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Models of Money with Spatially Separated Agents

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1. Introduction

This paper presents three models which explain the observation that money is used in payment for commodities and barter is not prevalent. In each of these models money is intrinsically useless, inasmuch as it does not enter directly into either utility functions or production functions, and inconvertible, inasmuch as no one stands ready to convert money into anything else.[1] Moreover, money does not enter any of the models by way of legal restrictions[2] or by way of a requirement that commodities cannot be acquired without it, à la Clower (1967).[3] Rather, money is explained in the sense that the following procedure is adopted. First, the environment is specified carefully and completely--the agents of the model, their preferences and endowments, and most important, who can communicate with whom. It is then established that there exists a monetary equilibrium, that is, a competitive equilibrium in which a fixed-supply money has value.

It is widely accepted that money cannot be explained in this sense in a standard, general equilibrium model. (See, for example, Hahn (1965).) In a Walrasian model, at least, money cannot facilitate exchange; the nonmonetary competitive equilibria are Pareto optimal. (Note again that a distinction is maintained here between money, fiat money that is, and private credit.) Thus, to get money into a model something must inhibit the operation of markets.[4] Moreover, if terminal conditions are to be avoided, time must be infinite.

In the model of Samuelson (1958), some markets are precluded in an obvious way: there can be no transactions between the current young and the young of the next generation--unborn individuals cannot trade. And in versions of this overlapping-generations model, there does exist a monetary equilibrium, one which improves upon the nonmonetary equilibrium. In fact, in some versions, the monetary equilibrium is itself optimal and is associated with nonbinding nonnegativity constraints on the holding of money balances. Yet one wonders whether the properties of the overlapping-generations model carry over to alternative models which explain money in the above sense.

In the models of this paper markets are precluded in another way, by spatially separating agents. Infinitely-lived agents who discount future over present consumption are allocated over time into distinct markets or islands. The crucial idea here is that markets must clear on each island in every time period; there can be no communication across islands, that is, there is no central market or exchange system. Certainly this way of decentralizing an economy is not new; Lucas (1972) uses such islands to explain the movement of economic aggregates.[5/] More to the point, the explicit pairing of agents is a scheme used in various recent microeconomic approaches to money, including Starr (1972), Ostroy (1973), Feldman (1973), Ostroy and Starr (1974), and Harris (forth). Yet, for the most part, these approaches are not really dynamic equilibrium theories. An important exception is Harris, but he is concerned with commodity money, not outside indebtedness.

In two of the models of this paper, the turnpike model of exchange (Section 2) and Lucas' version of the Cass-Yaari (1966) model (Section 4), there exists a monetary equilibrium, that is, a competitive equilibrium with valued money. So, as claimed, these models explain money. And, as in the overlapping-generations construct, this monetary equilibrium improves upon the nonmonetary equilibrium (autarky). Unlike the overlapping-generations model, however, the monetary equilibria in these models with spatially separated agents are nonoptimal and are associated with binding nonnegativity constraints on the holding of money balances Thus the decentralization of spatial separation is not completely overcome with money.[6/]

The argument as to why no optimal allocation can be supported as a monetary equilibrium in these models is fairly intuitive. Suppose all agents are of the same age and all discount the future at the same rate, as in the turnpike model and Lucas' version of the Cass-Yaari model. Then, if an optimal allocation is to be supported, there must be a rate of deflation equal to the common discount rate. But, in the absence of taxes, such a deflation is inconsistent with individual maximization, as real wealth (real money balances) would be unbounded. That no stationary monetary equilibrium can be optimal is an immediate corollary.

As Grandmont and Youness (1972, 1973) point out, this latter conclusion appears frequently in the literature (see, for example, Clower (1970), Friedman (1969), Johnson (1970), and Samuelson (1968), (1969)) where the argument turns on a divergence between the positive marginal utility of real money balances and the zero marginal cost of creating them. Grandmont and Youness are critical of this literature, and they are not alone; Clower (1970), for example, argues quite forcefully that these welfare questions must be addressed in a model which makes explicit the monetary exchange process. Grandmont and Youness do establish the aforementioned conclusion rigorously in a general equilibrium, monetary economy. Yet, as the authors note, even in their model money is introduced ... "in a very crude way, by imposing contraints on transactions". That is, in contrast to the models of this paper, their's is not a model which explains money. In this respect, at least, this paper may be viewed as an important extension of Grandmont ans Youness and of this literature.

If lump-sum taxes on money balances <u>are</u> permitted, then the models with spatially separated agents of this paper can also produce Friedman's (1969) conclusions on the optimal quantity of money. That is, optimal allocations can be supported in an interventionist monetary equilibrium, with lump-sum taxes, in which the rate of interest on money equals the common discount rate and in which agents are satiated with money balances, i.e., the nonnegativity constraints on money balances are nonbinding. (Also see Grandmont and Youness (1972), and Bewley (forth).)

Speaking rather loosely, the overlapping-generations model overturns these welfare results by pairing agents of different ages and therefore different rates of discount.[7/] This is argued more fully in Section 3, where the turnpike model is modified to incorporate finitely-lived agents and hence becomes an overlapping-generations model. Hopefully an essential feature of the overlapping-generations model is revealed.

As noted above, monetary economics necessarily involves the economics of infinity. In the overlapping-generations model there is an infinite number of generations, though a finite number of agents alive at any one date. In the turnpike model, and in the Cass-Yaari model presented here, there is an infinite number of agents alive at any one date. This specification ensures that no private debt is traded, so that its exclusion is endogenous, i.e., not imposed by the modeler. Section 5 offers some preliminary comments on private debt (inside money) in the context of a modified turnpike model, one without the contemporaneous infinity.

Finally, a caveat is in order. The intent in what follows is to understand the implications of various exchange structures for monetary equilibria. Thus, speaking rather loosely, preferences and endowments are held fixed across models as the exchange structure is varied. To this end, maximum generality is not pursued within the context of each model. Agents are assumed throughout to have preferences and endowments of a very special form. Moreover, certain strong symmetry conditions (on the class of allocations under consideration) are imposed exogenously, without elaboration. Finally it may be noted that the models of this paper are successful in explaining money without the introduction of uncertainty;

it remains an open question as to whether these models can approximate economies in which moral hazard and bankruptcy play a crucial rule, as Brunner and Meltzer (1971) have emphasized.

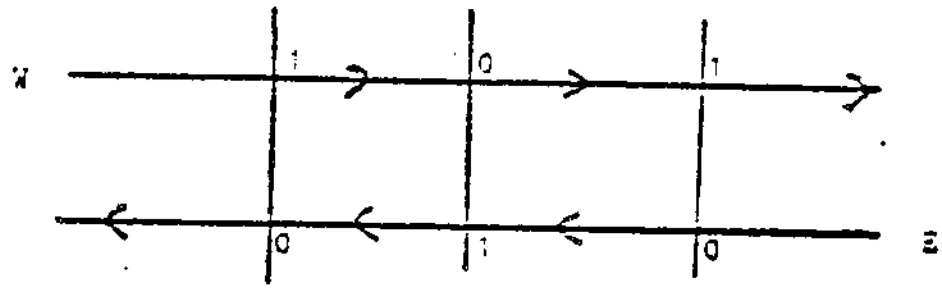

Figure 1: The Turnpike Model

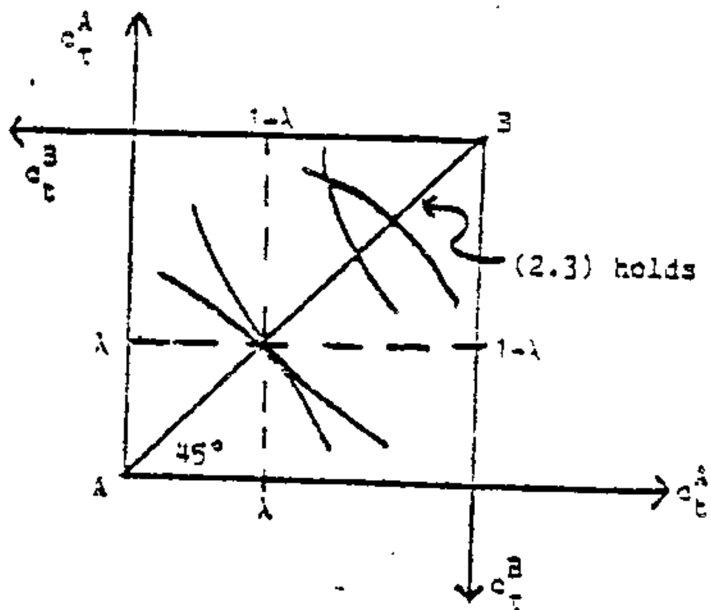

Figure 2: Optimal Allocations in the Turnpike Model

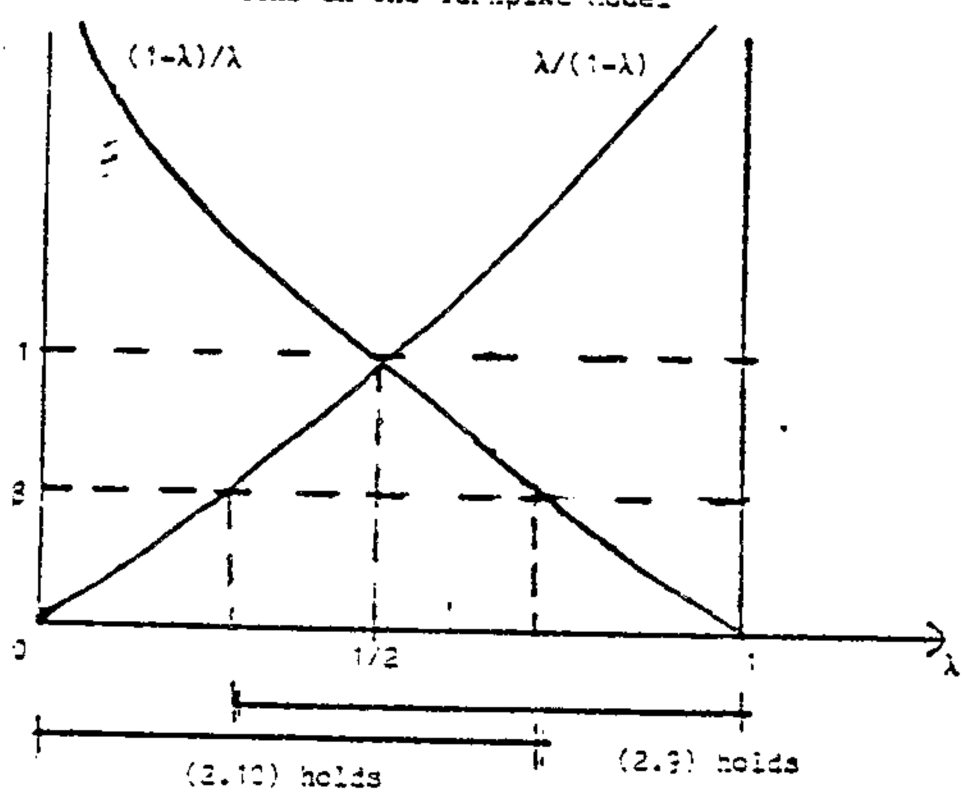

Figure 3: The Relationship Between λ and β

3. A Turnpike Model of Exchange

In the turnpike model each of a countably infinite number of agents is allocated into one of a countably infinite number of spatially distinct markets or islands in each period of his life. The exogenous allocation procedure is such that any two agents are paired at most once during their lifetime, and, moreover, they share no common third agent as a trading partner.[7] All agents are born at time zero, so that at any time t all agents are of the same age. Each lives forever and faces a sequence of endowments of the single consumption good of the model which alternates between zero and one unit. At any time $t \geq 0$ an agent who has an endowment of one unit is paired with an agent who has an endowment of zero. The consumption good cannot be stored.

An economy with these characteristics is depicted in Figure 1. Each agent is imagined to be travelling on a turnpike, either east or west. The arrows indicate the direction of travel, and the spikes indicate the markets. The numbers 0, 1 index the endowment of an agent located at the indicated position. Initially, at $t = 0$, there is one agent at each position. It should be emphasized here that each agent has no control over his lifetime itinerary. Each agent moves forward one market in each period. Also, these markets are isolated one from another; there can be no transaction or communication among them at any time.

Each agent has preferences over his (infinite) lifetime consumption sequence $\{c_t\}_{t=0}^{\infty}$ as described by the utility function $\sum_{t=0}^{\infty}\beta^t U(c_t)$ where $c_t \geq 0$, $0 < \beta < 1$, and $U(\cdot)$ is strictly concave, strictly increasing, bounded, and continuously differentiable with $U'(0) = \infty$. Thus all agents have the same time separable utility function of a rather special form, and in particular all discount future over present consumption at the same rate, β.[8]

This model displays a well-known property, the absence of double-coincidence of wants. At each time t, considered in isolation, there can be no

Pareto improving bilateral trade; there is only one consumption good, and more is preferred to less. One may ask, of course, whether borrowing and lending might not improve matters. Section 5 below is devoted entirely to this question in a slightly modified context, but it should be noted here that, in a sense which will be made precise, there can be no private debt in the present model. For consider an agent at time t who has zero units of the consumption good. Such an agent might wish to issue an IOU, to be honored in better times, when he has one unit of the consumption good. Similarly, the agent with whom he is paired, who has one unit, might be inclined to accept such an IOU. Yet the model is constructed in such a way that the IOU can never be redeemed by the issuer; the pair will never meet again, and the purchaser of such an IOU can only pass it along to an agent "behind" the issuer. Thus <u>if</u> one takes as a <u>defining characteristic</u> of private debt that it ultimately be redeemed by the issuer, there can be no private debt in this model.[9/]

Having specified the environment for this model, the next step is to characterize Pareto optimal allocations. For this purpose a strong symmetry condition is imposed, that in any allocation, agents cannot be distinguished by their initial market position. That is, any agent who begins his life with zero units of the consumption good must be treated the same way as any other agent who begins with zero units, independent of his initial location. All such agents are hereafter referred to as agents of type A. A similar restriction is placed on those who begin with one unit, agents of type B. It bears repeating here that when an allocation is termed optimal below, it is only established to be optimal in the class of symmetric allocations; there remains the possibility of a nonsymmetric allocation which is Pareto superior.

