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ESTIMATING LINEAR FILTERS WITH ERRORS IN VARIABLES USING THE HILBERT TRANSFORM

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Abstract—
In this paper we present a consistent estimator for a linear filter (distributed lag) when the independent variable is subject to observational error. Unlike the standard errors-in-variables estimator which uses instrumental variables, our estimator works directly with observed data. It is based on the Hilbert transform relationship between the phase and the log gain of a minimum phase-lag linear filter. The results of using our method to estimate a known filter and to estimate the relationship between consumption and income demonstrate that the method performs quite well even when the noise-to-signal ratio for the observed independent variable is large. We also develop a criterion for determining whether an estimated phase function is minimum phase-lag.

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0. Introduction

This paper presents a novel approach to the problem of estimating the discrete–time impulse response $\{h(k)\}$ of a linear filter when both the input signal $x(t)$ and the output signal

$$y(t) = \sum_{k=0}^{\infty} h(k)x(t-k) \tag{0.1}$$

are corrupted by additive noises whose spectra are of unknown shape. The corrupted signals are $\{x'(t) = x(t) + \nu(t)\}$ and $\{y'(t) = y(t) + \epsilon(t)\}$, respectively. The noises satisfy the following three assumptions: (1) $\{\nu(t)\}$ is uncorrelated with $\{\epsilon(t)\}$, (2) $\{\nu(t)\}$ is uncorrelated with $\{x(t)\}$, and (3) $\{\epsilon(t)\}$ is uncorrelated with $\{y(t)\}$.

The problem of estimating $\{h(k)\}$ is an important one in several areas of geophysical signal processing. One is magnetotelluric resistivity measurements. If (0.1) is a model of magnetotelluric measurements of electromagnetic impedance, then $\{h(k)\}$ is the impulse response of the earth, $\{y(t)\}$ is the electric field, and $\{x(t)\}$ is the magnetic field at a point on the earth's surface. There are several sources for the noises. The primary ones are the electronic noises in the instruments measuring reflection responses or electric and magnetic fields. The instrumentation noise is usually independent of the signals, and the noise in one instrument is usually independent of that in another, so that the three assumptions above are likely to be satisfied.

If the input noise can be observed, its spectrum can be estimated, and the standard cross spectrum approach will yield an asymptotically unbiased estimate of the frequency transfer function, which can then be Fourier transformed to yield an estimated impulse response. This method is equivalent to fitting the impulse response by least squares. In addition, if the input process is nongaussian and has a non zero skewness, then it is possible to estimate the impulse response of a non–minimum phase filter using the cross bispectrum between input and output as described by Hinich and Wilson (1992). The cross bispectrum approach works as follows: the cross bispectrum between input and output is estimated along with the bispectrum of the input. The ratio of the cross bispectrum to the input bispectrum yields an asymptotically unbiased and asymptotically gaussian estimate of the frequency transfer function.

When the noises cannot be observed, the standard cross spectrum approach to estimating the filter's response will yield a biased estimate since the noise covariances confound the true signal covariances in the covariance function of the observed time series. The cross bispectrum approach will also yield a biased estimate of the transfer function since the bispectrum in the ratio will be the sum of the input signal's bispectrum and the bispectrum of the additive noise in the input.

It is not possible to observe the noise alone in most applications. In magnetotelluric measurements, for example, the input and output signals are the magnetic and electrical fields, and neither one can be turned off to observe the noise. The term errors–in–variables is used in the statistics literature to designate a linear model where one or more of the "independent" variates have errors with unknown variances. Our impulse–response model is a special case of errors–in–variables. This problem is intractable unless one makes some identification restrictions which can be checked.

One approach for obtaining asymptotically unbiased estimates of the impulse response where there are errors–in–variables is to find a series $\{z(t)\}$ that is correlated with $\{x(t)\}$ but is uncorrelated with the noise $\{\nu(t)\}$. The $\{z(t)\}$ is called an instrumental variable [Reiersol (1945)]. An asymptotically unbiased estimate of the impulse response can be obtained using the ratio of the cross spectrum of $\{z(t)\}$ and $\{y(t)\}$ to the cross spectrum of $\{z(t)\}$ and $\{x(t)\}$. We will present an instrumental variable for the special case when $\{\nu(t)\}$ is white noise, but show that it produces poor results.

In this paper, we present an alternative method for estimating the impulse response for noisy input and output measurements. Our method is a refinement of that developed by Boehl, Bostick, and Smith (1977); Clay and Hinich (1981); and Hinich (1983). It is based on the Hilbert transform relationship between the phase and log gain of a linear filter, and we will call our estimate the Hilbert Transform Estimate (HTE). We use the discrete–time Hilbert transform to develop a method that yields an asymptotically unbiased estimate of the impulse response up to a scale multiple under the critical assumption that the filter is minimum phase (invertible). If in addition there is a band of frequencies where both the input and output noises have no variance, then the model is fully identified and the scale factor is estimable.

Our approach also allows us to check the minimum phase assumption. Thus, the critical assumption may be rejected by data analysis. That is a plus for our method.

Since the HTE uses the frequency domain approach to the estimation of linear filters, a frequency domain representation of the errors in variable problem is presented in Section 1. The Hilbert transform relationship between the log gain and the phase is also presented in this section. The HTE is derived in Section 2. Two problems which arise in empirically implementing the HTE — namely, unwrapping the phase and determining whether an arbitrary phase is minimum phase — are discussed in Section 3. Section 4 presents the results of using the HTE with artificial data and compares these results with those obtained using instrumental variables. The final section contains a summary.

1. The Errors in Variables Problem and the Hilbert Transform

1.1. The Case of No Observational Errors

Assume that $\{h(k)\}$ is absolutely summable (the filter is stable). Thus, the transfer function of the filter,

$$H(\omega) = \sum_{k=0}^{\infty} h(k) \exp(-i\omega k)$$

exists for $0 \leq \omega < 2\pi$. $|H(\omega)|$ is the gain and $\phi(\omega) = \arctan[\mathrm{Im}H(\omega)/\mathrm{Re}H(\omega)]$ is its phase. Further, assume that $\{x(t)\}$ is a mean zero, stationary time series with an absolutely summable covariance function. Under these assumptions on $\{x(t)\}$, its spectrum $S(\omega|x)$ exists.

Thus, from (0.1) the cross spectrum between $\{x(t)\}$ and $\{y(t)\}$ is

$$S(\omega|x,y) = H(\omega)S(\omega|x)$$

which can be rearranged to yield

$$H(\omega) = S(\omega|x,y)/S(\omega|x). \tag{1.1}$$

That is, the transfer function of the filter is equal to the ratio of the cross spectrum of input and output to the own spectrum of the input.

This is the standard cross spectrum approach to the estimation of the transfer function. When $\{x(t)\}$ and $\{y(t)\}$ are observed without error (or, more precisely, when the covariance function of $\{x(t)\}$ and the cross covariance function of $\{x(t)\}$ with $\{y(t)\}$ are known), the transfer function of the filter can be recovered using (1.1).[1]

1.2. The Case of Observational Errors

The transfer function of the filter cannot be recovered without bias using (1.1) when the input and output are only observable with error. Specifically, assume that only the series $\{x'(t)\}$ and $\{y'(t)\}$ are observable and that, for all $t = 0, \pm 1, \pm 2, \ldots,$

$$x'(t) = x(t) + \nu(t) \tag{1.2}$$

and

$$y'(t) = y(t) + \epsilon(t) \tag{1.3}$$

where $\{\nu(t)\}$ and $\{\epsilon(t)\}$ are mean zero, stationary noise processes with absolutely summable covariance functions. Each is assumed to be uncorrelated with both $\{x(t)\}$ and $\{y(t)\}$, and they are assumed to be uncorrelated with each other.

Under these assumptions on $\{x(t)\}$, $\{y(t)\}$, $\{\nu(t)\}$, and $\{\epsilon(t)\}$,

$$S(\omega|x',y') = S(\omega|x,y).$$

That is, the cross spectrum of the observed input and output is the same as the cross spectrum of the true input and output. Since $S(\omega|\nu) \neq 0$, then in general,

$$S(\omega|x',y')/S(\omega|x') = H(\omega)/[1 + S(\omega|\nu)S(\omega|x)] \neq H(\omega).$$

This is the frequency domain analog of the well-known result that ordinary least squares (OLS) estimates are biased when there are errors in variables because the covariance matrix of the observed independent variables does not equal the covariance matrix of the true variables. This implies that if there were no errors in variables problem, the phase of estimated filter obtained with OLS would be similar to the phase of the cross spectrum [see Brillinger (1981, Section 6.3)]. Note that if $S(\omega|\nu) = 0$ for frequencies in a band $\omega_a < \omega < \omega_b$, then $S(\omega|x',y')/S(\omega|x') = H(\omega)$ in that band. We will use this result in Section 3.4.

