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Notes on Difference Equations and Lag Operators

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The backward shift or lag operator is defined by

$$L\, X_t = X_{t-1}$$

$$(1) \qquad L^n\, X_t = X_{t-n} \qquad \text{for } n = \ldots -2,-1,0,1,2,\ldots$$

Multiplying a variable X_t by L^n thus gives the value of X shifted back n periods. Notice that if $n < 0$ in (1), the effect of multiplying X_t by L^n is to shift X forward in time by (-n) periods.

We shall consider polynomials in the lag operator

$$\begin{aligned} A(L) &= a_0 + a_1\, L + a_2\, L^2 + \ldots \\ &= \sum_{j=0}^{\infty} a_j\, L^j, \end{aligned}$$

where the a_j's are constants and $L^0 \equiv 1$. Multiplying X_t by A(L) yields a moving sum of X's:

$$\begin{aligned} A(L)\, X_t &= (a_0 + a_1 L + a_2 L^2 + \ldots)\, X_t \\ &= a_0\, X_t + a_1\, X_{t-1} + a_2\, X_{t-2} + \ldots \\ &= \sum_{j=0}^{\infty} a_j\, X_{t-j}. \end{aligned}$$

It is generally convenient to work with polynomials A(L) that are "rational," meaning that they can be expressed as the ratio of two (finite order) polynomials in L:

$$A(L) = \frac{B(L)}{C(L)}$$

where

$$B(L) = \sum_{j=0}^{m} b_j L^j, \; C(L) = \sum_{j=0}^{n} C_j L^J \text{ where}$$

the b_j's and C_j's are constant. Assuming that $A(L)$ is rational amounts to imposing a more economical and restrictive parameterization on the a_j's.

To take the simplest example of a rational polynomial in L, consider

$$(2) \qquad A(L) = \frac{1}{1-\lambda L} .$$

For the scalar $|C| < 1$, we know that

$$(3) \qquad \frac{1}{1-C} = 1 + C + C^2 + \ldots .$$

This suggests treating λL of (2) exactly like the C of (3) to get

$$(4) \qquad \frac{1}{1-\lambda L} = 1 + \lambda L + \lambda^2 L^2 + \ldots,$$

an expansion which is sometimes only "useful" so long as $|\lambda| < 1$. To prove that the equality (4) is true, multiply both sides of (4) by $(1-\lambda L)$ to obtain

$$\frac{1-\lambda L}{1-\lambda L} = 1 = (1+\lambda L+\lambda^2 L^2+\ldots) - \lambda L(1+\lambda L+\lambda L^2+\ldots) = 1,$$

which holds for any value of λ, not just values of λ obeying $|\lambda| < 1$. The reason that sometimes we say that (4) is "useful" only if $|\lambda| < 1$ derives from the following argument. We intend often to multiply $1/(1-\lambda L)$ by X_t to obtain the infinite moving sum

$$(5) \qquad \frac{1}{1-\lambda L} X_t = (1+\lambda L+\lambda^2 L^2+\ldots) X_t$$

$$= \sum_{i=0}^{\infty} \lambda^i X_{t-i}.$$

Consider this sum for a path of X which is constant over time, so that $X_{t-i} = \bar{X}$ for all i and all t. Then the sum of (5) becomes

$$\frac{1}{1-\lambda L} X_t = \bar{X} \sum_{i=0}^{\infty} \lambda^i$$

The sum $\sum_{i=0}^{\infty} \lambda^i$ equals $(1/(1-\lambda))$ if $|\lambda| < 1$.

But if $|\lambda| \geq 1$ that sum is unbounded, being $+\infty$ if $\lambda \geq 1$. We will sometimes (though not always) be applying the polynomial in the lag operator (4) in situations in which it is appropriate to go infinitely far back in time; and we sometimes find it necessary to insist that in such cases the infinite sum in (5) exist where X has been constant through time. This is what leads to the requirement sometimes imposed that $|\lambda| < 1$ in (4). As we shall see, however, in standard analyses of difference equations, which take the starting point of all processes as some point only finitely far back into the past, the requirement that $|\lambda| < 1$ need not be imposed in (4).

