allocation were substituted for, say,

the laissez-faire equilibrium allocation in the first period (t = 1),
the welfare of the members of generation zero would decrease. The
substitution would leave unaffected their consumption in period zero
(t = 0); but it would decrease their consumption in the first period.

We perhaps did not have to be so long in making it, but
that is the essential point. If individuals live not just for one
period or for infinitely many, then in any period there are some, the
current old, in the last period of their respective lives, or for whom
there is no tomorrow.[18/] And as the traditional Pareto criterion demands,
account must be taken of them. For us, then, an allocation is Pareto-
optimal if there is no feasible Pareto-superior allocation, no feasible
allocation that increases the utility of some member(s) of generation
t $\geq$ 0 and leaves unchanged the utility of all other individuals, in-
cluding those who in the first period are of age two.

B.   It happens that the portfolio autarky equilibrium consumption
allocation is not Pareto-optimal and that the laissez-faire equilibrium
allocation is. To establish that, we now prove our

> Proposition 6: On our assumptions about resources
> and the utility and production functions, a stationary
> consumption allocation is Pareto-optimal if and only if:
>
> (a) $\sum_h c_{i1}^h(t) + \sum_h c_{i2}^h(t-1) = S_i(\bar{p})$
>
> for i = 1,2 and t $\geq$ 1;
>
> (b) $c_{1j}^h(t)/c_{2j}^h(t) = V^{-1}(\bar{p})$

for all h and j if t $\geq$ 1 and for j = 2 and all h if t = 0;

(c) $a_1 f'[c_{21}^h(t)\bar{g}]/a_2 f'[c_{22}^h(t)\bar{g}] = \lambda(t)$

for all h and t $\geq$ 1; and

(d) $\lambda(t) \geq 1$ for t $\geq$ 1.

In words, what condition (a) requires is that in every period total con-
sumption of good i equal the free-trade equilibrium output of that good.
What condition (b) requires is that in every period all individuals con-
sume the two goods in the same proportions. And with (b), what condition
(c) requires is that the consumption allocations of all the members of
any given generation (from the first on) be such as to yield the same
inter-temporal marginal rate of substitution for them. And, finally,
what condition (d) requires is that the reciprocal of the marginal rate
of substitution common to all members of any particular generation be
not less than unity.

To establish the sufficiency of conditions (a) - (d), we show
that a contradiction results from the assumption that there is an alloca-
tion A satisfying those conditions and another allocation B, not neces-
sarily stationary, that is Pareto-superior to A. Our proof is in two
parts. We first show that if there is a Pareto-superior allocation B,
then there exists an allocation C, perhaps the same as B, Pareto-superior
to A and satisfying conditions (a) - (c) of Proposition 6: that is to
say, either B satisfies conditions (a) - (c) or there is a different
allocation C, Pareto-superior to B, that does. Then, in the second

part of our sufficiency proof, we show that the existence of an allocation C, Pareto-superior to A and satisfying conditions (a) - (c), yields a contradiction; either that allocation is not Pareto-superior to A or it is not feasible.

We suppose to begin that the allocation B, although Pareto-superior to A, does not satisfy conditions (a) and (b) for some period $\bar{t} \geq 1$. But then there exists another allocation, denoted by B´, that is Pareto-superior to B and satisfies those conditions. For (and this we do not prove) those conditions are satisfied by the solution to the (within-period) problem of maximizing the utility of some member of generation $\bar{t}$ by the choice of a feasible consumption allocation, subject to the condition that consumption of every individual in every period $t \neq \bar{t}$ be equal to its B allocation value and the further condition that the utilities of all individuals alive in period $\bar{t}$ be not less than their respective B allocation values.

And if the B´ allocation does not satisfy condition (c) for some t, say $\bar{t} \geq 1$? Then there exists another allocation C, Pareto-superior to B´, that satisfies that condition and conditions (a) and (b) as well. For (and again we do not prove this) condition (c) is satisfied by the solution to the following (within-generation) maximization problem: maximize the utility of some member of generation $\bar{t}$ by the choice of a feasible consumption allocation, subject to conditions (a) and (b), the condition that all members of all generations other than $\bar{t}$ receive their B´ allocations and, lastly, the condition that the utilities of all members of generation $\bar{t}$ be at least as great as their respective B´

allocation values.

We thus have that if there is an allocation Pareto-superior to A, then there is a Pareto-superior allocation, denoted by C, that satisfies conditions (a) - (c). And what remains is for us to obtain the promised contradiction. In doing that, we make use of what we refer to as expansion paths ("income-expenditure path" might be a better phrase). So we pause briefly to explain what an expansion path is.

Any allocation satisfying condition (b), which is simply equation (16) relabeled, can be depicted in the (c_{22}, c_{21}) plane; any element of such an allocation, an individual's allocation, is a point in that plane. And if an allocation satisfies condition (c) as well, then the N consumption points (not necessarily distinct) of the members of generation $t \geq 1$ must lie on a curve or, as we refer to it, an expansion path. There are indifference curves on the (c_{22}, c_{21}) plane, the same for all individuals, and the expansion path is, so to speak, the locus of points on the several indifference curves where the slopes of those curves are the same, or where the slopes are equal to $-1/\lambda(t)$, the given or chosen common marginal rate of substitution for the members of generation t.

Since the function $f'(y)$ has an inverse for $y \geq 0$, condition (c) implies

$$(38) \quad c_{21}^h(t) = E(\lambda(t), c_{22}^h(t))$$

where the function E, which has partial derivatives $E_1 = \partial E / \partial \lambda(t)$ and $E_2 = \partial E / \partial c_{22}^h(t)$, gives the family of expansion paths. [In the (c_{22}, c_{21})

plane, there is a path for every choice of $\lambda(t) > 0$.] As is readily

verified by appeal to our assumptions about the function f, $E_2 > 0$

for $\lambda(t) > 0$. Also, along E, $c_{21}^h(t) \to 0$ as $c_{22}^h(t) \to 0$; and $c_{21}^h(t) \to \infty$

as $c_{22}^h(t) \to \infty$. And finally, for $c_{22}^h(t) > 0$, $E_1 < 0$. So if $\lambda_2(t) > \lambda_1(t)$,

then the $\lambda_2(t)$ expansion path lies everywhere below the $\lambda_1(t)$ expansion

path. More particularly, the two paths have no point in common.

Several possible expansion paths are shown in Figure III. And

as we may suppose, the one labeled λ_A is the A allocation expansion path.