1.3. The Hilbert Transform

Since it is not possible to recover $H(\omega)$ using (1.1) and the observed data when the input and output are subject to observational error, we adopt a different approach. It is based on two important results. First, the phase function of the filter is identical to the phase function of the cross spectrum of the observed series, $\phi_{x'y'}(\omega)$. That is

$$\phi(\omega) = \phi_{x'y'}(\omega).$$

This result follows immediately from (1.1) and the definition of the phase. Second, in a special case, knowledge of the phase of a linear filter is sufficient to determine the transfer function of the filter. This special case is that in which the filter is minimum phase; that is, it is stable and causal and its z-transform

$$H_C(z) = \sum_{k=0}^{\infty} h(k)z^k \qquad (1.4)$$

has no zeros on $|z| \leq 1$.

It is well known that the real and imaginary parts of an analytic function are related by the Hilbert transform.[2] When $H_C(z)$ has no zeros on $|z| < 1$,

$$\log H_C(z) = \log |H_C(z)| + i \arg H_C(z) \qquad (1.5)$$

exists. It is important to note that in (1.5) the complex logarithm is defined to be continuous, so that the real and imaginary parts of $\log H_C(z)$ will be analytic and satisfy the conditions of the Hilbert transform. We also normalize $\arg H_C(z)$, so that $H_C(0) = 1$.

Consequently, when a linear filter is minimum phase, its phase and log gain will be related by the Hilbert transform. Specifically, in this case

$$\log |H(\omega)| = (2\pi)^{-1} \int_0^{2\pi} \phi(\omega')\cot[(\omega-\omega')/2]d\omega' + c \qquad (1.6)$$

for $0 \leq \omega < 2\pi$, where c is a scaling constant which arises because $\{h(k)\}$ and $\{ch(k)\}$ have the same phases but different gains. The integrand in (1.6) has a singularity only at $\omega = \omega'$ where the principal value of the integral exists.

1.4. A Useful Result

When $H_C(z)$ has zeros on $|z| \leq 1$, (1.5) does not exist for all z. The Argument Principle [see Conway (1978, Section 5.3) or Titchmarch (1939, Section

3.41)], however, relates the number of zeros of $H_C(z)$ and the change in the phase of the filter on the unit circle. Specifically, $\phi(\pi) - \phi(0) = \pi n$, where n is the number of zeros of $H_C(z)$ on $|z| \leq 1$.

2. Derivation of the HTE

Let $\hat{\phi}(\omega)$ denote an asymptotically unbiased and asymptotically gaussian estimator of $\phi(\omega)$ which converges in mean square to the phase of the filter for any frequency in $(0,2\pi)$ and assume a sample of N observations which occur at times $t = 0, 1, \ldots, N-1$.[3] With this sample we can obtain discrete estimates of the phase at the angular frequencies $\omega_j = 2\pi j/K$ for $j = 0, 1, \ldots, K-1$ and $N/K \geq 1$ integer. This grid of frequencies is chosen because it can provide a sequence of asymptotically independent phase estimates (see Appendix A). We denote this estimated phase sequence by $\{\hat{\phi}(\omega_j)\}$ and note that $\hat{\phi}(\omega_j) = -\hat{\phi}(\omega_{K-j})$.

The discrete number of phase estimates means that we cannot use (1.6) directly with $\{\hat{\phi}(\omega_j)\}$ to estimate the log gain. Cizek (1970) derives a discrete sum approximation to (1.6) which he calls the discrete Hilbert transform. Let

$$\psi(k) = \begin{cases} 0 \text{ for } k = 0,\ K/2 \\ -i \text{ for } 0 < k < K/2 \\ i \text{ for } K/2 < k \leq K-1 \end{cases} \tag{2.1}$$

and

$$\theta(j) = K^{-1} \sum_{k=0}^{K-1} \psi(k) \exp(i\omega_k j). \tag{2.2}$$

Then the discrete Hilbert transform of $\{\hat{\phi}(\omega_k)\}$ is

$$\log|\hat{H}(\omega_k)| = \sum_{j=0}^{K-1} \hat{\phi}(\omega_j)\,\theta(j-k) \tag{2.3}$$

for $k = 0, \ldots, K-1$.[4]

Thus, $\{\log|\hat{H}(\omega_k)|\}$ is an estimator of $\{\log|H(\omega_k)|\}$ and, under our assumptions on $\{\tilde{\phi}(\omega_j)\}$, it will be an asymptotically unbiased estimator of $\{\log|H(\omega_k)| + c\}$. Consequently, $\{|\hat{H}(\omega_k)|\} = \{\exp[\log|\hat{H}(\omega_k)|]\}$ is an asymptotically unbiased estimator of $\{c|H(\omega_k)|\}$.

Define

$$\hat{H}(\omega_k) = |\hat{H}(\omega_k)|\,\exp[i\tilde{\phi}(\omega_k)] \tag{2.4}$$

for $k = 0, \ldots, K-1$. From above, $\{\hat{H}(\omega_k)\}$ is an asymptotically unbiased estimator of $\{H(\omega_k)\}$ up to a scalar multiple. Consequently, $\{h(k)\}$ can be estimated (up to a scalar multiple) from the observed processes by taking the inverse discrete Fourier transform of the estimator of the transfer function; that is

$$\hat{h}(k) = K^{-1} \sum_{j=0}^{K-1} \hat{H}(\omega_j)\,\exp(i\omega_j k) \tag{2.5}$$

for $k = 0, \ldots, K-1$. The estimator $\{\hat{h}(k)\}$ will be subsequently referred to as the Hilbert transform estimator (HTE). The large sample properties of the HTE are given by (A.5) in Appendix A.

3. Problems in Empirically Implementing the HTE

3.1. Unwrapping the Phase

The theoretical discussion of the Hilbert transform relationship between the phase and log gain assumed that the phase function was continuous with domain $(-\infty, \infty)$. When the phase of an arbitrary linear filter is calculated according to $\phi(\omega) = \arctan[\mathrm{ImS}(\omega|x',y')/\mathrm{ReS}(\omega|x',y')]$, however, the domain of the phase function is usually assumed to be $[-\pi, \pi)$ with the result that discontinuities will occur whenever $\phi(\omega) = \pi$ due to the modulo 2π operation of restricting $\phi(\omega)$ to its principal value.[5] This point is illustrated in Figure 1a where we plot the minimum phase filter $(1 - L + 0.99L^2)^2 = 1 - 2L + 2.98L^2 - 1.98L^3 + 0.9801L^4$, where L is the lag operator.

Removing these discontinuities to obtain a continuous phase function is known as <u>unwrapping the phase</u>. When $\phi(\omega)$ is a piecewise continuous function on $[-\pi, \pi)$, the phase can be unwrapped using an algorithm suggested by Lii and Rosenblatt (1982). They suggest constructing a continuous phase function $\tilde{\phi}(\omega)$ from $\phi(\omega)$ according to

$$\tilde{\phi}(\omega) = \phi(\omega) + 2\pi\rho(\omega) \tag{3.1}$$

where $\rho(\omega)$ is an integer multiple of an indicator function chosen to ensure continuity of the phase function. The unwrapped phase of the linear filter in Figure 1a is illustrated in Figure 1b.

As Figure 1 clearly illustrates, unwrapping a continuous phase function appears to be straightforward since the discontinuities will only occur when $\phi(\omega) = \pi$. However, we face the more difficult phase unwrapping problem of removing the discontinuities induced by the modulo 2π operation given only a <u>sequence</u> of K phases, $\{\phi(\omega_j)\}$, all of which are in $[-\pi, \pi)$.[6]

The approach we use to unwrap phase sequences is an adaptation of the Lii and Rosenblatt procedure (3.1). Specifically, we choose a sequence of integers $\{p(j)\}$ such that the unwrapped phase sequence $\{\tilde{\phi}(\omega_j) \mid \tilde{\phi}(\omega_j) = \phi(\omega_j) + 2\pi p(j)\}$ satisfies

$$|\tilde{\phi}(\omega_j) - \tilde{\phi}(\omega_{j-1})| \leq r \tag{3.2}$$

where the scalar $\pi \leq r < 2\pi$ and $\tilde{\phi}(0) = 0$. That is, $\{\tilde{\phi}(\omega)\}$ is obtained by adding integer multiples of 2π to $\{\phi(\omega_j)\}$ until the absolute difference between any adjacent elements of the resulting phase sequence is less than some value r.

