Federal Reserve Bank of Minneapolis
Research Department Working Paper

FORMULATING DYNAMIC LINEAR RATIONAL EXPECTATIONS MODELS BY MEANS OF PERIODIC-COEFFICIENT LINEAR STOCHASTIC DIFFERENCE EQUATIONS

Richard M. Todd

Working Paper 206
PACS File 3670

July 1982

Formulating Dynamic Linear Rational Expectations Models by Means of Periodic-Coefficient Linear Stochastic Difference Equations

Introduction

The techniques recently introduced into economics to make estimation of the parameters of dynamic systems tractable are not well-suited to most agricultural commodity systems. These techniques involve the explicit solution of certain time-invariant linear stochastic difference equations (LSDEs) that arise from the optimization of constant-coefficient quadratic objective functions subject to linear constraints. The constant-coefficient quadratic objective function is inappropriate for much of agriculture because, even under the assumption of a linear production function, a firm's objective function should include the cubic interaction of market price, the firm's inputs, and a seasonally varying average productivity. Hence the time-invariant linear-quadratic framework, though tractable enough for empirical work, seems to require agricultural analysts interested in sub-annual models to ignore the effects of variations in either market prices, firms' inputs, or average productivity on competitive equilibrium. Modeling agriculture on an annual basis, to avoid seasonal variation in average productivity, is not an attractive solution either, for the lengths of most of the biological lags prominent in agriculture are not integer multiples of a year. Economists interested in modeling dynamic equilibrium in agricultural systems seem to be faced with choosing between tractability and adequate realism.

In this paper I propose a compromise solution to the dilemma facing agricultural modelers. The key element of the compromise is to model average productivity as a seasonally varying but deterministic sequence. In addition, a restricted set of quadratic cost-of-adjustment terms is used.

Modeling average productivity as a periodic (e.g., seasonally varying) but deterministic sequence achieves at least the same degree of realism as in the more familiar time-invariant linear-quadratic models of dynamic economic equilibrium, where average productivity is also represented as nonstochastic (in fact, as a constant). The assumption of a periodic, deterministic average productivity can be viewed as the natural extension of the time-invariant models to agriculture. The restricted class of quadratic cost-of-adjustment terms used here does limit the modeler's flexibility in representing dynamic constraints. However, much can be done within this class, and it may be possible to broaden the class in later work.

This strategy also preserves the tractability of the linear-quadratic model--it allows closed-form expressions for firms' decision rules and competitive equilibrium laws of motion that are useful for parameter estimation. Under the assumption of periodic, deterministic average productivity, the model's objective functions become quadratic in stochastic and choice variables but with periodic coefficients rather than constant coefficients. Similarly, the LSDEs that arise in optimizing these periodic-coefficient objective functions subject to linear constraints also contain periodic coefficients in either their forcing terms, characteristic polynomials, or both. I shall show that the class of periodic-coefficient LSDEs that arises from the models proposed here can be solved explicitly, though the closed-form solutions are somewhat more complicated than for the corresponding constant-coefficient LSDEs.

In the first section of this paper I analyze a simple but typical dynamic agricultural model with firms' objective functions that (a) are quadratic in stochastic and choice variables, (b) contain only restricted forms of quadratic costs of adjustment, (c) have periodic coefficients, and (d) are maximized subject to linear constraints. The first-order necessary conditions (Euler equations) for maximizing the net present value of a representative firm in the industry are derived in the first section and turn out to include a forced LSDE with a constant-coefficient characteristic polynomial and peridoic-coefficient forcing terms. The first-order necessary conditions for maximizing the welfare criterion of a hypothetical social planner whose optimal decisions replicate competitive equilibrium are also derived in the first section; they include a forced LSDE with periodic-coefficient characteristic polynomial and forcing terms. The second section of the paper presents techniques for computing closed-form solutions to periodic-coefficient LSDEs of the types derived in the first section.

Section I. Deriving Linear Stochastic Difference Equations That Characterize Competitive Equilibrium in an Industry with Periodic Average Productivity

A description of the industry.

Consider a simple industry consisting of m identical firms each producing a single output from a single input and selling all output each period to consumers (no inventories are held).

The production function is linear in the input and contains an additive stochastic term, but the average productivity of the input varies periodically and deterministically. In particular,

$$(1) \qquad \tilde{q}_t = f_t \tilde{n}_t + (\tilde{e}_t/m),$$

where $\tilde{q}_t$ is output per firm, $\tilde{n}_t$ is input per firm, $\frac{\tilde{e}_t}{m}$ is each firm's share of the industry-wide production shock, and f_t, a strictly periodic and nonnegative sequence of period p, is the average productivity of the input. Here, and throughout this paper, variables are dated by time subscripts.

The firm's cost of production consists of the cost of purchasing the input plus two quadratic terms, one to reflect diseconomies of scale and another to reflect the cost of adjusting the firm's level of input use. The quadratic cost-of-adjustment term, which determines how the endogenous and exogenous variables in the system are linked dynamically, has the form of the square of a simple difference of input levels. I shall use a first difference of input levels for now, but Appendix A shows that higher order differences, perhaps more appropriate to agriculture, can also be used. Let the cost of production be

$$(2) \qquad \tilde{c}_t = \tilde{w}_t\tilde{n}_t + (\gamma/2)\tilde{n}_t^{\,2} + (\delta/2)(\tilde{n}_t - \tilde{n}_{t-1})^2,$$

where $\tilde{w}_t$ is the cost of acquiring one unit of input, $(\gamma/2)\tilde{n}_t^{\,2}$ reflects diseconomies of scale, and $(\delta/2)(\tilde{n}_t - \tilde{n}_{t-1})^2$ is the cost-of adjustment term. The constants γ and δ are positive.

There are m identical firms in the industry, so we can express aggregate output as

$$(3) \qquad \tilde{Q}_t = m\tilde{q}_t$$

and aggregate input as

$$(4) \qquad \tilde{N}_t = m\tilde{n}_t.$$

Each period firms sell all their output to consumers. I have assumed that the consumers' optimization problems do not generate dynamic decision rules. Instead, their decisions are summarized in the aggregate demand curve

$$(5) \qquad \tilde{P}_t = D_0 - D_1\tilde{Q}_t,$$

where D_0 and D_1 are positive constants.

The exogenous stochastic processes that determine the equilibrium price and quantity sequence in this industry are $\{\tilde{e}_t\}$ and $\{\tilde{w}_t\}$. Let these processes be jointly covariance stationary with laws of motion given by

$$(6) \qquad \tilde{\zeta}(L)\tilde{e}_t = \tilde{v}^e_t, \text{ with } \tilde{\zeta}(L) = 1-\tilde{\zeta}_1 L - \dots - \tilde{\zeta}_r L^r, \text{ and}$$

$$\tilde{\psi}(L)\tilde{w}_t = \tilde{v}^w_t, \text{ with } \tilde{\psi}(L) = 1 - \tilde{\psi}_1 L - \dots - \tilde{\psi}_q L^q,$$

where L is the lag operator defined by $Lx_t = x_{t-1}$. The white-noise processes $\{\tilde{v}^e_t\}$ and $\{\tilde{v}^w_t\}$ are the innovations of the joint $\{\tilde{e}_t, \tilde{w}_t\}$ process.

Deriving the firms' first-order necessary conditions

A representative firm in the industry is assumed to chose inputs to maximize the conditional expectation of its discounted net present value, which is given by

$$(7) \qquad \lim_{T\to\infty} E_t \sum_{j=0}^{T} \beta^j \{\tilde{P}_{t+j} f_{t+j} \tilde{n}_{t+j} + \tilde{P}_{t+j}(\tilde{e}_{t+j}/m) - \tilde{w}_{t+j}\tilde{n}_{t+j}$$
$$- (\gamma/2)\, \tilde{n}^2_{t+j} - (\delta/2)(\tilde{n}_{t+j} - \tilde{n}_{t+j-1})^2\},$$

subject to a given $\tilde{n}_{t-1}$ and equations (1) and (3) - (6). The discount factor β lies in the interval (0,1). The symbol E_t denotes expectation conditioned on information available to the firm at time t, which is $\tilde{\Omega}_t \cup \{\tilde{n}_{t-1}, \tilde{n}_{t-2}, \dots\}$, where $\tilde{\Omega}_t = \{f_t, f_{t+1}, \dots ; \tilde{N}_t, \tilde{N}_{t-1}, \dots ;$