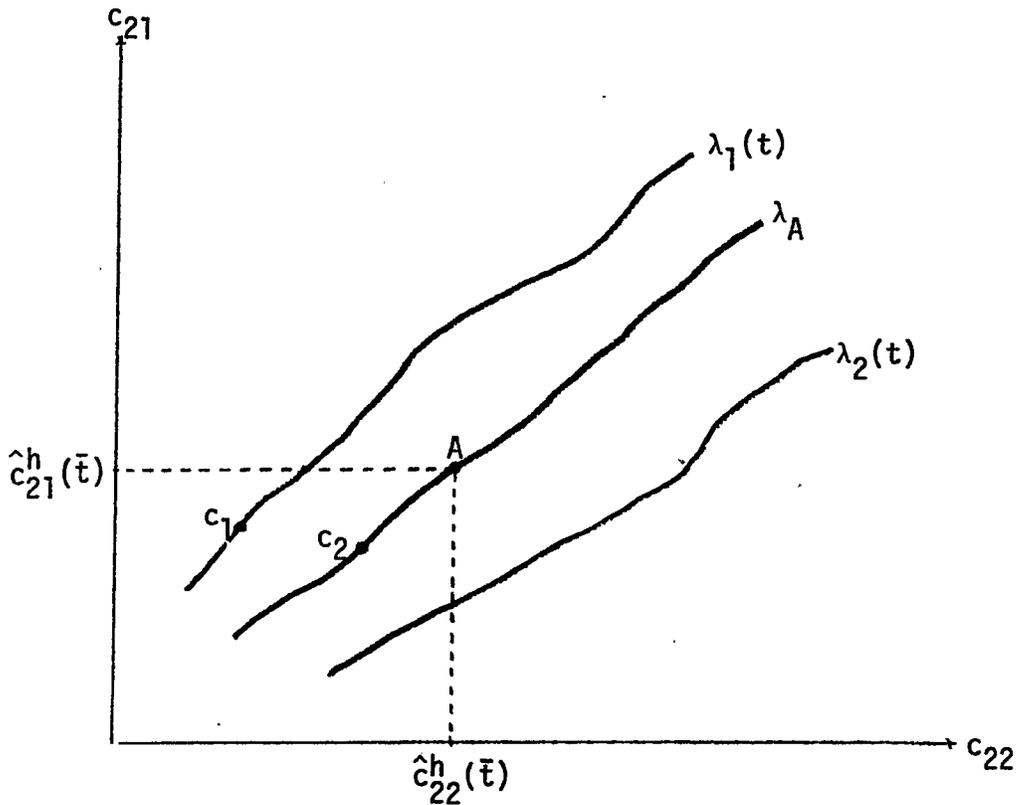By the stationarity of that allocation, there is only one. The consump-

FIGURE III

tion points of all members of generation t for all $t \geq 1$ lie on that path. For the C allocation, however, there may be many expansion paths. The consumption points of the members of the same generation t lie on an expansion path, for example, the one labeled $\lambda_1(t)$ in Figure III. But the C allocation of members of different generations may lie on different paths.

But now, if the C allocation is Pareto-superior to the A allocation, then there must be at least one member of a particular generation, say $\bar{t} \geq 0$, who is better off with his C allocation than with his A allocation. And the C allocation of that individual, taken as a point in the (c_{22}, c_{21}) plane, either lies above or on the A allocation expansion path (in Figure III, λ_A) or below it. There are no other possibilities. We consider the two possibilities (cases) separately, showing that in either event a contradiction results.

Case 1: The hypothesis of this case is that the C allocation consumption points of those individuals (there may be only one) who are better off with their C allocations than with their A allocations lie on the A allocation expansion path or above it, perhaps on the expansion path $\lambda_1(t)$ of Figure III. And we take these individuals as being members of generation $\bar{t} \geq 1$.[19] It follows from the hypothesis, though, that $\bar{c}_{21}^h(\bar{t}) > \hat{c}_{21}^h(\bar{t})$ for some members of generation $\bar{t}$, where $\bar{c}_{2j}^h(t)$ and $\hat{c}_{2j}^h(t)$ are respectively the C and A allocation values of $c_{2j}^h(t)$. If that inequality did not hold for some h, then there would be no member of generation $\bar{t}$ who preferred his C allocation to his A allocation.[20] Then

$\bar{c}_{21}^h(\bar{t}) \geq \hat{c}_{21}^h(\bar{t})$ for all h.[21] And hence by condition (a) of Proposition 6 $\bar{c}_{22}^h(\bar{t}-1) < \hat{c}_{22}^h(\bar{t}-1)$ for some h (for some members, that is, of generation $\bar{t}-1$). But since no individual can be worse off with his C allocation than with his A allocation, $\bar{c}_{21}^h(\bar{t}-1) > \hat{c}_{21}^h(\bar{t}-1)$ for some h. And it follows, by repeating the foregoing argument a finite number of times, that $\bar{c}_{22}^h(0) < \hat{c}_{22}^h(0)$ for some h and, therefore, that contrary to assumption the C allocation is not Pareto-superior to the A allocation.

Case 2: The hypothesis of this case is that the C allocation consumption points of those individuals, members of generation $\bar{t} \geq 0$, who prefer their C allocation lie below the A allocation expansion path, perhaps on the expansion path $\lambda_2(t)$ of Figure III. But that hypothesis also yields a contradiction. Letting $C_{2j}(t) = \sum_h c_{2j}^h(t)$, we show that

$$\bar{C}_{22}(t) - \hat{C}_{22} \geq \gamma(t)$$

for $t \geq \bar{t}$, where $\gamma(t) \to \infty$ as $t \to \infty$, and thus that the C allocation is not feasible.