Now let c_t^i denote the number of units of consumption of an agent of type i at time t. Then an allocation $\{c_t^A\}_{t=0}^{\infty}$, $\{c_t^B\}_{t=0}^{\infty}$ is said to be feasible if

$$(2.1)\qquad c_t^A + c_t^B \leq 1,\ c_t^A \geq 0,\ c_t^B \geq 0 \qquad \text{all } t \geq 0.$$

(An allocation is said to be interior if consumption is strictly positive for each agent type in each time period.) It may be assumed without loss of generality in what follows that resources are fully utilized. Then to determine an interior Pareto optimal allocation, it is enough to maximize a weighted average of the utilities of the two agent types, subject to the resource constraints, as is established below. This yields

Problem 1:

$$\max_{\{c_t^A\}_{t=0}^{\infty},\ \{c_t^B\}_{t=0}^{\infty}} w^A[\textstyle\sum_{t=0}^{\infty}\beta^t U(c_t^A)] + w^B[\textstyle\sum_{t=0}^{\infty}\beta^t U(c_t^B)]$$

subject to (2.1) where $w^A > 0$, $w^B > 0$, $w^A + w^B = 1$. Necessary and sufficient first-order conditions for problem (1) are

$$(2.2)\qquad w^i\beta^t U'(c_t^i) - \theta_t = 0,\ i=A, B \qquad t \geq 0$$

where the θ_t are positive Lagrange multipliers. Trivial manipulation of (2.2) yields

$$(2.3)\qquad \frac{U'(c_t^A)}{U'(c_\tau^A)} = \frac{U'(c_t^B)}{U'(c_\tau^B)} \qquad \text{all } t, \tau \geq 0.$$

Conditions (2.1) and (2.3) are fully equivalent with

$$(2.4)\qquad c_t^A = \lambda,\ c_t^B = 1 - \lambda,\ 0 < \lambda < 1 \qquad \text{all } t \geq 0.$$

(See Figure 2.) Thus a necessary and sufficient condition for a feasible interior allocation $\{c_t^A\}_{t=0}^{\infty}$, $\{c_t^B\}_{t=0}^{\infty}$ to be optimal is that each agent of type A receive λ units of the consumption good in each period t: That this condition is necessary for optimality follows from the obvious fact that if condition (2.3) is not satisfied for some periods t and τ, then there is a Pareto superior feasible allocation. That this condition is sufficient is also obvious. For suppose there exists a feasible allocation which is Pareto superior. Then it would satisfy constraint (2.1) and increase the value of the objective function in problem (1), a contradiction. Hereafter, then, reference will be made to an interior optimum λ in which both agents receive constant consumption.

The question may now be raised as to whether optimal allocations can be supported in competitive equilibria with valued fiat money. To do so one must discuss carefully what are meant by fiat money and competitive markets in the context of this model. A unit of fiat money is imagined to be a physical commodity, say a piece of paper, which may be carried costlessly by the agents as they travel among islands and used in exchange. As a commodity, the stock of it in the possession of any trader at any time cannot be negative. On each island and at each time period there is assumed to be a competitive market in which fiat money can be exchanged for the consumption good at a specified rate. That is, agents take the price of the consumption good as given and maximize utility by choice of the amount to consume and the amount of money balances to carry over into the next period. No attempt is made here to justify the price-taking assumption or defend the competitive equilibrium notion; to the extent that the mechanism which underlies the equilibrium notion requires a large (perhaps infinite) number of agents, each agent discussed above may be taken as representative of agents in identical situations.

Consistent with the symmetry assumption, attention will be restricted to monetary equilibria in which the price of the consumption good in terms of money at any time t is the same in each market. This price is denoted P_t and is assumed to be finite and strictly positive. Also, let M_t^i denote the number of units of fiat money chosen at time $t-1$ by agent i and carried over into period t,

z_t^i denote the number of units of a lump-sum tax on money balances (or subsidy, if negative) on agent type i at the beginning of period t, and y_t^i denote the endowment of agent type i at time t. Then taking as given the sequence $\{p_t\}_{t=0}^{\infty}$, $\{z_t^i\}_{t=0}^{\infty}$ and initial money balances M_0^i, each agent of type i is confronted with

<u>Problem (i)</u>:

$$\max_{\{c_t^i\}_{t=0}^{\infty},\ \{M_t^i\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t^i)$$

subject to

$$c_t^i \geq 0 \text{ all } t \geq 0, \qquad M_t^i \geq 0 \text{ all } t \geq 0$$

$$(b_t) \qquad p_t c_t^i + M_{t+1}^i \leq p_t y_t^i + M_t^i - z_t^i \qquad \text{all } t \geq 0$$

given $M_0^i \geq 0$, $z_0^i = 0$. Here (b_t) is the budget constraint which prevails in period t. With $U'(0) = \infty$ the nonnegativity constraint on consumption need not be made explicit; in contrast the nonnegativity constraint on money balances may be binding. Assuming without loss of generality that the budget constraint (b_t) holds as an equality, so that in effect only $\{M_t^i\}_{t=1}^{\infty}$ need be chosen, and making the obvious substitution for the c_t^i, one obtains necessary Euler conditions for a maximum

$$(2.5) \qquad -\frac{\beta^{t-1}U'(c_{t-1}^i)}{p_{t-1}} + \frac{\beta^t U'(c_t^i)}{p_t} + \theta_t^i = 0 \qquad \text{all } t \geq 1$$

where θ_t^i is the Lagrange multiplier associated with the nonnegativity constraint on money balances, that is,

$$\theta_t^i \geq 0,\ M_t^i \geq 0,\ \theta_t^i M_t^i = 0.$$

Thus,

$$\text{(2.6)} \qquad \frac{U'(c^i_{t-1})}{\beta U'(c^i_t)} \geq \frac{p_{t-1}}{p_t} \qquad \text{all } t \geq 1$$

where (2.6) must hold as an equality if $M^i_t > 0$ and (2.6) must hold as an inequality if and only if $\theta^i_t > 0$, that is, when the marginal utility of a unit of fiat money spent on period t-1 consumption exceeds the marginal utility of a unit of fiat money spent on period t consumption <u>and</u> there is no more fiat money to spend in period t-1.

A competitive equilibrium with valued fiat money may now be defined.

<u>Definition:</u> A <u>monetary equilibrium</u> is a sequence of finite positive prices $\{p^*_t\}^\infty_{t=0}$ and sequences of consumptions $\{c^{i*}_t\}^\infty_{t=0}$, money balances $\{M^{i*}_t\}^\infty_{t=0}$, and lump-sum taxes $\{z^{i*}_t\}^\infty_{t=0}$ for each agent type i = A, B such that

i) (maximization) the sequences $\{c^{i*}_t\}^\infty_{t=0}$, $\{M^{i*}_t\}^\infty_{t=1}$ solve problem (i) relative to $\{p^*_t\}^\infty_{t=0}$, $\{z^{i*}_t\}^\infty_{t=0}$, and M^{i*}_0, and

ii) (market clearing) $c^{A*}_t + c^{B*}_t = 1$, all $t \geq 0$.

One may now ask whether optimal allocations can be supported in a monetary equilibria without intervention. The answer is summarized in

<u>Proposition 2.1</u>: No interior optimum λ can be supported in a monetary equilibrium without intervention, i.e., with $z^{i*}_t \equiv 0$ for all i = A, B.

<u>Proof</u>: The proof is by contradiction. Thus suppose that the allocation $c^A_t = \lambda$, $c^B_t = 1 - \lambda$, all $t \geq 0$, can be supported in a monetary equilibrium without intervention. With $z^{i*}_t \equiv 0$, with $y^A_t = 0$ for t even and $y^B_t = 0$ for t odd, and with $U'(0) = \infty$, it is clear that the nonnegativity constraint on money balances cannot be binding for agent type A for choices made when t is odd or for agent type B for choices made when t is even. Thus from (2.6) equilibrium prices $\{p^*_t\}^\infty_{t=0}$ must satisfy

$$\frac{U'(\lambda)}{\beta U'(\lambda)} = \frac{p^*_{t-1}}{p^*_t} \qquad t \geq 2,\ t \text{ even}$$

$$\frac{U'(1-\lambda)}{\beta U'(1-\lambda)} = \frac{p^*_{t-1}}{p^*_t} \qquad t \geq 1,\ t \text{ odd.}$$

It follows that

$$(2.7) \qquad p^*_t = \beta p^*_{t-1} \qquad \text{all } t \geq 1,$$

i.e., <u>the rate of deflation must be 1-β</u>. Now consider the evolution of money balances of agent type B given the price sequence $\{p^*_t\}_{t=0}^{\infty}$ and the specified consumption sequence $c^B_t = 1 - \lambda$, all $t \geq 0$. Agent type B begins life with $M^{B*}_0 \geq 0$ units of fiat money, acquires $p^*_0\lambda$ units in period zero, and spends $p^*_1(1-\lambda)$ units in period one. Thus

$$(2.8) \qquad M^{B*}_2 - M^{B*}_0 = p^*_0\lambda - p^*_1(1-\lambda).$$

Clearly the increment to money balances from $t = 0$ to $t = 2$, the left-hand side of (2.8), is nonnegative if the right-hand side is nonnegative. Substituting from (2.7), the right-hand side is nonnegative if

$$(2.9) \qquad \frac{\lambda}{1-\lambda} \geq \beta.$$

In fact, one may readily verify that the increment to money balances is nonnegative for agent type B from t to $t+2$ for <u>all</u> t even if (2.9) holds. Similar calculations establish that the increment to money balances is nonnegative for agent type A from t to $t+2$ for <u>all</u> t odd if

$$(2.10) \qquad \frac{1-\lambda}{\lambda} \geq \beta.$$

The left-hand sides of inequalities (2.9) and (2.10) are graphed in Figure 3 as functions of the parameter λ. As Figure 3 makes clear, with the discount rate β fixed, $0 < \beta < 1$, at least one of the relationships (2.9) and (2.10) must hold as a strict inequality for any value of λ between zero and one. That is, at least one agent type will be accumulating money balances over time in the above sense. But then this cannot be an equilibrium. For if (2.9) holds as a strict inequality, for example, agent type B could spend these "excess balances" at $t \geq 1$, t odd, and improve upon the consumption sequence $c_t^{B*} \equiv 1 - \lambda$. This completes the proof.

Thus if an optimal allocation is to be attained in a monetary equilibrium, the rate of deflation must be $1 - \beta$ and, consequently, there must be some intervention by way of taxes and/or subsidies. That at least some optimal allocations can be supported in this way is established in

Proposition 2.2: Any interior optimum λ with $\beta \leq [\lambda/(1-\lambda)]$ and $\beta \leq [(1-\lambda)/\lambda]$ can be supported in a monetary equilibrium with rate of deflation $1 - \beta$; with $z_t^{B*} = p_{t-1}^*[\lambda-\beta(1-\lambda)] \geq 0$ for $t \geq 1$, t odd, and zero otherwise; and with $z_t^{A*} = p_{t-1}^*[(1-\lambda)-\lambda\beta] \geq 0$ for $t \geq 2$, t even, and zero otherwise.

Proof: See the appendix.

Given propositions (2.1) and (2.2) one may well ask whether there exist any monetary equilibria which do not require intervention. The search for such equilibria is facilitated by the following proposition which suggests that the search can be limited to equilibria in which the nonnegativity constraints on money balances play an important role.

Proposition 2.3: Any monetary equilibrium with nonbinding nonnegativity constraints on money balances on each agent in each period supports an optimal allocation and hence requires some intervention.

Proof: By hypothesis, $\theta_t^{i*} \equiv 0$. Thus from (2.5) it follows that

$$(2.11) \qquad \frac{\beta^{t-1}U'(c_{t-1}^{i*})}{\beta^t U'(c_t^{i*})} = \frac{p_{t-1}^*}{p_t^*} \quad i=A, B \quad \text{all } t \geq 1.$$

Manipulation of (2.11) yields

$$(2.12)\qquad \frac{\beta^{t}U'(c_t^{i*})}{\beta^{\tau}U'(c_\tau^{i*})} = \frac{p_t^*}{p_\tau^*}\quad i=A,\ B\qquad \text{all } t,\ \tau \geq 0.$$

As (2.12) holds for both i,

$$(2.13)\qquad \frac{U'(c_t^{A*})}{U'(c_\tau^{A*})} = \frac{U'(c_t^{B*})}{U'(c_\tau^{B*})}\qquad \text{all } t,\ \tau \geq 0.$$

Condition (2.13) and market-clearing condition (ii) of an equilibrium are sufficient for an optimum as discussed above. The conclusion of proposition follows from proposition (2.1). This completes the proof.

The search for a noninterventionist monetary equilibrium is also facilitated by the observation that, roughly speaking, a time trend to prices, say a constant rate of deflation $1 - \beta$ as in proposition (2.2), would seem to necessitate intervention in order to keep purchasing power constant. That is, one might search for an equilibrium in which prices remain constant over time, at some price $p^* > 0$. Thus it may be guessed that in a noninterventionist monetary equilibrium, each agent of type B will have one unit of purchasing power to be allocated over consumption in each pair of periods $(t, t+1)$, $t \geq 0$, t even, selling the consumption good for money in period t and spending all accumulated money balances in period $t+1$, with a binding nonnegativity constraint on money balances at $t+1$. This will generate consumptions c^* and c^{**}, as depicted in Figure 4. Of course, each agent of type A will be doing the same thing in each pair of periods $(t+1, t+2)$ $t \geq 0$, t even. This discussion is summarized in

<u>Proposition 2.4</u>: There exists a noninterventionist monetary equilibrium with constant prices, with binding nonnegativity constraints on money balances in every other period, and with alternating consumption sequences. In particular for taxes and prices,

$$z_t^{i*} \equiv 0, \; p_t^* \equiv p^* > 0;$$

for agent A, $M_0^{A*} = p^*c^{**}$, and

$$c_t^{A*} = c^{**}, \; M_{t+1}^{A*} = 0, \; \theta_{t+1}^{A*} > 0 \qquad t \geq 0, \; t \text{ even}$$

$$c_t^{A} = c^{*}, \; M_{t+1}^{A*} = p^*c^{**}, \; \theta_{t+1}^{A*} = 0 \qquad t \geq 1, \; t \text{ odd};$$

for agent B, $M_0^{B*} = 0$, and

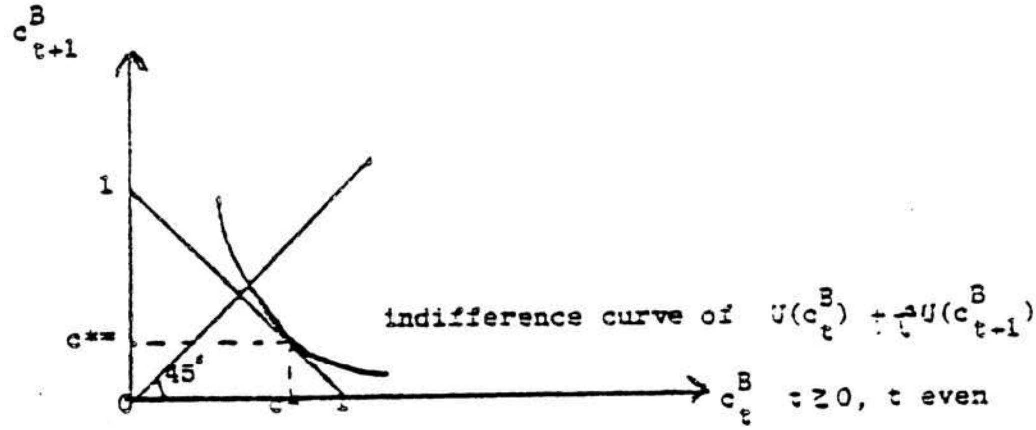

Figure 4: The Turnpike's Monetary Equilibrium

$$c_t^B = c^*,\ M_{t+1}^{B*} = p^*c^{**},\ \theta_{t+1}^{B*} = 0 \qquad t \geq 0,\ t \text{ even}$$

$$c_t^B = c^{**},\ M_{t+1}^B = 0,\ \theta_{t+1}^B > 0 \qquad t \geq 1,\ t \text{ odd};$$

and where c^* and c^{**} satisfy

$$\frac{U'(c^*)}{\beta U'(c^{**})} = 1,\ c^* + c^{**} = 1.$$

The equilibrium allocation is nonoptimal, but Pareto superior to autarky.