$\tilde{w}_t$, $\tilde{w}_{t-1}$, ... ; $\tilde{e}_t, \tilde{e}_{t-1}$, ... }. Note that the firm knows the entire future of the $\{f_t\}$ sequence with certainty (because it is deterministic and periodic). While equation (7) is a direct and natural way to write the firm's objective function, it turns out that the firm's behavior can be more conveniently analyzed if we rewrite (7) in an undiscounted form by transforming the variables. Let

$$\beta^{t/2} \{\tilde{q}_t, \tilde{n}_t, \tilde{e}_t, \tilde{Q}_t, \tilde{N}_t, \tilde{P}_t, \tilde{w}_t, \tilde{v}_t^e, \tilde{v}_t^w\} =$$

$$\{q_t, n_t, e_t, Q_t, N_t, P_t, w_t, v_t^e, v_t^w\},$$

and

$$D_t = \beta^t D_0 = \beta D_{t-1}.$$

Then we can express the firm's objective as maximizing, over choices of sequences of inputs n_t,

$$\text{(8)} \qquad \lim_{T\to\infty} E_t \sum_{j=0}^{T} \{P_{t+j} f_{t+j} n_{t+j} + P_{t+j}(e_{t+j}/m) - w_{t+j} n_{t+j} - (\gamma/2)\, n_{t+j}^2 - (\delta/2)(n_{t+j} - \beta n_{t+j-1})^2\},$$

subject to a given n_{t-1} and

$$\text{(9)} \qquad q_t = f_t n_t + (e_t/m),$$

$$\text{(10)} \qquad Q_t = m q_t,$$

$$\text{(11)} \qquad N_t = m n_t,$$

(12) $P_t = D_t - D_1 Q_t,$

and

(13) $\zeta(L)e_t = \tilde{\zeta}\ (\beta L)e_t = v_t^e$

and

$$\psi(L)w_t = \tilde{\psi}(\beta L)\ w_t = v_t^w.$$

In terms of the transformed variables, E_t represents expectations conditioned on $\Omega_t \cup \{n_{t-1}, n_{t-2}, \ldots\}$, where

$$\Omega_t = \{f_t, f_{t+1}, \ldots;\ N_t,\ N_{t-1}, \ldots;\ w_t, w_{t-1}, \ldots;\ e_t,\ e_{t-1}, \ldots\}.$$

By the principle of certainty equivalence [Bertsekas, p. 81] we can solve the firm's problem as though it were deterministic and then replace any variables outside the relevant information set with their conditional expectations. Ignoring uncertainty and differentiating (8) with respect to n_{t+j}, $j=0,1,\ldots,T$, gives the following system of first-order necessary conditions for the m firms in the industry:

(14) (Euler equation)

$$(1-\alpha L+L^2)n_{s+1} = [1/(\delta\beta)](w_s - f_s P_s),$$

for $s = t,\ t+1,\ \ldots,\ t+T-1$,

where $\alpha = \{(1/\beta) + \beta + [\gamma/(\delta\beta)]\}$, and

(15) (Transversality condition)

$$\lim_{T\to\infty} \left[\delta\beta n_{T-1} - (\delta+\gamma)n_T + f_T P_T - w_T\right] = 0.$$

Note that we can use (9) - (12) to rewrite (14) and (15) as

$$(16) \qquad (1-\alpha L+L^2)n_{s+1} = [1/(\delta\beta)](w_s - f_s D_s + f_s D_1 Q_s)$$

$$= [1/(\delta\beta)](w_s - f_s D_s + D_1 f_s^2 N_s + D_1 f_s e_s)$$

and

$$(17) \qquad \lim_{T\to\infty} [\delta\beta n_{T-1} - (\delta+\gamma)n_T + f_T D_T - f_T^2 D_1 N_T - f_T D_1 e_T - w_T] = 0.$$

The firm's first-order necessary condition (16) is thus a LSDE with time-invariant characteristic polynomial $(1-\alpha Z+Z^2)$ and periodic coefficients, f_s and f_s^2, on some of the exogenous (to the firm) forcing sequences that appear on the right side of the equal sign. A technique for solving LSDEs of this form, given a boundary condition such as (17), will be presented in Section II of this paper. Equation (16) is not yet in the form of a decision rule, as we shall also see in Section II, where the techniques for solving LSDEs will be used to explain how to derive the firm's decision rule.

<u>Deriving the linear stochastic difference equations that describe the behavior of aggregate endogenous variables in equilibrium.</u>

The most direct route to the LSDEs and boundary conditions that govern the aggregate behavior of endogenous variables in this simple model is to multiply both sides of (16) and (17) by m to get the following expression for the dynamic behavior of N_t:

$$(1-\alpha L+L^2)N_{s+1} = [m/(\delta\beta)](w_s - f_s D_s + D_1 f_s^2 N_s + D_1 f_s e_s),$$

or

$$(19) \qquad [1-(\alpha + \frac{mD_1 f_s^2}{\delta\beta})L+L^2]N_{s+1} = [m/(\delta\beta)](w_s - f_s D_s + D_1 f_s e_s),$$

for s = t, t+1, ..., t+T-1,

and

$$(20) \qquad \lim_{T\to\infty} [\delta\beta\ N_{T-1} - (\delta+\gamma+mf_T^2\ D_1)N_T + m(f_T D_T - f_T D_1 e_T - w_T)] = 0.$$

The LSDEs and boundary conditions governing Q_t and P_t can be derived from (19) and (20) by using (9) - (12); these equations will not be exhibited here.

Note that, according to (19), the equilibrium LSDE for N_t has a periodic-coefficient characteristic polynomial $\{1-[\alpha+(\delta\beta)^{-1}mD_1 f_s^2]Z + Z^2\}$ and a periodic coefficient, f_s, on some of the exogenous (to the industry) forcing terms. Techniques for solving LSDEs of the form shown in (19), given boundary conditions as in (20), are also presented in Section II and then used to explain how to solve for the equilibrium laws of motion for the aggregate endogenous variables.

While it turns out that (19) and (20) are indeed the LSDE and a boundary condition that characterize industry equilibrium, and while it was convenient to introduce them quickly for comparison with the LSDE and boundary condition from the firm's problem (equations (16) and (17)), I have not actually established that these equations do characterize competitive equilibrium in the industry. For that purpose I shall employ the artifice of a hypothetical social planner whose decision problem has been shown to generate the LSDEs and boundary conditions characteristic of competitive equilibrium [Sargent (1980), p20].

Let the planner choose aggregate input to maximize the discounted stream of consumer surpluses net of social costs of production, where consumers surplus at time t is given by

$$\text{(21)} \qquad \int_0^{\tilde{Q}_t} (D_0 - D_1 x)dx = D_0\tilde{Q}_t - (D_1/2)\tilde{Q}_t^{\,2}$$

and social costs of production are

$$\text{(22)} \qquad w_t\tilde{N}_t + [\gamma/(2m)]\tilde{N}_t^2 + [\delta/(2m)](\tilde{N}_t - \tilde{N}_{t-1})^2.$$

That is, the planner maximizes, with respect to sequences of $\tilde{N}_t$,

$$\text{(23)} \qquad \lim_{T\to\infty} E_t \sum_{j=0}^{T} \beta^j \{D_0(f_{t+j}\tilde{N}_{t+j} + \tilde{e}_{t+j})$$
$$- (D_1/2)(f_{t+j}^2\tilde{N}_{t+j}^2 + 2f_{t+j}\tilde{N}_{t+j}\tilde{e}_{t+j} + \tilde{e}_{t+j}^2)$$
$$-\tilde{w}_{t+j}\tilde{N}_{t+j} - [\gamma/(2m)]\tilde{N}_{t+j}^2 - [\delta/(2m)](\tilde{N}_{t+j} - \tilde{N}_{t+j-1})^2\},$$

subject to a given $\tilde{N}_{t-1}$ and equations (1) and (3) - (6).

It is convenient to use the same transformation of variables as before to write the planner's problem as maximizing, with respect to sequences of N_t,

$$\text{(24)} \qquad \lim_{T\to\infty} E_t \sum_{j=0}^{T} \{D_{t+j}(f_{t+j}N_{t+j} + e_{t+j})$$

$$-(D_1/2)(f^2_{t+j}N^2_{t+j} + 2f_{t+j}N_{t+j}e_{t+j} + e^2_{t+j})$$

$$- w_{t+j}N_{t+j}-[\gamma/(2m)]N^2_{t+j}-[\delta/(2m)](N_{t+j}-N_{t+j-1})^2\}$$

subject to a given N_{t-1} and equations (9) - (13).

Again ignoring uncertainty, the derivatives of (24) with respect to N_{t+j} give equations (19) and (20) as the first-order necessary conditions for the social planner's problem, thus establishing that this LSDE and boundary condition do characterize competitive equilibrium.

Section II. Methods for solving a class of second-order forced linear stochastic difference equations whose characteristic polynomial and/or forcing terms have periodic coefficients.