It is useful to note that there is an alternative expansion for the "geometric" polynomial $1/(1-\lambda L)$. For notice that

(6) $$\frac{1}{1-\lambda L} = \frac{-\frac{1}{\lambda L}}{1-\frac{1}{\lambda}L^{-1}}$$

$$= \frac{-1}{\lambda L}\left(1 + \frac{1}{\lambda}L^{-1} + \left(\frac{1}{\lambda}\right)^2 L^{-2} + \ldots\right)$$

$$= \frac{-1}{\lambda}L^{-1} - \left(\frac{1}{\lambda}\right)^2 L^{-2} - \left(\frac{1}{\lambda}\right)^3 L^{-3} - \ldots,$$

an expansion which is especially "useful" where $|\lambda| > 1$, i.e., where $|1/\lambda| < 1$. So (6) implies that

$$\frac{1}{1-\lambda L} X_t = -\frac{1}{\lambda} X_{t+1} - \frac{1}{\lambda 2} X_{t+2} - \ldots$$

$$= -\sum_{i=1}^{\infty} (\frac{1}{\lambda})^i X_{t+i},$$

which shows $(1/(1-\lambda L))\ X_t$ to be a geometrically declining weighted sum of future values of X. Notice that for this infinite sum to be finite for a constant time path $X_{t+i} = \bar{X}$ for all i and t, the series

$$-\sum_{i=1}^{\infty} (\frac{1}{\lambda})^i$$

must be convergent, which requires that $|\frac{1}{\lambda}| < 1$.

To illustrate how polynomials in the lag operator can be manipulated, consider the difference equation

(7) $$Y_t = \lambda Y_{t-1} + b\ X_t + a \qquad t = -\infty,\ldots,0,1,2,\ldots$$

where X_t is an exogenous variable and Y_t is an endogenous variable. Write the above equation as

$$(1-\lambda L)\ Y_t = a + b\ X_t.$$

Dividing both sides of the equation by $(1-\lambda L)$ gives

$$Y_t = \frac{a}{1-\lambda L} + \frac{b}{1-\lambda L} X_t.$$

(8) $$Y_t = \frac{a}{1-\lambda} + b \sum_{i=0}^{\infty} \lambda^i X_{t-i},$$

since $a/(1-\lambda L) = \sum_{i=0}^{\infty} \lambda^i a = a \sum_{i=0}^{\infty} \lambda^i = a/(1-\lambda)$ provided $|\lambda| < 1$. So the first-order difference equation (7) and the geometric distributed lag equation (8) are equivalent. Equation (8) can be regarded as the "solution" to (7), since it describes the entire path of Y associated with a given time path for X. Notice that for the Y_t defined by (8) to be finite, $\lambda^i X_{t-i}$ must be "small" for large i. More precisely, we require

(9) $$\lim_{n \to \infty} \sum_{i=n}^{\infty} \lambda^i X_{t-i} = 0, \text{ for all } t.$$

For the case of X constant for all time, $X_{t-i} = \bar{X}$ all i and t, this condition becomes

$$\lim_{n \to 0} \bar{X} \frac{\lambda^n}{1-\lambda} = 0,$$

which requires $|\lambda| < 1$. Notice also that the infinite sum $a \sum_{i=0}^{\infty} \lambda^i$ in (8) is finite only if $|\lambda| < 1$, in which case it equals $a/(1-\lambda)$, or if $a = 0$, in which case it equals zero regardless of the value of λ. We tentatively assume that $|\lambda| < 1$.

For analyzing difference equations with arbitrary initial conditions given, it is convenient to rewrite (8) for $t > 0$ as

$$Y_t = a \sum_{i=0}^{t-1} \lambda^i + a \sum_{i=t}^{\infty} \lambda^i + b \sum_{i=0}^{t-1} \lambda^i X_{t-i}$$

$$+ b \sum_{i=t}^{\infty} \lambda^i X_{t-i}.$$

$$= \frac{a(1-\lambda^t)}{1-\lambda} + \frac{a\lambda^t}{1-\lambda} + b \sum_{i=0}^{t-1} \lambda^i X_{t-i}$$

$$+ b \lambda^t \sum_{i=0}^{\infty} \lambda^i X_{0-i}$$

(10) $$Y_t = \frac{a(1-\lambda^t)}{1-\lambda} + b \sum_{i=0}^{t-1} \lambda^i X_{t-i}$$

$$+ \lambda^t \left\{\frac{a}{1-\lambda} + b \sum_{i=0}^{\infty} \lambda^i X_{0-i}\right\} . \qquad t \geq 1$$

The term in braces equals Y_0, as reference to expression (8) will confirm. So (10) becomes

$$Y_t = \frac{a(1-\lambda^t)}{1-\lambda} + b \sum_{i=0}^{t-1} \lambda^i X_{t-i} + \lambda^t Y_0$$

or

(11) $$Y_t = \frac{a}{1-\lambda} + \lambda^t \left(Y_0 - \frac{a}{1-\lambda}\right) + b \sum_{i=0}^{t-1} \lambda^i X_{t-i}. \qquad t \geq 1$$

Now textbooks on difference equations often analyze the special case in which $X_t = 0$ for all $t > 0$. Under this special circumstance (11) becomes

(12) $$Y_t = \frac{a}{1-\lambda} + \lambda^t\left(Y_0 - \left(\frac{a}{1-\lambda}\right)\right),$$

which is the solution of the first-order difference equation $Y_t = a + \lambda Y_{t-1}$ subject to the initial condition that Y equals the arbitrarily given value Y_0 at time. Notice that if $Y_0 = a/(1-\lambda)$, then (12) implies $Y_t = Y_0$ for all $t \geq 0$, which shows $a/(1-\lambda)$ to be a "stationary point" or long-run equilibrium value of Y. Notice also that if, as we are assuming, $|\lambda| < 1$, then (12) implies that

$$\lim_{t \to \infty} Y_t = \frac{a}{1-\lambda},$$

which shows that the system is "stable," tending to approach the stationary point as time passes.