Proof: See the appendix.

The proof of proposition (2.4) utilizes the fact that for agent type A

$$p_t c_t^A + p_{t+1} c_{t+1}^A = p_{t+1} y_{t+1}^A + M_t^A - M_{t+2}^A \qquad t \geq 0,\ t \text{ even} \tag{2.14}$$

$$p_t c_t^A \leq M_t^A \qquad t \geq 0,\ t \text{ even} \tag{2.15}$$

where (2.14) is the money balance accumulation equation, and (2.15) follows from the restriction that $M_{t+1}^A \geq 0$. Letting $c_t^A = c^1$, $c_{t+1}^A = c^2$, $p_t = p^1$, $p_{t+1} = p^2$, $y_{t+1}^A = y^2$, $M_t^A = M$, $M_{t+2}^A = M'$, equations (2.14) and (2.15) may be written as

$$p^1c^1 + p^2c^2 = p^2y^2 + M - M' \tag{2.16}$$

$$p^1c^1 \leq M. \tag{2.17}$$

Here then (2.16) appears as a money balance accumulation equation in a two-commodity model, and (2.17) is a semi-Clower constraint, that the valuation of consumption of commodity one not exceed initial money balance—this formulation leads one to inquire as to the effect of a more standard Clower constraint of the form

$$p^1c^1 + p^2c^2 \leq M, \tag{2.18}$$

that the <u>total</u> valuation of consumption be bounded by initial money balances. Constraint (2.18) is not derived entirely from the technology of exchange. Imposed in addition is the requirement that agents bid in competitive markets for their own production. That is, agent type A at time t+1 as a producer is required to place all production y_{t+1}^A on the market and pay cash in advance for any consumption c_{t+1}^A.

Motivated by the above discussion, consider the following

<u>Definition</u>: A <u>Clower-type monetary equilibrium</u> is a sequence of finite positive prices $\{p_t^*\}_{t=0}^{\infty}$ and sequences of consumptions $\{c_t^{i*}\}_{t=0}^{\infty}$ and money balances $\{M_t^{i*}\}_{t=0}^{\infty}$ for each agent type i = A, B such that

i) maximization for type A—the sequences $\{c_t^{A*}\}_{t=0}^{\infty}$, $\{M_t^{A*}\}_{t=1}^{\infty}$ solve

$$\max_{\{c_t^A\}_{t=0}^{\infty},\ \{M_t^A\}_{t=1}^{\infty}} \sum_{\substack{t\geq 0\\ t\ \text{even}}} \beta^{t}[U(c_t^A)+\beta U(c_{t+1}^A)]$$

subject to

$$p_t^* c_t^A + p_{t+1}^* c_{t+1}^A = p_{t+1}^* y_{t+1}^A + M_t^A - M_{t+2}^A \qquad t \geq 0,\ t \text{ even}$$

$$p_t^* c_t^A + p_{t+1}^* c_{t+1}^A \leq M_t^A \qquad t \geq 0,\ t \text{ even}$$

$$M_{t+1}^A = M_t^A - p_t^* c_t^A \qquad t \geq 0,\ t \text{ even}$$

given $M_0^{A*} \geq 0$,

ii) maximization for type B—the sequences $\{c_t^{B*}\}_{t=0}^{\infty}$, $\{M_t^{B*}\}_{t=1}^{\infty}$ solve

$$\max_{\{c_t^B\}_{t=0}^{\infty},\ \{M_t^B\}_{t=1}^{\infty}} \{U(c_0^B)+\sum_{\substack{t\geq 1\\ t\ \text{odd}}} \beta^{t-1}[\beta U(c_t^B)+\beta^2 U(c_{t+1}^B)]\}$$

subject to

$$p_t^* c_t^B + p_{t+1}^* c_{t+1}^B = p_{t+1}^* y_{t+1}^B + M_t^B - M_{t+2}^B \qquad t \geq 1,\ t \text{ odd}$$

$$p_t^* c_t^B + p_{t+1}^* c_{t+1}^B \leq M_t^B \qquad t \geq 1,\ t \text{ odd}$$

$$M_{t+1}^{B} = M_t^B - p_t^* c_t^B \qquad t \geq 1,\ t \text{ odd}$$

$$p_0^* c_0^B = p_0^* y_0^B + M_0^B - M_1^B$$

$$p_0^* c_0^B \leq M_0^B$$

given $M_0^{B*} \geq 0$, and

iii) market clearing—$c_t^{A*} + c_t^{B*} = 1,\ t \geq 0$. This leads to

Proposition 2.5: If there exists a Clower-type monetary equilibrium with constant prices, i.e., with $p_t^* \equiv p^* > 0$, and with a symmetric consumption sequence, i.e., with $c_{t+1}^{B*} = c_t^{A*}$ all $t \geq 0$, then $c_{t+1}^{B*} = c_t^{A*} = c^*$ and $c_{t+1}^{A*} = c_t^{B*} = c^{**}$ for $t \geq 0$, t even, where c^* and c^{**} are defined in proposition (2.4). This allocation is nonoptimal but Pareto superior to autarky.

Proof: The necessary conditions for a maximum include

$$\beta^t U'(c_t^{A*}) - \theta_t^A p^* - \gamma_t^A p^* = 0 \qquad t \geq 0,\ t \text{ even}$$

$$\beta^t \beta U'(c_{t+1}^{A*}) - \theta_t^A p^* - \gamma_t^A p^* = 0 \qquad t \geq 0,\ t \text{ even}$$

where $\theta_t^A > 0$ and $\gamma_t^A \geq 0$ are Lagrange multipliers. Thus

$$(2.19) \qquad \frac{U'(c_t^{A*})}{\beta U'(c_{t+1}^{A*})} = 1 \qquad t \geq 0,\ t \text{ even.}$$

Similarly for agent type B

$$(2.20) \qquad \frac{U'(c_t^{B*})}{\beta U'(c_{t+1}^{B*})} = 1 \qquad t \geq 1,\ t \text{ odd.}$$

Market clearing and the symmetry hypothesis imply

$$(2.21) \qquad c_{t+1}^{B*} + c_t^{B*} = 1 \qquad \text{all } t \geq 0.$$

The unique solution to (2.20) and (2.21) is $c_t^{B*} = c^*$ and $c_{t+1}^{B*} = c^{**}$, $t \geq 1$, t odd. And by the symmetry hypothesis $c_t^{A*} = c^*$, $c_{t+1}^{A*} = c^{**}$ for $t \geq 0$, t even. Thus by market clearing $c_0^{\beta *} = c^{**}$ also. It is obvious that this allocation is non-optimal. Note also that for agent type A, for example, the consumption pair (c^*, c^{**}) dominates the endowment pair (0, 1) in periods (t, t+1) for $t \geq 0$, t even. This completes the proof.

To be noted here is that the imposition of the full Clower constraint (2.18) reverses the consumption sequences from those of proposition (2.4). Yet in this model the "intervention" implicit in the Clower constraint is not enough to attain optimal allocations.

In closing this section it may be noted that as either the discount rate goes to zero or, equivalently, as the frequency of transactions (pairings) increases, the turnpike model comes close to producing the welfare result of the overlapping-generations construct, that there exists an optimal noninterventionist monetary equilibrium. To see this, note, for example, that the amount of taxation needed to support the optimal allocation $\lambda = 1/2$ goes to zero as $\beta \rightarrow 1$ (see proposition 2.2). Alternatively, note that the noninterventionist monetary equilibrium consumption sequences approach the constant $\lambda = 1/2$ as $\beta \rightarrow 1$ (see proposition 2.4 and Figure 4). It may well be that this welfare result holds exactly in the limit, at $\beta = 1$, if agents use the overtaking criterion to evaluate consumption paths.[10/]

3. A Generalized Overlapping-Generations Model

The turnpike model may be contrasted with the overlapping-generations model of Samuelson [1958]; as is well known, the overlapping-generations model yields, under specified assumptions, a noninterventionist monetary equilibrium which is optimal. It should prove useful then to discover those elements which lead to the different implications of the two models. The intent of this section, then, is to modify the turnpike model to make it more comparable to the standard overlapping-generations construct. Putting this another way, the overlapping-generations model is generalized; in so doing, its essential features are revealed.

The obvious modification of the turnpike model produces the model depicted in Figure 5; in effect, the turnpike model has been truncated at both ends. Here one agent is born in each period at the beginning of the eastern and western routes, and each agent lives four periods. Note, agents aged 0 and 3 periods are paired, as are agents aged 1 and 2 periods.

Preliminary work with this model indicated that an optimal allocation can be supported as a noninterventionist monetary equilibrium, as with the standard overlapping-generations model. In such an equilibrium prices first fall and then rise over each agent's lifetime. Moreover, the (even-aged) lifetimes of each agent can be made arbitrarily long, by truncating the model further out, without altering these conclusions. But it may be noted that the age-pairings in this class of models are extreme, the youngest trading with the oldest, the next-to-youngest trading with the next-to-oldest, and so on, whereas in the turnpike model one's trading partners are of the same age. Thus a more natural comparison would be to a modified model in which one's trading partners are more or less the same age. This produces the generalized overlapping-generations model which is examined in the remainder of this section.

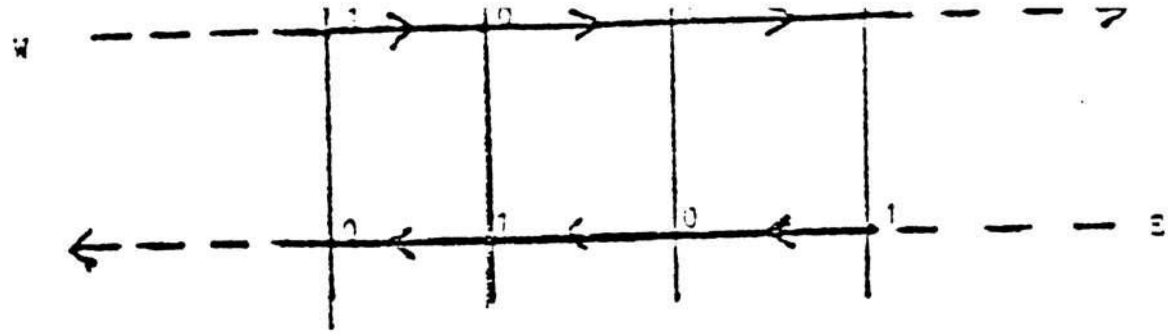

Figure 5: A Truncated Turnpike

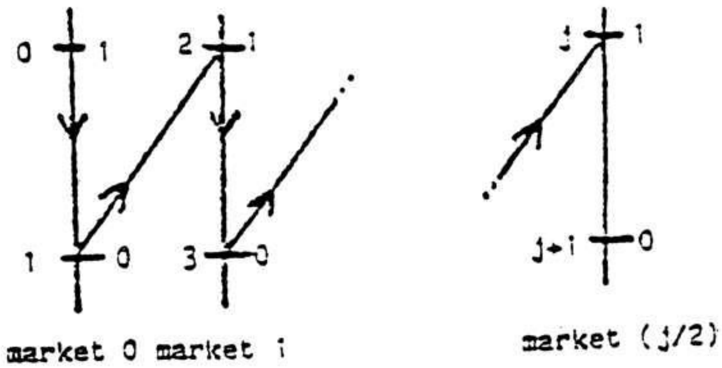

Figure 6: Generalized Overlapping Generations

As in the turnpike model, each of a countably infinite number of agents faces an endowment sequence of the single nonstorable consumption good over his infinite lifetime which alternates between one and zero. Yet here all agents are not of the same age; one representative trader is born in each period t, $t \geq 0$, and begins life with an endowment of one unit. Each agent is again allocated into one of a countably infinite number of spatially distinct markets in each period of his life, but here the allocation procedure is such that each agent is paired with an agent who is either one period older or one period younger. Figure 6 illustrates the scheme: The arrows indicate the direction of "travel," and the numbers on the right of market spikes indicate the endowment of an agent whose age is indicated on the left.

For our purpose the economy will be conceived of as beginning at time $t = 0$ but populated with agents born at times $t = -h$, $h \geq 1$. Thus at time $t = 0$, island $k = j/2$, $j \geq 0$, j even, is inhabited with two (representative) agents, one born at time $-j$ with an endowment of one unit and one born at time $-(j+1)$ with an endowment of zero units. At time $t = 1$, one new (representative) agent is born and enters market zero, while the other agents move forward as indicated, and so on. Note that if agents were to live two periods only, attention could be restricted to market zero alone, an economy which is identical to the simplest two-period overlapping-generations model. As will be shown, the present generalization retains the characteristics of that economy.