The optimization problems of a representative firm and a hypothetical social planner in the industry described in the first section gave rise to LSDEs with periodic coefficients in the characteristic polynomial and/or in the forcing function. Methods for solving such difference equations are presented in this section. It begins with a method for solving LSDEs of the same form as those in the first-order necessary conditions of firms in the industry described in section I. That is, these difference equations have a time-invariant characteristic polynomial but periodic-coefficient stochastic forcing functions. The second part of this section explains a method for solving LSDEs that resemble those in the first-order necessary conditions of the social planner described above. In particular, these difference equations have a periodic characteristic polynomial.

<u>Time-invariant characteristic polynomials with periodic-coefficient forcing functions.</u>

Consider the difference equation

(25) $$(1-\alpha L+L^2)n_{t+1} = f_t e_t \quad , \quad t \geq 0,$$

with n_{-1} given and transversality condition

(26) $$\lim_{T\to\infty} \left[\delta\beta n_{T-1} - (\delta+\gamma)n_T + f_T e_T\right] = 0,$$

where all the variables and coefficients are defined as in Section I. In particular, $\alpha \geq (1/\beta)+\beta$, $0 < \beta < 1$, $\zeta(L)e_t = v_t^e$, and f_t is a periodic sequence of period p. Note that since $0 < (1-\beta)^2 = 1-2\beta+\beta$, we have $2 = (2\beta/\beta) < (1+\beta/\beta) \leq \alpha$, or $\alpha > 2$.

The first step toward an explicit solution for n_t in terms of e_t that satisfies the transversality condition (26) is to factor the characteristic polynomial $(1-\alpha Z+Z^2)$. We seek coefficients ρ_1 and ρ_2 such that $(1-\rho_1 Z)(1-\rho_2 Z) = (1-\alpha Z+Z^2)$. That is, $\rho_1+\rho_2 = \alpha$ and $\rho_1\rho_2 = 1$, so that $\rho_1 = (1/\rho_2) = [1/(\alpha-\rho_1)]$. These relationships imply that we can choose ρ_1 to be between zero and one, for, if we let $g(x) = x - [1/(\alpha-x)]$ and recall $\alpha > 2$, we see that $g(0) < 0$ and $g(1) > 0$. Since g is continuous in $(-\infty,2)$, the intermediate value theorem implies $g(x)=0$ has a root in $(0,1)$. Let ρ_1 be that root. Then $\rho_2 = (1/\rho_1) > 1$.

We now use the factorization to further the development of an explicit solution for (25) and (26). For $t \geq 0$ we rewrite (25) as

$$(27) \qquad (1-\rho_1 L)(1-\rho_2 L)n_{t+1} = f_t e_t.$$

As demonstrated by Sargent [1979, Chapter IX], the transversality condition (26) requires solving the unstable root--ρ_2, the root greater than unity in modulus--forward and the stable root--ρ_1--backwards. That is, we invert $(1-\rho_2 L)$ to obtain

$$(28) \qquad (1-\rho_1 L)n_{t+1} = (1-\rho_2 L)^{-1} f_t e_t = -\rho_2^{-1} \sum_{j=0}^{\infty} \rho_2^{-j} f_{t+j+1} e_{t+j+1},$$

which simplifies to

$$(29) \qquad n_t = \rho_1 n_{t-1} - \rho_1 \sum_{j=0}^{\infty} \rho_1^j f_{t+j} e_{t+j}.$$

This completes an explicit solution to the deterministic differential equation (25) with boundary conditions n_{-1} and (26).

Equation (26) is not, however, in the form of a solution to the problem that faced the firms in the industry described in Section I, for the firms do not know future values of $\{e_t\}$ when they must choose n_t. According to the principle of certainly equivalence, the firms choose n_t not according to (29) but rather to satisfy

$$(30) \qquad n_t = \rho_1 n_{t-1} - \rho_1 \sum_{j=0}^{\infty} \rho_1^j f_{t+j} E_t e_{t+j},$$

where uncertainty has been reintroduced to the problem by replacing e_{t+j}, for $j=1,2,...$, by its conditional (on Ω_t) expectation, $E_t e_{t+j} = E(e_{t+j} \mid \Omega_t)$.

The final form of the representative firm's decision rule is then determined from (30) by replacing the summation on the right side by an equivalent expression involving only current and past values of $\{e_t\}$, i.e., e_t, $e_{t-1}, \ldots$. Hansen and Sargent [1980] have shown how sums of the form $\sum_{j=0}^{\infty} \lambda^j E_t e_{t+j}$ can be expressed as a weighted average of current and past values of $\{e_t\}$. By exploiting the strictly periodic nature of $\{f_t\}$, Hansen and Sargent's technique can be generalized to express $\sum_{j=0}^{\infty} \lambda^j f_{t+j} E_t e_{t+j}$ as a weighted average of current and past values of $\{e_t\}$, where we let $\lambda = \rho_1$ to match up with (30).

Recall that $\{f_t\}$ has period p and that $\zeta(L)e_t = v_t^e$. Let $f_{t+kp} = f_0$, $f_{t+kp+1} = f_1$, ..., $f_{t+kp+(p-1)} = f_{p-1}$, for any integer k. Then

$$(31) \qquad \sum_{j=0}^{\infty} \lambda^j f_{t+j} E_t e_{t+j} = \sum_{\ell=0}^{p-1} f_\ell \sum_{j=0}^{\infty} \lambda^{pj+\ell} E_t e_{t+pj+\ell},$$

and the problem is reduced to one of expressing $\sum_{j=0}^{\infty} \lambda^{pj+\ell} E_t e_{t+pj+\ell}$ as a weighted average of current and past values of the $\{e_t\}$ process.

To begin this next step in the solution write

$$(32) \qquad \sum_{j=0}^{\infty} \lambda^{pj+\ell} E_t e_{t+pj+\ell} = \lambda^\ell E_t[(1-\lambda^p L^{-p})^{-1} e_{t+\ell}].$$

As shown in Appendix A, the expression $(1-\lambda^p L^{-p})$ can be factored as $(1-\lambda^p L^{-p}) = (1-Z_0\lambda L^{-1})(1-Z_1\lambda L^{-1}) \ldots (1-Z_{p-1}\lambda L^{-1})$, where

$$(33) \qquad Z_k = e^{i(2k\pi/p)} = \cos(2k\pi/p) + i \sin(2k\pi/p), \text{ for } k=0,1,\ldots, p-1.$$

Then, by partial fractions,

(34) $$(1-\lambda^{P}L^{-P})^{-1} = \sum_{k=0}^{p-1} B_k(1-Z_k\lambda L^{-1})^{-1}, \text{ where}$$

(35) $$B_k = \prod_{\substack{j=0\\ j\neq k}}^{p-1} (1-\frac{Z_j}{Z_k}) = \prod_{\substack{j=0\\ j\neq k}}^{p-1} (1-e^{i2(j-k)\pi/p}).$$

Substituting (34) in (32) gives

(36) $$\sum_{j=0}^{\infty} \lambda^{pj+\ell}E_t e_{t+pj+\ell} = \lambda^{\ell} \sum_{k=0}^{p-1} B_k \sum_{j=0}^{\infty} Z_k^j \lambda^j E_t e_{t+j+\ell}.$$

The remain step, then, is to reduce $\sum_{j=0}^{\infty} Z_k^j \lambda^j E_t e_{t+\ell+j}$ to a weighted average of current and past values of $\{e_t\}$.

Appendix B shows that $\sum_{j=0}^{\infty} Z_k^j \lambda^j E_t e_{t+\ell+j}$ can be written as

(37) $$\{\tilde{\sigma}(L,Z_k\lambda,\ell,r)+[L^{-\ell}\mu_1(L,Z_k\lambda,\ell,r)]_+\}e_t;$$

where

$$\tilde{\sigma}(L,Z_k\lambda,\ell,r) = \sum_{j=0}^{r-1} [\sum_{h=j+1}^{\min(r,j+\ell)} \alpha_h \sum_{g=0}^{\min(\ell-h+j,r-1)} \mu_1^g a_{\ell-g-h+j+1}]L^j$$

and

$$[L^{-\ell}\mu_1(L,Z_k\lambda,\ell,r)]_+ = \begin{cases} -\zeta(Z_k\lambda)^{-1} \sum_{g=\ell}^{r-1} [\sum_{h=g+1}^{r} \zeta_h(Z_k\zeta)^{h-g}]L^{g-\ell}, \\ \quad \text{for } \ell=0,1,\ldots,r-1; \\ 0, \text{ for } \ell=r,r+1,\ldots; \end{cases}$$

with

$$\alpha_h = -\zeta_h, \; h=1,2,\ldots,r \;;$$

$$a_1 = 1 \text{ and } a_h = \sum_{j=1}^{\min(h-1,r)} \alpha_j a_{h-j} \quad \text{for } h=2,3,\ldots \;;$$

$$\mu_1^0 = \zeta(Z_k\lambda)^{-1};$$

and

$$\mu_1^g = -\zeta(Z_k\lambda)^{-1} \sum_{h=g+1}^{r} \zeta_h (Z_k\lambda)^{h-g}, \text{ for } g=2,3,\ldots,r-1.$$