Our proof is by induction. The Case 2 hypothesis implies $\bar{c}_{22}^h(\bar{t}) > \hat{c}_{22}^h(\bar{t})$ for some h and therefore for all h. [See footnotes 20 and 21.] So we have $\gamma(\bar{t}) > 0$, where by definition $\gamma(\bar{t}) = \bar{C}_{22}(\bar{t}) - \hat{C}_{22}$.[22]

Proceeding to the induction step, we suppose that

(39) $\bar{C}_{22}(t-1) - \hat{C}_{22} \geq \gamma(t-1) > 0$

where $t > \bar{t}$, and consider the following optimization problem: minimize $C_{22}(t)$ by the choice of $c(t) = (c^1(t), c^2(t), \ldots, c^N(t))$, subject to con-

ditions (a) and (b) of Proposition 6 and the additional constraints

$$(40) \quad \hat{c}_{21} - c_{21}(t) \geqq \gamma(t-1)$$

and

$$(41) \quad U[c^h(t)] \geqq U(\hat{c}^h).$$

We observe first that $\bar{c}(t)$ is a feasible solution to the problem, for the C allocation satisfies conditions (a) and (b). Also by inequality (39) and condition (a), $\bar{c}_{21}(t)$ satisfies constraint (40). And by the Pareto-superiority of the C allocation, of which the $\bar{c}^h(t)$ are elements, $\bar{c}(t)$ satisfies constraint (41). So we have the crucial inequality

$$(42) \quad \bar{c}_{22}(t) \geqq \tilde{c}_{22}(t)$$

where $\tilde{c}_{22}(t)$ is the minimizing value of $c_{22}(t)$. We make use of inequality (42) below.

But to get on. The solution to the above stated optimization problem is the unique and positive vector $\tilde{c}(t)$ which satisfies conditions (a) and (b), the equality versions of constraints (40) and (41) and condition (c) of Proposition 6 with $\lambda(t) = \tilde{\lambda}(t)$, where $\tilde{\lambda}(t) > 0$ is the optimizing value of the Lagrangian multiplier associated with constraint (40). And because $\tilde{c}(t)$ does satisfy the equality version of constraint (41), we may write

$$(43) \quad \tilde{c}_{22}^h(t) - \hat{c}_{22}^h = [\hat{c}_{21}^h - \tilde{c}_{21}^h(t)][\lambda_A + \zeta_h(\hat{c}_{21}^h - \tilde{c}_{21}^h(t))]$$

where by the strict concavity of the function f, the function ζ_h, the argument of which is the difference $\hat{c}_{21}^h - \tilde{c}_{21}^h(t)$, is such that $\zeta_h(0) = 0$

and $\zeta_h' > 0.$[23/] We may, however, also write

$$(44) \quad \hat{c}_{21}^h - \tilde{c}_{21}^h(t) = \theta_h(\gamma(t-1))$$

and, making use of equation (44), rewrite equation (43) as follows:

$$(45) \quad \tilde{c}_{22}^h(t) - \hat{c}_{22}^h = [\hat{c}_{21}^h - \tilde{c}_{21}^h(t)]\lambda_A + [\hat{c}_{21}^h - \tilde{c}_{21}^h(t)]\psi_h(\gamma(t-1))$$

where $\psi_h(\gamma(t-1)) = \zeta_h(\theta_h(\gamma(t-1)))$.

Now, $\theta_h(0) = 0$. For by the equality version of constraint (40), $\gamma(t-1) = 0$ implies $\sum_h[\hat{c}_{21}^h - \tilde{c}_{21}^h(t)] = 0$, which in turn, by the equality version of constraint (41) and condition (c), implies $\hat{c}_{21}^h - \tilde{c}_{21}^h(t) = 0$ for all h.[24/] Also, by a similar argument, $\theta_h' > 0$. Since the equality version of constraint (40) holds, an increase in $\gamma(t-1)$ implies a decrease in $\tilde{c}_{21}(t)$ and hence, for some h, a decrease in $\tilde{c}_{21}^h(t)$. But then, by the equality version of constraint (41) and condition (c), an increase in $\gamma(t-1)$ implies a decrease in $\tilde{c}_{21}^h(t)$ for all h. It follows that $\psi_h(0) = 0$ and that $\psi_h' > 0$.

We now define a new function $\psi(\gamma(t-1))$; its value at any $\gamma(t-1) > 0$ is the minimum over h of the values of the $\psi_h(\gamma(t-1))$. So ψ is a strictly increasing function and $\psi(0) = 0$. And we have from equation (45) that for $\gamma(t-1) \geq 0$

$$(46) \quad \tilde{c}_{22}^h(t) - \hat{c}_{22}^h \geq [\hat{c}_{21}^h - \tilde{c}_{21}^h(t)][\lambda_A + \psi(\gamma(t-1))].$$

Summing over h and making use of inequality (40), we get

$$\tilde{c}_{22}(t) - \hat{c}_{22} \geq \gamma(t-1)[\lambda_A + \psi(\gamma(t-1))]$$

and, making use of inequality (42) and condition (d) of Proposition 6, that

$$(47) \quad \bar{C}_{22}(t) - \hat{C}_{22} \geq \gamma(t-1)[1 + \psi(\gamma(t-1))] \equiv \gamma(t).$$

To complete our Case 2 proof, we have then only to show that the sequence $\{\gamma(t)\}$, where $t \geq \bar{t}$, is unbounded from above. By definition [see equation (47)]

$$\gamma(t)/\gamma(t-1) = 1 + \psi(\gamma(t-1)).$$

But since $\gamma(\bar{t}) > 0$, it follows that $\gamma(t-1) \geq \gamma(\bar{t})$. And therefore, since the function ψ is strictly increasing,

$$\psi(\gamma(t-1)) \geq \psi(\gamma(\bar{t}))$$

for $t > \bar{t}$. So we have

$$\gamma(t)/\gamma(t-1) \geq 1 + \psi(\gamma(\bar{t}))$$

and $\{\gamma(t)\}$ is indeed unbounded from above.

The conditions (a) - (d) of Proposition 6 are then sufficient. And to prove that they are necessary as well, we assume that there exists a stationary Pareto-optimal allocation D, not satisfying those conditions, and obtain a contradiction.

Recalling the first part of our sufficiency proof, we may suppose that the D allocation satisfies conditions (a) - (c). Thus, since the D allocation is stationary, the individual allocations for $t \geq 1$ are

$$c^h(t) = c^h = (c_{21}^h(V^{-1}(\bar{p}),1), c_{22}^h(V^{-1}(\bar{p}),1))$$

where c^h satisfies

$$(48) \quad a_1 f'[c_{21}^h \bar{g}]/a_2 f'[c_{22}^h \bar{g}] = \lambda_D.$$

And with the D allocation not satisfying condition (d), $\lambda_D < 1$.

Now, consider the allocations

$$c^h(\varepsilon) = ((c_{21}^h + \varepsilon)(V^{-1}(\bar{p}),1), (c_{22}^h - \varepsilon)(V^{-1}(\bar{p}),1)).$$

For such vectors satisfying equation (48), the stationary allocation version of condition (c),

$$\partial U[c^h(0)]/\partial \varepsilon \gtreqless 0 \text{ as } \lambda_D - 1 \gtreqless 0.$$

Thus, since $\lambda_D < 1$, there exists some $\varepsilon < 0$ which implies $U[c^h(\varepsilon)] > U[c^h(0)]$ for all h.