As in the turnpike model, there is a sense in which there can be no private debt in this model. Here, unlike the turnpike model, agents meet each other infinitely often; an agent born at time t is paired with an agent born at time t+1 when the former is of age 0, 2, 4, ..., and an agent born at time t is paired with an agent born at time t-1 when the former is of age 1, 3, 5, ... Yet when they meet, each of the pair has the same relative endowment position. An IOU issued by an agent of an odd age who has zero units of the consumption good can never be redeemed by the issuer—he will have zero units when the pair meets again.

Now to describe preferences, feasible allocations, and Pareto optimal allocations some additional notation is needed. Thus let $\{y_j\}_{j=0}^{\infty}$ denote the endowment sequence of a typical agent over his lifetime, where y_j is the endowment of an agent age j. Here then $y_j = 1$ for $j \geq 0$, j even; and $y_j = 0$ for $j \geq 1$, j odd. Let $c_j(t)$ denote the consumption of an agent born at time t who is of age j, $j \geq 0$, all t. Each agent has the same preferences as in the turnpike model. That is, the objective function of an agent born at time $t \geq 0$ is $\sum_{j=0}^{\infty}\beta^j U[c_j(t)]$ where $0 < \beta < 1$ and $U(\cdot)$ is strictly concave, strictly increasing, bounded, and continuously differentiable with $U'(0) = \infty$. The objective function of an agent born at time $-h < 0$ is $\sum_{j=h}^{\infty}\beta^{j-h}U[c_j(-h)]$. An allocation is a consumption sequence $\{c_j(t)\}_{j=0}^{\infty}$ for each agent born at time $t \geq 0$, and a consumption sequence $\{c_j(-h)\}_{j=h}^{\infty}$ for each agent born at time $-h \leq 0$. By construction there is only one unit of the consumption good among the two traders of any market at any point in time. Thus an allocation is said to be feasible if

$$(3.1)\quad \begin{aligned} &c_j(t) + c_{j+1}(t-1) \leq 1 \qquad && t \geq 0,\ j \geq 0,\ j \text{ even} \\ &c_j(-h) + c_{j+1}(-h-1) \leq 1 \qquad && h \geq 1,\ j \geq h,\ j \text{ even.} \end{aligned}$$

It will be assumed in what follows, without loss of generality, that these constraints must hold as equalities.

The next step is to define Pareto optimal allocations. For <u>this</u> purpose, a strong symmetry condition is imposed, namely, that agents of identical <u>ages</u> be treated identically, even though they can be distinguished by birthdate. That is,

$$(3.2)\qquad c_j(t) = c_j(\tau) = c_j \geq 0,\ j \geq 0 \qquad \underline{\text{all}}\ t,\ \tau.$$

Then an allocation $\{c_j\}_{j=0}^{\infty}$ is said to be <u>optimal</u> if there does not exist another allocation $\{\bar{c}_j\}_{j=0}^{\infty}$ with the property that

$$(3.3)\qquad \sum_{j=h}^{\infty} \beta^{j-h} U(\bar{c}_j) \geq \sum_{j=h}^{\infty} \beta^{j-h} U(c_j),\ h \geq 0$$

with strict inequality for at least one such h. Note that for $h = 0$, the terms in (3.3) represent the utility of one agent born at time $t \geq 0$, and for $h > 0$, the utility of an agent born at time $t = -h$. Thus the preferences of <u>all</u> agents are taken into account.

It is now claimed that the solution $\{c_j^*\}_{j=0}^{\infty}$ of the following problem is optimal in the above sense.

<u>Problem 2</u>:

$$\max_{\{c_j\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} \beta^j U(c_j)$$

subject to

$$c_j + c_{j+1} = 1 \qquad c_j \geq 0,\ j \geq 0,\ j \text{ even.}$$

To establish the claim note that, due to the time separable nature of the objective function and of the constraints, the unique solution to this problem is

$$c_j^* = c^*, \; c_{j+1}^* = c^{**} \qquad j \geq 0, \; j \text{ even}$$

where

$$\frac{U'(c^*)}{\beta U'(c^{**})} = 1, \; c^* + c^{**} = 1.$$

Now suppose there exists a feasible allocation $\{\bar{c}_j\}_{j=0}^{\infty}$ which Pareto dominates $\{c_j^*\}_{j=0}^{\infty}$. If an agent born at time $t \geq 0$ is to be better off under $\{\bar{c}_j\}_{j=0}^{\infty}$, then consumption must be increased for at least one element $\bar{c}_i$. Suppose $i \geq 0$ is even. Then feasibility requires that $\bar{c}_{i+1}$ be decreased. But c^* and c^{**} are chosen in such a way that such changes can only make the agent born at time t worse off, i.e.,

$$U(\bar{c}_i) + \beta U(\bar{c}_{i+1}) < U(c_i^*) + \beta U(c_{i+1}^*).$$

A similar argument applies for $i \geq 0$ and odd. Hence if $\{\bar{c}_j\}_{j=0}^{\infty}$ is to Pareto dominate $\{c_j^*\}_{j=0}^{\infty}$, it must make at least one agent born at time $-h < 0$ better off. By the above argument, an increase in utility is possible only for the relatively old person of some market at time $t = 0$, i.e., only if there is an increase in the element $c_h(-h)$, $h \geq 1$, h odd. But then the representative trader born at time $t \geq 0$ must be made worse off, and, by the above argument, there can be no compensating changes elsewhere. This establishes the claim.

Unlike the procedure in the turnpike model, no attempt is made here to characterize all possible Pareto optimal allocations in the restricted class. In the simple two-period overlapping-generations model other optimal allocations in the above sense do exist. Moreover, under specified assumptions, each of these can be supported in a monetary equilibrium with deflation and lump-sum taxation.

Further, there exist monetary equilibria with inflation and lump-sum subsidization which are nonoptimal. Analogues of these results could be sought here. Instead, attention will be limited to generalizing the well-known proposition mentioned at the beginning of this section, that there exists a noninterventionist monetary equilibrium which supports the above described optimal allocation.

To define a monetary equilibrium some additional notation is needed. Let p_t^k denote the price of the consumption good in market k at time t, $k \geq 0$, $t \geq 0$. Let $M_j(t)$ denote the money balances held by the agent born at time t at the beginning of period j of his life, chosen at age j-1. As attention is restricted to noninterventionist equilibria, no notation for lump-sum taxes is needed. As before, each agent takes initial money balances and the sequence of future prices as given and maximizes utility by choice of the sequences of money balances and consumptions over his infinite lifetime. Thus, for an agent born at time $t \geq 0$, consider

<u>Problem t:</u>

$$\max_{\{c_j(t)\}_{j=0}^{\infty},\ \{M_j(t)\}_{j=1}^{\infty}} \sum_{j=0}^{\infty} \beta^j U[c_j(t)]$$

subject to

$$c_j(t) \geq 0,\ M_j(t) \geq 0 \qquad j \geq 0$$

$$(b_j(t)) \quad p_{t+j}^k y_j + M_j(t) = p_{t+j}^k c_j(t) + M_{j+1}(t) \qquad j \geq 0$$

given

$$M_0(t) \geq 0,\ k = \begin{cases} \frac{j}{2} & j \text{ even, } j \geq 0 \\ \frac{(j-1)}{2} & j \text{ odd, } j \geq 1. \end{cases}$$

Similarly for an agent born at time $-h < 0$, consider

Problem $-h$:

$$\max_{\{c_j(-h)\}_{j=h}^{\infty},\ \{M_j(-h)\}_{j=h+1}^{\infty}} \sum_{j=h}^{\infty} \beta^{j-h} U[c_j(-h)]$$

subject to

$$c_j(-h) \geq 0,\ M_j(-h) \geq 0 \qquad j \geq h$$

$$(b_j(h)) \quad p_{j-h}^{k} y_j + M_j(-h) = p_{j-h}^{k} c_j(-h) + M_{j+1}(-h) \qquad j \geq h$$

given

$$M_h(-h) \geq 0,\ k = \begin{cases} \frac{j}{2} & j \text{ even},\ j \geq h \\ \frac{(j-1)}{2} & j \text{ odd},\ j \geq h. \end{cases}$$

One may now write out formally the following

Definition: A monetary equilibrium is a sequence of finite positive prices $\{p_t^{k*}\}_{t=0}^{\infty}$ for each market $k \geq 0$; sequences of consumptions $\{c_j^*(t)\}_{j=0}^{\infty}$ and money balances $\{M_j^*(t)\}_{j=0}^{\infty}$ for the agent born at each time $t \geq 0$; and sequences of consumptions $\{c_j^*(-h)\}_{j=h}^{\infty}$ and money balances $\{M_j^*(-h)\}_{j=h}^{\infty}$ for the agent born at each time $-h < 0$ such that

i) maximization for agent t — the sequences $\{c_j^*(t)\}_{j=0}^{\infty}$ and $\{M_j^*(t)\}_{j=1}^{\infty}$ solve problem t given $M_0^*(t)$,

ii) maximization for agent $-h$ — the sequences $\{c_j^*(-h)\}_{j=h}^{\infty}$ and $\{M_j^*(-h)\}_{j=h+1}^{\infty}$ solve problem $-h$ given $M_h^*(-h)$,

iii) market clearing —

$$c_j^*(t) + c_{j+1}^*(t-1) = 1 \qquad t \geq 0,\ j \geq 0,\ j \text{ even},$$

$$c_j^*(-h) + c_{j+1}^*(-h-1) = 1 \qquad h \geq 1,\ j \geq h,\ j \text{ even}.$$

To characterize one of the monetary equilibria of this model, return for a moment to problem (t). Differentiating with respect to $M_j(t)$, familiar necessary conditions for a maximum are obtained:

$$(3.4)\qquad -\frac{\beta^{j-1}U'[c_{j-1}(t)]}{p^k_{t+j-1}} + \frac{\beta^j U'[c_j(t)]}{p^k_{t+j}} + \theta_j(t) = 0 \qquad j \geq 1$$

where $\theta_j(t)$ is the nonnegative Lagrange multiplier associated with the constraint $M_j(t) \geq 0$. Expression (3.4) yields

$$(3.5)\qquad \frac{U'[c_{j-1}(t)]}{\beta U'[c_j(t)]} \geq \frac{p^k_{t+j-1}}{p^k_{t+j}} \qquad j \geq 1$$

where equality prevails if $M_j(t) > 0$. As before with $U'(0) = \infty$ and $y_j = 0$ for $j \geq 1$, j odd, it is obvious that <u>equality must prevail</u> for $j \geq 1$, j odd, <u>and all $t > 0$</u>.

Now suppose the optimal allocation $c^*_j = c^*$, $c^*_{j+1} = c^{**}$, $j \geq 0$, j even, were to be supported in a monetary equilibrium. Then with $j = 1$ in (3.5) as an equality,

$$1 = \frac{U'(c^*)}{\beta U'(c^{**})} = \frac{p^{0*}_t}{p^{0*}_{t+1}} \qquad \text{all } t \geq 0$$

where the equality on the left follows from the construction of c^* and c^{**}. That is, the price in market zero must remain constant over time. A similar argument yields the fact that the price of <u>each</u> market $k \geq 0$ must remain constant over time. Moreover, suppose (3.5) were to hold as an equality in such an equilibrium for j even as well. (That is, suppose the nonnegativity constraints on money balances were never binding.) Then with $j = 2$ in (3.5) as an equality,

$$\frac{1}{\beta^2} = \frac{U'(c^{**})}{\beta U'(c^*)} = \frac{p^{0*}_{t+1}}{p^{1*}_{t+2}} \qquad \text{all } t \geq 0$$

where again the equality on the left follows from the construction of c^* and c^{**}. That is, $p^{0*}_{t+1} > p^{1*}_{t+2}$ so that the price level would decrease as the agent born at time t moves across markets, from market zero at time t+1 to market one at time t+2. Again j even and $t \geq 0$ were arbitrary, so this relationship would hold across any two adjacent markets for any time period t.

Thus it may be guessed that the optimal allocation c^*, c^{**} can be supported in a monetary equilibrium with constant prices over time in each market, with deflation cross-sectionally over markets, and with deflation in every other period of each agent's lifetime. Before establishing this conjecture formally, it may be instructive to answer this question: How can it be that there exists a noninterventionist monetary equilibrium in this model with deflation but without taxation? The answer, of course, is that the price level stays constant in each market. In equilibrium the relatively old person of each market passes along all of his money holdings to the relatively young person, who then does the same in the next period. That is, money itself never moves across markets, and so real balances stay constant in each market. In equilibrium nominal money balances decline over markets with the price level; real balances stay constant over markets.

This discussion is now summarized in

Proposition 3.1: The optimal allocation c^*, c^{**} can be supported in a (noninterventionist) monetary equilibrium with constant prices over time in each market, with deflation rate $(1-\delta^2)$ across adjacent markets, and with nonbinding nonnegativity constraints on money balances for each agent of any age. In particular, for prices,

$$p_t^{k*} \equiv p^{k*} > 0 \qquad k \geq 0$$

$$p^{k*} = \delta^2 p^{k-1*} \qquad k \geq 1;$$

for the agent born at each time $t \geq 0$,

$$c_j^*(t) = c^*,\ M_{j+1}^*(t) = p^{k^*}c^{**} \qquad j \geq 0,\ j \text{ even}$$

$$c_j^*(t) = c^{**},\ M_{j+1}^*(t) = 0 \qquad j \geq 1,\ j \text{ odd}$$

where k is defined in problem (t); and for the agent born at each time $-h < 0$,

$$c_j^*(-h) = c^*,\ M_{j+1}^*(-h) = p^{k^*}c^{**} \qquad j \geq h,\ j \text{ even}$$

$$c_j^*(-h) = c^{**},\ M_{j+1}^*(-h) = 0 \qquad j \geq h,\ j \text{ odd}$$

where k is defined in problem (-h).

Proof: See the appendix.

Thus, it has been established that there exists an optimal allocation in this model which can be supported in a noninterventionist monetary equilibrium. Yet proposition (2.1) asserts that this is not possible in the turnpike model. Wherein lies the difference?

To be noted is that the allocation $c_j = c^*$, $c_{j+1} = c^{**}$ for $j \geq 0$, j even, is optimal here, in this generalized overlapping-generations model, but is not optimal in the turnpike model. (More specifically, the allocation $c_t^B = c^*$, $c_{t+1}^B = c^{**}$, $t \geq 0$, t even, is not optimal there.) This result turns on the fact that in the overlapping-generations model agents are paired at different ages. The optimal allocation takes into account that the young in each market prefer present over future consumption. Thus the age structure seems to be crucial.

