A Rational Expectations Model of Hog Cycles

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We shall specify a model of hog supply in which individual agents slaughter and breed stock based on rational expectations of the output prices and feed costs they will face in the future. Individual agents treat prices as exogenous to their individual decisions in line with the usual assumptions of perfect competition, yet prices are actually determined by the interaction of aggregate supply and demand.

This analysis results in the specification of restrictions on a vector autoregression of prices and output such that they are consistent with individual maximization, perfect competition, and market clearing. The free parameters of this system are then estimated by a maximum likelihood method. Using these parameter estimates, the effects of various policies can be evaluated by re-solving the maximization problems and deriving the forms of the processes under the new policy regimes.

Production in a Deterministic World

We consider the swine industry to be composed of n identical competitive firms each possessing a stock of animals of different ages. The firm breeds, feeds, and slaughters animals to maximize expected profit. We will first solve the problem of a producer facing nonstochastic, hence known, future prices of output and his one input, feed. It will be easy to move from this specification to one in which the prices are stochastic processes with known properties.

At the beginning of period t, a representative firm possesses a certain stock of mature animals (k_t) and a stock of newly-born animals (a_t). The firm faces known sequences of future output prices $\{p_{t+j}\}_{j=0}^{\infty}$ and feed costs $\{c_{t+j}\}_{j=0}^{\infty}$. The firm makes a decision to breed x_t of the k_t mature animals to produce $a_{t+1}=\ell x_t$ young animals at the beginning of the next period. The firm slaughters the animals not bred and feeds this period's newly-born barrows and gilts to maturity. The technology may thus be summarized by the following set of difference equations

$$a_{t+1} = \ell x_t$$

$$k_{t+1} = x_t + a_t .$$

where ℓ is the average litter size. The firm chooses a sequence $\{x_{t+j}\}_{j=0}^{\infty}$ to maximize its present value defined as

$$(1) \qquad v_t = \sum_{j=0}^{\infty} b^j \{p_{t+j}(k_{t+j}-x_{t+j})-c_{t+j}(x_{t+j}+a_{t+j})-\frac{d}{2}(x_{t+j}-x_{t+j-1})^2\}$$

$$0 < b < 1, \; d > 0$$

where b is a discount rate and p_{t+j} and c_{t+j} are the price of an animal slaughtered at the beginning of period t+j and the cost of feeding a living animal during period t+j respectively. We have assumed so far that the slaughter weight of mature animals and the feeding rates of mature and young animals are parameters outside of the control of the firm. We have also assumed that adult and newly-born animals have the same feed requirements--an assumption that is easily changed by multiplying a_{t+j} in equation (1) by a coefficient which expresses the different requirements of mature and young pigs. The third term in the expression enclosed in parentheses in equation (1) reflects the increasing costs of making large adjustments in the scale of operation in a single period. In order to facilitate solution of the firm's problem, it is convenient to rewrite equation (1) by expressing k_{t+j}'s and a_{t+j}'s in terms of x_t's using the substitutions

$$a_{t+j} = \ell x_{t+j-1}$$

$$k_{t+j} = x_{t+j-1} + \ell x_{t+j-2}$$

to yield

(2) $$v_t = \sum_{j=0}^{\infty} b^j\{p_{t+j}(\ell x_{t+j-2}+x_{t+j-1}-x_{t+j}) - c_{t+j}(\ell x_{t+j-1}+x_{t+j}) - \frac{d}{2}(x_{t+j}-x_{t+j-1})^2\}.$$

The maximizing sequence of x_t's must satisfy the following first-order conditions

(3) $$dbx_{t+j+1} - d(1+b)x_{t+j} + dx_{t+j-1} = (-b^2\ell p_{t+j+2}-bp_{t+j+1}+p_{t+j}) + (b\ell c_{t+j+1}+c_{t+j})$$

$$j=0,\ 1,\ \ldots,\ T-1$$

subject to the transversality condition

$$\lim_{T\to\infty} b^T\{-p_{t+T}-c_{t+T}-d(x_{t+T}-x_{t+T-1})\} = 0$$

and the value of x_{t-1}.

To find a solution to equation (3), we first rewrite it as

(4) $$(1-(1+\frac{1}{b})L+\frac{1}{b}L^2)x_{t+j+1} = A(L)p_{t+j+1} + D(L)c_{t+j+1}$$

where

$$A(L) = \frac{1}{bd}(-b^2\ell L^{-1}-b+L)$$

$$D(L) = \frac{1}{bd}(b\ell+L).$$

The polynomial in L on the left-hand side of (4) may be factored to yield

(6) $$(1-L)(1-\frac{1}{b}L)x_{t+j+1} = A(L)p_{t+j+1} + D(L)c_{t+j+1}.$$

If we multiply both sides of (6) by the "forward inverse" of $(1-\frac{1}{b}L)$ defined as

$$\frac{1}{(1-\frac{1}{b}L)} = \frac{-bL^{-1}}{1-bL^{-1}} \quad ,$$

we obtain

$$(1-L)x_{t+j+1} = -\frac{bL^{-1}}{1-bL^{-1}} \quad (A(L)p_{t+j+1}+D(L)c_{t+j+1})$$

or

(7) $$x_{t+j+1} = x_t - \frac{1}{1-bL^{-1}} \quad (bL^{-1}A(L)p_{t+j+1}+bL^{-1}D(L)c_{t+j+1}).$$

