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CORRESPONDENCE PRINCIPLES FOR CONCAVE ORTHOGONAL GAMES

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ABSTRACT

Silberberg [6] and Pauwels [2] have produced and clarified seminal results in the comparative statics of single-agent classical optimization problems. This paper extends Pauwels' method to derive analogous results for stable Nash equilibria in a subclass of the widely used class of concave orthogonal games defined by Rosen [3]. Application of these results to cost curve shifts in the asymmetric Cournot oligopoly immediately uncovers apparently new comparative statics results.

1. Introduction

Pauwels [2], in a seminal paper motivated by Silberberg [7], has succinctly derived general comparative statics (i.e., parametric sensitivity) results for the single-agent, finite dimensional, classical optimization problem:

$$\max_{x} f(x;\alpha) \tag{1}$$

$$\text{s.t. } g(x;\alpha) = 0,$$

where the decision vector $x \in R^n$, the parameter vector $\alpha \in R^m$, and g is a vector of r constraint functions. Symbols after semicolons indicate things treated parametrically by the agent. The general comparative static results for (1) have been used by Silberberg to easily derive all the known comparative static results in the theory of the consumer and the competitive firm, as well as some new results not uncovered by older methods. Comparative statics results are, of course, predictions often used to test the theories underlying them.

But no analogous general comparative statics results have been found in multi-agent, game-theoretic settings characteristic of oligopoly theory. The lack of such a general result hinders researchers seeking to test industry models against one another. Questions such as "Is the market competitive or not?," which are crucial in antitrust determinations, are hard to answer partly because of the lack of comparative statics results characterizing oligopoly models.

Perhaps it is unrealistic to expect that an easily derived, general comparative statics result would hold in a game-theoretic setting. After all, mathematical game-theoretic models of oligopoly are far more complex than (1), involving many agents' objectives and constraints. For example, even the existence of game-theoretic equilibria is a nontrivial question. One would hope, though, that plausible auxiliary assumptions, implying more than the mere existence of equilibria, would permit the derivation of a general comparative static result. Herein, I employ such assumptions in deriving general comparative statics results valid for oligopoly models falling in a subclass of the class of concave orthogonal games defined by Rosen [4], whose Nash equilibria are defined by:

$$\max_{x^i} f^i(x^i;x^{)i(},\alpha) \quad i = 1,\ldots,N \tag{2}$$
$$\text{s.t. } g^i(x^i;\alpha) = 0,$$

where $x^i \in R^n$, $\alpha \in R^m$, g^i is a vector of r_i constraint functions, and $x^{)i(} = (x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^N)$.

In addition to the usual regularity and concavity assumptions, two additional assumptions must be placed on (2), producing a subclass of games in which the results apply. These assumptions, while restrictive, do ensure local stability in a dynamic adjustment mechanism for (2). The latter interpretation of these assumptions means that the derived comparative statics results are correspondence principles (see Samuelson [5, Chapter 9]), which link assumptions guaranteeing stability to comparative statics results.

The restrictiveness of the two additional assumptions is evidence of a tradeoff researchers may have to accept when deriving game-theoretic extensions of Samuelson's program for finding the determinate, testable predictions of economic models. These results indicate that the scope of the correspondence principle may also be quite limited in game theoretic settings other than competitive equilibrium (see Quirk and Saposnik [3, chap. 6, sec. 4]).

This paper presents a brief review of Pauwels' work on the single agent problem (1), to motivate what follows. The correspondence principles in the multi-agent setting are then derived, followed by use of these results to examine the effects of cost curve shifts on Nash equilibria of asymmetric Cournot oligopolies. This application illustrates the power and simplicity of the correspondence principles derived here, as well as their propensity to yield both old and new comparative statics results. The application clearly illustrates the tradeoff between model generality and the desire to obtain determinate comparative statics from stable Nash equilibria.