The difficulties in choosing $\{p(j)\}$ can be illustrated once again using the filter $(1 - L + 0.99L^2)^2$. The sequence of discrete unwrapped phase points for this filter at the angular frequencies $\omega_j = 2\pi j/120$, $j = 0, \ldots, 60$ is shown in Figure 2a, and the correctly unwrapped phase sequence is shown in Figure 2b. Correctly unwrapping the phase sequence requires the choice of $p(j) = -1$ for $21 \leq j \leq 29$ and $p(j) = 0$, otherwise. This unwrapped phase sequence satisfies (3.2) for $r \geq 5\pi/4$.

Other choices of $\{p(j)\}$ will yield incorrectly unwrapped phases, however. One such case is shown in Figure 3. Here the phase sequence is unwrapped incorrectly with $p(j) = 1$, $j = 20$ and $j \geq 30$ and $p(j) = 0$ otherwise, yielding a phase sequence which satisfies (3.2) for $r \geq \pi$. Since the examples in Figures 2b and 3 clearly illustrate that the choice of $\{p(j)\}$ can affect the unwrapped estimated phase sequence used in the HTE and therefore the estimated transfer function, we suggest that investigators pay careful attention to this part of the estimation procedure especially if they have reason to believe that the filter might have roots close to the unit circle.

3.2. A Minimum Phase Criterion for Estimated Phase Sequences

The HTE requires that the unwrapped estimated phase sequence is that of a minimum phase filter. This point raises the question of whether there exist conditions which an arbitrary unwrapped estimated phase sequence must satisfy if it is to be that of a minimum phase filter. The result in Section 1.4 suggests a set of necessary conditions: The unwrapped estimated phase sequence $\{\tilde{\phi}(\omega_k)\}$ is that of a minimum phase filter if

$$\tilde{\phi}(0) = \tilde{\phi}(\pi) = 0. \tag{3.3}$$

We will refer to (3.3) as the minimum phase criterion.

There are two ways in which the minimum phase criterion can be used when the HTE is implemented empirically. The first occurs when theory delivers enough restrictions to determine a priori that the linear filter is minimum phase. In this case, since (3.3) determines the end points of the unwrapped estimated phase sequence, it may also help determine how the other points in the estimated phase sequence are unwrapped. To return to the example of the filter $(1 - L + 0.99L^2)^2$, knowledge that the linear filter is minimum phase would rule out the unwrapped phase in Figure 3.

The second occurs when theory does not deliver enough restrictions to determine a priori that the linear filter is minimum phase. In this case, the HTE should be used only when the unwrapped estimated phase sequence satisfies (3.3).

Of course, the choice of $\{p(j)\}$ will be important again since it can affect the determination of whether an estimated phase sequence is minimum phase. This is clearly illustrated by example in the previous section. When the phase sequence from the filter $(1 - L + 0.99L^2)^2$ was not unwrapped at all or was correctly unwrapped as in Figures 2a and 2b, then (3.3) would have been satisfied. However, if the phase sequence from this filter had been unwrapped incorrectly as in Figure 3, then (3.3) would not have been satisfied and the unwrapped estimated phase sequence would not have been identified as that of a minimum phase filter.

3.3. Screening for Poor Estimates

In Theorem B in Appendix B we prove that when a transfer function is obtained from a phase function using (1.6), the inverse Fourier transform of this transfer function will have the property that $h(0) = 1$. This result suggests that $\hat{h}(0)$ from the HTE should always equal unity.

We found, however, that when the HTE was used with estimated phase sequences which had large jumps, it yielded estimates of $\hat{h}(0)$ which were quite different from unity. We also found that in these cases, $\{\hat{h}(k)\}$ was not stable in the sense that $\Sigma_{k=0}^{T} |\hat{h}(k)|$ did not converge as $T \to K$. Thus, in our empirical analysis we use $|\hat{h}(0) - 1|$ as a criterion to screen for poor estimates.

3.4. Estimating the Filter's Scale Factor

The coherence function $\gamma(\omega|x,y)$ between a process $\{x(n)\}$ and $\{y(n)\}$ is defined as follows:

$$\gamma(\omega|x,y) = |S(\omega|x,y)|/[S(\omega|x)S(\omega|y)^{1/2}. \tag{3.4}$$

Since $S(\omega|y) = |H(\omega)|^2 S(\omega|x)$ and $S(\omega|x',y') = S(\omega|x,y)$, it follows from (3.4) that the squared coherence between $\{x'(t)\}$ and $\{y'(t)\}$ is

$$\gamma^2(\omega|x',y') = 1/[1 + \kappa(\omega|x)][1 + \kappa(\omega|y)] \tag{3.5}$$

where $\kappa(\omega|x) = S(\omega|\nu)/S(\omega|x)$ and $\kappa(\omega|y) = S(\omega|\epsilon)/S(\omega|y)$ are the noise–to–signal ratio functions for the input and output signals, respectively. Thus if $S(\omega|\nu) = S(\omega|\epsilon) = 0$ for frequencies ω in a band $\omega_a < \omega < \omega_b$, then $\gamma(\omega|x',y') = 1$ in the band.

The estimated squared coherence is obtained from the estimates of the cross spectrum $S(\omega|x',y')$ and the spectra of the observed signals. If the estimates of coherence is greater than 0.9 for at least one frequency ω_k, then the estimate of the gain $|H(\omega_k)|$ is a good estimate of the unknown scale factor which is not estimable from the Hilbert transform method.

Hopefully, there will be a number of frequencies where the estimated coherence will be high. If so, then the estimates of the gains for those frequencies should be averaged to obtain a more reliable estimate of the scale factor.

4. Results With Artificially Created Data

4.1 The Artificial Data

The artificial data used in the evaluation of the HTE and the comparison with instrumental variables estimates (IV) is generated as follows: First, an input series $\{x(t)\}$ is generated according to an AR(1) with a lag coefficient of 0.5 and a white noise error variance of unity. Second, an output series $\{y(t)\}$ is generated according to (1.1) with filter weights $h(0) = 1.0$, $h(1) = -1.5$, $h(2) = 1.0$, $h(3) = -0.5$, $h(4) = 0.25$, and $h(k) = 0$ for $k \geq 5$. Since this filter is stable and causal and has no zeros on $|z| \leq 1$, it is minimum phase. Finally, observational errors are added to $\{x(t)\}$ and $\{y(t)\}$ to obtain $\{x'(t)\}$ and $\{y'(t)\}$ [see (1.2) and (1.3)]. The assumed observational error processes $\{\nu(t)\}$ and $\{\epsilon(t)\}$ are white with variances equal to $\beta\,\mathrm{var}[x(t)]$ and $\beta\,\mathrm{var}[y(t)]$, respectively. The observation length of the series is N = 512,

4.2 The Bias in OLS Estimates

We first determine whether there is an errors in variables bias in the OLS estimates of the filter. The mean OLS estimates for 1000 trials and various β are plotted against the true values of the filter in Figure 4. The bias using OLS is marked even when the noise–to–signal ratio is as small as $\beta = 0.25$. Further, the bias increases with β.

4.3 HTE Estimates

The filter is estimated with the HTE as follows: First, a phase sequence of length N is estimated from $\{x'(t)\}$ and $\{y'(t)\}$ using the method discussed in section A.2 of Appendix A. This method is based on estimated cross spectra obtained by averaging M ($M \leq N/K$) adjacent cross periodogram ordinates. Next, a $\{p(j)\}$ is chosen so that the unwrapped estimated phase sequence satisfies (3.2) with $r = 3\pi/2$, and this sequence is checked to determine if the minimum phase criterion (3.3) is satisfied. If it is, an asymptotically independent estimated phase sequence of length K = 16 to be used in the HTE is obtained by selecting every N/K–th estimated phase from the original sequence. This estimated phase sequence is then used in (2.3) to obtain the HTE of the filter.

Several different experiments are run, each of which originally consists of 1,000 trials. Selected results are presented in Table 1 for those trials for which (3.3) is satisfied and for which $|\hat{h}(0) - 1| \leq 0.05$. A summary of what we find is as follows:

1. The estimated filter weights track the pattern of actual filter weights

quite well. This can be seen both in the actual sample estimates and in the small root mean square errors (RMSEs) of the estimates. Note that this holds even when $\beta = 1$, that is, even when the variance of the noise equals that of the signal. The results also show that the estimates of the sum of the filter weights are biased upward with the bias increasing as β increases.