By using the definitions of A(L) and D(L) and passing back to explicit notation for leads and lags (7) can be rewritten as

(8) $$x_{t+j+1} = x_{t+j} - \frac{1}{d}\sum_{i=0}^{\infty} b^i\{-b^2\ell p_{t+j+i+3}-bp_{t+j+i+2}+p_{t+j+i+1}$$

$$+b\ell c_{t+j+i+2}+c_{t+j+i+1}\}$$

which holds for all j=0, 1, ... (It can be shown that (8) holds for j=-1 as well.) Equation (8) is the breeding decision rule of an optimizing producer facing known future sequence $\{p_t\}$ and $\{c_t\}$.

This decision rule accords well with our intuition about the producers problem. All feed costs enter equation (8) with a negative sign. Hence, an increase in any current or future feed cost has the effect of decreasing breeding (and increasing current slaughter) because of the reduced profit which can be realized in future periods by enlarging the herd today. The current price of slaughtered animals, p_{t+j+1}, enters the breeding schedule with a negative sign indicating that a higher price for today's slaughter will, with future prices unchanged, call forth more current slaughter.

This decision rule is even easier to understand if we simplify (8) still further to yield

$$(9) \qquad x_{t+1} = x_t - \frac{1}{d}p_{t+1} + \frac{1}{d}\sum_{i=0}^{\infty} b^i\{b^2\ell p_{t+i+3} - b\ell c_{t+i+2} - c_{t+i+1}\}.$$

This can be transformed to yield

$$(10) \qquad d(x_{t+1}-x_t) = -p_{t+1} + \sum_{i=0}^{\infty} b^i\{b^2\ell p_{t+i+3} - b\ell c_{t+i+2} - c_{t+i+1}\}.$$

The left side of equation (10) is the marginal adjustment cost of breeding an additional unit. The right side of (10) is the net change in the present value of the producer's profit stream resulting from foregoing current slaughter to breed an additional animal. The first term on the right side of (10) is the income foregone by not slaughtering; the summation on the right side of (10) represents the present value of the profits made from each future litter of the animal being bred.

Production Under Uncertainty

We shall now consider the problem of a producer operating under the same technology as before but faced with uncertain prices for this output and his input, feed. The individual producer considers the $\{p_t\}$ and $\{c_t\}$ processes as exogenous to his decision. That is, the producer assumes that his decision will have no impact on the future values of p_t and c_t. Because of our linear-quadratic setting of the problem, we can express the producers optimal breeding decision as a function of his expected values of p and c in future periods.

The producer seeks to maximize

$$(11)\qquad w_t = \underset{t}{E}\sum_{j=0}^{\infty} b^j\{p_{t+j}(k_{t+j}-x_{t+j})-c_{t+j}(x_{t+j}+a_{t+j})-\frac{d}{2}(x_{t+j}-x_{t+j-1})^2\}$$

where $\underset{t}{E}(z)=Ez|I_t$ is the mathematical expectation based on unspecified information set I_t. We assume that $\{p_t\}_{t=0}^{\infty}$ and $\{c_t\}_{t=0}^{\infty}$ are stochastic processes of exponential order less than (1/b), i.e., there exist constants M>0 and N>0 such that

$$|\underset{t}{E}p_{t+j}| < M(\frac{1}{b})^k$$

$$|\underset{t}{E}c_{t+j}| < N(\frac{1}{b})^k$$

for all j>0 and for all t.

The first-order conditions for the maximization are

$$(12)\qquad db\underset{t+j}{E}x_{t+j+1} - d(1+b)x_{t+j} + dx_{t+j-1} =$$

$$- b^2\ell\underset{t+j}{E}p_{t+j+2} - b\underset{t+j}{E}p_{t+j+1} + p_{t+j} + b\ell\underset{t+j}{E}c_{t+j+1} + c_{t+j}$$

$$j=0,\ 1,\ \ldots.$$

The two boundary conditions are provided by the value of x_{t-1} and the transversality condition

$$(13)\qquad \lim_{T\to\infty} b^T\underset{t}{E}\{-p_{t+T}-c_{t+T}-d(x_{t+T}-x_{t+T-1})\} = 0.$$

We may solve (12) by defining a new operator B by the condition that

$$B^{-1}\underset{t}{E}z_{t+k} = \underset{t}{E}z_{t+k+1}.$$

Equation (12) may be rewritten as

$$(14) \qquad (1+(1+\frac{1}{b})B+(\frac{1}{b})B^2) \underset{t+j}{E}\, x_{t+j+1} = A(B) \underset{t+j}{E}\, p_{t+j+1} + D(B) \underset{t+j}{E}\, c_{t+j+1}$$

where

$$A(B) = \frac{1}{bd}(-b^2\ell B^{-1}-b+B)$$

and

$$D(B) = \frac{1}{bd}(b\ell+L).$$

The polynomial on the left side of (14) factors just as before to yield

$$(15) \qquad (1-B)(1-\frac{1}{b}B) \underset{t+j}{E}\, x_{t+j+1} = A(B) \underset{t+j}{E}\, p_{t+j+1} + D(B) \underset{t+j}{E}\, c_{t+j+1}.$$