2. Comparative Statics of the Classical Optimization Problem

Forming the Lagrangian for (1),

$$L(x,\lambda;\alpha) = f(x;\alpha) + \lambda' g(x;\alpha), \tag{3}$$

where the apostrophe denotes the vector transpose of the r-vector of Lagrange multipliers, Pauwels makes five assumptions:

The maps f and g are twice differentiable. (A1)

For the parameter vector α, there exist $\hat{x}$, $\hat{\lambda}$ such that:

$$L_x(\hat{x},\hat{\lambda};\alpha) = f_x(\hat{x};\alpha) + g_x'(\hat{x};\alpha)\hat{\lambda} = 0 \tag{A2}$$

$$L_\lambda(\hat{x},\hat{\lambda};\alpha) = g(\hat{x};\alpha) = 0,$$

where a subscripted function denotes its gradient with respect to the subscripted vector, i.e., $f_x = (\partial f/\partial x_i)$, and where a subscripted vector of functions denotes its Jacobian matrix with respect to the subscript, i.e., $g_x = (\partial g^i/\partial x_j)$.

$$v' L_{xx} v < 0 \text{ for all } v \in R^n,\ v \neq 0, \text{ which satisfy } g_x(\hat{x};\alpha)v = 0. \qquad \text{(A3)}$$

Assumption (A3) is summarized by saying that L_{xx} is negative definite subject to constraint. Then, $\hat{x}$ is a regular, strict, local maximum. Pauwels implicitly assumes global conditions on f and g so that:

$$\hat{x} \text{ satisfying (A2) and (A3) is a solution of (1).} \qquad \text{(A4)}$$

Then, to examine the comparative statics of $\hat{x}$ in response to infinitesimal changes in α, one makes an additional assumption:

$$\text{The rank of } g_x = r < n, \text{ i.e., } g_x \text{ is of full rank.} \qquad \text{(A5)}$$

One then applies the implicit function theorem to obtain the comparative statics variational equation:

$$\begin{bmatrix} \hat{\lambda}_\alpha \\ \hat{x}_\alpha \end{bmatrix} = - \begin{bmatrix} 0 & g_x \\ g_x' & L_{xx} \end{bmatrix}^{-1} \begin{bmatrix} g_\alpha \\ L_{x\alpha} \end{bmatrix} \qquad (4)$$

where $L_{x\alpha} = (\partial^2 L / \partial x_i \partial \alpha_j)$.

Pauwels [2, pp. 484-486] then derives his fundamental results. First, he shows the following:

(CS1) <u>The symmetric matrix</u> $[g_\alpha' \ L_{x\alpha}'] \begin{bmatrix} \hat{\lambda}_\alpha \\ \hat{x}_\alpha \end{bmatrix}$ <u>is positive semidefinite on the null space of</u> g_α <u>and is positive definite off its subspace of vectors</u> u <u>for which</u> $L_{x\alpha}u$ <u>belongs to the space spanned by the row vectors of</u> g_x.[1/]

Suppose that (1) is augmented to incorporate another s-vector of "just binding" constraints

$$g^{+}(\hat{x};\alpha) = 0. \tag{5}$$

Assuming the analogous conditions (A1)-(A5) for the augmented set of r + s constraints, Pauwels derives a general Le Chatelier Principle:

(CS2) <u>The matrix</u> $[g_\alpha^{'}\ g_\alpha^{+'}\ L_{x\alpha}^{'}]\begin{bmatrix} \hat{\lambda}_\alpha^{+} - \hat{\lambda}_\alpha \\ \hat{\mu}^{+} \\ \hat{x}_\alpha^{+} - \hat{x}_\alpha \end{bmatrix}$ <u>is symmetric and negative semidefinite</u>,

where $\hat{\mu}^{+}$ denotes the s-vector of Lagrange multipliers on (5) and plus signs denote the new solutions for the augmented constraint set. Somewhat less precise versions of (CS1) and (CS2) have been used by Silberberg [7] to derive quickly, "on the back of an envelope," most of the known and some new results in the competitive theories of the firm and consumer. These results are also useful in providing econometrically testable restrictions on data.

3. Comparative Statics of Nash Equilibria in Concave Orthogonal Games

A Nash equilibrium for α is an Nn vector $\hat{x} = (\hat{x}^1,\ldots,\hat{x}^N)$ solving (2) for $i = 1,\ldots,N$. Assume conditions analogous to (A1)-(A5), i.e., assume the following for $i = 1,\ldots,N$:

The maps f^i and g^i are twice differentiable. (A1')

There exist $\hat{x}$ and $\hat{\lambda} = (\hat{\lambda}^1,\ldots,\hat{\lambda}^N)$ such that:

$$L^i_{x^i}(\hat{x}^i,\hat{\lambda}^i;\hat{x}^{)i(},\alpha) = f^i_{x^i}(\hat{x}^i;\hat{x}^{)i(},\alpha) + g^{i'}_{x^i}\hat{\lambda}^i = 0 \tag{A2'}$$

$$L^i_{\lambda^i}(\hat{x}^i,\hat{\lambda}^i;\hat{x}^{)i(},\alpha) = g^i(\hat{x}^i;\alpha) = 0$$

$$v^{i'} L^i_{x^i x^i} v^{i'} < 0 \text{ for all } v^i \varepsilon R^n,\ v^i \neq 0, \tag{A3'}$$

which satisfy $g^i_{x^i}(\hat{x}^i;\alpha)v^i = 0$.

