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NOTES FOR A FUTURE PAPER  
ON COMMODITY MONEY SYSTEMS

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These pages describe and partially analyze a set of models that are designed to help understand a variety of issues about commodity money systems, and to highlight the aspects of technologies and preferences which determine the answers to a number of positive and normative questions about how commodity money systems operate.

We begin with a list of questions, not all on the same level of abstraction.

1. Are commodity money systems generally inefficient--i.e., is it generally possible to Pareto dominate them with an alternative system using a well managed fiat currency? (Friedman seems to suggest so in A Program for Monetary Stability, Chapter 1.)
2. Does the degree of inefficiency depend on the particular commodity "chosen" for the currency? (Friedman seems to suggest that it does not.) What assumptions about technologies determine this?
3. In what sense is there a choice about the commodity to be used as the commodity money? In the absence of legal restrictions does one commodity or bundle of commodities "emerge" as a natural commodity money because of its conditions of production, physical depreciation characteristics, and so on? What sorts of legal restrictions or government open market operations or fiscal policies does one have in mind when one speaks about the government choosing a standard? (Note that by "technologies" and "conditions of production" we include objective randomness.)
4. What kinds of technologies and preferences give rise to models in which "Gresham's law" holds, there being a natural or market-determined rate of exchange between gold and silver, which the government cannot influence through open market operations without some legal

- restrictions? What is the role of legal restrictions, e.g., monopoly of coinage, in permitting there to be discrepancy between the "mint price" and market price of gold relative to silver?
5. Are there technologies-preferences for which bimetallism is potentially feasible? If so, what kinds of government actions could be used to influence the rate of exchange between gold and silver?
  6. In what sense does adopting a commodity standard impose fiscal discipline?
  7. Can seignorage be raised under a commodity money system?
  8. How do the answers to (6) and (7) interact with the question of efficiency of a commodity money standard? That is, in the alternative fiat money regime, is seignorage being raised through the force of legal restrictions, or by exploiting the natural role for a fiat money in an economy prone to capital or gold overaccumulation?
  9. Does the quantity theory of money apply under a commodity standard? If so, what aspects of technology-preferences does it depend on?
  10. How would adoption of a commodity standard influence the "Phillips curve?"

These notes present parametric versions of overlapping generations models designed to highlight features of commodity money standards. For the first set of models, there is a single nonstorable consumption good. Let  $w_t^h(s)$  be the endowment of the  $s$ -period consumption good of agent  $h$  born in period  $t$ . There is no uncertainty. Let  $c_t^h(s)$  be the consumption of the  $s$ -period consumption good of agent  $h$  born in period  $t$ . We assume that each individual  $h$  maximizes

$$\ln c_t^h(t) \ln c_t^h(t+1)$$

subject to the intertemporal budget constraint

$$c_t^h(t) + \frac{c_t^h(t+1)}{R(t)} \leq w_t^h(t) + \frac{w_t^h(t+1)}{R(t)}$$

where  $R(t)$  is the maximum attainable (nonstochastic) gross real rate of return on saving between dates  $t$  and  $t + 1$ , measured in units of  $t + 1$ -period good per unit of  $t$ -period good. This gives rise to the saving function

$$w_t^h(t) - c_t^h(t) = \left\{ w_t^h(t) - \frac{w_t^h(t+1)}{R(t)} \right\} / 2.$$

We'll call the nonstorable consumption good "bread." In generation  $t$ , there exist  $N(t)$  young people.

We shall consider several alternative models, which differ with regard to the technologies for transforming bread into storable "metals," which while not in people's utility functions, are potentially valued as stores of value.

The economy starts at time  $t = 1$  and continues forever. At  $t = 1$  there are  $N(0)$  old people, endowed in the aggregate with various metals, as detailed below. At each date  $t \geq 1$ ,  $N(t)$  young people are born. People of each generation live for two periods.

#### Model 1: A Reversible Gold Technology

In addition to bread, there exists "gold," which can be converted into bread and back at constant costs. In particular, one unit of bread produces  $\phi$  units of gold, and one unit of gold produces  $\phi^{-1}$  units of bread. Gold is perfectly storable between periods, and neither appreciates nor depreciates physically. Let  $p(t)$  be the bread price of a unit of gold, measured in

units of bread per unit of gold. Evidently, if both bread and gold are to be produced in period  $t$ , we must have  $p(t) = \phi$ . Note that  $p(t)$  is the reciprocal of the "price level."