Operating on (15) with the forward inverse of $(1-\frac{1}{b}B)$ yields relation

$$(16) \qquad (1-B) \underset{t+j}{E}\, x_{t+j+1} = \frac{-bB^{-1}}{1-bB^{-1}} (A(B) \underset{t+j}{E}\, p_{t+j+1}+D(B) \underset{t+j}{E}\, c_{t+j+1})$$

or

$$\underset{t+j}{E}\, x_{t+j+1} = x_{t+j} - \frac{b}{1-bB^{-1}} (B^{-1}A(B) \underset{t+j}{E}\, p_{t+j+1}+B^{-1}D(B) \underset{t+j}{E}\, c_{t+j+1}).$$

More explicitly

$$(17) \qquad \underset{t+j}{E}\, x_{t+j+1} = x_{t+j} - \frac{1}{d} \underset{t+j}{E}\, p_{t+j+1} + \frac{1}{d}\sum_{i=0}^{\infty} b^i \underset{t+j}{E} (b^2\ell p_{t+j+i+3}-b\ell c_{t+j+i+2}-c_{t+j+i+1}).$$

The solution for x_{t+j+1} is derived by expanding the information set in (17) to include all information actually available at the beginning of period t+j+1 when x_{t+j+1} is chosen.

$$x_{t+j+1} = x_{t+j} - \frac{1}{d} p_{t+j+1} + \frac{1}{d} \sum_{i=0}^{\infty} b^{i} \underset{t+j+1}{E} (b^{2} \ell p_{t+j+i+3} - b\ell c_{t+j+i+2} - c_{t+j+i+1}). \tag{18}$$

Equation (18) is the stochastic version of the deterministic breeding rule (10) in the previous problem. The producer balances expected future profits from breeding against the known price of currently slaughter.

The Rational Expectations Equilibrium

To this point, we have not derived any testable restrictions on data. Equation (18) is the relationship between the breeding level of an optimizing agent and his expectations of future prices and costs. The linear-quadratic technology we have assumed allows us to express the agents decision as a linear function of the means of his subjective forecasts. Although E was defined as the mathematical expectation conditioned on some data set, everything in the derivation of (18) would be valid if E were any (linear) procedure used by the agent to forecast the $\{p_t\}$ and $\{c_t\}$ processes. In order to construct a model which restricts data, we must specify the agents method of forecasting and several other aspects of the markets in which he deals.

Several attributes of these markets must be specified and combined with our model of individual production to yield testable implications for time series data. First, we will assume that agents behave as if they make optimal or rational forecasts of future prices based on all of the relevant price and production data available. Second, we shall derive the aggregate supply behavior of a large number of identical small producers. Third, we will specify the demand curve for industry output and the supply curve for the input feed. After adding all of these elements, we can solve the model to generate restrictions on the stochastic processes for feed costs, output prices, and production.

a. The Rest of the Model

The demand curve for the output of the industry is assumed to have the form

(19) $$p_t = A_0 - A_1 q_t + u_t \text{ for all } t$$

where q_t is the quantity of slaughtered pork, u_t is a (possibly serially correlated) random shock and A_0 and A_1 are positive scalars. This is a downward-sloping linear demand schedule which is subject to parallel shifts caused by a random variable u_t perhaps most realistically visualized as a business cycle or income shock.

The supply curve for feed faced by the industry is assumed to be upward sloping of the form

(20) $$c_t = C_0 + C_1 f_t + e_t \text{ for all } t$$

where f_t is the quantity of feed supplied, e_t is a serially correlated random shock, and C_0 and C_1 are both positive. It is probably most natural to think of e_t as either the effect of weather on feed harvests or perhaps the price effect of demand by domestic and foreign purchasers of feed grains and close substitutes.

The technology of the preceding sections will remain as before except that we will assume that production of the pig crop is stochastic and is described by the relationship

(21) $$a_t = \ell x_{t-1} + n_t$$

where a_t is the number of pigs born at the beginning of period t, x_{t-1} is the number of sows bred at time t-1, ℓ is the average litter size and, n_t is a (possibly serially correlated) random shock. Perhaps a more natural way to model stochastic production would be to assume that the production disturbance is multiplicative, i.e.,

$$(22) \qquad a_t = (\ell+n_t)x_t.$$

In the current task, we will prefer to use (21) because it, while less intuitive, it preserves the linearity which helps to simplify the solution of the current model.

We assume that the industry is composed of N individuals who are identical in endowments, technical skills, and information. The output of the industry can be expressed as

$$(23) \qquad q_t = Nh(k_t-x_t)$$

where h is the average slaughter weight expressed in the units of q_t. If we express k_t in terms of past values of the breeding variable x_t and substitute equation (23) into equation (19) to eliminate q_t, we obtain the demand curve for industry output in terms of the individual decision variable x_t.

$$(24) \qquad p_t = A_0 - A_1'(-x_t+x_{t-1}+\ell x_{t-2}+n_{t-1}) + u_t$$

where $A_1' = A_1Nh$. From now on A_1 should be interpreted as A_1' from equation (24) instead of A_1 from equation (19).