And clearly for that ε the allocation is feasible. Indeed, for any ε

$$\sum_h (c_{21}^h + \varepsilon) + \sum_h (c_{22}^h - \varepsilon) = S_2(\bar{p})$$

if the c_{2j}^h are those of the D allocation, which is stationary and satisfies condition (a).

The Pareto-superior allocation is achieved by transferring the consumption bundle $\varepsilon(V^{-1}(\bar{p}),1)$ from member h of generation t to member h of generation t-1. (Necessarily then the transfer is made in period t.) The members of generation zero are thus affected, but favorably.[25] And so, contrary to assumption, the D allocation is not Pareto-optimal.

It is immediate from Proposition 6, which has now been proved, that the portfolio autarky equilibrium consumption allocation is not in general Pareto-optimal. That allocation satisfies condition (c) if and only if $\bar{\beta}_1 = \bar{\beta}_2$. But the laissez-faire equilibrium consumption allocation, since it satisfies all the conditions of the Proposition, is Pareto-optimal.

C.   If the portfolio autarky equilibrium discount rates, $\bar{\beta}_1$ and $\bar{\beta}_2$, are the same, there exist no consumption allocations that are Pareto-superior to the equilibrium allocation of that regime. But if the rates are not the same, then such allocations must exist. And it would seem natural to ask whether the laissez-faire equilibrium allocation is one of those Pareto-superior allocations. We provide part of the answer to that question by proving our

> **Proposition 7:** If the taxes ν_k, μ_k, $k = 1,2$, are the same for both regimes, then the laissez-faire equilibrium allocation is not Pareto-superior to the portfolio autarky allocation.

We say that an allocation is symmetric if all the country k residents of a given generation receive the same allocations. So the laissez-faire and portfolio autarky equilibrium allocations are not only stationary, but symmetric as well. For, as we remarked above (p. 19), whichever regime obtains, there are for each j only two consumption functions, the ϕ_j^k; one for the residents of the first country and one for the residents of the second. But if the μ_k are the same

for both regimes, then the $\bar{w}_k^*$ are too and the ϕ_j^k are independent of regime.

Thus, on the assumption of Proposition 7, the consumption vector of any country k member of generation $t \geq 1$ is

$$c^k(\beta_k) = (\phi_1^k(\beta_k)(V^{-1}(\bar{p}),1),\phi_2^k(\beta_k)(V^{-1}(\bar{p}),1)).$$

And that being so

$$\partial U[c^k(\beta_k)]/\partial\beta_k = \bar{g}\sum_j a_j f'[\phi_j^k(\beta_k)\bar{g}](\phi_j^k)'.$$

Or, by equation (19)

$$\partial U[c^k(\beta_k)]/\partial\beta_k = a_1\bar{g}f'[\phi_1^k(\beta_k)\bar{g}][(\phi_1^k)' + \beta_k(\phi_2^k)'] < 0$$

where the inequality follows from differentiation of equation (20) with respect to β.[26/] And Proposition 7 follows then from Proposition 4, in which $\hat{\beta}$ was shown to be bounded by the $\bar{\beta}_k$.

D.   As might be expected, though, there are laissez-faire tax rates, different from the portfolio autarky rates, that yield a laissez-faire equilibrium consumption allocation Pareto-superior to any (non-optimal) portfolio autarky equilibrium allocation. In point of fact, if the portfolio autarky equilibrium allocation is non-optimal ($\bar{\beta}_1 \neq \bar{\beta}_2$), then there is a particular set of Pareto-superior allocations, to be defined presently, and any allocation in that set can be achieved by imposing the appropriate tax rates, head tax and land tax rates, on the laissez-faire regime. That is what we show in this subsection.

We proceed by first proving a preliminary proposition. There

is a subset P of all the allocations that are Pareto-superior to the portfolio autarky equilibrium allocation. It contains all those allocations that are Pareto-optimal, stationary and symmetric and gives to the old of period one their portfolio autarky equilibrium allocations. And what we prove is

> Proposition 8: Suppose that $\bar{\beta}_1 \neq \bar{\beta}_2$. Then there
> exists an interval $[0, \delta^*]$, where $\delta^* > 0$, such that
> for any δ on that interval the solution to the below-
> stated optimization problem, augmented by the port-
> folio autarky equilibrium allocation for the old of
> period one, is an allocation in P. And any alloca-
> tion in P is an augmented solution to that problem
> for δ on the interval $[0, \delta^*]$.

The optimization problem is as follows: maximize $U(c^1)$ by the choice of

$$(49) \quad c^k = (c_{21}^k(V^{-1}(\bar{p}),1), c_{22}^k(V^{-1}(\bar{p}),1))$$

for $k = 1,2$, subject to

$$(50) \quad U(c^2) \geq U[c^2(\bar{\beta}_2)] + \delta$$

and

$$(51) \quad \sum_k N_k c_{2j}^k \leq \sum_k N_k c_{2j}^k(\bar{\beta}_k)$$

where $j = 1,2$ and $c^k(\bar{\beta}_k)$ is the portfolio autarky equilibrium allocation of the country k members of generation $t \geq 1$. And δ^* is defined to be

the value of δ that gives a solution for c^1, namely $\tilde{c}^1(\delta)$, such that $U[\tilde{c}^1(\delta)] = U[c^1(\bar{\beta}_1)]$.

What Proposition 8 gives is a characterization of the set P; for the case $\bar{\beta}_1 \neq \bar{\beta}_2$, there is a one-to-one correspondence between the allocations of P and values of δ on the interval $[0,\delta*]$. To prove the proposition, we first observe that by the strict concavity of the function U there is a unique and positive solution to the above-stated maximization problem. Therefore, that solution, denoted by $\tilde{c}(\delta) = (\tilde{c}^1(\delta),\tilde{c}^2(\delta))$, satisfies the equality versions of constraints (50) and (51). It also satisfies condition (c) of Proposition 6 for $\lambda(t) = \tilde{\lambda}$. And it is such that $\partial U[\tilde{c}^1(\delta)]/\partial\delta < 0$.