It may be noted in passing that the allocation c^*, c^{**} can be supported in a noninterventionist monetary equilibrium in both models, yet the equilibria seem qualitatively different. In the generalized overlapping-generations model, the nonnegativity constraints on money balances are never binding, whereas in the turnpike model they are binding in alternate periods. This may lead one to question whether in the turnpike model equilibria with nonbinding constraints have been ruled out in some way. In particular, prices have been restricted to be constant across markets, whereas in the equilibrium of proposition (3.1) prices fall over markets. Yet Figure 1 reveals that in the turnpike model prices cannot fall over markets in the right way for all agents. Falling prices for agents travelling west imply rising prices for agents travelling east, for example.

4. The Lucas Version of the Cass-Yaari Model

Thus far attention has been restricted to models which have the property that money allows the economy to achieve a Pareto superior allocation of goods over time, relative to autarky. For the individual, money plays a role in equating, at least partially, intertemporal marginal rates of substitution. This has lead some to claim that money in such models serves as a store of value rather than as a medium of exchange. This section presents a third model with spatially separated agents in which money plays a role in achieving intratemporal efficiency (as well). In essence, the model is the well-known Cass-Yaari circle, but with trader pairs and a timing of transactions as suggested by Lucas.[11/]

The model consists of a countably infinite number of households and a countably infinite number of perishable commodities. Each (representative) household consists of a pair of agents and is imagined to be located on the real line, say one household per integer. See Figure 7. Each household i lives forever and faces an endowment sequence of commodity i which is constant, say one unit in each period $t \geq 0$. In each period t, each member of household i is capable of moving one-half the distance to one of the two adjacent integers, $(i+1)$ and $(i-1)$. Thus, in each period t, each household i is physically capable of carrying out transactions with households $(i-1)$ and $(i+1)$ in two spatially separated markets. There is no storage.

Household i cares only about commodities i and $(i+1)$, and discounts future over present consumption. Thus letting $c_{it}(i)$ and $c_{i+1,t}(i)$ denote the number of units of consumption by household i at time t of commodities i and $(i+1)$, respectively, the preferences of household i are represented by the utility function $\sum_{t=0}^{\infty}\beta^{t}V[c_{it}(i),c_{i+1,t}(i)]$, $0 < \beta < 1$. Here also $V[\cdot,\cdot]$ is strictly concave, strictly increasing, bounded, and continuously differentiable

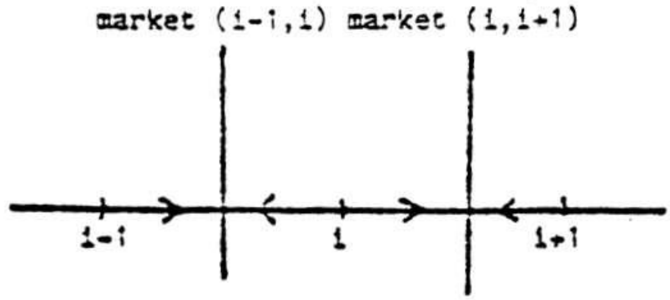

Figure 7: The Cass-Yaari Model

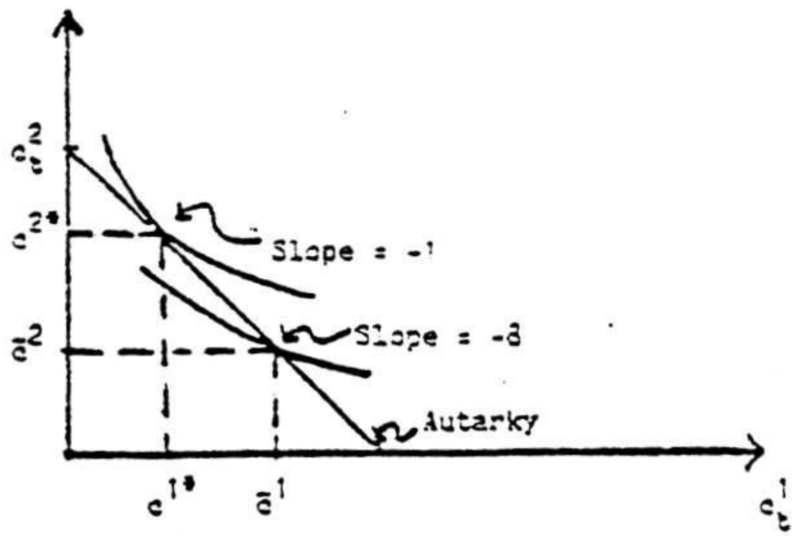

Figure 8: Equilibria in Lucas' Cass-Yaari Model

with indifference curves which are asymptotic to the axes. (A particular functional form will be assumed for some purposes in what follows.)

As Cass and Yaari note, this model displays the absence of double-coincidence of wants. At each time t each household i can trade with household (i+1), but i has no commodity (i+1) wants. It also should be noted that this model reverses the construction of Cass and Yaari, breaking their circle at some point and spreading it back out over the real line, with infinite extensions.[12/] As in the turnpike model, this serves to eliminate the possibility of private debt. Household i may issue an IOU to household (i+1) in exchange for commodity (i+1), but this IOU can be returned to household i only by household (i+1), and, as noted, i has no commodity (i+1) wants.

The next step in the analysis is to define feasible allocations and characterize those which are Pareto optimal. Without loss of generality attention is restricted to those allocations in which each household receives at most those commodities which enter its utility function. Thus an <u>allocation</u> is a sequence of consumptions $\{c_{it}(i), c_{i+1,t}(i)\}_{t=0}^{\infty}$ for each household i. An allocation is said to be <u>feasible</u> if

$$(4.1)\quad \begin{aligned} &c_{it}(i) + c_{it}(i-1) \leq 1 && t \geq 0, \text{ all integers } i \\ &c_{it}(i) \geq 0,\ c_{it}(i-1) \geq 0 && t \geq 0, \text{ all integers } i. \end{aligned}$$

Also, without loss of generality, the resource constraint in (4.1) is assumed to hold as an equality. Now in order to characterize Pareto optimal allocations, a strong symmetry condition is imposed—that in any feasible allocation each household i be treated identically with respect to "own" consumption, of commodity i, and "other" consumption, of commodity i+1. That is, an allocation $\{c_{it}(i), c_{i+1,t}(i)\}_{t=0}^{\infty}$ for all i, is said to be <u>symmetric</u> if

$$(4.2)\quad c_{it}(i) = c_t^1,\ c_{i+1,t}(i) = c_t^2 \qquad t \geq 0, \text{ all integers } i.$$

Within the class of such symmetric allocations, then, feasibility is equivalent with

$$(4.3) \qquad c_t^1 + c_t^2 = 1, \; c_t^1 \geq 0, \; c_t^2 \geq 0, \; t \geq 0.$$

It is now claimed that, subject to this symmetry restriction, the unique Pareto optimal allocation may be found as the solution to

<u>Problem 3</u>:

$$\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t V[c_t^1, c_t^2]$$

subject to (4.3). As the objective function and constraints sets are time separable, it is obvious that the unique solution $\{c_t^{1*}, c_t^{2*}\}_{t=0}^{\infty}$ to this problem satisfies

$$c_t^{1*} = c^{1*}, \; c_t^{2*} = c^{2*} \qquad \text{all } t \geq 0$$

where

$$(4.4) \qquad \frac{V_1(c^{1*}, c^{2*})}{V_2(c^{1*}, c^{2*})} = 1, \qquad c^{1*} + c^{2*} = 1.$$

(See Figure 8.)

Any symmetric feasible allocation which is supposed to improve upon this solution must satisfy (4.3) and increase utility in some period t. The choice of c^{1*} and c^{2*} makes this impossible. Similarly, any symmetric feasible allocation which differs from this solution can be improved upon, and hence is not an optimum. Finally, note that the unique Pareto optimum is defined completely by intratemporal considerations.

As before, one now seeks to discover the relationship between optimal allocations and monetary equilibria. Thus, suppose at each time $t \geq 0$ that households i and (i+1) meet in a competitive market in which commodity (i-1) can be exchanged for fiat money. Thus, let $p_{i+1,t}$ denote the price of commodity $(i+1)$ in terms of fiat money at time $t \geq 0$. Also, let $M_t(i)$ denote the number of units of fiat money held by household i at the beginning of period t, and let $z_t(i)$ denote the lump-sum tax. Finally, let $y_{it}(i)$ denote the endowment of commodity i of household i at time t, so that $y_{it}(i) \equiv 1$. At the beginning of each period t, one member of household i travels to the market $(i, i+1)$ with some of the beginning-of-period money balances and purchases commodity $(i+1)$ at the price $p_{i+1,t}$. Similarly, the other member of household i travels to the market $(i-1, i)$ with some of the endowment of commodity i and sells it for fiat money at the price p_{it}. At the end of each period t, both members of household i return to their original location and consume. Thus, taking the price sequence $\{p_{it}, p_{i+1,t}\}_{t=0}^{\infty}$ and the tax sequence $\{z_t(i)\}_{t=0}^{\infty}$ as given, each household i is confronted with

<u>Problem (i)</u>:

$$\max_{\{c_{it}(i), c_{i+1,t}(i)\}_{t=0}^{\infty}, \{M_t(i)\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t V[c_{it}(i), c_{i+1,t}(i)]$$

subject to

$$c_{it}(i) \geq 0,\ c_{i+1,t}(i) \geq 0,\ M_t(i) \geq 0 \qquad t \geq 0$$

$$(b_t(i)) \qquad p_{it}y_{it}(i) + M_t(i) - z_t(i) =$$

$$p_{it}c_{it}(i) + p_{i+1,t}c_{i+1,t}(i) + M_{t+1}(i) \qquad t \geq 0$$

$$(a_t(i)) \qquad p_{i+1,t}c_{i+1,t}(i) \leq M_t(i) \qquad t \geq 0$$

given $M_0(i) \geq 0$, $z_0(i) = 0$.

Here $(b_t(i))$ is the money balance accumulation equation, and $(a_t(i))$ is the constraint that the valuation of consumption of commodity $(i+1)$ by household i is bounded by beginning-of-period money balances. Thus $a_t(i)$ is very much in the spirit of a Clower constraint. <u>But here this constraint is generated by the underlying exchange technology of the model.</u>[13/]

In what follows, attention will be restricted to equilibria which are symmetric across households in that $\{p_{it}\}_{t=0}^{\infty} = \{p_t\}_{t=0}^{\infty}$ for all commodities i; and $\{z_t(i)\}_{t=0}^{\infty} = \{z_t\}_{t=0}^{\infty}$, $\{M_t(i)\}_{t=0}^{\infty} = \{M_t\}_{t=0}^{\infty}$, and $\{c_{it}(i), c_{i+1,t}(i)\}_{t=0}^{\infty} = \{c_t^1, c_t^2\}_{t=0}^{\infty}$ for all households i. Under these symmetry restrictions the problem of each household i is the same, namely, the problem of the representative household,

<u>Problem R</u>:

$$\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}, \{M_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t V[c_t^1, c_t^2]$$

subject to

$$c_t^1 \geq 0,\ c_t^2 \geq 0,\ M_t \geq 0 \qquad t \geq 0$$

$$(b_t) \qquad p_t y_t + M_t - z_t = p_t c_t^1 + p_t c_t^2 + M_{t+1}$$

$$(a_t) \qquad p_t c_t^2 \leq M_t$$

given $M_0 \geq 0$, $z_0 = 0$ with $y_t \equiv 1$.

The above discussion leads to the following

<u>Definition</u>: A <u>symmetric monetary equilibrium</u> is a sequence of finite positive prices $\{p_t^*\}_{t=0}^{\infty}$, and sequences of consumptions $\{c_t^{1*}, c_t^{2*}\}_{t=0}^{\infty}$, money balances $\{M_t^*\}_{t=0}^{\infty}$, and taxes $\{z_t^*\}_{t=0}^{\infty}$ such that

i) maximization—the sequences $\{c_t^{1*}, c_t^{2*}\}_{t=0}^{\infty}$ and $\{M_t^*\}_{t=1}^{\infty}$ solve problem (R) relative to $\{p_t^*\}_{t=0}^{\infty}$, $\{z_t^*\}_{t=0}^{\infty}$, and M_0^*, and

ii) market clearing—$c_t^{1*} - c_t^{2*} = 1$, all $t \geq 0$.

In order to discover the relationship between symmetric monetary equilibria and optimal allocations it is useful to consider the necessary Euler conditions for a maximum to problem (R). Assuming nonbinding nonnegativity constraints on money balances (and consumption) and following Lucas and Palmon [1978], these are of the form

$$(4.5)\qquad -\beta^t V_1(c_t^1, c_t^2) + \beta^t V_2(c_t^1, c_t^2) - p_t^* \theta_t = 0 \qquad t \geq 0$$

$$(4.6)\qquad V_1(c_{t-1}^1, c_{t-1}^2) = (p_{t-1}^*/p_t^*)\beta V_2(c_t^1, c_t^2) \qquad t \geq 1$$

where θ_t is the nonnegative Lagrange multiplier associated with the constraint (a_t). One implication is almost immediate,

<u>Proposition 4.1</u>: The optimal allocation c^{1*}, c^{2*} cannot be supported in a noninterventionist symmetric monetary equilibrium, i.e., with $z_t^* \equiv 0$.

<u>Proof</u>: Suppose the contrary. Then it follows from (4.6) and the construction of the optimum (4.4) that in such an equilibrium the rate of deflation must be $1 - \beta$, i.e.,

$$(4.7)\qquad p_t^* = \beta p_{t-1}^* \qquad t \geq 1.$$

Also, from the money balance accumulation equation (b_t) and feasibility of the optimum,

$$(4.8)\qquad M_{t+1}^* - M_t^* = p_t^*(y_t - c^{1*} - c^{2*}) = 0 \qquad t \geq 1.$$

Now consider constraint (a_t) at $t = 0$,

$$(4.9)\qquad p_0^* c^{2*} \leq M_0^*.$$

Repeated substitution of (4.7) and (4.8) into (4.9) yields

$$(4.10)\qquad p_t^* c^{2*} < M_t^* \qquad \text{all } t \geq 1.$$

Then holding the consumption sequence $\{c_t^1\}_{t=0}^{\infty}$ fixed identically at c^{1*}, the representative household could increase consumption of c_t^2 over c^{2*} in every period $t \geq 1$ by spending the "surplus" money balances. This is the desired contradiction, and it completes the proof. (For an alternative argument see Locay and Palmon.)