The amount of feed used by the N identical producers may be expressed as

$$(25) \qquad f_t = Nh'(x_t+a_t)$$

where h' is the average amount of feed consumed per period by each animal. Substituting for a_t in equation (25) and then for f_t in equation (20) yields the supply curve for feed in terms of x_t.

$$(26) \qquad c_t = C_0 + C_1'(x_t + \ell x_{t-1} + n_t) + e_t.$$

As with A_1' in equation (24), we will drop the prime (') in subsequent uses of C_1.

b. Rational Expectations

We shall maintain that agents treat the $\{p_t\}$ and $\{c_t\}$ processes as exogenous to their individual problem. We shall further maintain that agents act as if responding to the optimal linear forecasts of future prices based on all information on past prices and the current state. Actually the prices of output and feed will be determined by the intersection of the supply and demand curves for industry, output, and feed.

The rational expectations equilibrium of this system is a triple of stochastic processes $\{x_t^*\}$, $\{p_t^*\}$, and $\{c_t^*\}$ such that $\{x_t^*\}$ maximizes expected profit given $\{p_t^*\}$ and $\{c_t^*\}$ and also that $\{p_t^*\}$ and $\{c_t^*\}$ clear the pork and feed markets if production process $\{x_t^*\}$ is followed.

We can derive a representation of this trivariate stochastic process in terms observable values of its own past and past value of the three exogenous shock process $\{u_t\}$, $\{e_t\}$, and $\{n_t\}$ by completing four steps. First, we substitute equations (24) and (26) into equation (14) to obtain a representation of x_t in terms of its own past and the past, present, and expected future values of the u_t, e_t, and n_t. Second, we specify a stochastic structure for u_t, e_t, and n_t and develop closed form expressions for expected future values of them in terms of current and lagged values. Third, we substitute the forecasting relations from step two into the equation derived in step one in order to express x_t only in terms of current and lagged shocks. Finally, we cast the process in the form

$$(27)\qquad \begin{bmatrix} x_t \\ p_t \\ c_t \end{bmatrix} = V(L) \begin{bmatrix} x_{t-1} \\ p_{t-1} \\ c_{t-1} \end{bmatrix} + \begin{bmatrix} w_{1t} \\ w_{2t} \\ w_{3t} \end{bmatrix} + W(L) \begin{bmatrix} w_{1t-1} \\ w_{2t-1} \\ w_{3t-1} \end{bmatrix}$$

where V(L) and W(L) are matrices whose elements are polynomials in nonnegative powers of the lag operator L and the w_{it} are stochastic processes of shocks. In this form, the parameters of the model can be estimated by the maximum likelihood method of Wilson [ ].

1. Express x_t in terms of shocks

We will manipulate the three equations which are reproduced here for convenience.

$$(14)\qquad (db-d(1+b)B+dB^2) \underset{t+j}{E}\, x_{t+j+1} =$$

$$(-b^2\ell B^{-1}-b+B) \underset{t+j}{E}\, p_{t+j+1} + (b\ell+B) \underset{t+j}{E}\, c_{t+j+1}$$

$$(24)\qquad p_{t+j} = A_0 - A_1(-1+B+\ell B^2)x_{t+j} - A_1 n_{t+j-1} + u_{t+j}$$

$$(26)\qquad c_{t+j} = C_0 + C_1(1+\ell B)x_{t+j} + C_1 n_{t+j} + e_{t+j}.$$

Equation (14) is the Euler equation of the firm's maximization problem; equation (24) is the demand curve for output; and equation (26) is the supply curve for feed.

Substitution of (24) and (26) into (14) yields

$$(g_0+g_1B+g_2B^2+g_3B^3+g_4B^4) \underset{t+j}{E}\, x_{t+j+2} = \tag{28}$$

$$(-b^2\ell-bB+B^2) \underset{t+j}{E}\, u_{t+j+2} + (b\alpha\ell B+B^2) \underset{t+j}{E}\, e_{t+j+2}$$

$$+ (f_1B+f_2B^2+f_3B^3) \underset{t+j}{E}\, n_{t+j+2} + D_0.$$

where

$$\begin{aligned} g_0 &= A_1b_2\ell \\ g_1 &= A_1(b^2\ell-b) - C_1b\alpha\ell + db \\ g_2 &= -A_1(-b^2\ell^2-b-1) - C_1(b\alpha^2\ell^2+1) - d(1+b) \\ g_3 &= -A_1(b\ell-1) - C_1\alpha\ell + d \\ g_4 &= A_1\ell \end{aligned} \tag{29}$$

and

$$\begin{aligned} f_1 &= b^2\ell A_1 + b\alpha\ell C_1 \\ f_2 &= bA_1 + C_1 \\ f_3 &= -A_1 \end{aligned} \tag{30}$$

and

$$D_0 = A_0(-b^2\ell-b+1) + C_0(b\alpha\ell+1). \tag{31}$$

In order to solve equation (28), we wish to factor the fourth-order polynomial in B on the left side of the equation. To insure that the transversality condition (13) holds, we will solve the unstable roots of that polynomial into the future to express x_t in terms of some future values of the right-hand side variables.