Now, then, if $\delta = 0$ the vector $(c^1(\bar{\beta}_1),c^2(\bar{\beta}_2))$ is a feasible solution to the problem of Proposition 8. Consequently, $U[\tilde{c}^1(0)] \geq U[c^1(\bar{\beta}_1)]$. And if $U[\tilde{c}^1(0)] = U[c^1(\bar{\beta}_1)]$, then $\tilde{c}(\delta) = (c^1(\bar{\beta}_1),c^2(\bar{\beta}_2))$, for as observed above our problem has a unique solution. It follows that $(c^1(\bar{\beta}_1),c^2(\bar{\beta}_2))$ satisfies condition (c) of Proposition 6 and, more particularly, $\bar{\beta}_1 = \bar{\beta}_2$. Thus, $\bar{\beta}_1 \neq \bar{\beta}_2$ implies $U[\tilde{c}^1(0)] > U[c^1(\bar{\beta}_1)]$ and, since $\partial U[\tilde{c}^1(\delta)]/\partial\delta < 0$, that $\delta* > 0$.

Since $\delta* > 0$, it follows that for any δ in the interval $[0,\delta*]$, either $U[\tilde{c}^1(\delta)] > U[c^1(\bar{\beta}_1)]$ or $U[\tilde{c}^2(\delta)] > U[c^2(\bar{\beta}_2)]$ or both. So $\tilde{c}(\delta)$ is Pareto-superior to $(c^1(\bar{\beta}_1),c^2(\bar{\beta}_2))$; and the augmented allocation is as well. Also, by construction, $\tilde{c}(\delta)$ is stationary and symmetric; and hence the augmented allocation is too. It therefore has only to be shown that the augmented allocation is Pareto-optimal.

Since $\tilde{c}(\delta)$ and $(c^1(\bar{\beta}_1),c^2(\bar{\beta}_2))$ satisfy the equality version of constraint (51), condition (a) of Proposition 6 is satisfied by the

augmented allocation. So it is feasible. By equation (49), it satisfies condition (b). And as was noted above, $\tilde{c}(\delta)$ satisfies condition (c). Consequently, the augmented allocation does too. Finally, since $\tilde{c}(\delta)$ satisfies condition (c), $\tilde{c}^1(\delta)$ and $\tilde{c}^2(\delta)$ lie on a common expansion path, the $\tilde{\lambda}$ expansion path. And that path lies between the portfolio autarky expansion paths, one of which is the $\bar{\lambda}_1$ expansion path, where $\bar{\lambda}_1 = 1/\bar{\beta}_1$, and the other of which is the $\bar{\lambda}_2$ path, where $\bar{\lambda}_2 = 1/\bar{\beta}_2$. By the Pareto-superiority of $\tilde{c}(\delta)$, to assume otherwise implies either $\tilde{c}_{22}^k > c_{22}(\bar{\beta}_k)$ for $k = 1,2$ or $\hat{c}_{21}^k > c_{21}(\bar{\beta}_k)$ for $k = 1,2$ and, hence, that constraint (51) is violated. It follows that $\tilde{\lambda} > 1$, since $\bar{\beta}_k < 1$ for $k = 1,2$, and that the augmented allocation satisfies condition (d) of Proposition 6.

We go on now to the converse of the proposition that was just proved: if an allocation is in P, then it is an augmented solution to the maximization problem of Proposition 8 for δ on the interval $[0,\delta^*]$. Any allocation in P, being Pareto-optimal, satisfies condition (b); and hence for members of generation $t \geq 0$ the individual allocations are vectors of the form of the problem of Proposition 8. Then, too, any allocation in P satisfies constraint (51) for all $t \geq 1$. Any such allocation gives the old of period one their portfolio autarky allocations; so by condition (a) of Proposition 6, constraint (51) is satisfied for $t = 0$. And by the stationarity of the P allocations, that constraint is therefore satisfied for all t. Lastly, any allocation in P maximizes $U(c^1)$. If it did not, it would not be Pareto-optimal. Thus, any P allocation is a solution for some δ. But since the P allocations are Pareto-superior, only for δ on the interval $[0,\delta^*]$.

E.   Having shown that any allocation in P is an element of the solution set for the optimization problem of Proposition 8, we now establish our

Proposition 9: For any allocation in P, there exist tax rates μ_k and ν_k, $k = 1,2$, satisfying constraints (3) and (4), that yield a laissez-faire equilibrium consumption allocation which is the chosen P allocation.

Associated with the chosen P allocation is the solution $\tilde{c}(\delta)$. So we let the country k head tax be given by

$$(52) \quad \mu_k = \bar{w}_k - \bar{P}[\tilde{c}_{21}^k(\delta) + \tilde{c}_{22}^k(\delta)/\tilde{\lambda}(\delta)].$$

Since the second term on the RHS of equation (52) is necessarily positive, μ_k satisfies constraint (4); that is, $\mu_k < \bar{w}_k$. And it follows from equation (52) that $\tilde{c}^k(\delta)$ satisfies equations (19) and (20) if land prices and land tax rates are such as to imply $\beta = 1/\tilde{\lambda}(\delta)$. We therefore require that

$$(53) \quad T_k/(T_k + \bar{r}_k^*) = 1/\tilde{\lambda}(\delta)$$

for $k = 1,2$.

But any allocation in P gives the old of period one their portfolio autarky allocations; for every country k member of generation zero, the P allocation consumption vector is $c_{22}^k(\bar{\beta}_k)(V^{-1}(\bar{p}),1)$. So we also insist that T_k and ν_k satisfy

$$(54) \quad \overline{P}c_{22}^{k}(\overline{\beta}_k) = (L_k/N_k)(T_k + \overline{r}_k^*)$$

where L_k/N_k is the per capita land purchased by country k members of generation zero under portfolio autarky.[27] Equation (54) is simply a version of equality (2). And thus, if T_k and v_k satisfy it, the old of period one will under laissez-faire receive their respective portfolio autarky equilibrium allocations.

From equations (53) and (54), then

$$(55) \quad T_k = \overline{P}c_{22}^{k}(\overline{\beta}_k)(N_k/L_k)/\tilde{\lambda}(\delta) = b_k \text{(constant)} > 0$$

and

$$(56) \quad v_k = \overline{r}_k - b_k[\tilde{\lambda}(\delta) - 1].$$

Since $\tilde{\lambda}(\delta) > 1$, the v_k of equation (56) satisfies constraint (4); that is, $v_k < \overline{r}_k$.