Assume additional global conditions on all f^i and g^i so that the regular, strict, local maxima guaranteed by (A2') and (A3') are global, i.e., assume the analog of (A4):

$$\hat{x} \text{ satisfying (A2') and (A3') is a Nash equilibrium for } \alpha. \tag{A4'}$$

We also need the regularity condition analog of (A5):

$$\text{The rank of } g^i_{x^i} = r_i < n. \tag{A5'}$$

Formally differentiate (to be justified momentarily) the Nash equilibrium (by A4') conditions (A2') to obtain the partitioned differential:

$$\left[\begin{array}{cccc|cccc}
0 & \cdot & \cdot & 0 & g^1_{x^1} & 0 & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & 0 & g^2_{x^2} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & g^N_{x^N} \\
\hline
g^{1'}_{x^1} & 0 & \cdot & 0 & L^1_{x^1x^1} & f^1_{x^1x^2} & \cdot & f^1_{x^1x^N} \\
0 & \cdot & \cdot & \cdot & f^2_{x^2x^1} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & g^{N'}_{x^N} & f^N_{x^Nx^1} & \cdot & \cdot & L^N_{x^Nx^N}
\end{array}\right]
\left[\begin{array}{c}
\hat{\lambda}^1_\alpha \\ \cdot \\ \cdot \\ \hat{\lambda}^N_\alpha \\ \hline \hat{x}^1_\alpha \\ \cdot \\ \cdot \\ \hat{x}^N_\alpha
\end{array}\right]
= -
\left[\begin{array}{c}
g^1_\alpha \\ \cdot \\ \cdot \\ g^N_\alpha \\ \hline L^1_{x^1\alpha} \\ \cdot \\ \cdot \\ L^N_{x^N\alpha}
\end{array}\right]$$

where the square matrices $f^k_{x^k x^l} = (\partial^2 f^k/\partial x^k_i \partial x^l_j)$, with similar notation for the diagonal blocks $L^k_{x^k x^k}$, and $L^k_{x^k \alpha} = (\partial^2 L^k/\partial x^k_i \partial \alpha_j)$. This is written more compactly as:

$$\begin{bmatrix} 0 & g_x \\ g_x' & L_{xx} \end{bmatrix} \begin{bmatrix} \hat{\lambda}_\alpha \\ \hat{x}_\alpha \end{bmatrix} = - \begin{bmatrix} g_\alpha \\ L_{x\alpha} \end{bmatrix} \tag{6}$$

To obtain the desired analog of (CS1), further assumptions, which have no analog in the single agent problem, are needed:

The Nn-order square matrix L_{xx} is symmetric (A6')

and

The matrix L_{xx} is negative definite on the null space of g_x. (A7')

Because (A1') guarantees the symmetry of $L^i_{x^i x^i}$, Assumption (A6') reduces to the assumption that, for all $i \neq j$, the matrix $f^i_{x^i x^j}$ equals the matrix $f^j_{x^j x^i}$. Although this is a very restrictive condition, it is implied by and is far less restrictive than the assumption of identical agents--symmetric Nash equilibrium--that is used in many studies (see Stutzer [9]). Rather, (A6') could be dubbed "second-order symmetry," since it only requires that the marginal effect of agent j's decision x^j on agent i's first-order Lagrangian conditions is the same as the marginal effect that agent i's x^i has on agent j's Lagrangian conditions.