2. The estimated standard errors of the filter coefficients are close to the sample standard deviations of the estimators when $\beta = 0.25$ or $\beta = 0.5$. When $\beta = 1$, the estimated standard errors are always somewhat larger than the sample standard deviations. Thus, the approximate large sample variance of the HTE given by (A.5) appears to be an upper bound on the actual standard errors of the estimates of the filter weights.

3. As the number of weights used to smooth the periodograms (M) increases, the estimated coefficients are more likely to be biased toward zero. Further, the bias is larger, the larger is β. However, a bias–variance tradeoff is also evident. As M increases, the sample standard deviations of the estimates decrease. Additionally, the RMSEs of the estimates decrease as M increases presumably due to the fact that the estimates of h(5) through h(15) are closer to zero the larger M.

4.4 Comparison of HTE and IV Estimates

Using the same data we compare the HTE estimates with IV estimates. We use $x'(t-j-1)$ as the instrument for $x(t-j)$. Since the errors in $\{x'(t)\}$ are white, this procedure should produce asymptotically unbiased estimates of the filter coefficients.

The comparison of the HTE estimates with the IV estimates is given in Figure 5 for $\beta = 0.25$ and $\beta = 0.5$. (Since the HTE estimates for these value of β given in Table 1 are very close to each other, we have only plotted the HTE estimates for $M = 27$.) This Figure shows that the HTE estimates outperform the IV estimates. This is confirmed by the RMSEs. For the IV estimates, the RMSEs are 2.04 when $\beta = 0.25$ and 2.36 when $\beta = 0.5$. In contrast, for the HTE estimates for these β's are 0.093 and 0.139, respectively. Additionally, the sample standard deviations of the HTE estimates were never larger than one–third those of the IV estimates.

4.5 HTE Estimates with Colored Noise

We also evaluate the HTE when the noise contaminating the $\{x'(t)\}$ process is colored. Specifically, we generated $\{x(t)\}$ and $\{y(t)\}$ as above and added an AR(1) error with $\rho = 0.5$ for $\{\nu(t)\}$. The observational error on $\{y(t)\}$ was white noise.

The results are presented in Table 2 for those trials for which (3.3) is satisfied and for which $|\hat{h}(0) - 1| \leq 0.05$. The results are quite similar to those when all observational errors were white: the HTE estimates track the actual pattern of filter weights quite well and the estimated standard errors are close the the sample standard errors. If fact, comparing the results in Tables 1 and 2, the HTE track the filter weights better for colored noise than for white noise.

4.6 Ability to Detect Nonminimum Phase Filters

This section presents some evidence on the significance level and power of a hypothesis test of whether a phase function is minimum phase based on the minimum phase criterion (3.3). Specifically, we reject the null hypothesis that the actual phase function is minimum phase if the estimated phase sequence does not satisfy (3.3).

To obtain some evidence on the significance level of such a test, we determine the number of times the null hypothesis is rejected in the experiments discussed above. It is rejected 5 times in 1,000 trials when $\beta = 0.25$, 29 times in 1,000 trials when $\beta = 0.5$, and 102 times in 1,000 trials when $\beta = 1.0$.

To obtain some evidence on the power of such a test, we generate a new $\{y'(t)\}$ series using the procedure discussed above but substituting for $\{h(k)\}$ the stable, causal filter $h'(0) = 1.0$, $h'(1) = -2.1$, $h'(2) = 1.6$, $h'(3) = -0.7$, $h'(4) = 0.5$, and $h'(k) = 0$ for $k \geq 5$. Since $\{h'(k)\}$ has two zeros on $|z| \leq 1$, it is not minimum phase. The filter $\{h'(k)\}$ is selected since it has approximately the same gain as $\{h(k)\}$ but a different phase. With these new data, the null hypothesis is correctly rejected 960 times in 1,000 trials when $\beta = 0.25$, 835 times in 1,000 trials when $\beta = 0.5$, and 628 times in 1,000 trials when $\beta = 1.0$.

To obtain further evidence on the power of such a test, we generated another $\{y'(t)\}$ series using a maximum phase filter $\{h''(k)\}$ with a gain approximately the same as that of $\{h(k)\}$. A maximum phase filter is one which has the number of zeros on $|z| \leq 1$ equal to the number of its nonzero coefficients. Specifically, we used the filter $h''(0) = 1.0$, $h''(1) = -2.0$, $h''(2) = 4.0$, $h''(3) = -6.0$, $h''(4) = 4.0$, and $h''(k) = 0$ for $k \geq 5$. With these data the null hypothesis is correctly rejected in all 1,000 trials when $\beta = 0.25$, 997 times in 1000 trials when $\beta = 0.5$, and 967 times in 1,000 trials when $\beta = 1.0$. Thus, we conclude that this test of the hypothesis that the true filter is minimum phase has good power, especially when the true filter is maximum phase or the noise–to–signal ratio is not too large.

We also have done some experimentation to determine how the results are affected by changing M, the number of weights used to smooth the cross–

periodogram or by changing r, the maximum absolute difference allowed between adjacent elements of the unwrapped phase sequence. Not unexpectedly, we find that increasing M or increasing r raises the significance level but reduces the power of the test.

5. Summary

In this paper we have presented a method for estimating linear filter models when both the dependent and independent variables are observed with error. This method is based on the Hilbert transform relationship between the phase and the log gain of a minimum phase filter. We have demonstrated that this estimator is asymptotically unbiased and have presented its approximate large sample variance. Tests with artificially created data showed that our estimator tracked a known filter quite well and that its estimated standard errors were, for the most part, upper bounds on the sample standard deviations.

The limitation of the Hilbert transform estimator is, of course, that it requires the linear filter to be invertible (minimum phase). While this restriction may be expected to be satisfied a priori in many natural science applications, the same is not true in economic applications where theory often does not provide enough restrictions on the linear filter. Nonetheless, because we have also developed a criterion for determining whether an estimated phase sequence is minimum phase, our technique can have applications even if it is not possible to determine a priori that the filter to be estimated is invertible.

Endnotes

[1] For further discussion of this no observational errors case, see Brillinger (1981) or Jenkins and Watts (1968).

[2] See Tretter (1976, section 4.10) or Papoulis (1962, section 10.2) for a derivation and discussion of the Hilbert transform.

[3] There are several related approaches to obtaining consistent, asymptotically gaussian, and mean square convergent estimators of $\phi(\omega)$ for the observed processes. One such approach is discussed in Appendix A. Consistent estimators and their asymptotic properties are also discussed in Brillinger (1981, chap. 5).

[4] In order to investigate the properties of (2.3) as an approximation to the discrete Hilbert transform (1.7), for each of the angular frequencies ω_j we obtained the true gain and phase of the AR(1) series

$$x(t) = 0.5x(t-1) + \eta(t)$$

for $t = 0, \ldots, N-1$. Next, we calculated a log gain using (2.3) and the true phase. Finally, we computed the RMSE between this gain and the true gain. The results showed that (2.3) is an extremely good approximation to (1.7) for reasonable sample sizes.

[5] This discussion ignores the case in which the filter has one or more unit roots. In this case, the phase function will have a discontinuity of $\pi/2$ at each frequency for which a unit root occurs.

[6] In order to determine the effects of not unwrapping the estimated phase sequence before calculating the log gain with the discrete Hilbert transform, we used a wrapped phase sequence for the filter $(1 - .95L)^3$ in (2.3). The resulting log gain

had a cusp at both points where the phase sequence exhibited large jumps. These cusps were eliminated when the unwrapped phase sequence was used instead.

A.2. Cross Spectrum and Phase Estimators

Let the cross spectrum estimator at frequency $0 < \omega_0 \leq \pi$ be the average

$$\hat{S}(\omega_j | x,y) = M^{-1} \sum_{k=-m}^{m} DFT(j+k | \underline{x},N)^* \, DFT(j+k | \underline{y},N)$$

where $m = (M-1)/2$, $M << N$, and an asterisk denotes the complex conjugate. [If $0 < \omega_0 < \pi M/N$, then average the $DFT(j+k | \underline{x},N)^* \, DFT(j+k | \underline{y},N)$ for $0 < \omega_0 < \omega_0 + \pi M/N$. A similar constriction of bandwidth holds when $\pi(1 - M/N) < \omega_0 < \pi$.] From Theorems 7.3.1 and 7.3.2 in Brillinger (1981), the expected value of $\hat{S}(\omega_0 | x,y)$ is $S(\omega_0 | x,y) + O(M/N)$ and its variance is $O(M^{-1})$.