Now consider the first-order system (7) under the assumption that $a = 0$, so that $a \sum_{i=0}^{\infty} \lambda^i$ equals zero regardless of the value of λ. Then the appropriate counterpart to (10) is

$$Y_t = b \sum_{i=0}^{t-1} \lambda^i X_{t-i} + \lambda^t b \sum_{i=0}^{\infty} \lambda^i X_{0-i}.$$

Assuming that condition (9) is met even where $|\lambda| > 1$ (so that the second term in the equation is finite), the above equation becomes

$$Y_t = b \sum_{i=0}^{t-1} \lambda^i X_{t-i} + \lambda^t Y_0 \qquad t \geq 1$$

As before we analyze the special case where $X_t = 0$ for all $t > 0$. Then the above equation becomes

$$Y_t = \lambda^t Y_0. \qquad t \geq 1$$

The stationary point of this solution is zero, since if $Y_0 = 0$, Y will remain equal to zero forever, regardless of the value of λ. However, if $|\lambda| > 1$, the system will diverge farther and farther from this stationary point if either $Y_0 > 0$, or $Y_0 < 0$. If $\lambda > 1$, Y_t will tend toward $+\infty$ as $t \to \infty$ provided $Y_0 > 0$; Y_t will tend toward $-\infty$ as $t \to \infty$ if $Y_0 < 0$. If $\lambda < -1$, Y_t will display explosive oscillations of periodicity two time periods.

Where an arbitrary initial condition finitely far back in time is not supplied, so that the process is thought of as starting up infinitely far back in time, equation (8) is the solution to (7) provided that $|\lambda| < 1$ and that condition (9) is met. (We require $|\lambda| < 1$ so that a $\sum_{i=0}^{\infty} \lambda^i$ be finite.) It may seem that to the right side of (8) we could add a term $\alpha\lambda^t$, where α is arbitrary, to get

$$(8') \qquad Y_t = \frac{a}{1-\lambda} + b \sum_{i=0}^{\infty} \lambda^i X_{t-i} + \lambda^t \alpha,$$

which seems to be a solution of (7). To see this, notice that (8') implies

$$\lambda Y_{t-1} = \frac{a\lambda}{1-\lambda} + b \sum_{i=1}^{\infty} \lambda^i X_{t-i} + \lambda^t \alpha.$$

Subtracting the above equation from (8') yields equation (7), so that (8') is indeed a solution of (7). However, notice that we are requiring (8') to be a solution for all t. But if $\alpha > 0$, for example, then

$$\lim_{t \to -\infty} \lambda^t \alpha = \lim_{i \to \infty} \lambda^{-i} \alpha = \infty$$

if $0 < \lambda < 1$. (If $1 < \lambda < 0$, the λ^{-i} will display unbounded, undamped oscillations as $t \to -\infty$). Thus, if we require that the limit of $|Y_t|$ must be finite as $t \to -\infty$, we must in general have that $\alpha = 0$ in (8'), so that (8') collapses to (8). The condition that $\lim_{t \to -\infty} |Y_t| < \infty$ is, in effect, an initial condition that we are imposing on the solution.

If $|\lambda| > 1$, then (8) is not the appropriate solution for (7). A solution can be obtained by solving in the forward direction, using equation (6). The solution to (7) is then

$$(8'') \qquad Y_t = - \sum_{i=1}^{\infty} \left(\frac{1}{\lambda}\right)^i X_{t+i} - \frac{\frac{a}{\lambda}}{1-\frac{1}{\lambda}}$$

where we require that

$$(9') \qquad \lim_{n \to \infty} \sum_{i=n}^{\infty} \left(\frac{1}{\lambda}\right)^i X_{t+i} = 0,$$

so that the above infinite sum is finite.

As before, (8'') remains a solution to (7) if the term $\alpha\lambda^t$, α arbitrary, is added to the right side of (8''):

$$Y_t = - \sum_{i+1}^{\infty} \left(\frac{1}{\lambda}\right)^i X_{t+i} - \frac{\frac{a}{\lambda}}{1- \frac{1}{\lambda}} + \alpha\lambda^t.$$

To see this, subtract λY_{t-1} from both sides of the above equation. But since $|\lambda| > 1$, if $\alpha \neq 0$, then the above equation implies that for many X paths satisfying condition (9') (e.g., a path for which X is constant for all times),

$$\lim_{t \to \infty} |Y_t| = \infty.$$

This occurs because for $\alpha \neq 0$,

$$\lim_{t \to \infty} |\alpha \lambda^t| = \infty,$$

since $|\lambda| > 1$. Since we want Y to be finite for all t, we will impose the requirement

$$\lim_{t \to \infty} |Y_t| < \infty$$

which implies that $\alpha = 0$. So (8'') is the solution to (7) for $|\lambda| > 1$ that satisfies the "terminal condition" summarized by the above inequality.