Assume there is an initial stock of gold of  $G(0)$  units, all held by the current old at time  $t = 1$ . For  $t \geq 1$ , we assume that the young of generation  $t$  are endowed with  $(w_t^h(t), w_t^h(t+1))$  units of bread, where  $\sum_h w_t^h(t) > \sum_h w_t^h(t+1) > 0$  establishing a motive for generation  $t$  as a whole to save at a gross rate of return of unity. As in Wallace [ ] and Sargent and Wallace [ ], this condition still permits there to be individuals for whom  $w_t^h(t) < w_t^h(t+1)$  who prefer to borrow at a gross rate of return of unity. Below we will sometimes comment on the different preferences of borrowers and lenders as among alternative monetary regimes.

In this set-up, the only store of value is gold. It's gross rate of return is

$$R(t) = p(t+1)/p(t).$$

In this model, there is a unique equilibrium in which

$$p(t) = \phi$$

$$R(t) = 1.$$

Letting  $G(t)$  be the stock of gold stored between  $t$  and  $t + 1$  by generation  $t$ , we have that  $G(t)$  is determined by

$$\left\{ \sum_h w_t^h(t) - \sum_h w_t^h(t+1) \right\} / 2 = G(t) \phi$$

$$t \geq 1.$$

The reason that  $p(t) = \phi$  is that the endowments have been rigged so that each generation wants to consume bread at  $t$  (which implies that  $p(t) \leq \phi$ ) and to store gold at  $t$  (which implies that  $p(t) \geq \phi$ ).

In this model, the amount of resources denoted to "mining" new gold is determined by the above equation. It depends on (a) the saving (portfolio?) preferences of agents, and (b) the economy's rate of growth. In terms of bread, the amount of resources denoted to mining gold does not depend on  $\phi$ . (The left-hand side of the above equation is independent of  $\phi$ .) This is consistent with the statements of Milton Friedman (A Program for Monetary Stability, p. 4-5.)

The equilibrium in this model is inefficient.

#### Model 2: Reversible Gold and Silver Technologies

We modify model 1 by now assuming that there are two metals, gold and silver. One unit of bread produces  $\phi_g$  units of gold. One unit of bread produces  $\phi_s$  units of silver. Both gold and silver are perfectly physically storable with neither physical appreciation nor depreciation occurring. Both technologies are reversible. We let  $p_g(t)$  be the price of gold measured in units of bread per unit of gold, and  $p_s(t)$  be the price of silver measured in units of bread per unit of silver. Preferences and endowments are as in model 1. The current old at  $t = 1$  are in the aggregate endowed with  $S(0)$  units of silver and  $G(0)$  units of gold.

The model has a unique equilibrium with  $p_g(t) = \phi_g$ ,  $p_s(t) = \phi_s$ ,  $R(t) = 1$ , and aggregate gold storage at  $t$ ,  $G(t)$ , and silver storage at  $t$ ,  $S(t)$ , determined by

$$\left( \sum_h w_t^h(t) - \sum_h w_t^h(t+1) \right) / 2 = \phi_g \left( G(t) + \frac{\phi_s}{\phi_g} S(t) \right).$$

This equilibrium has a common gross rate of return of 1 on gold and silver. The model cannot determine the equilibrium amounts of  $G(t)$  and  $S(t)$  separately, but only the total  $G(t) + \frac{\phi_s}{\phi_g} S(t)$ . Thus, for  $t \geq 1$ , there is one equilibrium with  $S(t) = 0$  for  $t \geq 1$ , another with  $G(t) = 0$  for  $t \geq 1$ , and a continuum of intermediate equilibria with both  $S(t) > 0$  and  $G(t) > 0$  for  $t \geq 1$ .

This model provides an example of Friedman's assertions that:

"Interestingly enough, the amount of resources required to provide for growth does not depend on the commodity or commodities used as the standard but only on the cash balance preferences of the public and on the rate of growth of the economy." (A Program for Monetary Stability, p. 5.)

"The commodity in question might be gold or silver or copper or bricks or some combination of these or of other goods in fixed proportions, as under any of the variety of symmetrical or commodity reserve standards that have been proposed. The amount of the commodity in use as money would depend in its cost of production relative to other goods, and on the fraction of their wealth people want to hold in the form of money; additions to the stock of money could come from production by private enterprise; changes in the rate of production would reflect changes in the relative value placed on the monetary commodity and other goods or in the relative costs of producing the one and the other." (A Program for Monetary Stability, p. 4-5.)