Proposition (4.1) of this model is the analogue of proposition (2.1) in the turnpike model. And it seems that propositions (2.2) and (2.4) of the turnpike model have analogues here as well; that is, the optimal allocation can be supported in an interventionist monetary equilibrium, and there exists a noninterventionist monetary equilibrium which is nonoptimal but Pareto superior to autarky.[14/] For according to Locay and Palmon, the necessary transversality condition for the maximization problem confronting the representative household is

$$(4.11) \qquad \lim_{t\to\infty} \frac{\beta^t M_t V_2(c_t^1, c_t^2)}{p_t} = 0.$$

Then for the interventionist monetary equilibrium which is to support the optimal allocation c^{1*}, c^{2*}, consider the following specification. Let $M_0^* = p_0^* c^{2*}$ so that the representative household spends all <u>initial</u> money balances on the consumption good with which it is not endowed. Similarly, <u>in each period t</u> let the representative household spend all after-tax money holdings on this commodity, acquiring additional money from the sale of the endowment commodity, $(y_t - c^{1*})$. Also, let the rate of deflation be $1-\beta$. In summary, then, let

$$(4.12) \qquad M_{t+1}^* = p_t^*(y_t - c^{1*}) \qquad t \geq 0$$

$$(4.13) \qquad z_t^* = (M_t^* - p_t^* c^{2*}) > 0 \qquad t \geq 1$$

$$(4.14) \qquad p_t^* = \beta p_{t-1}^* \qquad t \geq 1.$$

It is apparent that this specification satisfies the necessary and sufficient conditions for a maximum, (4.5), (4.6), and (4.11), with nonbinding constraints (a_t), i.e., $\theta_t \equiv 0$.

For the noninterventionist monetary equilibrium consider the following specification. First, let prices be constant; then, motivated by (4.6), let $c_t^1 \equiv \bar{c}^1$, $c_t^2 \equiv \bar{c}^2$, where $\bar{c}^1$ and $\bar{c}^2$ are uniquely defined by

$$(4.15) \qquad \frac{V_1(\bar{c}^1,\bar{c}^2)}{V_2(\bar{c}^1,\bar{c}^2)} = \beta, \qquad \bar{c}^1 + \bar{c}^2 = 1.$$

Again, suppose that <u>all</u> beginning-of-period money balances are spent on the "other" consumption good, these being replenished from the sale of the "own" consumption good. That is, let

$$(4.16) \qquad M_{t+1}^* = p_t^*(y_t-\bar{c}^1)$$

$$(4.17) \qquad M_t^* = p_t^*\bar{c}^2$$

$$(4.18) \qquad p_t^* \equiv p^* > 0.$$

Again, the necessary and sufficient first-order conditions for a maximum are satisfied, this time with binding constraints (a_t), i.e., $\theta_t > 0$. It is clear that this consumption sequence is nonoptimal but Pareto superior to autarky (see Figure 8).

The reader may be struck by the similarity of the above results to those of the turnpike model. To repeat, optimal allocations cannot be supported in a noninterventionist monetary equilibrium, but there exists a monetary equilibrium with constant prices and binding constraints which is Pareto superior to autarky.[13/] Yet here, unlike the turnpike model, the imposition of a stronger Clower-type constraint may be sufficient to generate a monetary equilibrium without taxation which is optimal. In fact, the imposition of such a constraint can convert the

Cass-Yaari model into Lucas' [1979] model of money with certainty. These results are now established.

The above scheme is modified in two ways. First, the utility function $V[\cdot,\cdot]$ is assumed to be of the form

$$V[c^1,c^2] = U[(c^1/\alpha_1)^{\alpha_1}(c^2/\alpha_2)^{\alpha_2}]$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$, and where $U(\cdot)$ satisfies all the assumptions of the previous two sections. Second, the constraint (a_t) in problem (R) is strengthened to

$$(a_t') \qquad p_t c_t^1 + p_t c_t^2 \leq M_t.$$

As in Section 2, the idea here is that the member of household i who travels to the market (i-1, i) with the endowment $y_{it}(i) = 1$ must pay cash in advance for any units of commodity i which he is to take home. And again, a Clower-type symmetric monetary equilibrium may be defined in the obvious way, with (a_t') replacing (a_t) in problem (R), and $z_t^* \equiv 0$. This leads to

Proposition 4.2: The optimal allocation c^{1*}, c^{2*} can be supported in a Clower-type symmetric monetary equilibrium with constant prices. In particular, $c_t^{1*} \equiv c^{1*}$, $c_t^{2*} \equiv c^{2*}$, $p_t^* \equiv p^* > 0$, and $M_t^* = p^* y_t$ for all $t \geq 0$.

Proof: First let c_t denote real consumption expenditures in period t, i.e.,

$$(4.19) \qquad p^* c_t^1 + p^* c_t^2 = p^* c_t.$$

Substitution of (4.19) into the budget constraint (b_t) yields

$$(4.20) \qquad p^* y_t + M_t - M_{t+1} = p^* c_t.$$

Now fixing M_t and M_{t+1}, the intratemporal period t decision problem of the representative household is of the form

$$\max_{c_t^1 \geq 0,\ c_t^2 \geq 0} U[(c_t^1/\alpha_1)^{\alpha_1}(c_t^2/\alpha_2)^{\alpha_2}]$$

subject to

$$p^*c_t^1 + p^*c_t^2 = p^*c_t.$$

The unique solution to this problem is

$$c_t^1 = \alpha_1 c_t,\ c_t^2 = \alpha_2 c_t,$$

so the indirect utility as a function of c_t is just $U(c_t)$. Hence the problem of the representative household is reduced to

$$\max_{\{M_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

subject to

$$M_t \geq 0,\ c_t \geq 0 \qquad t \geq 0$$

$$p^*y + M_t - M_{t+1} = p^*c_t$$

$$p^*c_t \leq M_t$$

given $M_0^* = p^*y$ where $y = 1$. Lucas [1979] establishes that $M_t \equiv p^*y$ is the unique solution to this problem. Thus $p^*c_t = p^*y$ for all $t \geq 0$, and so the solution to the intratemporal problem must be $c_t^{*1} = c^{1*}$, $c_t^{*2} = c^{2*}$ for all $t \geq 0$, the optimum. This completes the proof.

That the imposition of a strong Clower-type constraint generates an optimal allocation in this model, whereas this is not so of the turnpike model, is somewhat puzzling. This result seems to turn on the fact that there is only one representative agent in this model, whereas there are two representative agents in the turnpike model, and that optimal allocations are defined accordingly.

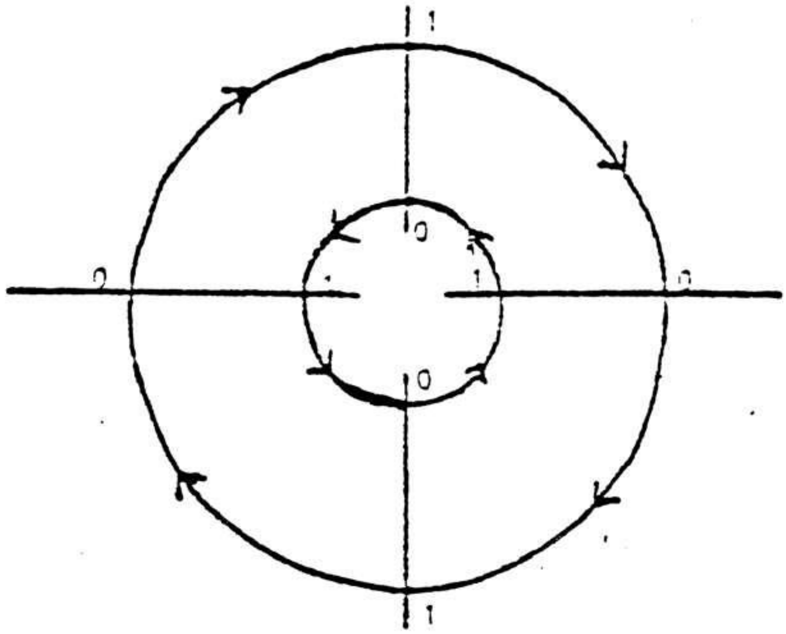

Figure 9: The Turnpike Circle

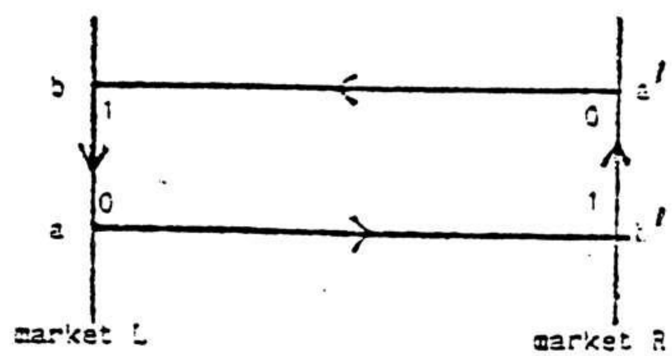

Figure 10: The Debt Model

5. Circles and Private Debt

As noted in the introduction, there are an infinite number of agents alive at any one date in both the turnpike model and in the version of the Cass-Yaari model just presented. This specification ensured that the exclusion of private debt was indeed endogenous. With the removal of this contemporaneous infinity, the role of private debt can be analyzed. This section is intended to be illustrative of the kind of analysis which may be undertaken.

The contemporaneous infinity is removed from the turnpike model by converting it into a circle. This is done in Figure 9 for an economy with eight agents. As before, arrows indicate the direction of travel, spikes indicate islands or markets, and numbers index the endowment of the agent at the indicated position.

Focusing on the pairings of agents in this model, it becomes clear that the set of agents can be partitioned into two groups or subeconomies, where agents in a subeconomy trade only with other agents of that subeconomy. Thus the essential features of the model depicted in Figure 9 can be captured by the simpler model depicted in Figure 10. Here there are only two markets, labelled L and R, respectively, and four agents, labelled a, a', b, and b', at their initial positions. To understand the way agents are paired over time, consider the itinerary of one of the agents. Agent a, of type A, begins in period zero with zero units of the consumption good and is paired in market L with agent b, of type B, who has one unit. In period one, agent a is allocated to market R and has one unit, being paired with agent b'. Continuing, agent a stays in market R in period two and finally moves back to market L in period three. Period four is the same as period zero.

The fact that agents meet repeatedly in this version of the turnpike model has no bearing on the determination of optimal allocations. Under the symmetry condition imposed in Section 2, an (interior) optimum has the property that each agent of type A receives λ units of the consumption good in each period. In fact, <u>all the propositions of Section 2 apply to this economy if one accepts the exogenous exclusion of debt</u>. Yet now there may be private debt equilibria. That is, <u>debt may be used as a means of payment</u>.

For the purpose of discussing private debt in this economy, attention is restricted first to the obvious four-period version of this model. (Again this has no effect on the properties of optima.) A particular scheme is considered. In the initial period, $t = 0$, each agent of type A is permitted to issue IOUs, where one such IOU is a promise to pay to the holder $(1+r)$ units of consumption good in period three. Both the interest rate r and the price $p_0 > 0$ of the consumption good, in terms of such IOUs in period zero, are taken as given by agents a and a'. Thus, the problem confronting each agent of type A in period zero is

<u>Problem A(0,3)</u>:

$$\max_{B_0^A \geq 0,\ c_0^A \geq 0,\ c_3^A \geq 0} U(c_0^A) + \beta^3 U(c_3^A)$$

subject to

$$(5.1) \qquad p_0 c_0^A \leq B_0^A$$

$$(5.2) \qquad c_3^A \leq y_3^A - (B_0^A - b_3^A)(1+r)$$

where B_0^A is the number of IOUs issued by agent type A and $-z_3^A$ is a lump-sum forgiveness (subsidy) of debt in period three. The above two budget constraints may be assumed to hold as equalities. Here the nonnegativity constraints may be ignored, yielding the necessary first-order condition

$$(5.3) \qquad \frac{U'(c_0^A)}{p_0} = \beta^3 U'(c_3^A)(1+r).$$

In periods one and two the debt issued by agents a and a' is traded in markets L and R, respectively. In particular, agent a can purchase the debt (of a') in market R in period one and sell the debt in market R in period two. Letting p_1 and p_2 denote the price of the consumption good in terms of IOUs in periods one and two, respectively, the problem confronting each agent type A in period one is

<u>Problem A(1,2)</u>:

$$\max_{B_2^A \geq 0,\ c_1^A \geq 0,\ c_2^A \geq 0} \beta U(c_1^A) + \beta^2 U(c_2^A)$$

subject to

$$(5.4) \qquad p_1 c_1^A \leq p_1 y_1^A - B_2^A$$

$$(5.5) \qquad p_2 c_2^A \leq B_2^A - z_2^A$$

where B_2^A is the number of IOUs acquired by agent type a in period one and z_2^A is a lump-sum tax (confiscation) of IOUs in period two. With $z_2^A \geq 0$ the nonnegativity constraints may be ignored, yielding the necessary first-order condition

$$(5.6) \qquad \frac{\beta U'(c_1^A)}{p_1} = \frac{\beta^2 U'(c_2^A)}{p_2}.$$

It is now obvious that the problem confronting each agent type B in period zero is

Problem B(0,1):

$$\max_{B_1^B \geq 0,\ c_0^B \geq 0,\ c_1^B \geq 0} U(c_0^B) + \beta U(c_1^B)$$

(5.7) $$p_0 c_0^B \leq p_0 y_0^B - B_1^B$$

(5.8) $$p_1 c_1^B \leq B_1^B - z_1^B$$

where B_1^B is number of IOUs acquired by agent type B in period zero and $z_1^B \geq 0$ is the lump-sum confiscation in period one. The necessary first-order condition is

(5.9) $$\frac{U'(c_0^B)}{p_0} = \frac{\beta U'(c_1^B)}{p_1}.$$

Similarly one obtains

Problem B(2,3):

$$\max_{B_3^B \geq 0,\ c_2^B \geq 0,\ c_3^B \geq 0} \beta^2 U(c_2^B) + \beta^3 U(c_3^B)$$

(5.10) $$p_2 c_2^B \leq p_2 y_2^B - B_3^B$$

(5.11) $$c_3^B \leq (B_3^B - z_3^B)(1+r)$$

with necessary first-order condition

(5.12) $$\frac{\beta^2 U'(c_2^B)}{p_2} = \beta^3 U'(c_3^B)(1+r).$$

These procedures lead to the following

Definition: A private debt equilibrium is an interest rate r^*, a sequence of finite positive prices $\{p_t^*\}_{t=0}^2$, and sequences of lump-sum taxes $\{z_t^{A*}\}_{t=2,3}$, $\{z_t^{B*}\}_{t=1,3}$, consumptions $\{c_t^{A*}\}_{t=0}^3$, $\{c_t^{B*}\}_{t=0}^3$, and debt decisions $\{B_t^{A*}\}_{t=0,2}$, $\{B_t^{B*}\}_{t=1,3}$ such that

i) maximization for A—c_0^{A*}, c_3^{A*}, B_3^{A*} solve problem A(0,3) relative to r^*, p_0^*, z_3^{A*}; and c_1^{A*}, c_2^{A*}, B_2^{A*} solve problem A(1,2) relative to p_1^*, p_2^*, z_2^{A*}:

ii) maximization for B—c_0^{B*}, c_1^{B*}, B_1^{B*} solve problem B(0,1) relative to p_0^*, p_1^*, z_1^{B*}; and c_2^{B*}, c_3^{B*}, B_2^{B*} solve problem B(2,3) relative to p_2^*, r^*, z_3^{B*}; and

iii) market clearing—$c_t^{A*} + c_t^{B*} = 1$, t=0, 1, 2, 3.