Let

$$(38) \qquad \sigma(L,Z_k\lambda,\ell,r) \equiv \tilde{\sigma}\,(L,Z_k\lambda,\ell,r) + [L^{-\ell}\mu_1(L,Z_k\lambda,\ell,r)]_+,$$

so that we have

$$(39) \qquad \sum_{j=0}^{\infty} Z_k^j \lambda^j E_t \; e_{t+\ell+j} = \sigma(L,Z_k\lambda,\ell,r)e_t.$$

Substituting (39) into (36) and the resulting expression into (31) gives

$$(40) \qquad \sum_{j=0}^{\infty} \lambda^j f_{t+j} \; E_t \; e_{t+j} = [\sum_{\ell=0}^{p-1} f_\ell \lambda^\ell (\sum_{k=0}^{p-1} B_k \; \sigma(L,Z_k\lambda,\ell,r))]e_t,$$

where $\sigma(L,Z_k\lambda,\ell,r)$ comes from equation (38) and the Z_k and B_k come from equations (33) and (35), respectively. (It is not too difficult to program an algorithm that will compute the Z_k and B_k and then express $\sigma(L,Z_k\lambda,\ell,r)$ in terms of λ and Z_k and the coefficients of $\zeta(L)$. Such an algorithm can be extended to calculate $\sum_{j=0}^{\infty} \lambda^j f_{t+j}E_t e_{t+j}$ as a weighted average of $e_t,e_{t-1},\ldots,e_{t-r+1}$, where the weights depend explicitly on $p,f_0,f_1,\ldots,f_{p-1}$, and λ.)

Letting $\rho_1=\lambda$ and substituting (40) in (30) puts the solution to (25) in a form that involves only current and past values of the forcing variable e_t. This solution, or

$$(41)\qquad n_t = \rho_1 n_{t-1} - \rho_1 \left[\sum_{\ell=0}^{p-1} f_\ell \rho_1^\ell \left(\sum_{k=0}^{p-1} B_k \sigma(L, Z_k\rho_1, \ell, r)\right)\right] e_t,$$

is in the form of a decision rule that a firm with first-order necessary conditions given by (25) and (26) could actually apply to information available at time t to determine its input n_t.

Periodic-coefficient characteristic polynomials.

Consider the following set of periodic-coefficient LSDEs that have the form of the social planner's Euler equations presented in Section I:

$$(42)\qquad n_{t+1} = \alpha_t n_t - n_{t-1} + f_t e_t,$$

where $\{\alpha_t\}$ and $\{f_t\}$ are deterministic sequences of period p and $\alpha_t>2$ for all t. As before, $\zeta(L)e_t = v_t^e$. This set of LSDEs must be solved subject to a given n_{-1} and a transversality condition, analogous to (20), given by

$$(43)\qquad \lim_{T\to\infty} \left[\delta\beta\, n_{T-1} - \delta\beta(\alpha_T - \beta)n_T + f_T e_T\right] = 0.$$

The key to solving this set of LSDEs is to rewrite it in terms of 2p new parameters, as follows:

(45) $$n_{t+1} = \rho_t n_t + \gamma_t (n_t - \rho_{t-1} n_{t-1}) + f_t e_t,$$

where $\{\rho_t\}$ and $\{\gamma_t\}$ are also periodic of period p with

(46) $$\rho_t + \gamma_t = \alpha_t$$

and

(47) $$\rho_t \, \gamma_{t+1} = 1.$$

It is possible to choose the (at most) 2p distinct values of the periodic $\{\rho_t\}$ and $\{\gamma_t\}$ sequences to satisfy (46) and (47) as well as $0<\rho_t<1<\gamma_t<\alpha_t$ (see Appendix C).

Provided the $\{e_t\}$ process goes to zero as t goes to infinity, the property $0<\rho_t<1<\gamma_t<\alpha_t$ of the parameters of the rewritten system guarantees the existence of a solution that satisfies the transversality condition (43). Rewriting (45) for $n_{t+1}, n_{t+2}, \ldots, n_{t+p}$ gives

(48) $$(n_{t+1} - \rho_t n_t) = \gamma_t (n_t - \rho_{t-1} n_{t-1}) + f_t e_t$$

$$(n_{t+2} - \rho_{t+1} n_{t+1}) = \gamma_{t+1} (n_{t+1} - \rho_t n_t) + f_{t+1} e_{t+1}$$

$$\vdots$$

$$(n_{t+p-1} - \rho_{t+p-2} n_{t+p-2}) = \gamma_{t+p-2} (n_{t+p-2} - \rho_{t+p-3} n_{t+p-3}) + f_{t+p-2} e_{t+p-2}$$

$$(n_{t+p} - \rho_{t+p-1} n_{t+p-1}) = \gamma_{t+p-1} (n_{t+p-1} - \rho_{t+p-2} n_{t+p-2}) + f_{t+p-1} e_{t+p-1}.$$

Repeated substitution gives

$$(n_{t+p}-\rho_{t+p-1}n_{t+p-1}) = \gamma_{t+p-1}(n_{t+p-1}-\rho_{t+p-2}n_{t+p-2})+f_{t+p-1}e_{t+p-1}.$$

Repeated substitution gives

(49) $$(n_{t+p}-\rho_{t+p-1}n_{t+p-1}) = \Lambda(n_t-\rho_{t-1}n_{t-1})+b_t,$$

where

(50) $$\Lambda = \gamma_{t+p-1}\gamma_{t+p-2} \cdots \gamma_{t+1}\gamma_t = \gamma_{p-1}\gamma_{p-2} \cdots \gamma_1\gamma_0,$$

and

(51) $$b_t = f_{t+p-1}e_{t+p-1}+\gamma_{t+p-1}f_{t+p-2}e_{t+p-2}+\gamma_{t+p-1}\gamma_{t+p-2}f_{t+p-3}e_{t+p-3}+ \cdots +\gamma_{t+p-1}\gamma_{t+p-2} \cdots \gamma_{t+2}\gamma_{t+1}f_te_t.$$

Since $\rho_{t+p-1} = \rho_{t-1}$, equation (49) can be rewritten as

(52) $$(1-\Lambda L^p)(n_{t+p}-\rho_{t+p-1}n_{t+p-1}) = b_t,$$

or

(53) $$n_{t+p} = \rho_{t+p-1}n_{t+p-1}+(1-\Lambda L^P)^{-1}b_t.$$

The property $1<\gamma_t$, for all t, implies $1<\Lambda$ and suggests that an explicit solution to the system of LSDEs given by (42) and (43) can be found by applying the forward inverse of $(1-\Lambda L^p)$, or $(1-\Lambda^{-1}L^{-p})^{-1}(-\Lambda^{-1}L^{-p})$, to b_t in (53) to get

(54) $$n_{t+p} = \rho_{t+p-1}n_{t+p-1}-(1/\Lambda) \sum_{j=0}^{\infty} (1/\Lambda)^j b_{t+p+pj}.$$

$$(55)\qquad n_t-\rho_{t-1}n_{t-1} = \gamma_{t-1}(n_{t-1}-\rho_{t-2}n_{t-2})+f_{t-1}e_{t-1}.$$

According to (54),

$$(56)\qquad (n_t-\rho_{t-1}n_{t-1}) = (-1/\Lambda)\sum_{j=0}^{\infty} (1/\Lambda)^j b_{t+pj}$$

and

$$(57)\qquad \gamma_{t-1}(n_{t-1}-\rho_{t-2}n_{t-2})+f_{t-1}e_{t-1} = \left(\frac{-\gamma_{t-1}}{\Lambda}\right)\left[\sum_{j=0}^{\infty} (1/\Lambda)^j b_{t-1+pj}\right]$$

$$+ f_{t-1}e_{t-1}.$$

Subtracting (57) from (56) gives

$$(58)\qquad (n_t-\rho_{t-1}n_{t-1})-\gamma_{t-1}(n_{t-1}-\rho_{t-2}n_{t-2})-f_{t-1}e_{t-1} =$$

$$(-1/\Lambda)\sum_{j=0}^{\infty} (-1/\Lambda)(b_{t+pj}-\gamma_{t-1}b_{t-1+pj})-f_{t-1}e_{t-1}.$$