In this model, Gresham's law obtains, with money that is overvalued at the mint driving out money that is undervalued at the mint. Suppose that the government imposes a legal restriction that states that only metals stamped as "dollars" can be stored. (Evidences of private indebtedness, however, are permitted to be held.) The government sets up a mint, which operates costlessly, and issues (or stamps or mints) dollars according to the following rules. It offers freely to coin dollars out of silver that is brought to it at the price of  $d_s$  dollars per unit of silver, and freely to coin dollars out of gold at the price of  $d_g$  dollars per unit of gold.

Such an ostensible attempt by the government to set up a bimetallic standard will only have the effects of defining a dollar and of determining which one of the two metals is stored as money. If  $d_g/d_s > P_g/P_s = \phi_g/\phi_s$ , then silver is "overvalued at the mint" and only silver will be minted and stored. Only if  $d_g/d_s = P_g/P_s$  is the indeterminacy about which metal will be stored preserved.

In this model, so far as concerns the consumption allocations, the institution of "bimetallism" in this form is innocuous and has no effects.

Model 2a: Reversible Gold and Silver Technologies  
With a Trend in Productivity

We modify model 2 by now assuming that  $\phi_s$  has a downward trend over time, so that at time  $t$  one unit of bread produces  $\phi_s(t)$  units of silver, with the process being reversible and with  $\phi_s(t+1) < \phi_s(t)$ . All other aspects of the model remain as they were, including the assumption that the gold technology parameter  $\phi_g$  is constant over time.

The equilibrium of this model has  $P_g(t) = \phi_g$ ,  $P_s(t) = \phi_s(t)$ . The gross rate of return in gold is unity, but now the gross rate of return on silver is  $P_s(t+1)/P_s(t) = \phi_s(t+1)/\phi_s(t) < 1$ , so that gold dominates silver as an asset. Therefore, only gold is stored, with the equilibrium condition that determines the amount of gold stored at  $t$  being

$$\sum_h (w_t^h(t) - w_t^h(t+1))/2 = \phi_g G(t).$$

Now suppose that the government institutes the kind of bimetallic coining scheme that we described above. The government requires that the only metals that people can store are those stamped dollars, which it stands ready to coin at the rate of  $d_s$  dollars per unit of silver and/or at the rate of  $d_g$  dollars per unit of gold. The mint prices  $d_s$  and  $d_g$  are constant over time.



We assume that  $d_g/d_s < P_g(1)/P_s(1) = \phi_g/\phi_s(1)$ , so that initially silver is undervalued at the mint implying that only gold is minted and stored. We assume that there is a finite  $T > 1$  such that  $d_g/d_s < \phi_g/\phi_s(t)$  for  $t < T$  but that  $d_g/d_s > \phi_g/\phi_s(t)$  for  $t \geq T$ . For  $t < T$ , only gold is stored. For  $t < T$ , the institution of bimetallism is innocuous since agents would freely choose to store gold anyway since it dominates silver in terms of its rate of return. For  $t \geq T$ , only silver is stored, since from  $T$  on it becomes overvalued at the mint due to the results of the fall over time in  $\phi_s(t)$ . For  $t \geq T$ , the institution of bimetallism makes a difference, since lenders would like to store gold but are prevented from doing so by the legal restriction requiring that only minted coins be stored. The gross rate of return on loans and money now becomes  $R^S(t) = \frac{\phi_s(t+1)}{\phi_s(t)}$ . The equilibrium condition determining the amount of silver stored and minted is, for  $t \geq T$ ,

$$\left( \sum_h w_t^h(t) - \frac{\phi_s(t)}{\phi_s(t+1)} \sum_h w_t^h(t+1) \right) / 2 = \phi_s(t) S(t).$$

We note that borrowers are better off after silver displaces gold, since both loans and silver bear a lower rate of return than when gold is the standard. There is an inflation after "silver drives out gold."

In this setup, after  $\phi_s$  has fallen enough by date  $T$  to make silver cheap enough to function as money, there is a force which hitherto had not been present for lenders and holders of money to evade or agitate against the institutions surrounding coinage that operate to depress the rate of return on assets. In the United States, the lenders foresaw this situation and perpetuated the "Crime of '73."

Model 3: An Irreversible Gold Technology

The model is identical with model 1, except that bread can no longer be converted into gold. One unit of gold can still be converted into  $\phi^{-1}$  units of bread, but the technology is irreversible. As before, the current old at  $t = 1$  are in the aggregate endowed with  $G(0)$  units of gold. All succeeding generations are endowed with bread in the patterns  $(w_t^h(t), w_t^h(t+1))$ . This set-up is designed to represent the notion of a perfectly inelastic stock supply of gold together with the existence of an "industrial" use for gold.