A major point of this section is that the <u>decentralization of the turnpike model cannot be overcome with private debt</u> alone. To see this, suppose for the moment that all four agents of the above model were in the same market in each of the four periods. Then there is a (centralized) Arrow-Debreu competitive equilibrium with

$$p_t^* = \delta p_{t-1}^* \qquad t=1,2 \qquad 1 - r^* = \frac{1}{\delta^3} \quad .$$

$$c_t^{A*} \equiv \frac{\delta}{1+\delta} \qquad c_t^{B*} \equiv \frac{1}{1+\delta} \qquad t=0,\ 1,\ 2,\ 3.$$

Of course this allocation is optimal. Yet it turns out that neither this allocation nor any other optimal allocation can be achieved in the decentralized economy under a private debt equilibrium without taxation. More formally, consider

<u>Proposition 5.1</u>: No interior optimum λ can be supported in a private debt equilibrium without taxation, i.e., with $z_t^{i*} \equiv 0$ for i = A, B.

<u>Proof</u>: Suppose the contrary. Then from problem B(0,1) and (5.9), $p_1^* = \delta p_0^*$. Budget constraint (5.7) as an equality yields $B_1^{B*} = p_0^*\lambda$. Substitution into (5.8) yields $\delta = \lambda/(1-\lambda)$. From problem A(1,2) and (5.6), $p_2^* = \delta p_1^*$. Budget constraint (5.4) as an equality yields $B_2^{A*} = p_1^*(1-\lambda)$. Substitution into (5.5) yields $\delta = (1-\lambda)/\lambda$. Figure 3 makes clear with $\delta < 1$ that these two specifications of δ are inconsistent. This completes the proof.

Proposition (5.1) and its analogue, proposition (2.1), suggest that inside money in the turnpike model acts very much like outside money. In fact, the analogue of proposition (2.2) may be obtained as well.

Proposition 5.2: Any interior optimum λ with $\beta \leq [\lambda/(1-\lambda)]$ and $\beta \leq [(1-\lambda)/\lambda]$ can be supported in a private debt equilibrium with lump-sum taxation and forgiveness of debt.

Proof: From problem A(0,3) and (5.3) let $(1+r^*)p_0^*\beta^3 = 1$. From (5.1) let $p_0^*\lambda = B_0^{A*}$, and from (5.2) let $z_3^{A*} = (1-\lambda)\beta^3 p_0^* - \lambda p_0^*$. From problem A(1,2) and (5.6) let $p_2^* = \beta p_1^*$. From (5.4) let $B_2^{A*} = p_1^*(1-\lambda)$, and from (5.5) let $0 \leq z_2^{A*} = p_1^*[(1-\lambda)-\beta\lambda]$. From problem B(0,1) and (5.9) let $p_1^* = \beta p_0^*$. From (5.7) let $B_1^{B*} = \lambda p_0^*$, and from (5.8) let $0 \leq z_1^{B*} = p_0^*[\lambda-(1-\lambda)\beta]$. Finally, from problem B(2,3) and (5.12) let $\beta p_2^*(1+r^*) = 1$. From (5.10) let $B_3^{B*} = p_2^*\lambda$, and from (5.11) let $0 \leq z_3^{B*} = p_2^*[\lambda-(1-\lambda)\beta]$. Now by construction, all the first-order conditions for maxima are statisfied, with the budget constraints as equalities in every period. This is sufficient for the proposed solution to satisfy the maximizing conditions (i) and (ii) of an equilibrium. Market-clearing condition (iii) is satisfied by construction also. Finally, it may be noted as a check on the above procedure that the sum of the confiscations of debt equals the lump-sum forgiveness in period three. This completes the proof.

Thus proposition (5.2) establishes that an optimal allocation can be achieved with nontrivial intervention in private credit markets. At this point one may well ask whether there exists a private debt equilibrium without such lump-sum taxation and forgiveness of debt which is Pareto nonoptimal but Pareto superior to autarky. In particular, can the allocations of the noninterventionist monetary equilibrium of proposition (2.4) be achieved? Perhaps it is now obvious from Section 2 and the above analysis that this question may be answered in the affirmative if one is willing to impose an upper bound on the issue of IOUs. That is, in problem A(0,3) impose the <u>additional exogenous</u> constraint that $B_0^A \leq d$ for some constant d, and define a <u>constrained private debt equilibrium</u> in the obvious way. There follows

<u>Proposition 5.3</u>: There exists a constrained private debt equilibrium with a binding constraint on the issue of IOUs in period zero. In particular,

$$r^* = 0; \; p_t^* = 1, \; t=0, 1, 2; \; z_t^{i*} \equiv 0, \; i=A, B.$$

$$c_t^{A*} = c^{**}, \; c_{t+1}^{A*} = c^*, \; B_t^{A*} = c^{**} = d, \; t=0, 2$$

$$c_t^{B*} = c^*, \; c_{t+1}^{B} = c^{**}, \; B_{t+1}^{B*} = c^{**}, \; t=0, 2.$$

<u>Proof</u>: The relevant first-order conditions and budget constraints are satisfied for problems A(1,2), B(0,1), B(2,3), and modified problem A(0,3). This completes the proof.

Proposition (5.3) turns on the fact that the constraint on the issue of inside money plays the role of a nonnegativity constraint on money balances in the same economy with fiat money. This along with proposition (2.3) may lead one to the conjecture that there does not exist a private debt equilibrium without taxation and without such exogenously imposed constraints. Yet it can be established that for the simple four-period economy described above there does exist at least one such equilibrium.[16/] And clearly one may introduce private debt into an infinite-period economy by duplication of the four-period scheme every four periods. What is not yet clear is the extent to which such equilibria rest on the rather special assumptions which have been loaded into the four-period scheme: that only agents of type A can issue debt in every fourth period, that this debt can only be redeemed four periods after it is issued, and so on. It would seem that a completely unrestricted private debt economy would be plagued by Ponzi schemes. An open and intriguing question is whether relatively unrestricted debt and fiat money can coexist; this is the subject of ongoing research.

6. Concluding Remarks

The contention of this paper is obvious: models of money with spatially separated agents should be taken seriously as models of money. Certainly these communication-cost models explain money in a rigorous way, at least subject to the implicit restrictions of the competitive paradigm. But more research is needed. Remaining to be investigated, for example, are the issues of asset dominance and capital over accumulation when storage is allowed. To be looked at also is the problem of multiple monetary equilibria, especially without all the exogenously imposed symmetry restrictions.

Ultimately, though, it is difficult to make judgements on the relative merits of models in the abstract, without reference either to actual observations or to policy questions. One would like to know, for example, whether models with spatially separated agents can be modified to explain the existence of both inside and outside money. Wallace [1978] has established that the overlapping-generations construct is not subject to this criticism. As for policy issues, the overlapping-generations construct has been shown by Bryant and Wallace [1977], [1978] and by Kareken and Wallace [1977], [1978] to have strong policy implications for both open market operations and international financial arrangements, respectively. It remains to be seen whether other models can do as well on this account, and, if so, whether the implications will be the same.

In closing let us return to the claim that the three models of this paper expain the use of money. This claim is equivalent with the statement that in each of the models there exists a (noninterventionist) monetary equilibrium, one in which money has value. Thus the approach of this paper relies heavily on the competitive paradigm. Ideally, though, competitive equilibrium allocations should be viewed as the outcome of an explicit game or mechanism, e.g., see Shubik [1972], Prescott and Townsend [1978], or Townsend [1978], but this raises an obvious question: why has the competitive mechanism been imposed as opposed to some other? In this regard, consider the welfare theorems of this paper. These theorems are consistent with the view that the operation of competitive markets is possible though direct redistribution of endowments is not, or at least that the first scheme is less onerous than the second. Putting this in another way, if the agents of the model could agree to direct redistribution of the endowments, then Pareto optimal allocations could be achieved without the use of money. The welfare theorems of this paper are also consistent with the view that the operation of competitive markets along with lump-sum taxation of money is more appealing than direct redistribution. Clearly this second view is even more tenuous than the first. Finally, it may be noted that in Lucas' version of the Cass-Yaari model, optimal allocations can be achieved with either lump-sum taxation or the imposition of a Clower constraint, requiring the use of money to purchase commodities. Is there any sense in which one of these schemes is preferable to the other? The point of this discussion is that in the context of the specified economic environments of the models of this paper, any criterion used to select from among various schemes is ad hoc and thus unsatisfactory. <u>What is needed is theory in which the choice of social arrangements or games is endogenous</u>. That is, the <u>environment</u> of the model should be sufficiently rich that certain <u>games</u> or constraints are either technically infeasible, or too costly (if not impossible) to enforce. Models with moral hazard and asymmetric information may be needed, as was suggested at the outset (c.f., Harris and Townsend [1978], [1979] or Townsend [1979]). It would seem to be particularly important in monetary economics to make the choice of joint arrangements endogenous, i.e., to solve Shubik's [1978] start-up problem. As Bryant [1979] has emphasized, the seigniorage associated with the issue of money must be allocated.

Appendix

<u>Proof of Proposition 2.2:</u>

First, let $p_t^* = \beta p_{t-1}^*$, all $t \geq 1$. Next, for agent type A let $c_0^{A*} = \lambda$, $(M_0^{A*}/p_0^*) = \lambda$, and $M_1^{A*} = 0$ so that agent type A spends all of his initial money balances on consumption. Subsequently, tax as needed to maintain the consumption sequence $c_t^{A*} \equiv \lambda$ with money balances returning to zero in every other period:

$$y_t^A = 1,\ c_t^{A*} = \lambda,\ z_t^{A*} = 0,\ M_{t+1}^{A*} = p_t^*(1-\lambda) \qquad t \geq 1,\ t \text{ odd}$$

$$y_t^A = 0,\ c_t^{A*} = \lambda,\ z_t^{A*} = p_t^*[\frac{1-\lambda}{\beta} - \lambda] \geq 0,\ M_{t+1}^{A*} = 0 \qquad t \geq 2,\ t \text{ even.}$$

Similarly, for agent type B let $M_0^{B*} = 0$ and

$$y_t^B = 1,\ c_t^{B*} = 1 - \lambda,\ z_t^{B*} = 0,\ M_{t+1}^{B*} = p_t^*\lambda \qquad t \geq 0,\ t \text{ even}$$

$$y_t^B = 0,\ c_t^{B*} = 1 - \lambda,\ z_t^{B*} = p_t^*[\frac{\lambda}{\beta} - (1-\lambda)] \geq 0,\ M_{t+1}^{B*} = 0 \quad t \geq 1,\ t \text{ odd.}$$

By construction (market clearing) condition (ii) is satisfied, so it remains to verify that the above specification constitutes a solution to the maximization problem confronting each agent type i given M_0^{i*}, $\{p_t^*\}_{t=0}^{\infty}$, $\{z_t^{*i}\}_{t=0}^{\infty}$. This will be done explicitly for agent type B; the argument for agent type A follows immediately.

Consider first <u>any</u> consumption sequence $\{\hat{c}_t^B\}_{t=0}^{\infty}$ and associated money balance sequence $\{\hat{M}_t^B\}_{t=1}^{\infty}$ which are supposed to solve the maximization problem of agent type B. With $p_t^* y_t^B - z_t^{B*} \leq 0$, all $t \geq 1$, t odd, it follows that $\hat{M}_t^B > 0$ for $t \geq 1$, t odd. Hence (2.6) must hold as an equality at such times, i.e.,

$$\text{(A.1)} \qquad \frac{U'(\hat{c}_{t-1}^B)}{\beta U'(\hat{c}_t^B)} = \frac{p_{t-1}^*}{p_t^*} = \frac{1}{\beta} \qquad t \geq 1,\ t \text{ odd.}$$

It follows that

$$\text{(A.2)} \qquad \hat{c}_{t-1}^B = \hat{c}_t^B \qquad t \geq 1,\ t \text{ odd.}$$

Next, convert the problem of agent type B into real terms. In particular, let $w_t^B = (p_t^* y_t^B - z_t^{B*})/p_t^*$ so that

$$w_t^B = 1 \qquad t \geq 0,\ t \text{ even}$$

$$w_t^B = -[\frac{\lambda}{\beta} - (1-\lambda)] \qquad t \geq 1,\ t \text{ odd.}$$

Also let $m_t^B \equiv M_t^B/p_t^*$ denote real money balances held by agent type B at the beginning of period t. From the budget constraint (b_t) as an equality and utilizing the fact that $p_t^* = \beta p_{t-1}^*$, all $t \geq 1$, it follows that

$$\text{(A.3)} \qquad c_t^B + \beta m_{t+1}^B = w_t^B + m_t^B \qquad \text{all } t \geq 0.$$

Then, from (A.2), setting $c_t^B = c_{t+1}^B$ for $t \geq 0$, t even, and solving for m_{t+1}^B one obtains

$$\text{(A.4)} \qquad (1+\beta)m_{t+1}^B = w_t^B - w_{t+1}^B + m_t^B - \beta m_{t+2}^B \qquad t \geq 0,\ t \text{ even}$$

$$\text{(A.5)} \qquad (1+\beta)c_t^B = (1+\beta)c_{t+1}^B = w_t^B + \beta w_{t+1}^B + m_t^B - \beta^2 m_{t+2}^B \qquad t \geq 0,\ t \text{ even.}$$

Following the methods of Lucas and Prescott [1971] it can be established that there exists a bounded continuous function $V(\cdot)$ satisfying the functional equation

$$V(m_0^B) = \max_{m_2^B}\{[U(c_0^B)+\beta U(c_1^B)]+\beta^2 V(m_2^B)\}$$

subject to

i) $0 \leq m_2^B \leq (w_1^B - w_0^B - m_0^B)/\beta$,

ii) c_0^B and c_1^B satisfy (A.5) at $t = 0$, given $m_0^B \geq 0$.