The roots of an arbitrary fourth-order polynomial may be all stable, all unstable, or any combination of the two types of roots. However, the pattern of the g_i's gives us a good clue to the size and location of the roots of G(B). The coefficients are "almost" symmetrical, i.e.,

$$g_0 = b^2 g_4 \text{ and } g_1 = bg_3. \tag{32}$$

In such a situation, the roots of G(B)=0 occur in two pairs such that the product of each pair of roots is b, i.e., G(B) may be expressed as

$$G(B) = (1-\lambda_1 B)(1- \frac{b}{\lambda_1}B)(1-\lambda_2 B)(1- \frac{b}{\lambda_2}B). \tag{33}$$

Note: This is easily validated. For concreteness, we'll consider the case where G(B) is fourth-order and restriction (32) on the coefficients hold. Let us take any root $\lambda \neq 0$, i.e.,

$$g_0 + g_1\lambda + g_2\lambda^2 + g_3\lambda^3 + g_4\lambda^4 = 0. \tag{34}$$

We wish to show that

$$g_0 + g_1(\frac{b}{\lambda}) + g_2(\frac{b}{\lambda})^2 + g_3(\frac{b}{\lambda})^3 + g_4(\frac{b}{\lambda})^4 = 0. \tag{35}$$

Multiplying (35) by $(\frac{\lambda^4}{b^2})$ with $b \neq 0$, yields

$$\frac{g_0}{b^2}\lambda^4 + \frac{g_1}{b}\lambda^3 + g_2\lambda^2 + g_3 b\lambda + g_4 b^2 = 0. \tag{36}$$

But $g_0/b^2 = g_4$, etc., so substituting from (32), (36) becomes

$$g_4\lambda^4 + g_3\lambda^3 + g_2\lambda^2 + g_1\lambda + g_0 = 0. \tag{37}$$

If the discount factor b is equal to one, then the roots of G(B)=0 would either all be on the unit circle or would occur in reciprocal pairs, one member of each pair inside the unit circle and one member outside. For b<1, the roots may be paired (one inside, one outside) for a considerable range of values. We shall continue to solve the model under the assumption that the two smaller roots of G(B)=0 are both less than b, i.e.,

$$\lambda_1 < b < 1 \text{ and } \lambda_2 < b < 1.$$

This assumption assures that $\frac{b}{\lambda_1}$, and $\frac{b}{\lambda_2}$ are both greater than 1. If this assumption doesn't hold, the derivation of the ensuing expressions would be different. After estimation, we can inspect the estimates of λ_1, λ_2, and b to see if the following expansion is the appropriate one.

Under this assumption about λ_1 and λ_2 we solve the unstable roots into the future as follows. Since G(B) can be factored as seen above, we may write (28) as

$$(38) \qquad (1-\lambda_1 B)(1-\frac{b}{\lambda_1}B)(1-\lambda_2 B)(1-\frac{b}{\lambda_2}B) \underset{t+j}{E} x_{t+j+2} = z_{t+j}$$

where we define z_{t+j} as the entire expression on the right side of (28). If we operate on both sides of (38) by the forward inverses of $(1-b/\lambda_1)$ and $(1-b/\lambda_2)$, we get

$$(39) \qquad (1-\lambda_1 B)(1-\lambda_2 B) \underset{t+j}{E} x_{t+j+2} = \frac{-\frac{\lambda_1}{b_1}B^{-1}}{1-\frac{\lambda_1}{b}B^{-1}} \; \frac{-\frac{\lambda_2}{b}B^{-1}}{1-\frac{\lambda_2}{b}B^{-1}} \; z_{t+j}$$

or

$$(40) \qquad (1-\lambda_1 B)(1-\lambda_2 B) \underset{t+j}{E} x_{t+j+2} = \frac{\lambda_1 \lambda_2}{b^2}B^{-2} \frac{1}{(1-\frac{\lambda_1}{b_1}B^{-1})(1-\frac{\lambda_2}{b}B^{-1})} z_{t+j}.$$

Using the fact that for any $\theta_1 \neq \theta_2$

$$(41) \qquad \frac{1}{(1-\theta_1 B^{-1})(1-\theta_2 B^{-1})} = \frac{1}{\theta_1-\theta_2} \left[\frac{\theta_1}{1-\theta_1 B^{-1}} - \frac{\theta_2}{1-\theta_2 B^{-1}} \right]$$

we may simplify (41) to yield

$$(42) \qquad (1-\lambda_1 B)(1-\lambda_2 B) \underset{t+j}{E} x_{t+j+2} = \frac{\lambda_1\lambda_2}{b(\lambda_1-\lambda_2)} B^{-2}$$

$$\frac{\frac{b}{\lambda_1}}{1-\frac{b}{\lambda_1}B^{-1}} - \frac{\frac{b}{\lambda_2}}{1-\frac{b}{\lambda_2}B^{-1}} \quad z_{t+j}.$$

This may be rewritten as

$$(43) \qquad \underset{t+j}{E} x_{t+j+2} = (\lambda_1+\lambda_2) \underset{t+j}{E} x_{t+j+1} - \lambda_1\lambda_2 x_{t+j} +$$

$$\sum_{i=0}^{\infty} h_i \underset{t+j}{E} z_{t+j+i}$$

where

$$(44) \qquad h_i = \frac{\lambda_1\lambda_2}{b(\lambda_1-\lambda_2)} \quad (\frac{\lambda_1}{b})^{i+1} - (\frac{\lambda_2}{b})^{i+1} \qquad .$$