We have still to show, though, that μ_k of equation (52) and v_k of equation (56) imply constraint (3) or that there is government budget balance. Multiplying equations (52) and (56) by, respectively, N_k and L_k, summing the k = 1 and k = 2 versions of both equations and then adding the resulting equations, we get

$$(57) \quad \sum_k N_k\mu_k + \sum_k L_k v_k = \overline{P}[\sum_k N_k \tilde{c}_{21}^{k}(\delta) + \sum_k N_k \tilde{c}_{22}^{k}(\delta)]$$

$$- (\sum_k N_k \overline{w}_k + \sum_k L_k \overline{r}_k)$$

$$+ \overline{P}[\sum_k N_k \tilde{c}_{22}^{k}(\delta) - \sum_k N_k c_{22}^{k}(\overline{\beta}_k)]/\tilde{\lambda}(\delta).$$

But $\tilde{c}(\delta)$ satisfies the equality version of constraint (51), so the third

term on the RHS of equation (57) vanishes. And the first and second terms are equal. For by the homogeneity of the F_i functions, total world factor payments equals the value of world output; and since $(c^1(\bar{\beta}_1), c^2(\bar{\beta}_2))$ is an equilibrium allocation, it follows from the equality version of constraint (51) that the first term on the RHS of equation (57) also equals the value of world output.

To complete our proof of the existence of appropriate laissez-faire tax rates, we show that the T_k of equation (55) are equilibrium prices for the laissez-faire regime. From equations (24) and (27), the equilibrium restriction can be written

$$(58) \quad \bar{P}\phi_2[T_2/(T_2 + \bar{r}_2^*)] = \sum_k L_k r_k^* + \sum_k L_k T_k.$$

But by the definition of the function ϕ_2 and equation (53), the LHS of equation (58) is

$$\bar{P}\sum_k N_k \tilde{c}_{22}^k(\delta).$$

And by equations (55) and (56), the RHS is

$$\bar{P}\sum_k N_k c_{22}^k(\bar{\beta}_k).$$

So that the T_k of equation (55) are laissez-faire equilibrium prices follows from the equality version of constraint (51).

V. CONCLUSION

We have come to the end. Or almost, for before stopping we comment briefly on our assumptions. We chose them to ensure that the

equilibria of the two regimes would be stationary and recursive (implying an equilibrium goods price independent of regime). It is natural that we should have wanted stationary equilibria, particularly since we took the second factor of production to be non-reproducible. And that we should have wanted the equilibria to be recursive is too, at least in a way. Those equations of ours that for both regimes determine, inter alia, the equilibrium goods price are precisely those of the traditional Heckscher-Ohlin representation of the world. So with our recursive structure, many of the traditional barter-trade theory results obtain under our representation. Of the two equilibrium properties that in effect we imposed, stationarity would seem the more important. That is to say, we are confident we could have obtained our main welfare result (the laissez-faire equilibrium, unlike the portfolio autarky equilibrium, is Pareto-optimal) even if we had to contend with different or regime-specific equilibrium goods prices.

We could, though, have made weaker assumptions and still ended up with stationary and recursive equilibria. Thus, we could have managed with more than two goods and countries. And although for a stationary equilibrium all generations have to be the "same," we could have assumed a kind of genetic sameness; in generation $t+1$, there is at least one member who has the same tastes as some member of generation t. For recursive equilibria, it is a requirement that utility functions be separable in the consumptions of different periods, if perhaps not additively separable, and that the functions of within-period consumption goods be homothetic and imply the same marginal-rate-of-substitution functions for

all individuals and for consumption in different periods. We could then
have allowed a different f function for each genetic line and for con-
sumption in different periods.

But could we have managed if, being true to the modern growth-
theory version of barter-trade, we had taken the second factor of produc-
tion as being reproducible, some kind of man-made capital? We happen to
think that it is reasonable, maybe even (heaven forbid) realistic, to
assume a non-reproducible factor of production. Nonetheless, the question
would also seem to be worth considering. We believe that we could have
managed, but that we would have come to different conclusions. We could
certainly still have posed the portfolio autarky issue: Does it matter
whether the residents of country k are allowed to own assets (reproducible
capital) located in other countries? Of course, if the second factor of
production is reproducible and (as it probably should be assumed to be)
physically mobile, then the issue may be put another way: Does it matter
whether the residents of country k are allowed to hire labor located in
other countries? Or does it matter whether "foreign investment" is
allowed?[28/] In any event, our guess is that we would have come to the
following conclusion: If tastes are the same, then for the steady state
it makes no difference whether there is portfolio autarky, or whether
foreign investment is allowed; that is to say, the steady states are the
same. But for some arbitrary initial condition, the equilibrium paths
for the two regimes are of course non-stationary and may not be the same.
We suspect that they are not the same and, further, that the laissez-faire
regime path is better than the portfolio autarky path. Yet, we are not

all that confident of our conjectures. So it would perhaps be worthwhile
to analyze portfolio autarky using the assumption that the second factor
of production is reproducible and physically mobile as well. Kemp [4]
has made a start. But his world economy is populated by individuals who
live for only one period or, alternatively, infinitely many. And he does
not evaluate non-stationary equilibrium paths.

APPENDIX

A.   There is a third possible economic policy regime, a complete
autarky regime, distinguished by prohibitions, applicable world-wide,
on international exchanges of assets and goods. And in this appendix
we compare the equilibrium consumption allocation of that regime with
the equilibrium allocation of the portfolio autarky regime. We show
by producing a counter-example that the portfolio autarky allocation
is not in general Pareto-superior to the complete autarky allocation.
For some choice of utility and production functions and factor endow-
ments, it is not Pareto-superior to the complete autarky allocation.

        We suppose here, as we did in the text, that all individuals
have the same tastes and that all country k residents are similarly
endowed (all have one unit of first-period labor services). But when
tastes and endowments of all the residents of each of the several
countries are the same, then in the traditional analysis everyone bene-
fits from a change to free trade. So the example of a world economy
offered in this appendix would seem to be of some interest.