From the definition of b_t in (51) and $\gamma_{t-1} = \gamma_{t-1+p}$,

$$(59)\qquad b_t-\gamma_{t-1}b_{t-1} = (f_{t+p-1}e_{t+p-1})-(\gamma_{t-1+p}f_{t-1+p-1}e_{t-1+p-1})$$

$$+(\gamma_{t+p-1}f_{t+p-2}e_{t+p-2})-(\gamma_{t-1+p}\gamma_{t-1+p-1}f_{t-1+p-2}e_{t-1+p-2})+$$

$$(\gamma_{t+p-1}\gamma_{t+p-2}f_{t+p-3}e_{t+p-3})-\cdots-(\gamma_{t-1+p}\gamma_{t-1+p-1}\cdots\gamma_{t+1}f_te_t)$$

$$+(\gamma_{t+p-1}\gamma_{t+p-2}\cdots\gamma_{t+1}f_te_t)-(\gamma_{t-1+p}\gamma_{t-1+p-1}\cdots\gamma_{t+1}\gamma_tf_{t-1}e_{t-1})$$

$$= f_{t+p-1}e_{t+p-1} - \Lambda f_{t-1}e_{t-1}.$$

Substituting this result in (58) gives

(60) $$(n_t-\rho_{t-1}n_{t-1})-\gamma_{t-1}(n_{t-1}-\rho_{t-2}n_{t-2})-f_{t-1}e_{t-1}$$

$$= \frac{-1}{\Lambda} \sum_{j=0}^{\infty} (\frac{1}{\Lambda})^j \; (f_{t+pj+p-1}e_{t+pj+p-1}-\Lambda \; f_{t+pj-1}e_{t+pj-1})-f_{t-1}e_{t-1}$$

$$= \frac{-1}{\Lambda} [f_{t+p-1}e_{t+p-1}-\Lambda \; f_{t-1}e_{t-1}+(\frac{1}{\Lambda})f_{t+2p-1}e_{t+2p-1}$$

$$- f_{t+p-1}e_{t+p-1}+(\frac{1}{\Lambda})^2 \; f_{t+3p-1}e_{t+3p-1}-(\frac{1}{\Lambda}) \; f_{t+2p-1}e_{t+2p-1}$$

$$+ (\frac{1}{\Lambda})^3 \; f_{t+4p-1}e_{t+4p-1}-(\frac{1}{\Lambda})^2 \; f_{t+3p-1}e_{t+3p-1}+ \ldots] - f_{t-1}e_{t-1}$$

$$= (\frac{-1}{\Lambda})(-\Lambda f_{t-1}e_{t-1}) - f_{t-1}e_{t=1} = 0,$$

which verifies that (54) satisfies (42).

Multiplying both sides of (54) by L^p and taking mathematical expectations conditioned on information available at time t gives the solution to the equivalent stochastic problem as

(61) $$n_t = \rho_{t-1}n_{t-1}-(1/\Lambda) \sum_{j=0}^{\infty} (1/\Lambda)^j \; E_t b_{t+pj}.$$

Appendix A shows that we can replace $(1/\Lambda)$ by ϕ^{pj} in the sum on the right side of (61), where ϕ^{-1} is the positive p^{th} root of Λ. Then the sum becomes

(62) $$\sum_{j=0}^{\infty} (1/\Lambda)^j \; E_t b_{t+pj} = \sum_{j=0}^{\infty} \phi^{pj} \; (f_{t+pj+p-1})E_t e_{t+pj+p-1}$$

$$+ \sum_{j=0}^{\infty} \phi^{pj} \; (\gamma_{t+pj+p-1}f_{t+pj+p-2}) \; E_t e_{t+pj+p-2}$$

$$+ \sum_{j=0}^{\infty} \phi^{pj} \; (\gamma_{t+pj+p-1}\gamma_{t+pj+p-2}f_{t+pj+p-3})E_t e_{t+pj+p-3}$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$+ \sum_{j=0}^{\infty} \phi^{pj} \quad (\gamma_{t+pj+p-1}\gamma_{t+pj+p-2} \cdots \gamma_{t+pj+1}f_{t+pj}) \; E_t e_{t+pj} .$$

Since the coefficients $(f_{t+pj+p-1})$, $(\gamma_{t+pj+p-1}f_{t+pj+p-2})$, $\cdots$, $(\gamma_{t+pj+p-1}\gamma_{t+pj+p-2} \cdots \gamma_{t+pj+1}f_{t+pj})$ are each periodic of period p, each of the sums on the right side of (62) is of the form treated in the first part of this section. Applying the methods explained there to the sums on the right side of (52) gives a realizable solution to the system given by (42) and (43).

Such a solution has the form of a decision rule for the social planner's problem that was described in Section I. Note that the planner would actually have up to p distinct decision rules, each derived from equation (54), the (at most) p distinct values of ρ_t, and the (at most) p distinct values of γ_t. The planner would use the p rules successively and then begin again with the first one, etc.

Appendix A: Factoring $\left(1-\lambda^p L^{-p}\right)$, and Related Topics

Factoring $\left(1-\lambda^p L^{-p}\right)$

Theorem: $\left(1-\lambda^p L^{-p}\right)$ can be factored as

$$\text{(A1)} \qquad \left(1-\lambda^p L^{-p}\right) = \left(1-Z_0\lambda L^{-1}\right)\left(1-Z_1\lambda L^{-1}\right) \ldots \left(1-Z_{p-1}\lambda L^{-1}\right),$$

where

$$\text{(A2)} \qquad Z_k = e^{i(2k\pi/p)} = \cos(2k\pi/p) + i \sin(2k\pi/p),$$

for $k = 0, 1, 2, \ldots, p-1$.

Proof: Let Z denote (λ/L), so that $\left(1-\lambda^p L^{-p}\right) = (1-Z^p)$. The equation $1-Z^p = 0$ has the p distinct complex roots, Z_0, Z_1, ..., Z_{p-2}, and Z_{p-1} given by equation (A2) [Churchill, Brown, and Verhey, pp. 15-16]. According to the fundamental theorem of algebra and its corollaries [Shanahan, pp. 204-205], $(1-Z^p)$ can be expressed in terms of its roots as

$$\text{(A3)} \qquad (1-Z^p) = b(Z-Z_0)(Z-Z_1) \ldots (Z-Z_{p-1}),$$

where b is a complex constant. In this case, $b = -1$, the coefficient on Z^p in $(1-Z^p)$. In addition, $b(-Z_0)(-Z_1) \ldots (-Z_{p-1}) = 1$, the constant term in $(1-Z^p)$. Hence $(-Z_0)(-Z_1) \ldots (-Z_{p-1}) = -1$. Rearranging (A3) gives

$$(1-Z^p) = \left[b(-Z_0)(-Z_1) \ldots (-Z_{p-1})\right]\left(1-Z_0^{-1}Z\right)\left(1-Z_1^{-1}Z\right) \ldots \left(1-Z_{p-1}^{-1}Z\right),$$

or

(A4) $$(1-Z^p) = (1-Z_0^{-1}Z)(1-Z_{p-1}^{-1}Z)(1-Z_{p-2}^{-1}Z) \dots (1-Z_1^{-1}Z).$$

Since $e^{i2\pi} = 1$ and $Z_0^{-1} = Z_0 = 1$, we can rewrite (A4) as

(A5) $$(1-Z^p) = (1-Z_0Z)(1-Z_{p-1}^{-1}e^{i2\pi}Z) \dots (1-Z_1^{-1}e^{i2\pi}Z).$$

But $Z_{p-k}^{-1}e^{i2\pi} = e^{i\left(\frac{2(k-p)\pi}{p}\right)}\ e^{i2\pi} = e^{i\left(\frac{2k\pi}{p}\right)} = Z_k$, so that

(A6) $$(1-Z^p) = (1-Z_0Z)(1-Z_1Z) \dots (1-Z_{p-1}Z)$$

Recalling that $Z = (\lambda/L)$ gives the desired result, equation (A1).

Factoring $(1-\lambda L^p)$

In the solution of LSDEs of the form $(1-\alpha_t L+L^2)n_t = f_t e_t$, it is necessary, in effect, to factor $(1-\lambda Z^p)$, where $\lambda > 0$. This can be done in the following way. Let s^{-1} be the positive p^{th} root of λ. If Z_k, $k = 0, 1, 2, \dots, p-1$, are the roots of $(1-Z^p)$ defined above, then sZ_k^{-1} are the roots of $(1-\lambda Z^p)$. That is,

$$[1-\lambda(sZ_k^{-1})^p] = [1-\lambda(\tfrac{1}{\lambda})] = 0, \text{ for } k = 0, 1, 2, \dots, p-1.$$

Thus, $(1-\lambda Z^p) = -\lambda(Z-sZ_0^{-1})(Z-sZ_1^{-1}) \dots (Z-sZ_{p-1}^{-1})$

$$= [(-\lambda)(-sZ_0^{-1})(-sZ_1^{-1}) \dots (-sZ_p^{-1})]\ (1-s^{-1}Z_0Z)(1-s^{-1}Z_1Z) \dots (1-s^{-1}Z_{p-1}Z).$$

In effect, we have written $(1-\lambda Z^p) = \left(1-(\sqrt[p]{\lambda}\ Z)^p\right)$ and factored as before. By this device, we can express the sums on the right side of equation (61) in terms of current and past values of the $\{e_t\}$ sequence. We merely let ϕ^{-1} be the positive ρ^{th} root of Λ, substitute $(\phi^{-1})^{pj}$ for $(1/\Lambda)^j$, and proceed as described above (see equation (31) and the pages that follow it).