The estimator of the phase at ω_0 is given by

$$\hat{\phi}(\omega_0) = \arctan[\mathrm{Im}\, \hat{S}(\omega_0 | x,y) / \mathrm{Re}\, \hat{S}(\omega_0 | x,y)]. \tag{A.1}$$

Expanding (A.1) in a Taylor series about $S(\omega_0 | x,y)$ yields

$$\hat{\phi}(\omega_0) = \phi(\omega_0) + e(\omega_0)$$

where the expected value of the error in the estimated phase, $e(\omega_0)$, is of order M/N. The large sample variance of the phase estimator is

$$\mathrm{var}[\hat{\phi}(\omega_0)] = (2M)^{-1} [\gamma^{-2}(\omega_0) - 1] + O(M^{-2}) \tag{A.2}$$

where

$$\gamma^2(\omega_0) = |S(\omega_0 | x,y)|^2 \, [S(\omega_0 | x) \, S(\omega_0 | y)]^{-1} \tag{A.3}$$

denotes the squared coherency [see Hinich and Clay (1968)]. Let $M = N^{\alpha}$ with $0 < \alpha < 1$. Then (A.2) and (A.3) demonstrate that $\hat{\phi}(\omega_0)$ is an asymptotically unbiased estimator of $\phi(\omega_0)$ as $N \to \infty$ if $\gamma^2(\omega_0) > 0$.

Assume for convenience that $K = N/M$ is an integer, and let $\omega_k = 2\pi kM/N$ (or $2\pi k/K$). The correlation between $\hat{\phi}(\omega_j)$ and $\hat{\phi}(\omega_k)$, $j \neq k$, is $O(1/MN)$ since the discrete Fourier transform for the frequency grid $\{2\pi n/N\}$ have cross correlations of order $O(1/|n-n'|N)$ for $n \neq n'$ if the two time series have well behaved cumulants [Brillinger (1981, Theorem 4.4.2)]. Thus, we can obtain a set of $(K/2 + 1)$ approximately uncorrelated phase estimates from a sample of size N. Denote this estimated phase sequence as $\{\hat{\phi}(\omega_j)\}$. Since $M = N^{\alpha}$ for $0 < \alpha < 1$, $K = N^{1-\alpha} \to \infty$ as $N \to \infty$.

A.3. Variance of the Log Gain

The estimated phase sequence $\{\hat{\phi}(\omega_j)\}$ can be transformed to obtain a discrete approximation to the log gain with the discrete Hilbert transform. That is,

$$\log|\hat{H}(\omega_k)| = \mathrm{DHT}(k|\hat{\underline{\phi}},K),$$

which is (2.3) rewritten, and

$$\mathrm{vec}\{\log|\hat{H}(\omega_k)| - \log|H(\omega_k)|\} = \mathrm{vec}\{\mathrm{DHT}(k|\underline{e},K)\} = \underline{Be}.$$

Thus,

$$\mathrm{cov}[\mathrm{vec}\{\log|\hat{H}(\omega_k)|\}] = \underline{BVB}'$$

where $\underline{V} = E[\underline{ee}'] = \text{cov}[\underline{\hat{\phi}}]$ is the covariance matrix of the errors in the estimated phase. Specifically,

$$\underline{V} = \| v_{jk} \| = \begin{cases} \text{var}[\hat{\phi}(\omega_{j-1})], \; j = K-k \neq 1, \;\; K/2+1 \\ -\text{var}[\hat{\phi}(\omega_{j-1})], \; j = k \neq 1, \; K/2+1 \\ 0, \; \text{otherwise} \end{cases}$$

A.4. Variance of the Coefficients

The estimate of the complex gain for frequency ω_k using this approach is

$$\hat{H}(\omega_k) = \exp[\log|\hat{H}(\omega_k)| + i\hat{\phi}(\omega_k)]. \tag{A.4}$$

Expanding (A.4) in a Taylor's series about $H(\omega_k)$ yields

$$\hat{H}(\omega_k) - H(\omega_k) = H(\omega_k)[\text{DHT}(k|\underline{e},K) + ie(k)].$$

Therefore,

$$\text{vec}\{\hat{H}(\omega_k) - H(\omega_k)\} = \text{vec}\{H(\omega_k)[\text{DHT}(k|\underline{e},K) + ie(k)]\}$$

$$= \underline{\Gamma}(\underline{B} + i\underline{I})\underline{e}$$

where $\underline{\Gamma} = \text{diag}[H(\omega_k)]$.

Since

$$\hat{h}(n) - h(n) = K^{-1} \sum_{k=0}^{K-1} [\hat{H}(\omega_k) - H(\omega_k)] \exp(i\omega_n k),$$

$$\text{vec}\{\hat{h}(n) - h(n)\} = K^{-1}\underline{F}^{\dagger}\text{vec}\{\hat{H}(\omega_k) - H(\omega_k)\} = K^{-1}\underline{F}^{\dagger}\underline{\Gamma}(\underline{B} + i\underline{I})\underline{e}.$$

The elements of $\text{vec}\{\hat{h}(n) - h(n)\}$ are real. Therefore, the covariance matrix of the filter weights is

$$\underline{W} = \text{cov}[\text{vec}\{\hat{h}(n)\}] = E[\text{vec}\{\hat{h}(n) - h(n)\}\ \text{vec}\{\hat{h}(n) - h(n)\}']$$

$$= K^{-1}\underline{F}^{\dagger}\underline{\Gamma}(\underline{B} + i\underline{I})\underline{V}(\underline{B} + i\underline{I})'\underline{\Gamma}^{\dagger}\underline{F}. \tag{A.5}$$

A.5. A Conjecture

Let P be the length of a linear filter. That is, when the linear filter is written as a polynominal in the lag operator, $P - 1$ is the largest power of L with a nonzero coefficient. Then, we conjecture:

> If $(P + K/2 - 2) \geq K$, then $w_{ii} \neq 0$ for all i. Otherwise, $w_{11} = 0$ and $w_{ii} = 0$ for $i \geq (P + K/2)$.

This conjecture arises from calculations of $\underline{W}$ for various known filters. We have been unable to prove it. Nonetheless, it is consistent with the findings with artificial data both in this paper and in Hinich and Weber (1984). Further, there are intuitive reasons why there might be no more than $(P + K/2 - 2)$ nonzero variances. Since the phase function is odd and constained to be zero at $\omega = 0, \pi$, there are only $K/2 - 1$ discrete phase points which are being estimated, and since h(0) is constrained to be zero by the Hilbert transform (see Appendix B), there are only $P - 1$ filter weights to be estimated.

Appendix B: Proof that h(0) = 1

The following two lemmas are used in the proof of the theorem:

Lemma B.1. $\int_0^{2\pi} \cot[(\lambda - \omega)/2] d\lambda = 0$ for every ω.

Proof. Consider the minimum phase filter $h(0) = 2$ and $h(k) = 0$ for $k > 0$. This filter's transfer function is the constant 2, and thus its phase is zero for all ω. From (1.7),

$$0 = \int_0^{2\pi} \log|2| \cot[\pi(\lambda - \omega)/2] d\lambda$$

and the result then follows.□

Lemma B.2. $\int_0^{2\pi} \log|H(\omega)| d\omega = 0$ for a minimum phase filter.

Proof. Reversing the order of integration when integrating both sides of (1.7),

$$\int_0^{2\pi} \log|H(\omega)| d\omega = 2\pi^{-1} \int_0^{2\pi} \int_0^{2\pi} \cot[(\omega - \lambda)/2]\ \phi(\lambda)\ d\omega\ d\lambda = 0$$

from Lemma B.1.□

Theorem B. When $\log|H(\omega)|$ is obtained from the phase function $\phi(\omega)$ by (1.7),

$H(\omega) = \exp\{\log|H(\omega)| + i\phi(\omega)\}$, and

$$h(k) = (2\pi)^{-1} \int_0^{2\pi} H(\omega) \exp(i\omega k) d\omega,$$

then $h(0) = 1$.

Proof. Since $H_C(z)$ has no zeros or poles in $|z| \leq 1$,

$$\log|H_C(0)| = \int_0^{2\pi} \log|H(\omega)| d\omega$$

by Jensen's theorem [Titchmarsh (1939, p. 125)]. From Lemma B.2, $\log|H_C(0)| = 0$ and thus $\log|h(0)| = 0$.□

References

Boehl, J. E.; F. X. Bostick, Jr.; and H. W. Smith. 1977. An application of the Hilbert transform to the magnetotelluric method. Working paper. Elect. Engr. Rsch. Lab., University of Texas, Austin, Texas.

Brillinger, D. R. 1981. Time series: Data analysis and theory. Expanded ed. San Francisco: Holden–Day.