By substituting for z_{t+j} we obtain

$$(45) \qquad \underset{t}{E}x_{t+2} = (\lambda_1+\lambda_2)\underset{t}{E}x_{t+1} - \lambda_1\lambda_2 x_t$$

$$+ \sum_{i=0}^{\infty} h_i \{-b^2 \ell \underset{t}{E}u_{t+2+i} - b\underset{t}{E}u_{t+1+i} + \underset{t}{E}u_{t+i}\}$$

$$+ \sum_{i=0}^{\infty} h_i \{b\alpha\ell \underset{t}{E}e_{t+3+i} + \underset{t}{E}e_{t+2+i}\}$$

$$+ \sum h_i \{v_0 \underset{t}{E}n_{t+1+i} + v_1 \underset{t}{E}n_{t+i} + v_2 \underset{t}{E}n_{t-1+i}\} + D_1$$

where

$$v_0 = b\ell(bA_1+\alpha C_1)$$

$$(46) \qquad v_1 = bA_1 + C_1$$

$$v_2 = -A_1$$

and

$$D_1 = \frac{\lambda_1 \lambda_2 D_0}{b(\lambda_1 - \lambda_2)(1 - \frac{\lambda_1}{b})(1 - \frac{\lambda_2}{b})} .$$

By passing to the information set as of time t+2 and then relabelling the time axis we express x_t as

$$\begin{aligned}(47) \qquad x_t = {} & (\lambda_1+\lambda_2)x_{t-1} - \lambda_1\lambda_2 x_{t-2} \\ & + \sum_{i=0}^{\infty} h_i \{-b^2 \ell E_t u_{t+2+i} - bE_t u_{t+1+i} + E_t u_{t+i}\} \\ & + \sum_{i=0}^{\infty} h_i \{b\alpha\ell E_t e_{t+1+i} + E_t e_{t+i}\} \\ & + \sum_{i=0}^{\infty} h_i \{v_0 E_t n_{t+1+i} + v_1 E_t n_{t+i} + v_2 E_t n_{t+i-1}\} + D_1.\end{aligned}$$

Equation (47) expresses x_t in terms of its own past and the past, present, and expected future values of the disturbance process $\{u_t\}$, $\{e_t\}$, and $\{n_t\}$.

2. Develop closed-form forecasting equation

We must now specify the stochastic structure of the shocks $\{u_t\}$, $\{e_t\}$, and $\{n_t\}$ and develop explicit forecasting equations for their future values. We assume that each can be expressed in mth order autoregressive form with independent, while disturbances, i.e.,

$$\text{(a)} \quad u_t = \sum_{i=1}^{m} p_i u_{t-i} + \varepsilon_{1t}$$

$$\text{(48) (b)} \quad e_t = \sum_{i=1}^{m} \psi_i e_{t-i} + \varepsilon_{2t}$$

$$\text{(c)} \quad n_t = \sum_{i=1}^{m} \theta_i n_{t-i} + \varepsilon_{3t}$$

where ε_{it}'s are independent and

$$\underset{t}{E}\varepsilon_{it+k} = 0 \text{ for } i=1, 2, \text{ or } 3 \text{ and for all } k\geq 1.$$

For ease of exposition, we shall maintain that the order of each process is m. The same calculations are possible if the orders differ. The process (a) may be expressed as a first-order system in the form

$$(49)\qquad \begin{bmatrix} u_t \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_{t-n+1} \end{bmatrix} = \begin{bmatrix} p_1\ p_2\ p_3\ p_{n-1} & p_n \\ & 0 \\ I_{n-1} & \cdot \\ & \cdot \\ & \cdot \\ & 0 \end{bmatrix} \begin{bmatrix} u_{t-1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_{t-n} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

where I_{n-1} is an (n-1) identity matrix. We may rewrite (49) as

$$(50)\ (a)\quad z_{1t} = A_1 z_{1t-1} + \varepsilon_{1t}$$

where

$$z_1 = \begin{bmatrix} u_t \\ \cdot \\ \cdot \\ \cdot \\ u_{t-n+1} \end{bmatrix} \quad \text{and } A_1 = \begin{bmatrix} p' & \\ & \\ I_{n-1} & 0 \end{bmatrix}$$

where p' is the row vector $(p_1 \ldots p_m)$. Similarly, we may express the e_t, and n_t processes as

$$(50)\ (b)\quad z_{2t} = A_2 z_{2t-1} + \varepsilon_{2t}$$

$$(c)\quad z_{3t} = A_3 z_{3t-1} + \varepsilon_{3t}$$

where z_{2t} and z_{3t} are defined in terms of e_t and n_t and A_2 and A_3 are defined in terms of ψ_i's and θ_i's, respectively.