Condition (A7') can be motivated either as necessary for our comparative statics result or by considering the dynamic adjustment mechanism:

$$dx^i/dt = L^i_{x^i}(x^i;x^{(i)},\alpha);\ i = 1,\ldots,N;\ x(0) \text{ given,} \tag{7}$$

which is related to that used by Rosen [4, p. 529]. The i^{th} agent is thus presumed to adjust its decision vector x^i at a rate equal to the gradient of its Lagrangian; i.e., steepest ascent is employed. Assume that (7) operates at least locally near a steady state $\hat{x}$ satisfying (A2') and (A3'), which by (A4') is a Nash equilibrium. Then, (A7') implies that the Jacobian at $\hat{x}$ of the right-hand side of (7) is negative definite on the null space of g_x. In conjunction with (A6'), (A7') thus implies that all eigenvalues of (7) associated with feasible directions are negative at $\hat{x}$, thus guaranteeing local stability of a Nash equilibrium $\hat{x}$.

The analog to (CS1) is now easily obtained. Assumptions (A5'), (A6'), and (A7') imply the existence (as has been assumed in the formal differentiation) and symmetry of $\begin{bmatrix} 0 & g_x \\ g_x' & L_{xx} \end{bmatrix}^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}' & C_{22} \end{bmatrix}$. The corollary to Lemma I in Pauwels then implies that the null space of C_{22} is spanned by the rows of g_x. Following Pauwels [2, p. 484], invert (6), premultiply both sides of the result by $[g_\alpha' \;\; L_{x\alpha}']$, and derive the resulting quadratic form on the null space of g_α. Then, apply Lemma II in Pauwels to obtain the analog of (CS1):

(CS1') <u>The symmetric matrix</u> $[g_\alpha' \;\; L_{x\alpha}']\begin{bmatrix} \hat{\lambda}_\alpha \\ \hat{x}_\alpha \end{bmatrix}$ <u>is positive semidefinite on the null space of</u> g_α, <u>and is positive definite off its subspace of vectors</u> u <u>for which</u> $L_{x\alpha}u$ <u>belongs to the space spanned by the row vectors of</u> g_x.

To obtain the analog of (CS2), suppose the constraints in (2) are augmented by s_i-vectors of additional "just binding" constraints

$$g^{i+}(\hat{x}^i;\alpha) = 0; \; i = 1,\ldots,N. \tag{8}$$

Assuming the analogous conditions (A1')-(A7') for (2) with the augmented set of constraints, Pauwels' proof method trivially yields the analog of (CS2):

(CS2') <u>The matrix</u> $[g'_\alpha \;\; g^{+'}_\alpha \;\; L'_{x\alpha}] \begin{bmatrix} \hat{\lambda}^+_\alpha - \hat{\lambda}_\alpha \\ \hat{\mu}^+_\alpha \\ \hat{x}^+_\alpha - \hat{x}_\alpha \end{bmatrix}$ <u>is symmetric and negative semi-definite</u>,

where the Jacobian of the additional Lagrange multipliers $\hat{\mu}^+_\alpha = (\hat{\mu}^{1+}_\alpha \ldots \hat{\mu}^{N+}_\alpha)'$, the matrix $\hat{\mu}^{k+}_\alpha = (\partial\hat{\mu}^{k+}_i/\partial\alpha_j)$ and $g^+_\alpha = (g^{1+}_\alpha \ldots g^{N+}_\alpha)'$, and the matrix $g^{k+}_\alpha = (\partial g^{k+}_i/\partial\alpha_j)$.

4. An Application: The Effects of Cost Curve Shifts on Cournot Oligopoly

There are N firms, the i^{th} of which produces an output level x^i of the homogeneous good and possesses a twice-differentiable cost function $c^i(x^i)$. The twice-differentiable inverse market demand curve is denoted by $p(\sum_{k=1}^{N} x^k)$. Let α_i denote a positive cost curve shift parameter for firm i. It might represent the effects of a tax proportional to cost, a costly regulation, or some other cost-changing phenomenon. Then, (2) becomes:

$$\max_{x^i} f^i(x^i;x^{)i(},\alpha) = x^i p(x^i + \sum_{j\neq i} x^j) - \alpha_i c^i(x^i); \; i = 1,\ldots,N. \qquad (9)$$

Assumption (A1') is satisfied, as is (A5') vacuously. We follow Okuguchi [1, pp. 6-9] in making structural assumptions guaranteeing the satisfaction of (A2')-(A4'). They are:

(i) $c^i(0) = 0$

(ii) $dp/d(\cdot) \triangleq p_x < 0.$

There exist M_i such that for all $x \in \prod_{i=1}^{N} [0,M_i]$: (10)

(iii) $\quad c^i_{x^i}(0) < p(\sum_{j \neq i} x^j)$

(iv) $\quad M_i p_x(M_i + \sum_{j \neq i} x^j) + p(M_i + \sum_{j \neq i} x^j) < c^i_{x^i}(M_i)$

(v) $\quad f^i$ is globally strictly concave in x^i for any $x^{)i(}$, α.