Cizek, V. 1970. Discrete Hilbert transform. IEEE Transactions on Audio and Electroacoustics AU–18 (December): 340–43.

Clay, C. S., and M. J. Hinich. 1981. Estimating the Earth's impedance function when there is noise in the electric and magnetic signals. In Applied time series analysis II, ed. D. F. Findley, 189–219. New York: Academic Press.

Conway, J. B. 1978. Functions of one complex variable. 2nd ed. New York: Springer–Verlag.

Hinich, M. J. 1983. Estimating the gain of a linear filter from noisy data. In Time series in the frequency domain, ed. D. R. Brillinger and P. R. Krishnaiah, chap. 8. Handbook of Statistics 3. Amsterdam: North–Holland.

Hinich, M.J., and C.S. Clay. 1968. The application of the discrete Fourier transform in the estimation of power spectra, coherence, and bispectra of geophysical data. Reviews of geophysics. 6 (3): 347–363.

Hinich, M. J., and W. E. Weber. 1984. A Hilbert transform method for estimating distributed lag models with randomly missed or distorted observations. In Time series analysis of irregularly observed data, ed. E. Parzen, 134–57. Lecture Notes in Statistics 25. New York: Springer–Verlag.

Hinich, M.J., and G.R. Wilson. 1992. Time delay estimation using the cross bispectrum. IEEE Transactions on signal processing. 40 (1): 106–113.

Jenkins, G. M., and D. G. Watts. 1968. Spectral analysis and its applications. Oakland: Holden–Day.

Lii, K. S., and M. Rosenblatt. 1982. Deconvolution and estimation of transfer function phase and coefficients for nonGaussian linear processes. Annals of statistics 10 (4): 1195–1208.

Papoulas, A. 1962. The Fourier integral and its applications. New York: McGraw–Hill.

Reiersol, O. 1945. Confluence analysis by means of instrumental sets of variables. In Arkiv for matematik, astronomi och fysik, 1–119. Uppsala: Almquist and Wicksells Boktryckeri–AB.

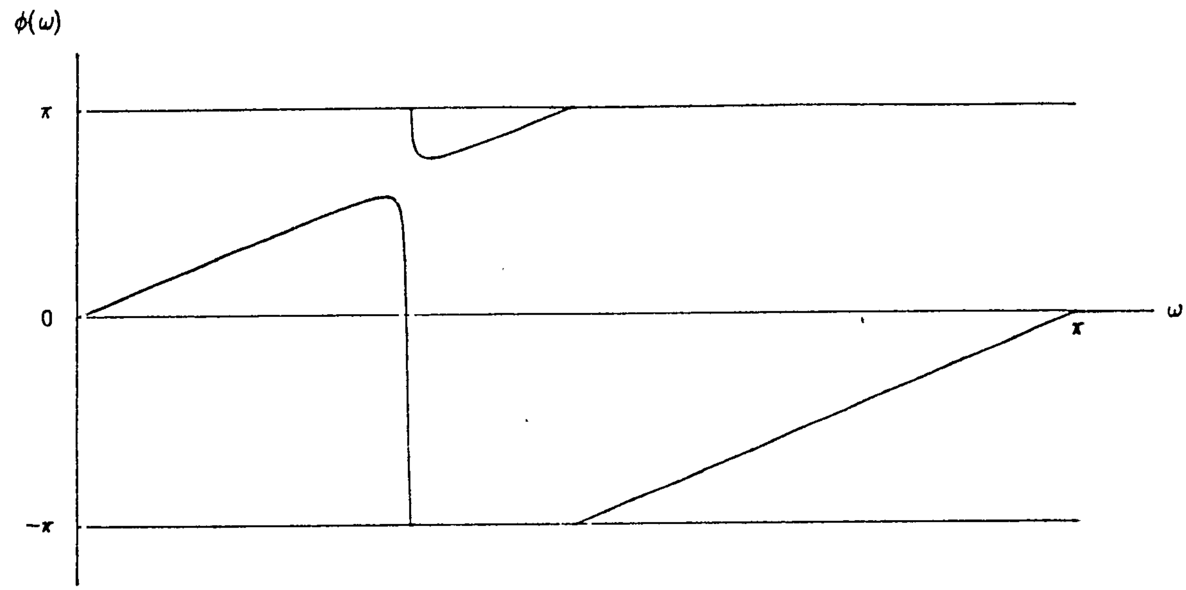
Titchmarsh, E. C. 1939. The theory of functions. 2d ed. Oxford: Oxford University Press. Reprint 1979.

Tretter, S. A. 1976. Introduction to discrete–time signal processing. New York: Wiley.

Figure 1

Phase of the Filter $(1 - L + .99L^2)^2$

(a) Before unwrapping



(b) After unwrapping

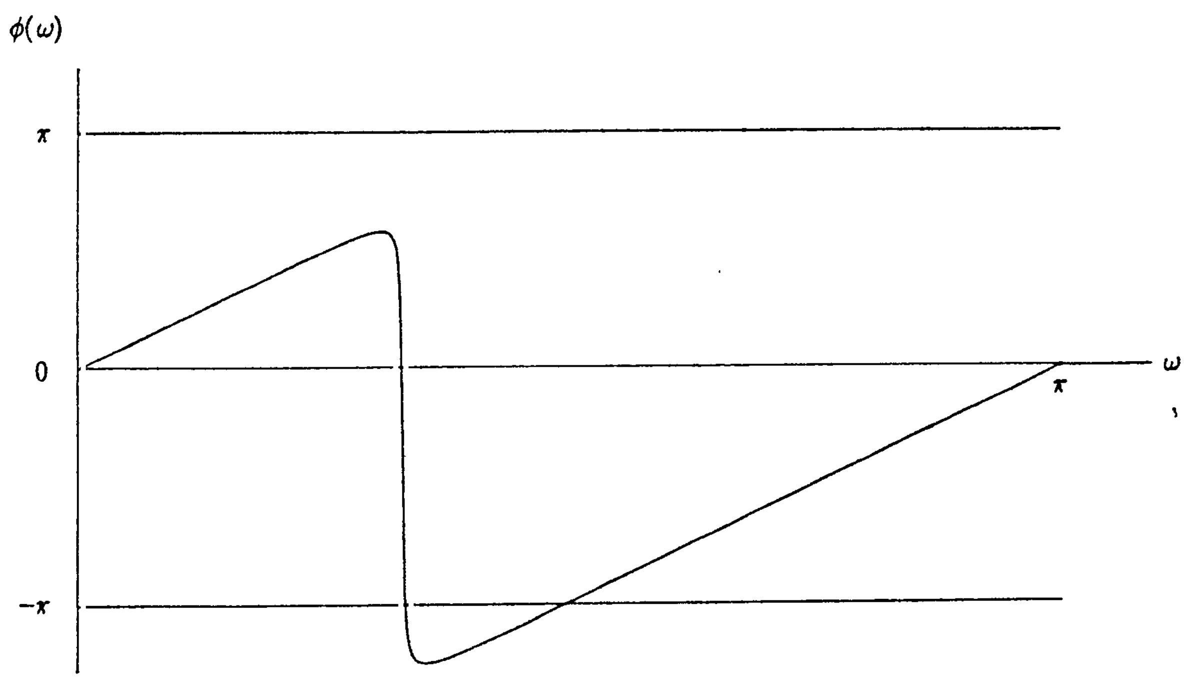
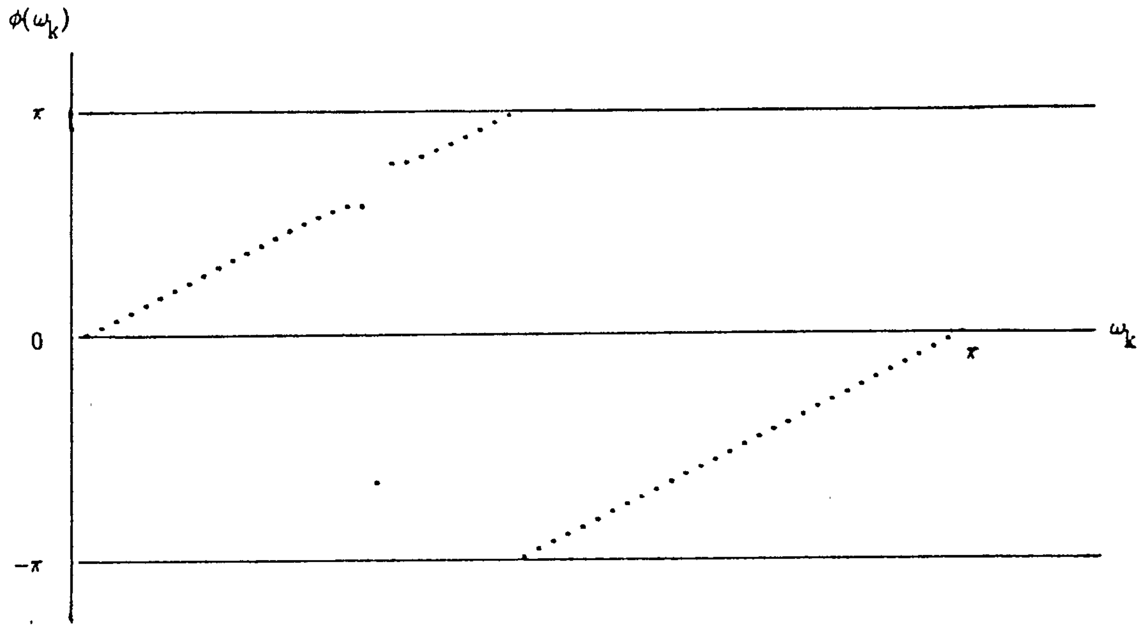