Simple higher-order difference equations

The body of this paper deals with second-order LSDEs derived from first-order quadratic costs of adjustment, $(\delta/2)(n_t-n_{t-1})^2$. The methods for solving these LSDEs are easily adapted to solving the higher-order LSDEs derived from quadratic costs of adjustment of the form $(\frac{\delta}{2})\ (n_t-n_{t-q})^2$, where q is any positive integer. Costs of adjustment of this form lead to LSDEs having the form

$$\text{(A7)} \qquad (1-\alpha_t L^q+L^{2q})n_{t+q} = f_t e_t,$$

where all variables are as previously defined.

First consider the case where $\alpha_t = \alpha$, for all t. Then, with ρ_1 and ρ_2 defined as before, we can rewrite (A7) as

$$\text{(A8)} \qquad (1-\rho_1 L^q)(1-\rho_2 L^q)n_{t+q} = f_t e_t,$$

or, using $(1-\rho_2 L^q)^{-1} = -\rho_2^{-1}L^{-q}\ (1-\rho_2^{-1}L^{-q}) = \rho_1 L^{-q}(1-\rho_1 L^{-q})$ and applying the operator algebra formally,

$$\text{(A9)} \qquad n_t = \rho_1 n_{t-1} - \rho_1 \sum_{j=0}^{\infty} \rho_1^j\ f_{t+jq}\ e_{t+jq}$$

In this appendix, we have already seen that by letting $\rho_1^j = [\sqrt[q]{\rho_1}]^{qj}$ we can use methods described in this paper to express the sum in (A9) in terms of current and past values of the $\{e_t\}$ sequence.

The case where α_t is periodic of period p is a bit more complicated but entirely analogous to the case presented in the text. Defining ρ_t and γ_t so that $\rho_t + \gamma_t = \alpha_t$ and $\gamma_t \rho_{t-q} = 1$, we have that

(A10) $$n_{t+q} = \alpha_t n_t - n_{t-q} + f_t e_t$$

becomes

(A11) $$n_{t+q} = \rho_t n_t + \gamma_t (n_t - \rho_{t-q} n_{t-q}) + f_t e_t$$

and

$$n_{t+2q} = \rho_{t+q} n_{t+q} + \gamma_{t+q}(n_{t+q} - \rho_t n_t) + f_{t+q} e_{t+q}$$

$$\vdots$$

$$_{t+\tilde{p}q} = \rho_{t+\tilde{p}q-q} n_{t+\tilde{p}q-q} + \gamma_{t+\tilde{p}q-q}\left(n_{t+\tilde{p}q-q} - \rho_{t+\tilde{p}q-2q} n_{t+\tilde{p}q-2q}\right)$$

$$+ f_{t+\tilde{p}q-q} e_{t+\tilde{p}q-q},$$

where $\tilde{p}$ is the smallest integer such that $\tilde{p}q$ is evenly divisible by p. Then, proceeding as in the text, we get

(6) $$(1-\tilde{\lambda}L^{\tilde{p}q})(n_{t+\tilde{p}q} - \rho_{t+\tilde{p}q-q} n_{t+\tilde{p}q-q}) = \tilde{b}_t,$$

where

$$(7) \qquad \tilde{\Lambda} = \gamma_{t+\tilde{p}q-q}\,\gamma_{t+\tilde{p}q-2q} \cdots \gamma_{t+q}\,\gamma_t$$

and

$$(8) \qquad \tilde{b}_t = f_{t+\tilde{p}q-q}\,e_{t+\tilde{p}q-q} + \gamma_{t+\tilde{p}q-q}\,f_{t+\tilde{p}q-2q}\,e_{t+\tilde{p}q-2q}$$

$$+ \gamma_{t+\tilde{p}q-q}\,\gamma_{t+\tilde{p}q-2q}\,f_{t+\tilde{p}q-3q}\,e_{t+\tilde{p}q-3q} + \cdots$$

$$+ \gamma_{t+\tilde{p}q-q}\,\gamma_{t+\tilde{p}q-2q} \cdots \gamma_{t+q}\,f_t\,e_t .$$

Using $(1-\tilde{\Lambda}L^{\tilde{p}q})^{-1} = (-1/\tilde{\Lambda})L^{-\tilde{p}q}\,[1-(1/\tilde{\Lambda})L^{-\tilde{p}q}]^{-1}$, we get

$$(9) \qquad n_t = \rho_{t-q}\,n_{t-q} - (1/\tilde{\Lambda}) \sum_{j=0}^{\infty} (1/\tilde{\Lambda})^j\, \tilde{b}_{t+\tilde{p}qj} .$$

Let ϕ^{-1} be the positive $\tilde{p}q$ th root of $\tilde{\Lambda}$. Then we can substitute $\phi^{\tilde{p}qj}$ for $(1/\tilde{\Lambda})^j$ and proceed as described above.

Appendix B: Extending the Weiner-Kolmogorov Prediction Formulas Derived by Hansen and Sargent

Larry Christiano developed the results in this appendix in the summer of 1980 and gave me a photocopy of his manuscript in early 1981. I have only slightly altered the notation and wording of his manuscript in preparing this appendix.

For $\ell = 0, 1, 2, \ldots$, let

$$\text{(B1)} \qquad y_t^{\ell} \equiv E_t \sum_{j=0}^{\infty} \lambda^j \, e_{t+\ell+j},$$

where $\zeta(L)e_t = \Gamma(L)v_t$, $\zeta(L) = 1 + \zeta_1 L + \ldots + \zeta_r L^r$, $\Gamma(L) = \Gamma_0 + \Gamma_1 L + \ldots + \Gamma_q L^q$, v_t is fundamental for e_t, and $\lambda < 1$. The operator E_t indicates linear least squares projection on the set $\{e_t, e_{t-1}, \ldots\}$.

Hansen and Sargent showed that

$$\text{(B2)} \qquad y_t^0 \equiv E_t \sum_{j=0}^{\infty} \lambda^j e_{t+j} = \mu_1(L,\lambda)e_t + \mu_2(L,\lambda)v_t,$$

where

$$\text{(B3)} \qquad \mu_1(L,\lambda) = \zeta(\lambda)^{-1} - \zeta(\lambda)^{-1} \sum_{k=1}^{r-1} \Big[\sum_{h=k+1}^{r} \zeta_h \lambda^{h-k} \Big] L^k$$

$$= \mu_1^0 + \mu_1^1 L + \ldots + \mu_1^{r-1} L^{r-1},$$

and

(B4) $$\mu_2(L,\lambda) = \zeta(\lambda)^{-1} \sum_{k=0}^{q-1}\Big[\sum_{h=k+1}^{q} \Gamma_h \lambda^{h-k}\Big] L^k$$

$$= \mu_2^0 + \mu_2^1 L + \ldots + \mu_2^{q-1} L^{q-1}.$$

The strategy for extending their formulas to y_t^ℓ, $\ell = 1,2,\ldots$, is to use the law of iterated projections to calculate $y_t^\ell = E_t\, y_{t+\ell}^0$ after substituting $\mu_1(L,\lambda)e_{t+\ell} + \mu_2(L,\lambda)v_{t+\ell}$ for $y_{t+\ell}^0$. Applying the law of iterated projections and the Weiner-Kolmogorov prediction formula to $y_{t+\ell}^0$ gives

(B5) $$y_t^\ell = E_t\, y_{t+\ell}^0 = E_t\big[\mu_1(L,\lambda)e_{t+\ell} + \mu_2(L,\lambda)v_{t+\ell}\big],$$

or

(B6) $$y_t^\ell = E_t\, \tilde{\mu}_1(L,\lambda)e_{t+\ell} + \big[L^{-\ell}\mu_1(L,\lambda)\big]_+ e_t + \big[L^{-\ell}\mu_2(L,\lambda)\big]_+ v_t,$$

where $\tilde{\mu}_1(L,\lambda) = \mu_1^0 + \mu_1^1 L + \ldots + \mu_1^n L^n$ and $n = \min(\ell-1, r-1)$.