In (10), (i) ensures that each firm's profit is bounded below by zero. Assumption (ii) is the usual downward sloping demand. Assumption (iii) holds that each firm's profit is increasing at zero output, whereas (iv) ensures that it eventually starts to decrease past some positive output level, M_i. By continuity of $f^i_{x^i}$, there must exist $\hat{x}^i < M_i$, satisfying $f^i_{x^i} = 0$, for any $x^{)i(}$, α. This is a global maximum due to (v), which also guarantees the satisfaction of (A3'). The usual fixed-point argument then guarantees that (A2') and (A4') are also satisfied.

Because $r_i = 0$ and $n = 1$, (6) simplifies to $f_{xx} \hat{x}_\alpha = -f_{x\alpha}$, where:

$$f^i_{x^i x^i} = 2p_x + \hat{x}^i p_{xx} - \alpha_i c^i_{x^i x^i} \tag{11}$$

$$f^i_{x^i x^j} = p_x + \hat{x}^i p_{xx}; \; j \neq i$$

$$f^i_{x^i \alpha_i} = -\, c^i_{x^i}$$

$$f^i_{x^i \alpha_j} = 0; \; j \neq i.$$

The second-order symmetry condition (A6') will be satisfied when $f^i_{x^i x^j} = f^j_{x^j x^i}$. There are two alternate conditions satisfying (A6'):

All firms are identical, i.e., the equilibrium is symmetric (12)

or

$p_{xx} = 0$, i.e., the inverse market demand curve is locally linear at a Nash equilibrium $\hat{x}$. (13)

Condition (12), while quite severe, is often imposed to examine the comparative statics of oligopoly models (see Seade [6]) and in other noncooperative games (see Stutzer [9]). Of course, comparative statics analysis becomes quite simple (and less interesting) in this "symmetric," identical agents case because the equilibrium conditions can be reduced to a single equation. Condition (13) seems less severe, and a stronger global linearity condition is also frequently used in oligopoly models (e.g., see Spence [8]). To make matters interesting, we will assume only second-order symmetry (13), rather than symmetry (12), in what follows.

Assumption (A7') can be satisfied by plausibly assuming that there are no increasing returns to scale left unexploited in equilibrium, i.e., that each firm's marginal cost is nondecreasing at equilibrium. To see this, let a vector $u \neq 0$ and note from (11) that $u'f_{xx}u = \sum_i \sum_j f^i_{x^i x^j} u_i u_j = p_x(2\sum u_i^2 + \sum\sum_{i \neq j} u_i u_j) - \sum_i \alpha_i c^i_{x^i x^i} u_i{}^2 = p_x(\sum u_i{}^2 + (\sum u_i)^2) - \sum \alpha_i c^i_{x^i x^i} u_i^2$. Because (10ii) makes the first term negative, a sufficient condition for (A7') is thus

$$c^i_{x^i x^i}(\hat{x}^i) > 0;\ i = 1,\ldots,N. \tag{14}$$

We can now apply (CS1'). Because there are no constraints, (CS1') implies that the matrix $f'_{x\alpha}\hat{x}_\alpha$ is positive definite. Using (11) and (13), the (i,j) element of $f'_{x\alpha}\hat{x}_\alpha$ is:

$$f'_{x\alpha}\hat{x}_\alpha(i,j) = -c^i_{x^i}\hat{x}^i_{\alpha_j}. \tag{15}$$

Positive definiteness of (15) implies that the diagonal elements of (15) are positive, which implies that

$$\hat{x}^{i}_{\alpha_i} < 0; \; i = 1,\ldots,N. \tag{16}$$

Thus, a proportional, upward shift in a firm's cost curve results in its output diminishing.