The equilibrium condition for this model is

$$\left( \sum_h w_t^h(t) - \frac{p(t)}{p(t+1)} \sum_h w_t^h(t+1) \right) / 2 = G(0)p(t)$$

where  $p(t) \geq \phi$  is the bread price of gold and  $R(t) = p(t+1)/p(t)$ . We require that for  $t \geq 1$   $p(t) \geq \phi$ , or else some of the gold would be converted into bread and eaten. Dividing both sides of the above equation by  $p(t)$  and rearranging gives the difference equation

$$\frac{1}{p(t)} = \frac{2 G(0)}{\sum_h w_t^h(t)} + \frac{\sum_h w_t^h(t+1)}{\sum_h w_t^h(t)} \frac{1}{p(t+1)} .$$

The nature of the dependence of prices on  $G(0)$  depends on the magnitude of  $G(0)$ ,  $\phi$ , and the endowment patterns. For example, specialize the model by assuming

$$\frac{\sum_h w_t^h(t+1)}{\sum_h w_t^h(t)} = \lambda$$

$$\sum_h w_t^h(t) = W(1)\eta^t$$

where  $\eta > 1$  and  $\frac{\lambda}{\eta} < 1$ . Then we have

$$\frac{1}{p(t)} = \frac{2 G(0)}{W(1)\eta^t} + \lambda \frac{1}{p(t+1)} \quad t \geq 1$$

The solution of this difference equation is

$$\frac{1}{p(t)} = \frac{2 G(0)}{W(1)(1 - \frac{\lambda}{\eta})\eta^t} .$$

This equation can be rearranged to be of the quantity theory form

$$G(0)p(t) = W_1 \eta^t (1 - \frac{\lambda}{\eta}) .$$

If this equation implies that  $p(t) \geq \phi$  for all  $t \geq 1$ , then it gives the unique solution of the model. Note that it is then the unique solution of the difference equation for which  $p(t) \geq \phi$  for all  $t$ .

However, if the above equation implies  $p(1) < \phi$ , then the equilibrium is given by  $p(1) = \phi$ , with  $\tilde{G}(0) < G(0)$  gold being stored from time 1 to 2 where

$$\tilde{G}(0) \phi = W_1 \eta (1 - \frac{\lambda}{\eta}) .$$

The old at time 1 convert  $G(0) - \tilde{G}(0)$  into bread. Then the price of gold for  $t \geq 1$  is given by

$$\tilde{G}(0) p(t) = W_1 \eta^t (1 - \frac{\lambda}{\eta}) .$$

In summary, if  $G(0) < \frac{W_1 \eta (1 - \frac{\lambda}{\eta})}{\phi}$ , the initial price level  $\frac{1}{p(1)}$  varies in quantity theory fashion with small variations in the initial stock of gold  $G(0)$ . In this quantity theory case, gold is sufficiently rare that none is converted into bread.

In the regions of its parameter space to which the quantity theory holds, this model is compatible with the following remarks of Paul Samuelson:

Given physical amounts of tobacco, food, ballet, etc., have significance in terms of the want pattern of the consumer, but it is not possible to attach similar significance to a given number of physical units of money, say to a number of ounces of gold. It would be otherwise in the case of gold which was to be used to fill teeth, but such uses of gold in the industrial arts we purposely neglect. The amount of money which is needed depends upon the work that is to be done, which in turn depends upon the prices of all goods in terms of gold. (Foundations of Economic Analysis, pp. 118-119.)

Neglecting the industrial uses of gold corresponds to our assuming that gold is sufficiently rare that its value is higher than  $\phi$ .

This equilibrium is always inefficient.

Model 4: A Reversible Silver Technology, an Irreversible Gold Technology

This model is identical with model 2, except that the gold technology is now assumed irreversible. One unit of bread can be transformed into  $\phi_s$  units of silver, and the process is reversible. One unit of gold can be transformed into  $\phi_g^{-1}$  units of bread but this process is irreversible. The initial aggregate supplies of gold and silver, assumed to be in the hands of the old at  $t = 1$ , are  $G(0)$  and  $S(0)$ , respectively. As above, we let  $p_s(t)$  be the bread price of silver and  $p_g(t)$  be the bread price of gold.