If we let $\alpha(L) \equiv \alpha_1 + \alpha_2 L + \ldots + \alpha_r L^{r-1} \equiv (\zeta_1 + \zeta_2 L + \ldots + \zeta_r L^{r-1})$, then

$$e_{t+1} = \alpha(L)e_t + \Gamma(L)v_{t+1}$$

$$e_{t+2} = \{\alpha_1\alpha(L) + [L^{-1}\alpha(L)]_+\}e_t + [\alpha_1 L + 1]\Gamma(L)v_{t+2}$$

$$e_{t+3} = \{(\alpha_1^2+\alpha_2)\alpha(L) + \alpha_1[L^{-1}\alpha(L)]_+ + [L^{-2}\alpha(L)]_+\}e_t$$

$$+ \{(\alpha_1^2+\alpha_2)L^2 + \alpha_1 L + 1\}\Gamma(L)v_t.$$

$$\vdots$$

(B7) $$e_{t+j} = \{\sum_{h=1}^{j} a_h [L^{h-j}\alpha(L)]_+\} e_t + \{\sum_{h=1}^{j} a_h L^{h-1}\}\Gamma(L) v_{t+j},$$

where $a_1 = 1$ and $a_h = \sum_{j=1}^{\min(h-1,r)} \alpha_j a_{h-j}$, for $h = 2, 3 \ldots.$

Applying the linear projection operator E_t to (B7) and substituting the result into the first term of (B6) gives

$$E_t\, \tilde{\mu}_1(L,\lambda) e_{t+\ell} = \mu_1^0 E_t e_{t+\ell} + \mu_1^1 E_t e_{t+\ell-1} + \cdots + \mu_1^n\, E_t e_{t+\ell-n}$$

$$= \{\mu_1^0 \sum_{h=1}^{\ell} a_i\, [L^{h-\ell}\alpha(L)]_+ + \mu_1^1 \sum_{h=1}^{\ell-1} a_h\, [L^{h+1-\ell}\alpha(L)]_+ + \mu_1^2 \sum_{h=1}^{\ell-2} a_h$$

$$[L^{h+2-\ell}\alpha(L)]_+ + \ldots + \mu_1^n \sum_{h=1}^{\ell-n} a_h [L^{h+n-\ell}\alpha(L)]_+\} e_t$$

$$+ \{\mu_1^0 [L^{-\ell}\Gamma(L) \sum_{h=1}^{\ell} a_h L^{h-1}]_+ + \mu_1^1 [L^{1-\ell}\Gamma(L) \sum_{h=1}^{\ell-1} a_h L^{h-1}]_+$$

$$+ \mu_1^2 [L^{2-\ell}\Gamma(L) \sum_{h=1}^{\ell-2} a_h L^{h-1}]_+ + \cdots + \mu_1^n [L^{n-\ell}\Gamma(L) \sum_{h=1}^{\ell-n} a_h L^{h-1}]_+ v_t,$$

or

(B8) $$E_t \tilde{\mu}_1(L,\lambda) e_{t+\ell} = \tilde{\sigma}_1(L,\lambda) e_t + \tilde{\sigma}_2(L,\lambda) v_t,$$

where

(B9) $$\tilde{\sigma}_1(L,\lambda) \equiv \sum_{k=0}^{n} \mu_1^k \sum_{h=1}^{\ell-k} a_h [L^{h+k-\ell}\alpha(L)]_+$$

and

(B10) $$\tilde{\sigma}_2(L,\lambda) \equiv \sum_{k=0}^{n} \mu_1^k [L^{k-\ell}\Gamma(L) \sum_{h=1}^{\ell-k} a_h L^{h-1}]_+ .$$

We can simplify $\tilde{\sigma}_1(L,\lambda)$ by noting that

(B11) $$[L^{h+k-\ell}\alpha(L)]_+ = \begin{cases} \alpha_{\ell-k-h+1} + \alpha_{\ell-k-h+2}L + \dots + \alpha_r L^{r-1-\ell+h+k}, \\ \quad \text{for } (r-1) \geq (m-h-k); \\ 0, \text{ for } (r-1) < (m-h-k). \end{cases}$$

Then

(B12) $$\tilde{\sigma}_1(L,\lambda) = \sum_{k=0}^{n} \mu_1^n \sum_{h=1}^{\ell-k} a_h \sum_{j=0}^{r-1-\ell+h+k} \alpha_{\ell-k-h+1+j} L^j ,$$

where the final summation is defined to be zero when $r-1-\ell+h+k < 0$. Rearrangement of (B12) gives

(B13) $$\tilde{\sigma}_1(L,\lambda) = \sum_{j=0}^{r-1} [\sum_{h=j+1}^{\min(r,j+\ell)} \alpha_h \sum_{k=0}^{\min(\ell-h+j,r-1)} \mu_1^k a_{\ell-k-h+j+1}] L^j .$$

To simplify $\tilde{\sigma}_2$ (L,λ), note that

$$\Gamma(L)\sum_{h=1}^{\ell-k} a_h L^{h-1} = (a_1 + a_2 L + a_3 L^2 + \dots + a_{\ell-k} L^{\ell-k-1})$$

$$(\Gamma_0 + \Gamma_1 L + \dots + \Gamma_q L^q)$$

$$= a_1\Gamma_0 + (a_1\Gamma_1 + a_2\Gamma_0)L + (a_1\Gamma_2 + a_2\Gamma_1 + a_3\Gamma_0)L^2 + \dots$$

$$+ a_{\ell-k}\Gamma_q L^{q+\ell-k-1},$$

or

$$\text{(B14)} \qquad \Gamma(L) \sum_{h=1}^{\ell-k} a_h \, L^{h-1} = \sum_{h=0}^{q+\ell-k-1} b_h^k \, L^h,$$

where $b_h^k = \sum_{j=\max(h-q,0)}^{\min(h,\ell-k-1)} a_{j+1}\Gamma_{h-j}$. From (B14) we get

$$\text{(B15)} \qquad [L^{k-\ell}\Gamma(L)\sum_{h=1}^{\ell-k} a_h L^{h-1}]_+ = b_{\ell-k}^k + b_{\ell-k+1}^k L + \dots + b_{q+\ell-k-1}^k L^{q-1}$$

$$= \sum_{h=0}^{q-1} b_{\ell-k+h}^k L^h,$$

for k = 0, 1, ..., ℓ-1. Finally, substituting (B15) into (B10) we get

$$\text{(B16)} \qquad \tilde{\sigma}_2(L,\lambda) = \sum_{k=0}^{n} \mu_1^k \sum_{h=0}^{q-1} b_{\ell-k+h}^k L^h = \sum_{h=0}^{q-1} [\sum_{k=0}^{n} \mu_1^k b_{\ell-k+h}^k] L^h$$

$$= \sum_{h=0}^{q-1} [\sum_{k=0}^{\min(\ell-1,r-1)} \mu_1^k \sum_{j=\max(\ell-k+h-q,0)}^{\ell-k-1} a_{j+1}\Gamma_{\ell-k+h-j}] L^h.$$

Now substitute (B16) and (B13) into (B8) and use the resulting expression in (B6) to get

(B17) $$y_t^{\ell} = \sigma_1(L,\lambda)e_t + \sigma_2(L,\lambda)v_t,$$

where

(B18) $$\sigma_1(L,\lambda) = \tilde{\sigma}_1(L,\lambda) + [L^{-\ell}\mu_1(L,\lambda)]_+$$

(B19) $$\sigma_2(L,\lambda) = \tilde{\sigma}_2(L,\lambda) + [L^{-\ell}\mu_2(L,\lambda)]_+.$$

Equation (B17) is the solution we sought. To summarize our results, note that we have shown

$$E_t \sum_{j=0}^{\infty} \lambda^j e_{t+\ell+j} = \sigma_1(L,\lambda)e_t + \sigma_2(1,\lambda)v_t,$$

where

$$\sigma_1(L,\lambda) = \{\sum_{j=0}^{r-1}[\sum_{h=j+1}^{\min(r,j+\ell)} \alpha_h \sum_{k=0}^{\min(\ell-h+j,r-1)} \mu_1^k a_{\ell-k-h+j+1}]L^j\}$$
$$+ [L^{-\ell}\mu_1(L,\lambda)]_+,$$

and

$$\sigma_2(L,\lambda) = \{\sum_{h=0}^{q-1}[\sum_{k=0}^{\min(\ell-1,r-1)} \mu_1^k \sum_{j=\max(\ell-k+h-q,0)}^{\ell-k-1} a_{j+1}\Gamma_{\ell-k+h-j}]L^h\}$$
$$+ [L^{-\ell}\mu_2(L,\lambda)]_+,$$

with

$$\zeta(L)e_t \equiv (1 + \zeta_1 L + \dots + \zeta_r L^r)e_t$$

$$= (\Gamma_0 + \Gamma_1 L + \dots + \Gamma_q L^q)v_t \equiv \Gamma(L)v_t;$$

$$\alpha_h = -\zeta_h,\ h = 1, 2, \dots, r;$$

$$a_1 = 1 \text{ and } \quad a_h = \sum_{j=1}^{\min(h-1,r)} \alpha_j a_{h-j}, \text{ for } h = 2, 3, \dots;$$

$$\mu_1(L,\lambda) = \zeta(\lambda)^{-1} - \zeta(\lambda)^{-1} \sum_{k=1}^{r-1} [\sum_{h=k+1}^{r} \zeta_h \lambda^{h-k}] L^k;$$

$$\mu_2(L,\lambda) = \zeta(\lambda)^{-1} \sum_{k=0}^{q-1} [\sum_{h=k+1}^{q} \Gamma_h \lambda^{h-k}] L^k;$$

$$\mu_1^0 = \zeta(\lambda)^{-1};$$

$$\mu_1^k = -\zeta(\lambda)^{-1} \sum_{h=k+1}^{r} \zeta_h \lambda^{h-k}, \text{ for } k = 2, 3, \dots, r-1.$$