(Here the upper bound on m_2^B follows from (A.4) at $t = 0$ and the constraint $m_1^B \geq 0$. Note that the constraint set on m_2^B is compact and the objective function, in brackets above, as an indirect function of m_2^B, is bounded and continuous.) Here then the solution $m_2^B = \Psi(m_0^B)$ is the stationary policy function which solves

$$\max_{\{m_t^B\}_{t=2}^{\infty}} \sum_{\substack{t\geq 0 \\ t \text{ even}}} \beta^t [U(c_t^B)+\beta U(c_{t+1}^B)]$$

subject to

i) $0 \leq m_{t+2}^B \leq (w_{t+1}^B - w_t^B - m_t^B)/\beta$,

ii) c_t^B and c_{t+1}^B satisfy (A.5), given $m_0^B \geq 0$.

Thus there does exist at least one solution to the problem confronting agent type B.

Clearly, the proposed solution

$$c_t^B \equiv 1 - \lambda, \; m_t^{B*} = 0 \qquad t \geq 2, \; t \text{ even}$$

$$m_t^{B*} = \frac{\lambda}{\beta} \qquad t \geq 1, \; t \text{ odd}$$

satisfies (2.6) with equality in every period. By construction the budget constraint (b_t) is also satisfied as an equality in every period. Now suppose there exists a consumption sequence $\{\hat{c}_t^B\}_{t=0}^{\infty}$, and associated real money balance sequence $\{\hat{m}_t^B\}_{t=1}^{\infty}$, which does better than the proposed solution, and consider the first-period τ at which $\hat{c}_\tau^B \neq (1-\lambda)$. (Note from (A.2) that τ must be even.) Clearly, $\hat{c}_\tau^B > (1-\lambda)$ is not feasible, for with $\hat{c}_{\tau+1}^B > (1-\lambda)$ also, one obtains $\hat{m}_{\tau+2}^B < 0$. Nor is $\hat{c}_\tau^B < (1-\lambda)$ possible. For in this case $\hat{c}_{\tau+1}^B < (1-\lambda)$, so $\hat{m}_{\tau+2}^B > 0$. Thus (2.6) would hold as an equality at $t = \tau + 2$, so that $\hat{c}_{\tau+2}^B < (1-\lambda)$ also, and so on. That is, the consumption path would be maintained below the proposed solution for all $t \geq \tau$, and this cannot improve matters. Hence the proposed solution is indeed maximizing.

A virtually identical argument establishes that given $c_0^{A*} = \lambda$, $m_0^{A*} = \lambda$, and $m_1^{A*} = 0$, the sequences $\{c_t^{A*}\}_{t=1}^{\infty}$ and $\{m_t^{A*}\}_{t=2}^{\infty}$ solve the problem of agent type

A <u>from t = 1 onward</u>. In particular, by the principle of optimality, at t = 2 given

$$m_2^{A*} = \frac{(1-\lambda)}{\beta},\ w_2^A = -[\frac{(1-\lambda)}{\beta} - \lambda],\ m_2^{A*} + w_2^A = \lambda,$$

the sequences $\{c_t^{A*}\}_{t=2}^{\infty}$, $\{m_t^{A*}\}_{t=3}^{\infty}$ solve the problem of agent type A. But given $m_0^{A*} = \lambda$ this implies that $\{c_t^{A*}\}_{t=0}^{\infty}$ and $\{m_t^{A*}\}_{t=1}^{\infty}$ solve the problem of agent type A, as desired.

<u>Proof of Proposition 2.4</u>:

By construction market-clearing condition (ii) of an equilibrium is satisfied, so it remains to verify that the specification of the proposition is consistent with maximization. This will be done explicitly for agent type A.

Consider first <u>any</u> consumption sequence $\{c_t^A\}_{t=0}^{\infty}$ and associated money balance sequence $\{M_t\}_{t=1}^{\infty}$ which satisfy the budget constraints (b_t) as an equality, i.e.,

(A.6) $\quad M_t^A = p^*c_t^A + M_{t+1}^A \qquad t \geq 0$, t even

(A.7) $\quad p^*y_{t+1}^A + M_{t+1}^A = p^*c_{t+1}^A + M_{t+2}^A \qquad t \geq 0$, t even.

Solving (A.6) for M_{t+1}^A and substituting into (A.7) yields

(A.8) $\quad p^*c_t^A + p^*c_{t+1}^A = p^*y_{t+1}^A + M_t^A - M_{t+2}^A \qquad t \geq 0$, t even.

From (A.6) also, with $M_{t+1}^A \geq 0$,

(A.9) $\quad p^*c_t^A \leq M_t^A \qquad t \geq 0$, t even.

Again following the methods of Lucas and Prescott, it can be established that there exists a continuous bounded function $V(\cdot)$ which satisfies the functional equation

$$V(M_0^A) = \max_{M_2^A,\ c_0^A} \{U(c_0^A) + \beta U[\frac{M_0^A + p^* y_1^A - M_2^A - p^* c_0^A}{p^*}] + \beta^2 V(M_2^A)\}$$

subject to

i) $0 \leq p^* c_0^A \leq M_0^A$

ii) $0 \leq M_2^A \leq p^* y_1^A + M_0^A - p^* c_0^A$, given $M_0^A \geq 0$.

Here, then, the solution $(M_2^A, c_0^A) = \phi(M_0^A)$ is the stationary policy function which solves

$$\max_{\{c_t^A\},\ \{M_t^A\}} \sum_{\substack{t \geq 0 \\ t \text{ even}}} \beta^t [U(c_t^A) + \beta U(\frac{M_t^A + p^* y_{t+1}^A - M_{t+2}^A - p^* c_t^A}{p^*})]$$

subject to

i) $0 \leq p^* c_t^A \leq M_t^A$

ii) $0 \leq M_{t+2}^A \leq p^* y_{t+1}^A + M_t^A - p^* c_t^A$, given $M_0^A \geq 0$.

Thus there does exist a solution to the problem of agent A.

Clearly, the proposed solution satisfies (2.6) as an equality for $t \geq 0$, t even, and as an inequality for $t \geq 1$, t odd, i.e.,

$$\text{(A.10)} \qquad \frac{U'(c_{t-1}^{A*})}{\beta U'(c_t^{A*})} = \frac{U'(c^{**})}{\beta U'(c^*)} = \frac{1}{\beta^2} > 1 \qquad t \geq 1,\ t \text{ odd}.$$

Also, the budget constraint is satisfied as an equality in every period. Now fix $\hat{c}_0^A = c_0^{A*}$, $\hat{M}_1^A = M_1^{A*}$, and suppose there exists a consumption sequence $\{\hat{c}_t^A\}_{t=1}^{\infty}$, and associated money balance sequence $\{\hat{M}_t^A\}_{t=2}^{\infty}$, which does better than the proposed solution from $t = 1$ onward. Consider the first-period τ for which $\hat{c}_\tau^A \neq c_\tau^{A*}$. As (2.6) will be satisfied as an equality for $t \geq 2$, t even, it follows that $\tau \geq 1$ and is odd. Clearly, $\hat{c}_\tau^A > c_\tau^{A*}$ is not feasible, for with $\hat{c}_{\tau-1}^A > \hat{c}_{\tau-1}^{A*}$, also, one obtains $\hat{M}^A_{\tau+2} < 0$. Nor is $\hat{c}^A_\tau < c^{A*}_\tau$ possible. For in this case $\hat{c}^A_{\tau+1} < c^{A*}_{\tau+1}$ also, and $\hat{M}^A_{\tau+2} > 0$. Thus (2.6) must hold as an equality at $t = \tau + 2$, so from (A.10), $\hat{c}^A_{\tau+2} < c^{A*}_{\tau+2}$, and so on. This cannot be an improvement. Thus $\{c^{A*}_t\}_{t=1}^{\infty}$, $\{M^{A*}_t\}_{t=2}^{\infty}$ is indeed maximal for agent A from $t = 1$ onward, and so, by the principle of optimality, is optimal from $t = 2$ onward with c^{A*}_1 and M^{A*}_2 given. But this implies $\{c^{A*}_t\}_{t=0}^{\infty}$, $\{M^{A*}_t\}_{t=1}^{\infty}$ is maximal for agent A at $t = 0$, as claimed. A virtually identical argument (without the last step) establishes that the specified solution is maximal for agent type B as well.

Finally, note that for agent A, for example, from (A.10) and (2.5), $\theta^{A*}_t > 0$ for $t \geq 1$, t odd. Similarly, $\theta^{B*}_t > 0$ for $t \geq 0$, t even.

That the equilibrium allocation is nonoptimal is obvious from the fact that the consumption sequences are not constant. That it is Pareto superior to autarky is also obvious, but it is instructive to note that for agent A, for example, c^{**} dominates 0 in period 0, and the consumption pair (c^*, c^{**}) dominates the endowment pair (1,0) in periods $(t, t+1)$ $t \geq 1$, t odd.

Proof of Proposition 3.1:

It is first established that the above specification is maximizing for the agent born at time $t \geq 0$. Consider first *any* consumption sequence $\{c_j(t)\}_{j=0}^{\infty}$ and associated money balance sequence $\{M_j(t)\}_{j=1}^{\infty}$ which satisfy the budget constraint $(b_j(t))$ and $(b_{j+1}(t))$ as equalities, $j \geq 0$, j even. Substitution for $M_{j+1}(t)$ yields

$$\text{(A.11)} \qquad M_j(t) + p^{k*}_{t+j}\gamma_j - M_{j+2}(t) = p^{k*}_{t+j}c_j(t) + p^{k*}_{t+j+1}c_{j+1}(t).$$

Defining real money balances $m_i(t) = M_i(t)/p^{k*}_{t+i}$, $i = j, j+2$, and recalling the specified relationship

$$p^{k*}_{t+j} = p^{k*}_{t+j+1} = (1/\beta^2)p^{k*}_{t+j+2},$$

(A.11) then yields

$$m_j(t) + y_j = c_j(t) + c_{j+1}(t) + \beta^2 m_{j+2}(t) \qquad j \geq 0,\ j \text{ even}.$$

Now holding $m_j(t)$ and $m_{j+2}(t)$ fixed, define real disposable income $d_j(t)$ by

$$\text{(A.12)} \qquad d_j(t) = m_j(t) + y_j - \beta^2 m_{j+2}(t) \qquad j \geq 0,\ j \text{ even}$$

and consider in isolation the following problem:

$$\max_{c_j(t) \geq 0,\ c_{j+1}(t) \geq 0} \{U[c_j(t)] + \beta U[c_{j+1}(t)]\}$$

subject to

$$c_j(t) + c_{j+1}(t) = d_j(t).$$

Solving for the maximizing $c_j(t)$ and $c_{j+1}(t)$ as continuous functions of $d_j(t)$, substitution into the objective function then yields the bounded, continuous indirect utility function, denoted here by $W[d_j(t)]$. Thus problem (t) reduces to

$$\max_{\{m_j(t)\}_j} \sum_{\substack{j\geq 0 \\ j \text{ even}}} \beta^j W[d_j(t)]$$

subject to (A.12) and

$$m_j(t) \geq 0 \qquad j \geq 2,\ j \text{ even given } m_0(t) \geq 0.$$

Again the functional equation approach yields a stationary policy which solves this problem.

It is clear from the discussion preceding the theorem that the specified solution to problem (t) satisfies the necessary first-order conditions (3.4) as equalities. (The budget constraints also hold as equalities.) Now suppose there exists a consumption sequence $\{\hat{c}_j(t)\}_{j=0}^{\infty}$, and associated money balance sequence $\{\hat{m}_j(t)\}_{j=1}^{\infty}$, which does better than the proposed solution and consider the first age g for which $\hat{c}_g(t) \neq c_g^*(t)$. A now familiar argument leads to a contradiction.

It follows from the principle of optimality that for any $h \geq 1$, the sequences $\{c_j^*(t)\}_{j=h}^{\infty}$, $\{m_j^*(t)\}_{j=h+1}^{\infty}$ are maximal for agent t given $m_h^*(t)$. But then by symmetry the sequences $\{c_j^*(-h)\}_{j=h}^{\infty}$, $\{m_j^*(-h)\}_{j=h+1}^{\infty}$ are maximal for the agent born at each period -h, given $m_h^*(-h)$, as we needed to show.

FOOTNOTES

1/ The terminology here is Wallace's (1978).

2/ But see the concluding section for a qualification to this statement.

3/ The implications of such Clower constraints, over and above the constraints implied by the technology of exchange, are examined, however.

4/ See Wallace (1978) and the discussion in Hahn (1973) on inessential money.

5/ A case can be made that models of money with spatially separated agents are of interest in their own right, quite apart from providing an alternative to overlapping-generations.

6/ On an a priori basis this should not be a surprise; indeed, versions of the overlapping-generations model have been criticized for producing optimal monetary equilibria. See Wallace (1978).

7/ Cass, Okuno, and Zelcha (1979) have argued that the inefficiency of monetary equilibrium emerges in the overlapping-generations model under alternative assumptions.

8/ Recall the caveat at the end of the introduction.

9/ If there are limitations on the issue of IOUs, there can exist equilibria in which IOUs have value and are never redeemed. Such equilibria are virtually indistinguishable from equilibria with valued fiat money, as defined below.

10/ This is left as an open question. It may be noted, however, that Grandmont and Youness (1972) do establish certain results in the limit, at $\beta = 1$, using the overtaking criterion.

11/ For the most part I am reporting in this section on some results known to Lucas and his students and suggested to me by Lucas in various conversations; the interested reader is urged to consult Locay and Palmon (1978) on which this section draws heavily. The model is presented here both because it does not seem to be known generally and because it offers a natural comparison with the other two models.

12/ Lucas' version of the Cass-Yaari model retains the circle.

13/ It is curious to note that $a_-(i)$ corresponds to the transactions constraint in Grandmont and Youness (1972), (1973) for $k = 0$, a case which is not really analyzed there.

14/ Grandmont and Youness (1972), (1973) establish all these results for the case $0 < k < 1$ in their transactions constraint.

15/ Again one obtains asymptotic welfare results as $\beta \to 1$ (c.f., the discussion at the end of Section 2).

Footnotes cont.

16/ The example is due to Neil Wallace.

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