The symmetry of (15) implies the reciprocity relations:

$$c^{i}_{x^i}\,\hat{x}^{i}_{\alpha_j} = c^{j}_{x^j}\,\hat{x}^{j}_{\alpha_i}; \; i,j = 1,\ldots,N, \tag{17}$$

which permits one to derive $\hat{x}^{j}_{\alpha_i}$ from $\hat{x}^{i}_{\alpha_j}$ and the ratio of marginal costs of the two firms in equilibrium. If firm i has higher marginal cost than firm j in equilibrium, then an upward shift in firm i's costs has a greater impact on firm j's output than vice versa.

Finally, positive definiteness also implies that the upper left, second- and higher-order principal minors of (15) are positive. The economic meaning of these determinantal inequalities is somewhat obscure, but let us examine the upper left, second-order principal minor of (15) and conclude:

$$c^{1}_{x^1}c^{2}_{x^2}\,\hat{x}^{1}_{\alpha_1}\hat{x}^{2}_{\alpha_2} - c^{1}_{x^1}c^{2}_{x^2}\,\hat{x}^{1}_{\alpha_2}\hat{x}^{2}_{\alpha_1} > 0,$$

and because both marginal costs are positive,

$$\hat{x}^{1}_{\alpha_1}\hat{x}^{2}_{\alpha_2} > \hat{x}^{1}_{\alpha_2}\hat{x}^{2}_{\alpha_1}. \tag{18}$$

In other words, the "own effects" of firm cost curve shifts on their own outputs "dominate" the "cross effects" the shifts have on the outputs of other firms. Readers are free to amuse themselves by deriving and interpreting the other higher-order minor inequalities. In conjunction with (16), and the previously unknown (17) and (18), these inequalities might prove useful in the estimation and testing of the asymmetric oligopoly model restricted by (13) and (14). In any event, their discovery has apparently escaped those employing traditional comparative statics methods in oligopoly models.

Result (CS2') can also be fruitfully applied. As before, suppose that government regulations raise the costs of firms. Worried about the possible output reductions, the government also enacts a regulation forcing (the politically weakest) firm i to maintain its former output level. Formally, government has added the "just binding" constraint:

$$g^{i+}(x^i;\alpha) \equiv \hat{x}^i - x^i = 0. \tag{19}$$

Then, because there were no constraints initially, (CS2') yields the simpler result that:

$$[g^{+\prime}_{\alpha} \;\; f^{\prime}_{x\alpha}] \begin{bmatrix} \hat{\mu}^{+}_{\alpha} \\ \hat{x}^{+}_{\alpha} - \hat{x}_{\alpha} \end{bmatrix} \text{ is symmetric and negative semidefinite} \tag{20}$$

where, because (19) does not contain α, $g^{+\prime}_{\alpha}$ is a column vector of N zeros; and $\hat{\mu}^{+}_{\alpha} = (\hat{\mu}^{+}_{\alpha_1},\ldots,\hat{\mu}^{+}_{\alpha_N})$, the gradient of the Lagrange multiplier of (19). Negative semidefiniteness implies that the diagonal elements of (20) are nonpositive, which simplifies to:

$$c^j_{x^j}\hat{x}^j_{\alpha_j} < c^j_{x^j}\hat{x}^{j+}_{\alpha_j}; \; \underline{\text{all}} \; j = 1,\ldots,N. \tag{21}$$

Equations (21) and (16) imply that the negative response of firm j to an upward shift on its cost curve will be no more negative if one constrains <u>any</u> firm i's output from falling. Symmetry of (20) simplifies to

$$c^i_{x^i}\hat{x}^{i+}_{\alpha_j} = c^j_{x^j}\hat{x}^{j+}_{\alpha_i}, \tag{22}$$

which shows that (17) still holds in the constrained equilibrium. A simple exercise for the reader will also show that (21) and (22) are still true if any number $r < N$ of the firms are subject to (19).

Whether or not determinate results, like (16)-(19), (21), and (22) could be derived without the restrictive second-order symmetry assumption (A6'), is an open question for future research. It may very well be that restrictive assumptions like it are an unavoidable tradeoff which must be accepted in order to derive determinate results in game-theoretic settings. Failure to recognize the likelihood of the tradeoff may lead researchers to falsely conjecture that results derived in symmetric set-ups would continue to hold in more general asymmetric settings.

Footnote

[1]/In fact $u'[g'_\alpha \; L'_{x\alpha}] \begin{bmatrix} \hat{\lambda}_\alpha \\ \hat{x}_\alpha \end{bmatrix} u = 0$ <u>on</u> that subspace.

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