Thus,

$$[L^{-\ell}\mu_1(L,\lambda)]_+ = \begin{cases} -\zeta(\lambda)^{-1} \sum_{k=\ell}^{r-1} [\sum_{h=k+1}^{r} \zeta_h \lambda^{h-k}] L^{k-\ell}, \\ \quad \text{for } \ell + 0, 1, 2, \dots, r-1; \\ 0, \text{ for } \ell = r, r+1, \dots; \end{cases}$$

and

$$[L^{-\ell}\mu_2(L,\lambda)]_+ = \begin{cases} \zeta(\lambda)^{-1} \sum_{k=1}^{q-1} [\sum_{h=k+1}^{q} \Gamma_h \lambda^{h-k}] L^{k-\ell}, \\ \quad \text{for } \ell = 1, 2, \dots, q-1; \\ 0, \text{ for } \ell = q, q+1, \dots . \end{cases}$$

To match the results of this appendix with the formula given in equation (36), note that in the body of this paper we have $\Gamma(L)v_t^e = v_t^e$, or q=0. Hence $\sigma_2(L,\lambda) = 0$ for that prediction problem.

References

Hansen, Lars Peter and Thomas J. Sargent (1981), "A Note on Weiner-Kolmogorov Prediction Formulas for Rational Expectations Models." Research Department Staff Report 69, Federal Reserve Bank of Minneapolis, September.

Appendix C: Properties of the Parameters Used to Rewrite the System of LSDEs with Periodic-Coefficient Characteristic Equations

The results in the second part of Section II depend upon being able to rewrite (42) as (45), where the 2p new parameters of the periodic (of period p) $\{\rho_t\}$ and $\{\gamma_t\}$ sequences satisfy (46), (47), and $0 < \rho_t < 1 < \gamma_t < 1$ for all t. Here we shall show that (46) and (47) uniquely determine these 2p new parameters.

Represent the system (42) as

(C1) $$n_{t+1} = \alpha_\ell n_t - n_{t-1} + f_t e_t,$$

where $t = kp+\ell$, $k = -1,0,1,2, \ldots,$ and $\ell \in (0,1,2, \ldots, p-1)$.

Then we seek to rewrite the system as

(C2) $$n_{t+1} = \rho_\ell n_t + \gamma_\ell (n_t - \rho_{\ell-1} n_{t-1}) + f_t e_t,$$

where

(C3) $$\rho_\ell + \gamma_\ell = \alpha_\ell$$

and

(C4) $$\rho_{\ell-1}\gamma_\ell = 1,$$

with $\rho_{-1} = \rho_{p-1}$ when $\ell = 0$. Equation (C2) corresponds to equation (45). In addition, we need to show that $\rho_0, \rho_1, \ldots, \rho_{p-1}, \gamma_0, \gamma_1, \ldots, \gamma_{p-1}$ can be uniquely chosen to satisfy $0 < \rho_\ell < 1 < \gamma_\ell < \alpha_\ell$, for $\ell = 0,1,2, \ldots, p-1$.

To begin, use (C3) and (C4) to express ρ_0 as an implicit function of $\alpha_0, \alpha_1, \ldots, \alpha_{p-1}$, or

(C5) $$\rho_0 = \cfrac{1}{\alpha_1 - \cfrac{1}{\alpha_2 - \cfrac{1}{\alpha_3 - \ddots \cfrac{1}{\alpha_{p-1} - \cfrac{1}{\alpha_0 - \rho_0}}}}} .$$

Define $F_\ell(x) \equiv (\alpha_\ell - x)^{-1}$, for $\ell = 0, 1, 2, \ldots, p-1$, and $\psi(x) = x - F_1(F_2(F_3(\ldots(F_{p-1}(F_0(x)))\ldots)))$. Then (C5) can be written as

(C6) $$\psi(\rho_0) = \rho_0 - F_1(F_2(F_3(\ldots(F_0(\rho_0))\ldots)))=0.$$

We can now state

Theorem 1: The function ψ defined above has a unique root in $(0,1)$.

Proof: Since $\alpha_\ell > 2$, F_ℓ is continuous on the domain $(-\infty,2]$, for $\ell = 0,1,2, \ldots, p-1$. Furthermore, $0 < F_\ell(x) < 1$ for $x \leq 1$ and $\ell = 0,1,2, \ldots, p-1$. Thus if $x \leq 1$, $0 < F_0(x) < 1$; $0 < F_{p-1}((F_0(x))) < 1$ and $F_{p-1} \circ F_0$ is continuous on $(-\infty,1]$; $0 < F_{p-2}(F_{p-1}(F_0(x))) < 1$ and $F_{p-2} \circ F_{p-1} \circ F_0$ is continuous on $(-\infty, 1]$; etc. Continuing in this fashion shows that $x \leq 1$ implies that $0 < F_1(F_2(F_3(\ldots(F_{p-1}(F_0(x)))\ldots))) < 1$ and that Ψ is continuous on $(-\infty, 1]$. Hence $\psi(0) < 0$ and $\psi(1) > 0$. By the continuity of ψ on $(-\infty, 1]$ and the intermediate value theorem, ψ has a root in $(0,1)$.

To see that the root is unique, note that $0 < \frac{d}{dx} F_\ell(x) < 1$ for $x \leq 1$ and $\ell = 0, 1, 2, \ldots, p-1$. Using this property and the chain rule of differentiation gives

(C7) $$0 < \frac{d}{dx} (F_1 o F_2 o \ldots o F_{p-1} o F_0)(x) < 1 \text{ for } x \leq 1.$$

Hence

$0 < \frac{d}{dx} \psi(x) = 1 - \frac{d}{dx} (F_1 o F_2 o \ldots o F_0)(x) < 1$ for $x \leq 1$, so ψ is monotone in (0,1) and thus has only one root in that interval. Q.E.D.

According to the theorem, there is one and only value of ρ_0 between 0 and 1 that satisfies equation (C5). Given this value, select $\rho_1, \rho_2, \ldots, \rho_{p-1}$ according to

(C8) $$\rho_{p-1} = \frac{1}{\alpha_0 - \rho_0} = F_0(\rho_0) < 1.$$

$$\rho_{p-2} = \frac{1}{\alpha_{p-1} - \rho_{p-1}} = F_{p-1}(F_0(\rho_0)) < 1$$

$$\vdots$$

$$\rho_1 = \frac{1}{\alpha_2 - \rho_2} = F_2(F_3(\ldots(F_{p-2}(F_{p-1}(F_0(\rho_0))))\ldots)) < 1.$$

Finally, set

(C9) $$\gamma_0 = 1/\rho_{p-1}$$

$$\gamma_\ell = 1/\rho_{\ell-1},\ \ell = 1, 2, \ldots, p-1.$$

Choosing according to (C9) guarantees that (C4) will be satisfied. To see that equation (C3) also holds, note that

$$\rho_0 + \gamma_0 = \rho_0 + (1/\rho_{p-1}) = \rho_0 + [1/((\alpha_0 - \rho_0) -1)] = \rho_0 + \alpha_0 - \rho_0 = \alpha_0,$$

and

$$\rho_\ell + \gamma_\ell = \rho_\ell + (1/\rho_{\ell-1}) = \rho_\ell + [1/((\alpha_\ell - \rho_\ell) -1)] = \rho_\ell + \alpha_\ell - \rho_\ell = \alpha_\ell,$$

for ℓ=1,2, ..., p-1. This shows how to select parameters ρ_0, ρ_1, ..., ρ_{p-1}, ρ_{p-1}, γ_0, γ_1, ..., γ_{p-1} that satisfy (C3) and (C4) as well as $0 < \rho_\ell < 1 < \gamma_\ell < \alpha_\ell$, for ℓ = 0, 1, 2, ..., p-1. To see that only this set of parameters can satisfy the conditions, recall that ρ_0 was unique and note that (C3) and (C4) imply that (C8) and (C9) must be satisfied.