Figure 2

Phase sequence of the filter $(1 - L + .99L^2)^2$

(a) Without unwrapping



(b) Correctly unwrapped

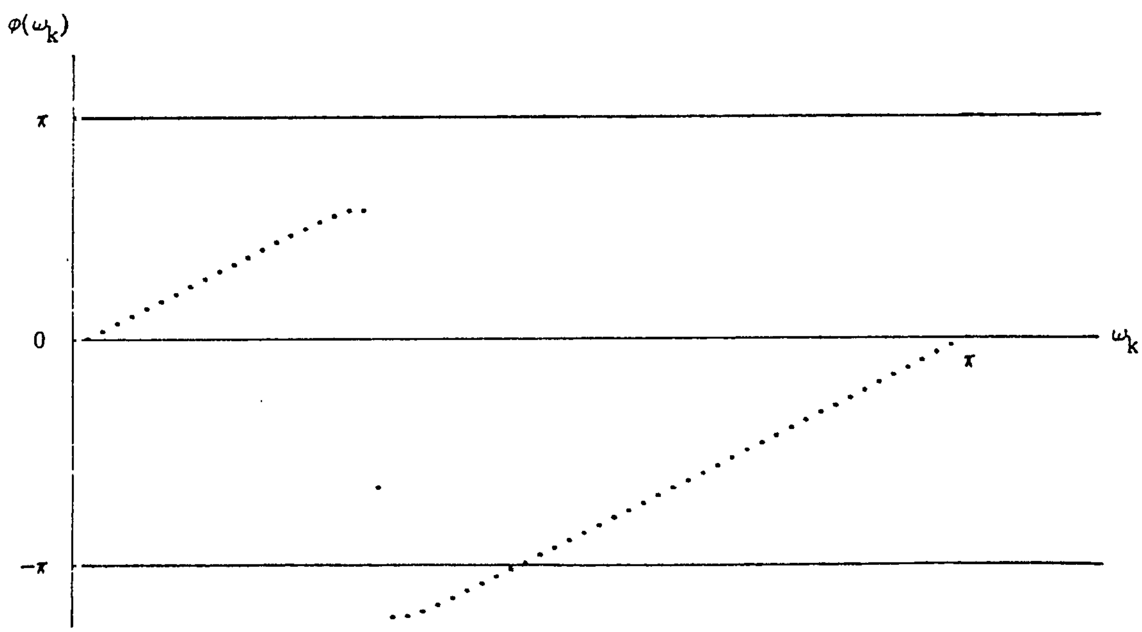


Figure 3
An incorrectly unwrapped phase sequence of the filter $(1 - L + .99L^2)^2$