For any of the z_{it}'s it can be shown by recursive substitutions that

(51) $$z_{it+k} = A_i^k z_{it} + \sum_{i=0}^{k-1} A_i^j \varepsilon_{it+k-j}.$$

By applying the $\underset{t}{E}$ operator to both sides of equation (51) and noting that $\underset{t}{E}\varepsilon_{it+k}=0$ for $k\geq 1$ we know that

(52) $$\underset{t}{E} z_{it+k} = A_i^k z_{it}.$$

The matrix A_i can be written in Jordan canonical form

$$A_i = P_i \Lambda_i P_i^{-1}$$

where P_i is the matrix of the eigenvectors and Λ_i is matrix with the eigenvalues of A_i on the diagonal, possibly some 1's on the next diagonal, and zeroes elsewhere. If we assume that all of the eigenvalues are distinct, then Λ_i is a diagonal matrix with the eigenvalues along the diagonal. It is easily seen that

$$A_i^2 = (P_i \Lambda_i P_i^{-1})(P_i \Lambda_i P_i^{-1}) = P_i \Lambda_i^2 P_i^{-1}$$

and, in general

$$A_i^j = P_i \Lambda_i^j P_i^{-1}.$$

Let us further define c as the row vector (1, 0, 0, ..., 0) of length m. Then the expectations u_t, e_t, and n_t may be expressed as

(53) $$\underset{t}{E} u_{t+j} = cP_1 \Lambda_1^j P_1^{-1} z_{1t}$$

$$\underset{t}{E} e_{t+j} = cP_2 \Lambda_2^j P_2^{-1} z_{2t}$$

$$\underset{t}{E} n_{t+j} = cP_3 \Lambda_3^j P_3^{-1} z_{3t}$$

for all $j\geq 1$.

These are exactly the forecasting relations we seek. Each expresses the expected value at some date in the future as a function of current and past values. The next step involves expressing equation (47) for x_t as a function of only current and lagged values of itself and the shocks.

3. Eliminate expectation terms from the x_t equation

If we use the relations in (53) and substitute into equation (47), we have

$$\begin{aligned}(54)\qquad x_t &= (\lambda_1+\lambda_2)x_{t-1} - \lambda_1\lambda_2 x_{t-z}\\ &+ \frac{\lambda_1\lambda_2}{b(\lambda_1-\lambda_2)}\sum_{i=0}^{\infty}\{((\frac{\lambda_1}{b})^{i+1}-(\frac{\lambda_2}{b})^{i+1})(-b^2\ell cP_1\Lambda_1^{i+2}P_1^{-1}-bcP_1\Lambda_1^{i+1}P_1^{-1}\\ &+ cP_1\Lambda_1^{i}P_1^{-1})\}z_{1t}\\ &+ \frac{\lambda_1\lambda_2}{b(\lambda_1-\lambda_2)}\sum_{i=0}^{\infty}\{((\frac{\lambda_1}{b})^{i+1}-(\frac{\lambda_2}{b})^{i+1})(ba\ell cP_2\Lambda_2^{i+1}P_2^{-1}+cP_2\Lambda_2^{i}P_2^{-1})z_{2t}\\ &+ \frac{\lambda_1\lambda_2}{b(\lambda_1-\lambda_2)}\sum_{i=0}^{\infty}\{((\frac{\lambda_1}{b})^{i+1}-(\frac{\lambda_2}{b})^{i+1})(v_0cP_3\Lambda_3^{i+1}P_3^{-1}+v_1cP_3\Lambda_3^{i}P_3^{-1}\\ &+ v_2cP_3\Lambda_3^{i-1}P_3^{-1})z_{3t} + D_1.\end{aligned}$$

By noting that

$$\sum_{i=0}^{\infty}\delta^i cP\Lambda^i P^{-1} = cP\{\sum_{i=0}^{\infty}(\delta\Lambda)^i\}P^{-1}$$

we can expand the z_{1t} term on the right-hand side of equation (54).

$$\begin{aligned}(55)\qquad &\frac{\lambda_1\lambda_2}{b(\lambda_1-\lambda_2)}\sum_{i=0}^{\infty}\{((\frac{\lambda_1}{b})^{i+1}-(\frac{\lambda_2}{b})^{i+1})(-b^2\ell cP_1\Lambda_1^{i+2}P_1^{-1}-bcP_1\Lambda 1^{i+1}P_1^{-1}\\ &+ cP_1\Lambda_1^{i}P_1^{-1})\}z_{1t}\end{aligned}$$

$$= \frac{\lambda_1\lambda_2}{b(\lambda_1-\lambda_2)}[-b^2\ell cP_1\{\frac{b}{\lambda_1}\sum_{i=0}^{\infty}(\frac{\lambda_1}{b})^{i+2}\Lambda_1^{i+2}\}P_1^{-1}$$

$$+ b^2\ell cP_1\{\frac{b}{\lambda_2}\sum_{i=0}^{\infty}(\frac{\lambda_2}{b})^{i+2}\Lambda_1^{i+2}\}P_1^{-1}$$

$$-bcP_1\{\sum_{i=0}^{\infty}(\frac{\lambda_1}{b})^{i+1}\Lambda_1^{i+1}\}P_1^{-1}$$

$$+ bcP_1\{\sum_{i=0}^{\infty}(\frac{\lambda_2}{b})^{i+1}\Lambda_1^{i+1}\}P_1^{-1}$$

$$+ cP_1\{(\frac{\lambda_1}{b})\sum_{i=0}^{\infty}(\frac{\lambda_1}{b})^{i}\Lambda_1^{i}\}P_1^{-1}$$

$$- cP_1\{(\frac{\lambda_2}{b})\sum_{i=0}^{\infty}(\frac{\lambda_2}{b})^{i}\Lambda_1^{i}\}P_1^{-1}]z_{1t}.$$