B. We assume here that

$$U[c^h(t)] = \sum_j a_j f((c_{1j}^h(t)c_{2j}^h(t))^{\frac{1}{2}});$$

that is, $g(c_{1j}^h(t),c_{2j}^h(t)) = (c_{1j}^h(t)c_{2j}^h(t))^{\frac{1}{2}}$. Thus, for prices that are constant over time, maximization of $U[c^h(t)]$, subject to the equality versions of constraints (1) and (2) and equation (6) for $k = 1,2$, implies

(1) $c_{11}^k = a_1 w_k/2(a_1 + a_2)$

(2) $c_{21}^k = c_{11}^k/p_k$

(3) $c_{12}^k = a_2 c_{11}^k(1 + R_k)/a_1$

(4) $c_{22}^k = a_2 c_{11}^k(1 + R_k)/a_1 p_k$

and, lastly;

(5) $R_k = r_k K_k/(w_k - 2c_{11}^k)$

where $K_k = L_k/N_k$ and where $R_k = r_k/T_k = (1 - \beta_k)/\beta_k$ is the country k interest rate, p_k is the country k goods price and w_k and r_k are the country k marginal products.[29/] The c_{ij}^k are the optimal consumption quantities for any country k member of generation t.[30/]

C. And here we take it that firms are constrained by Leontief-type production functions:

$$X_i = \min(b_i N_i, L_i)$$

where, for definiteness, $b_1 < b_2$. It follows that $\bar{X}^k = (\bar{X}_{1k}, \bar{X}_{2k})$, the

country k full-employment output vector, satisfies the following equations:

$$X_{1k} + X_{2k} = N_k K_k$$

(6)

$$X_{1k} + (b_2/b_1)X_{2k} = b_2 N_k.$$

Now, for $\overline{X}^k$ to be the equilibrium output vector, it is sufficient that

$$(7) \quad 1 \geq g_2(\overline{c}_{1j}^k, \overline{c}_{2j}^k)/g_1(\overline{c}_{1j}^k, \overline{c}_{2j}^k) \geq b_1/b_2$$

where $\overline{c}_{1j}^k = \overline{X}_{ik}/N_k$. But for our choice of a g function, $g_2/g_1 = \overline{c}_{1j}^k/\overline{c}_{2j}^k = \overline{X}_{1k}/\overline{X}_{2k}$. And therefore, by equations (6), inequalities (7) can be rewritten as

$$(8) \quad (b_1 + b_2)/2 \geq K_k \geq 2b_1 b_2/(b_1 + b_2).$$

We assume that the b_i and K_k satisfy inequalities (7) and have therefore that $\overline{X}^k$ is the equilibrium output vector for country k. If the complete autarky regime obtains, though, then the goods market equilibrium conditions are

$$N_k c_{i1}^k = \overline{X}_{ik}$$

from which it follows that

$$(9) \quad p_k = \overline{X}_{1k}/\overline{X}_{2k}$$

$$= (b_1/b_2)[(b_2 - K_k)/(K_k - b_1)]$$

where p_k is the complete autarky regime equilibrium goods price for country k. Further, since $\overline{X}^k > 0$,

$$\bar{X}_{1k} = w_k N_{1k} + r_k L_{1k}$$

and

$$p_k \bar{X}_{2k} = w_k N_{2k} + r_k L_{2k}.$$

That is, profit of the country k industry that produces good i must be zero.[31] But cost minimization implies $L_{ik} = X_{ik}$ and $N_{ik} = X_{ik}/b_i$; and hence

$$(10) \quad \begin{aligned} w_k &= b_1 b_2 (1 - p_k)/(b_2 - b_1) \\ r_k &= (b_2 p_k - b_1)/(b_2 - b_1) \end{aligned}$$

where w_k and r_k are the complete autarky regime equilibrium factor returns of country k.

And finally, since the b_i and K_k satisfy inequalities (7)

$$(11) \quad p = (\bar{X}_{11} + \bar{X}_{12})/(\bar{X}_{21} + \bar{X}_{22})$$

$$= b_1(b_2 - K)/b_2(K - b_1)$$

where $K = (L_1 + L_2)/N_1 + N_2)$ and where p is the portfolio autarky equilibrium goods price. The portfolio autarky equilibrium factor returns of country k are given by equations (10) with p_k replaced by p.

D. Now, then, consider the following parameter values: $a_1 = a_2 = 1$; $b_1 = 1$; $b_2 = 2$; $K_1 = 1.35$; and $K_2 = 1.45$. Those values satisfy inequalities (8) and imply $p_1 > p_2$. For the choice $f(y) = \ln y$, they also imply

$$\partial U(c^k)/\partial p < 0$$

and, hence, that a switch from the complete autarky regime to the portfolio autarky regime would increase the welfare of first country members of generation $t \geq 1$ and decrease the welfare of second country members.

FOOTNOTES

1. It is thus no part of our purpose to evaluate the equilibrium of a world wherein foreign investment, so-called, is prohibited, or wherein owners of physically mobile real assets are prohibited from employing them in foreign countries.

2. In [2, p. 129 ff.], which appeared after this paper had been pretty much completed, Gale provides a representation something like ours. His, though, would seem to be a representation of a one-good pure-exchange world. Nor is it entirely clear that there is an asset market in his representation. Although apparently not concerned about portfolio autarky, he does, however, prove propositions that are essentially the same as some of ours (see especially p. 134).

3. That country k has a land endowment is easily accepted. Because the number of age-one residents of country k is constant from period to period and every age-one individual supplies the same amount of labor, it is, however, also reasonable to regard country k as having a labor endowment.

4. With apologies to those who care, we take the individual as being not a "she" or an "it" but a "he."

5. Since the choice problem faced or solved by an individual is not independent of regime, there is a laissez-faire equilibrium and a portfolio autarky equilibrium. And of course, whichever regime

obtains, there is an equilibrium for every choice of tax rates.

6. It follows from $f'(y) > 0$ that $\tilde{\lambda}_j^h(t) \neq 0$.

7. Equation (10) is obtained by dividing the LHS (RHS) of the j = 1
    version of equation (8) by the LHS (RHS) of the j = 1 version of
    equation (9). Equation (11) is obtained by the same division,
    using the j = 2 versions of those equations.

8. To be quite precise, what we have proved is that for p > 0 there
    is one and only one p satisfying equation (15). But it is clear
    that zero is not a possible equilibrium value. Thus $S_2(0) = 0$;
    but by equation (1), if p = 0 there is no finite $\tilde{c}_{21}^h(t)$. We should
    add that in the appendix we use Leontief-type production functions.
    It is still true, though, even if the supply functions are set-
    valued, that there exists a unique equilibrium goods price, an
    equilibrium price that is independent of regime.

9. Suppose that for some member of generation t the k = 1 version holds
    with equality and the k = 2 version holds with strict inequality and,
    further, that for any other member the opposite is true. Then $\beta_1(t) >$
    $\beta_2(t)$ and $\beta_1(t) < \beta_2(t)$. Or suppose that for all members the k = 1
    version holds with, say, equality and the k = 2 version holds with
    strict inequality. On that assumption, $\tilde{q}_2^h(t) = 0$ for all h. But
    then for k = 2 equation (6) is not satisfied.