The technology in this model implies that  $p_s(t) = \phi_s$  and that  $p_g(t) \geq \phi_g$ . There are two kinds of equilibria. First, there is a class of equilibria in which both silver and gold are stored and both bear a common gross rate of return of unity. Second, there may be an equilibrium in which gold dominates silver as an asset, and only gold is stored.

Turning to the first class of equilibria, we seek an equilibrium in which  $p_g(t)$  is constant for all  $t \geq 1$ , and  $p_s(t) = \phi_s$  is constant for all  $t \geq 1$ , so that both bear a common rate of return  $R^g(t) = R^s(t) = 1$ . The equilibrium condition for the model is

$$\left( \sum_h w_t^h(t) - \sum_h w_t^h(t+1) \right) / 2 = p_g(t) \left( G(t) + \frac{p_s(t)}{p_g(t)} S(t) \right).$$

This equation is to be solved for a constant  $p_g = p_g(t)$ ,  $G(t)$  and  $S(t)$  for  $t \geq 1$ . Evidently, any  $p_g > \phi_g$  determines an equilibrium with  $G(t) = G(0)$ , since if  $p_g > \phi_g$  no gold would ever be converted into bread. There is a continuum of equilibria indexed by  $p_g > \phi_g$ . Given such a  $p_g$ , the above equation determines  $S(t)$ . In a growing economy, an increasing fraction of the "money stock"  $\left( G(t) + \frac{p_g}{p_s} S(t) \right)$  would consist of silver as time passes. Furthermore, the above equation is a version of the quantity theory equation. To take a special example, set  $w_t^h(t) = w_1$ ,  $w_t^h(t+1) = w_2$  for all  $t$  and  $h$ . Then the above equation becomes

$$\frac{(w_1 - w_2)}{2} = p_g \frac{\left( G(0) + \frac{p_s}{p_g} S(t) \right)}{N(t)},$$

which states that real balances per capita are a constant. In summary, in this model, any relative price of gold per unit silver  $p_s/p_g \leq \phi_s/\phi_g$  is an equilibrium relative price.

However, there in general exists another equilibrium in which  $R^g(t) > R^s(t) = 1$ . In particular, we seek an equilibrium in which no silver is stored as an asset. The equilibrium condition is

$$\left[ \sum_h w_t^h(t) - \frac{p_g(t)}{p_g(t+1)} \sum_h w_t^h(t+1) \right] / 2 = G(0) p_g(t).$$

Rearranging as in model 3 gives

$$\frac{1}{P_g(t)} = \frac{2 G(0)}{\sum_h w_t^h(t)} + \frac{\sum_h w_t^h(t+1)}{\sum_h w_t^h(t)} \frac{1}{P_g(t+1)} .$$

In section 3, we found that the solution to this equation in our special case was

$$\frac{1}{P_g(t)} = \frac{2 G(0)}{W(1)(1 - \frac{\lambda}{\eta})\eta^t}$$

which is the unique equilibrium if the  $p_g(1) \geq \phi_g$ . Note that in this case, the model implies  $p_g(t+1)/p_g(t) = \eta > 1$  in a growing economy. Thus, gold dominates silver as required in this kind of equilibrium.

If the above equation implies that  $p_g(1) < \phi_g$ , an equilibrium can be found in which enough of the initial gold stock is converted into bread to drive  $p_g(1)$  up to  $\phi_g$ .

We now use model 4 to consider again the institution of bimetallism which we studied above with model 2. As above, we consider a setup in which the government requires that only "dollars" be stored, that it monopolize the coining of dollars, and that it stand ready freely to coin new dollars out of silver at a rate of  $d_s$  dollars per unit of silver and at a rate of  $d_g$  dollars per unit of gold.

In model 4, the government is free to choose any ratio for the mint prices  $d_g/d_s$  that satisfies  $d_g/d_s \geq \phi_g/\phi_s$ . This will determine a unique equilibrium, from among members of the first class of equilibria, in which  $P_s(t) = \phi_s$  and  $P_g(t) = d_g \geq \phi_g$ . In this equilibrium, all of the gold stock will be used as money, and additional silver will be coined at a rate determined by the equation

$$\sum (w_t^h(t) - w_t^h(t+1))/2 = d_g \left( G(0) + \frac{\phi_s}{d_g} S(t) \right) .$$

The above equation implies that the higher the value of  $d_g$  selected, the less silver will be minted and stored.