It easily verified that if ψ_i's are the diagonal elements of Λ_1 and $\delta<1$ and j is some fixed integer, then

$$\sum_{i=0}^{\infty}\delta^{i+j}\Lambda_1^{i+j} = \frac{\delta^j\psi_i^j}{1-\delta\psi_i} \tag{56}$$

where the right-hand side of (56) is a diagonal matrix of dimension m whose ii-th element is

$$\frac{\delta^j\psi_i^j}{1-\delta\psi_i} .$$

Then (55) can be expressed as

$$\frac{\lambda_1\lambda_2}{b(\lambda_1-\lambda_2)}\cdot\{-b^2\ell c(\frac{b}{\lambda_1})P_1 \frac{\psi_i^2\lambda_1^2/b^2}{1-\psi_1\lambda_1/b} P_1^{-1} \tag{57}$$

$$+ b^2\ell c(\frac{b}{\lambda_2})P_1 \frac{\psi_i^2\lambda_2^2/b^2}{1-\psi_1\lambda_2/b} P_1^{-1}$$

$$- bcP_1 \frac{\psi_1\lambda_1/b}{1-\psi_1\lambda_1/b} P_1^{-1}$$

$$+ bcP_1 \frac{\psi_1 \lambda_2/b}{1-\psi_1 \lambda_2/b} P_1^{-1}$$

$$+ c(\frac{\lambda_1}{b})P_1 \frac{1}{1-\psi_1 \lambda_1/b} P_1^{-1}$$

$$- c(\frac{\lambda_2}{b})P_1 \frac{1}{1-\psi_1 \lambda_2/b} P_1^{-1}\}z_{1t}.$$

This expression is inner product of the first row of a complicated matrix and the vector z_{1t}. We may define this first row as $(\alpha_1, \ldots, \alpha_m)$ and then the entire expression may be written as αz_{1t}. If we simplify the expressions in z_{2t} and z_{3t} in a similar way, (54) may be written as

(58) $$x_t = (\lambda_1+\lambda_2)x_{t-1} - \lambda_1\lambda_2x_{t-2} + \alpha z_{1t} + \beta z_{2t} + \gamma z_{3t} + D_1.$$

Using the definitions of the z_i's we may write

(59) $$x_t = (\lambda_1+\lambda_2)x_{t-1} - \lambda_1\lambda_2x_{t-2} + \sum_{i=0}^{m} (\alpha_i u_{t-i}+\beta_i e_{t-i}+\gamma_i n_{t-i}) + D_1.$$

This is an equation that expresses x_t in terms of its own past and current and past values of the disturbances u, e, and n.

4. Express vector autoregression in estimatible form

We obtain a form of the model which can be estimated by combining equations (59), (24), and (26). By substituting equation (59) for the contemporaneous term in x_t in equations (24) and (26) we can express the system as a trivariate vector autoregression, e.g.,

(60) $$\begin{matrix} x_t \\ p_t \\ x_t \end{matrix} = \begin{matrix} \lambda_1 + \lambda_2 & 0 & 0 \\ A_1(\lambda_1+\lambda_2-1) & 0 & 0 \\ c_1(\lambda_1+\lambda_2+\ell) & 0 & 0 \end{matrix} \quad \begin{matrix} x_{t-1} \\ p_{t-1} \\ c_{t-1} \end{matrix}$$

$$\begin{matrix} +\lambda_1\lambda_2 & 0 & 0 \\ -A_1(\lambda_1\lambda_2+1) & 0 & 0 \\ c_1(\lambda_1\lambda_2) & 0 & 0 \end{matrix} \quad \begin{matrix} x_{t-2} \\ p_{t-2} \\ c_{t-2} \end{matrix}$$

$$+ H_0 \begin{matrix} u_t \\ e_t \\ n_t \end{matrix} + H_1 \begin{matrix} u_{t-1} \\ e_{t-1} \\ n_{t-1} \end{matrix} + H_2 \begin{matrix} u_{t-2} \\ e_{t-2} \\ n_{t-2} \end{matrix}$$

where H_0, H_1, and H_2 are all expressed as functions of α, β, γ, A, and c_1. If we define

$$\begin{matrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{matrix} = H_0 \begin{matrix} u_t \\ e_t \\ n_t \end{matrix}$$

then system (60) becomes

$$(61) \qquad \begin{matrix} x_t \\ p_t \\ c_t \end{matrix} = V_1 \begin{matrix} x_{t-1} \\ p_{t-1} \\ c_{t-1} \end{matrix} + V_2 \begin{matrix} x_{t-2} \\ p_{t-2} \\ c_{t-2} \end{matrix} + \begin{matrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{matrix} + H_1H_0^{-1} \begin{matrix} \varepsilon_{1t-1} \\ \varepsilon_{2t-1} \\ \varepsilon_{3t-1} \end{matrix}$$

$$+ H_2H_0^{-1} \begin{matrix} \varepsilon_{1t-2} \\ \varepsilon_{2t-2} \\ \varepsilon_{3t-3} \end{matrix}$$

where V_1 and V_2 are just the appropriate matrices from (60).

The parameters of the model can be estimated by a maximum likelihood method by minimizing the determinant of the variance-covariance matrix of the ε_{it}'s in system (61).