10. The second of the constraints on the $\hat{c}_{2j}^h(t)$, equation (20), is
    obtained by solving the equality version of constraint (1) for

$q_1^h(t)$ or $q_2^h(t)$ and substituting the resulting expression into the equality version of constraint (2).

11. We think of that condition as a kind of gross substitution condition, since it holds if and only if $(\phi_1^h)^- \geq 0$. Interestingly enough, it is the condition that ensures a unique equilibrium (see below, pp. 23-25).

12. In proving that proposition, what we rule out is any cyclical equilibrium that satisfies condition (a). Obviously, any sequence $\{T_k(t)\}$ that diverges is not a possible equilibrium sequence. Nor is any cyclical sequence that does not satisfy condition (a).

13. Evidently, to the left of $\hat{T}_2$ the line of the equation $T_2(t) = \hat{T}_2$ bounds any H function from above, although it may be that for some $T_2(t+1), H[T_2(t+1)] = \hat{T}_2$; and to the right of $\hat{T}_2$, that line bounds H from below. We use the negatively sloped 45° line as a bound, though, to make a point. Our gross substitution condition would seem to be in a sense too strong. Had we been able to find a weaker condition that could be given an economic interpretation, we should have been able to get by using it.

14. The uniqueness of the fixed point is what guarantees convergence to $\hat{T}_2$, since any limit of the sequence must be a fixed point of the function H.

15. Gale has proved an analogous proposition [see 2, Theorem 3] for a one-good world. It seems to us, though, that his one good can be interpreted as a composite good only if additional restrictions

are imposed on the utility functions. More restrictions than he has imposed are required for our Proposition 1 or an analogue.

16. We might just as well have chosen to depict a portfolio autarky consumption allocation. Whichever of our two regimes obtains, the argument that follows is valid. For the $\bar{\beta}_k$, like $\hat{\beta}$, are less than unity.

17. We have by the stationarity of the equilibrium allocation that $\hat{c}_{22}^h(t-1) = \hat{c}_{22}^h(t)$. And if $(c_{21}^h(t), c_{22}^h(t))$ lies on the line with slope -1, then $c_{21}^h(t) = \hat{c}_{21}^h(t) - \varepsilon^h$ and $c_{22}^h(t) = \hat{c}_{22}^h(t) + \varepsilon^h$, from which it follows that

$$\sum_h c_{21}^h(t) + \sum_h c_{22}^h(t) = S_2(\bar{p}).$$

18. Starrett [7] makes the point in a more general context.

19. The argument developed below (Case 2) covers the possibility that it is members of generation zero who are better off with their C allocations than with their A allocations.

20. To give a "proof," we make use of Figure III. The point A, which lies on the expansion path λ_A, is the A allocation of a member of generation $\bar{t}$ who supposedly prefers his C allocation. If the inequality does not hold, that individual's C allocation is, however, given by a point such as c_1 or c_2. And by our assumptions about utility, neither the point c_1 nor the point c_2 can be preferred to the point A.

21. Suppose that $\bar{c}_{21}^h(\bar{t}) < \hat{c}_{21}^h(\bar{t})$ for some h. Since no individual is worse off with his C allocation than with his A allocation, $\bar{c}_{22}^h(\bar{t}) > \hat{c}_{22}^h(\bar{t})$. But then the C allocation of the chosen member of generation $\bar{t}$ is a point lying below the A allocation expansion path. And since expansion paths do not intersect, the C allocation does not satisfy condition (c).

22. If the individuals who prefer their C allocations are members of generation zero, then $\gamma(0) > 0$. That must be, since $\bar{c}_{21}^h(0) = \hat{c}_{21}^h(0)$ for all h.

23. The two points $(\hat{c}_{22}^h, \hat{c}_{21}^h)$ and $(\tilde{c}_{22}^h(t), \tilde{c}_{21}^h(t))$ lie on the indifference curve $U(\hat{c}^h)$ and equation (43) results from application of the law of the mean. Recall that the slope of $U(\hat{c}^h)$ at the point $(\hat{c}_{22}^h, \hat{c}_{21}^h)$ is $-1/\lambda_A$.

24. Assuming that $\hat{c}_{21}^h - \tilde{c}_{21}^h(t)$ is positive for some values of h and negative for others, one comes to a contradiction: that expansion paths intersect.

25. The argument can be made using Figure II. Since $\lambda_D < 1$, the indifference curve passing through the D allocation point, say (c_{22}^h, c_{21}^h), has a slope less than -1 at that point. So there necessarily are preferred allocations southeast of (c_{22}^h, c_{21}^h) lying on a line that passes through that point and has a slope -1. And as remarked above, those preferred allocations are feasible (see footnote 17) and increase the welfare of, among others, the members of generation zero.

26. The β of equation (19) is without a subscript. But it is obviously permissible to think of there being two country-specific discount rates even when the laissez-faire regime obtains. If it does, though, then those rates are the same.

27. The appearance of L_k/N_k in equation (54) may require a few words of explanation. We assume that for $t < 1$ the portfolio autarky regime obtained. And consequently all members of generation zero start period one with their portfolio autarky land purchases. Obviously, if there is an arbitrary distribution of land holdings over the members of generation zero, then it is not in general possible to ensure their getting the appropriate consumption allocations by choosing two land tax rates, v_1 and v_2.

28. Traditionally, foreign investment has been thought of as shipping capital abroad, or sending it by rail or truck. In [3, p. 3], Jones is quite explicit: "Foreign investment involves a change in location, but not in ownership, of real capital equipment."

29. Because the complete autarky regime equilibrium, like the portfolio autarky regime equilibrium, is stationary, we are justified in taking prices as being constant over time. Tastes being the same, equation (6) implies that under either regime the land purchase of any resident of country k is L_k/N_k.

30. It will turn out a change of regimes can make either the first or the second country residents of generation t worse off. Since we are

only interested in finding a counter-example to the Pareto-superiority of the portfolio autarky consumption allocation, we may then limit ourselves to the members of generation t or, in other words, to comparing steady-state equilibrium solutions.

31. The total revenue and total cost curves are linear, so if total revenue is less than total cost, the optimal output is zero; and if total revenue is greater than total cost, there is no finite optimal output.

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