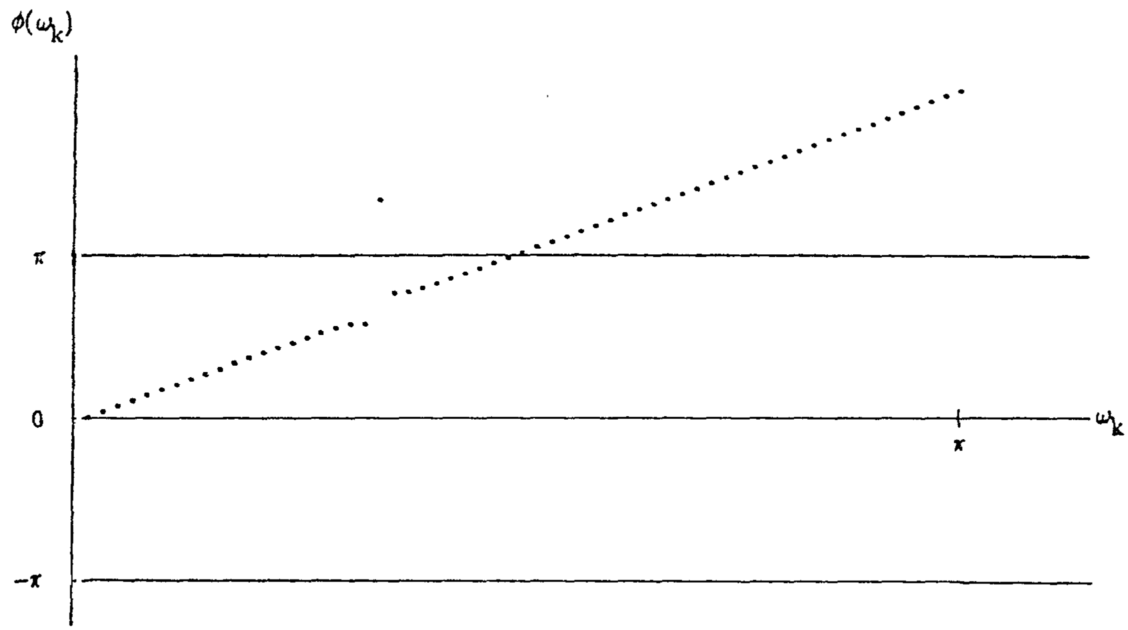


Figure 4: Bias in OLS Estimates for Various Betas

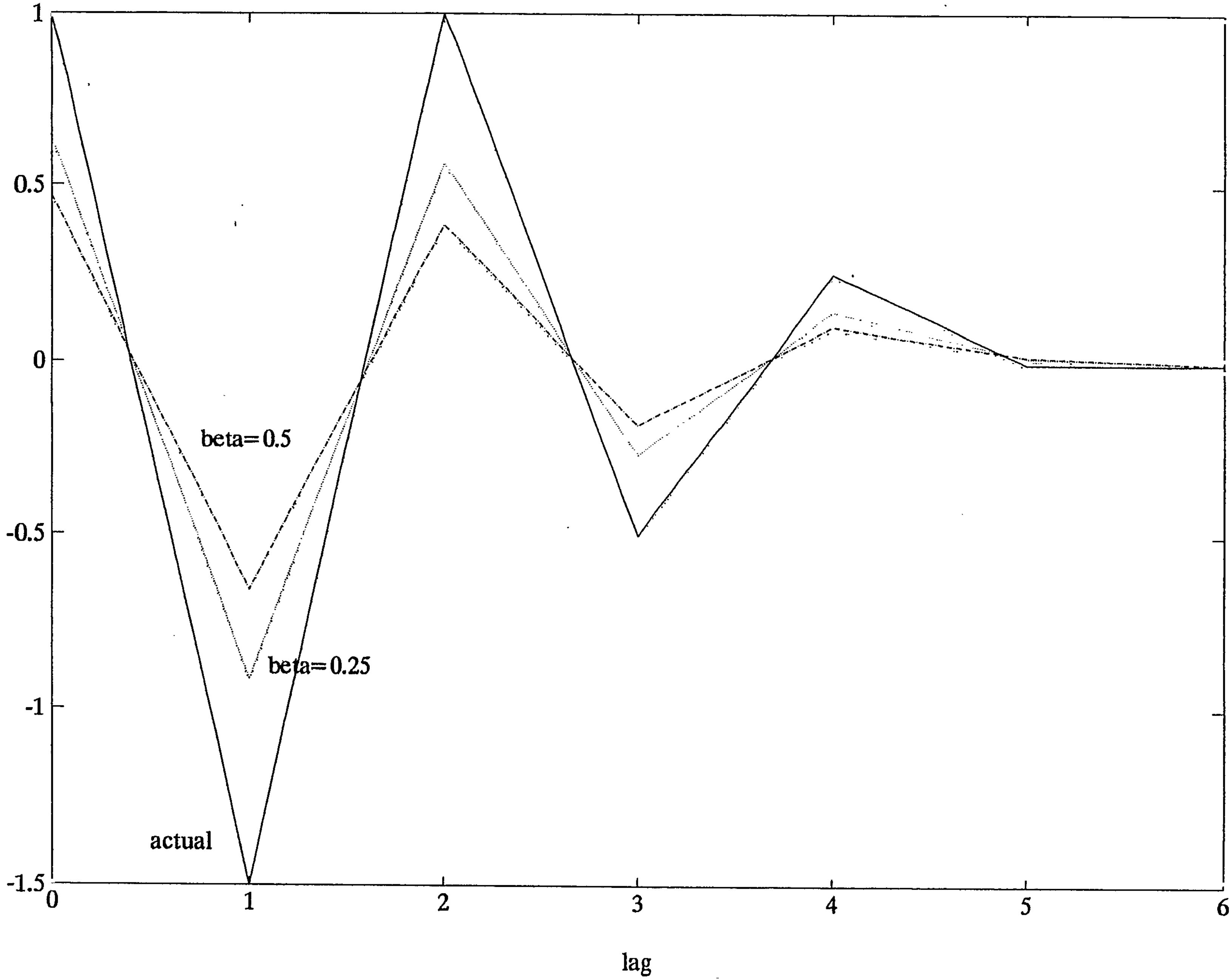


Figure 5: Comparison of HTE and IV Estimates

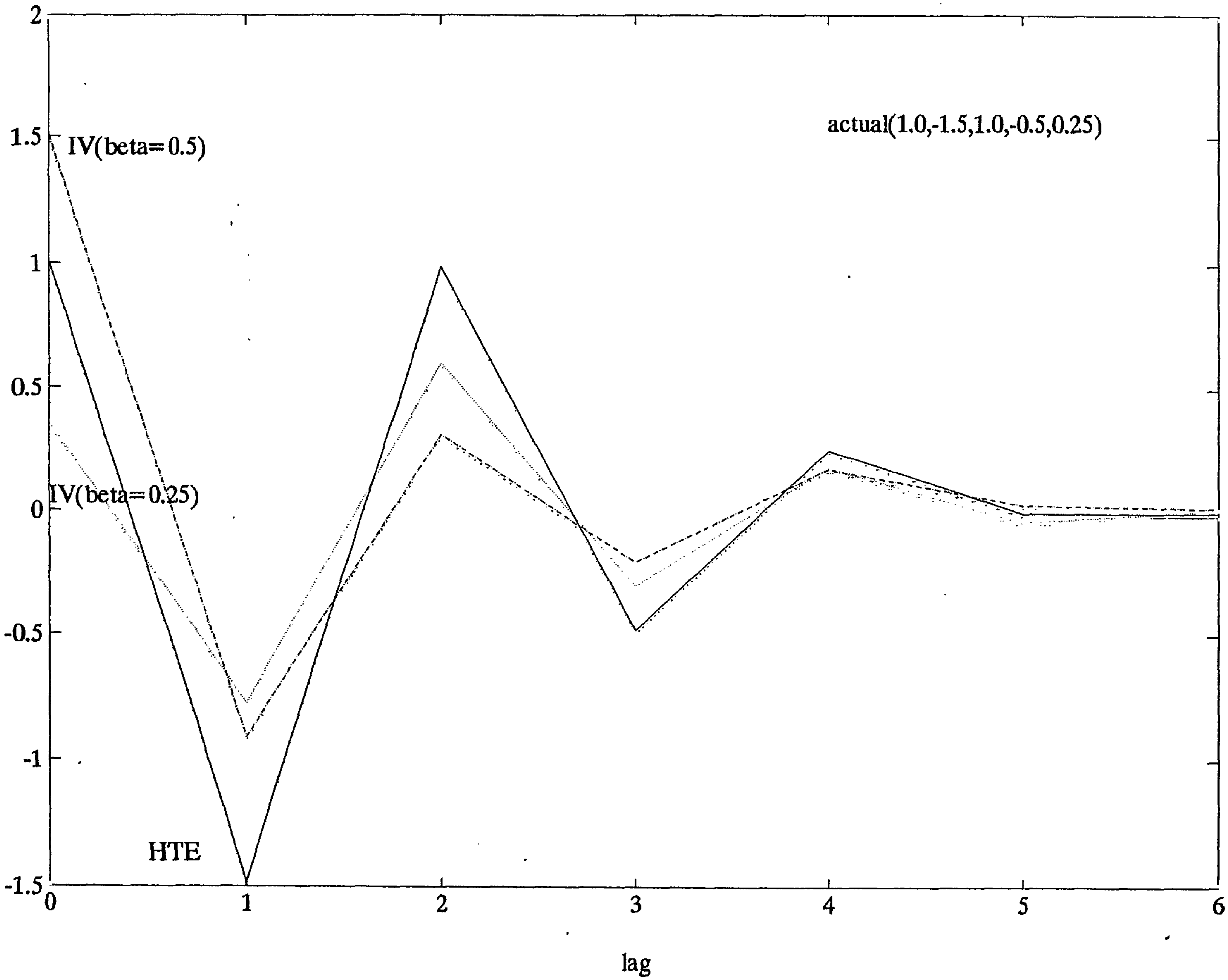


Table 1--HTE Estimates of a Known Filter
White Noise Errors

	Parameter or Statistic	Actual	Mean	Sample Std. Deviation	Mean Est. Std. Error
Coefficients	h(0)	1.0	1.01	0.014	0.015
			1.00	0.012	0.007
			1.01	0.017	0.034
			1.01	0.015	0.018
			1.01	0.022	0.084
			1.01	0.019	0.047
	h(1)	-1.5	-1.49	0.129	0.131
			-1.48	0.095	0.094
			-1.48	0.196	0.204
			-1.48	0.145	0.149
			-1.45	0.297	0.336
			-1.47	0.227	0.249
	h(2)	1.0	1.01	0.182	0.178
			0.990	0.132	0.130
			0.999	0.276	0.283
			0.992	0.200	0.202
			0.958	0.427	0.486
			0.980	0.304	0.345
	h(3)	-0.5	-0.496	0.189	0.194
			-0.480	0.142	0.142
			-0.488	0.276	0.300
			-0.481	0.213	0.216
			-0.483	0.427	0.475
			-0.469	0.320	0.358
	h(4)	0.25	0.259	0.196	0.199
			0.244	0.146	0.145
			0.255	0.287	0.308
			0.249	0.220	0.220
			0.266	0.430	0.485
			0.240	0.326	0.366
	h(5)	0.0	-0.002	0.197	0.201
			-0.003	0.146	0.146
			0.010	0.290	0.314
			-0.001	0.214	0.222
			0.001	0.429	0.515
			0.008	0.326	0.375
	h(6)	0.0	0.004	0.200	0.202
			-0.005	0.148	0.146
			0.004	0.291	0.318
			0.001	0.219	0.223
			0.022	0.447	0.512
			0.001	0.338	0.378
Sum of Coefficients	$\sum_{k=0}^{K-1} h(k)$	0.25	0.308	0.253	
			0.274	0.158	
			0.341	0.343	
			0.306	0.245	
			0.369	0.443	
			0.338	0.338	
RMSE			0.126	0.050	
			0.093	0.037	
			0.183	0.070	
			0.139	0.054	
			0.272	0.118	
			0.207	0.083	
β/M/ Number of Trials			0.25	15	949
			0.25	27	992
			0.50	15	756
			0.50	27	937
			1.0	15	405
			1.0	27	695

Table 2--HTE Estimates of a Known Filter
Colored Errors

	Parameter or Statistic	Actual	Mean	Sample Std. Deviation	Mean Est. Std. Error
Coefficients	h(0)	1.0	1.00 1.01	0.011 0.015	0.006 0.018
	h(1)	-1.5	-1.48 -1.48	0.095 0.148	0.094 0.160
	h(2)	1.0	0.991 0.996	0.118 0.176	0.115 0.181
	h(3)	-0.5	-0.479 -0.480	0.120 0.179	0.120 0.187
	h(4)	0.25	0.243 0.251	0.121 0.183	0.120 0.185
	h(5)	0.0	-0.002 -0.003	0.121 0.180	0.120 0.186
	h(6)	0.0	-0.006 0.002	0.121 0.182	0.120 0.186
Sum of Coefficients	$\sum_{k=0}^{K-1} h(k)$	0.25	0.280 0.318	0.177 0.279	
RMSE			0.081 0.121	0.029 0.045	
β/M/ Number of Trials			0.25 0.50	27 27	990 931