Notice that the institution of bimetallism prevents there from being an equilibrium of the kind described above in which  $R^G(t) > R^S(t) = 1$ . The legal restriction and the offer of free coinage of both gold and silver prevent  $R^G(t) > 1$  from being an equilibrium. Thus, in terms of model 4, the institution of bimetallism is a device for simultaneously eliminating the possibility of a gold-only equilibrium and picking out a unique equilibrium price ratio for gold and silver. In this model, Gresham's law fails to hold. Bimetallism "works" in the sense that the government is free to name a price for gold relative to silver and to dollars, and to make it stick. Gold and silver coexist as money.

In this model, borrowers prefer the bimetallic regime, since they face a lower rate of return than under the gold-only equilibrium. Lenders prefer the gold-only equilibrium.

Model 5: A Reversible, Constant Returns, Random Technology  
for Producing Gold

In this model, at time  $t$ , one unit of bread can be converted into  $\phi(t)$  units of gold, and the process is reversible. The technology parameter  $\phi(t)$  is a random variable with positive support that is distributed independently and identically through time with cumulative distribution function  $\text{Prob}(\phi(t) \leq \phi) = F(\phi)$ . The realization of  $\phi(t)$  becomes known at the beginning of period  $t$ . The old at  $t = 1$  are endowed in the aggregate with  $G(0)$  units of gold. Everything else about the model agrees with model 1.

In equilibrium, the bread price of gold  $p_g(t) = \phi(t)$ . The rate of return on gold will be varying over time. In general, the amount of gold stored will depend on the form of the utility function  $E u(c_t^h(t), u_t^h(t+1))$ , the endowment patterns,  $\phi(t)$ , and the distribution  $F$ . In general, positive and randomly time varying amounts of gold will be stored.

For example, take the utility function,  $E_t \{ \ln c_t^h(t) + \ln c_t^h(t+1) \}$ . Let  $\phi(t)$  have the discrete probability distribution

$$\text{Prob} \{ \phi(t) = \phi_i \} = f_i, \quad i = 1, \dots, I$$

$$\sum f_i = 1, \quad \phi_i > 0.$$

Let  $s(t,i)$  be the implicit price of a unit of bread at time  $t + 1$  in state  $i$  in terms of bread at time  $t$  (measured in units of bread at  $t$  per unit of bread at time  $t + 1$  in state  $i$ ). Then the equilibrium prices are  $s(t,i) = \phi(t)/\phi(t+1,i)$ . For the above utility function the individual saving function is  $[w_t^h(t) - w_t^h(t+1) \sum_i \phi(t)/\phi(t+1,i)] / 2$ . Thus, the condition that determines the equilibrium amount of gold stored at  $t$  is

$$\sum_h [w_t^h(t) - w_t^h(t+1) \sum_i \phi(t)/\phi(t+1,i)] / 2.$$

Thus, the condition that determines the equilibrium amount of gold stored at  $t$  is

$$\sum_h [w_t^h(t) - w_t^h(t+1) \sum_i \phi(t)/\phi(t+1,i)] = G(t)\phi(t).$$



Model 6: Reversible, Constant Returns Random Technologies  
for Producing Both Gold and Silver

This is a variant of model 2. Here one unit of bread can be transformed into  $\phi_g(t)$  units of gold or into  $\phi_s(t)$  units of silver. Both technologies are reversible. The terms  $(\phi_g(t), \phi_s(t))$  are independently drawn over time from a cumulative probability function  $\text{Prob}(\phi_g(t) < \phi_g; \phi_s(t) < \phi_s) = F(\phi_g, \phi_s)$  where  $F$  has strictly positive support. The realizations of  $\phi_g(t)$  and  $\phi_s(t)$  are known at the beginning of time  $t$ . Everything else about the model is the same as model 1, with the current old at  $t = 1$  being endowed in the aggregate with  $G(0)$  units of gold and  $S(0)$  units of silver.

There seem to be several possibilities in this model, depending on the nature of  $F$ . It seems possible that gold might dominate silver, might be dominated by silver, or the two might coexist for portfolio diversification reasons.

Model 7: A Reversible But Random Silver Technology,  
an Irreversible Nonrandom Gold Technology

This model is a mutation of model 4 in which  $\phi_s$  becomes a random variable  $\phi_g(t)$  with positive support. All other features of model 4 remain intact.

This model would seem to have several possibilities. Gold might dominate silver with there existing a unique equilibrium. There might be a distribution functions  $F$  for which multiple equilibria again arise, one in which gold dominates silver, some others in which silver coexists with gold.