Second-Order Difference Equations

Consider the second-order difference equation

$$Y_t = t_1 Y_{t-1} + t_2 Y_{t-2} + a + b X_t. \tag{13}$$

Using lag operators, (13) can be written as

$$(1-t_1 L - t_2 L^2) Y_t = a + b X_t$$

or

(14) $$Y_t = \frac{a}{1-t_1L-t_2L^2} + \frac{b}{1-t_1L-t_2L^2} X_t .$$

by long division it is easy to verify that

(15) $$\frac{b}{1-t_1L-t_2L^2} = \sum_{i=0}^{\infty} w_i L^i$$

where $w_0 = b_0$

$w_1 = b_0 t_1$

$w_j = t_1 \, w_{j-1} + t_2 \, w_{j-2}$ for $j \geq 2$.

That is,

$$\begin{array}{r|l} & 1+t_1L+(t_2+t_1^2)L^2 + (t_1(t_2+t_1^2)+t_1t_2)L^3 + \ldots \\ \hline 1-t_1L-t_2L^2 & 1 \\ & \underline{1-t_1L-t_2L^2} \\ & \quad t_1L + t_2L^2 \\ & \quad \underline{t_1L - t_1^2 \, L^2 - t_1t_2L^3} \\ & \qquad (t_2+t_1^2)L + t_1t_2L^3 \\ & \qquad \underline{(t_2+t_1^2)L - t_1(t_2+t_1^2)L^3 - t_2(t_2+t_1^2)L^4} \quad \ldots \end{array}$$

Notice that the weights in (15) follow a geometric pattern if $t_2 = 0$, as we would expect, since then (13) collapses to a first-order equation.

It is convenient to write the polynomial $(1-t_1 \, L-t_2L^2)$ in an alternative way, given by the "factorization"

(16) $$\begin{aligned} 1-t_1 \, L-t_2 \, L^2 &= (1-\lambda_1L)(1-\lambda_2L) \\ &= (1-(\lambda_1+\lambda_2)L + \lambda_1\lambda_2L^2), \end{aligned}$$

so that $\lambda_1 + \lambda_2 = t_1$ and $-\lambda_1\lambda_2 = t_2$. To see how λ_1 and λ_2 are related to the "roots" or "zeroes" of $(1-t_1 L-t_2 L^2)$, notice that

$$(1-\lambda_1 L)(1-\lambda_2 L) = \lambda_1\lambda_2(\frac{1}{\lambda_1} - L)(\frac{1}{\lambda_2} - L).$$

Therefore the equation

$$0 = (1-\lambda_1 L)(1-\lambda_2 L) = \lambda_1\lambda_2(\frac{1}{\lambda_1} - L)(\frac{1}{\lambda_2} - L)$$

is satisfied at the two "roots" $L = \frac{1}{\lambda_1}$ and $L = \frac{1}{\lambda_2}$. Given the polynomial $1-t_1 L-t_2 L^2$, the roots $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ are found from solving the "characteristic equation"

$$1-t_1 L - t_2 L^2 = 0 \text{ or } t_2 L^2 + t_1 L-1 = 0$$

for two values of L. The roots are given by the quadratic formula

$$L = \frac{-t_1 \pm \sqrt{t_1^2 + 4t_2}}{2t_2}. \tag{17}$$

Formula (17) enables us to obtain the reciprocals of λ_1 and λ_2 for given values of t_1 and t_2.

So without loss of generality, we can write the second-order difference equation as

$$(1-\lambda_1 L)(1-\lambda_2 L)Y_t = a + b X_t.$$

$$Y_t = \frac{a}{(1-\lambda_1 L)(1-\lambda_2 L)} + \frac{b}{(1-\lambda_1 L)(1-\lambda_2 L)} X_t. \tag{18}$$

Notice that if $\lambda_1 \neq \lambda_2$

$$\frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)} = \frac{1}{\lambda_1-\lambda_2} (\frac{\lambda_1}{1-\lambda_1 L} - \frac{\lambda_2}{1-\lambda_2 L}),$$

which can be verified directly. Thus (18) can be written

$$Y_t = \frac{a}{(1-\lambda_1)(1-\lambda_2)} + \frac{\lambda_1 b}{\lambda_1-\lambda_2} \cdot \frac{1}{1-\lambda_1 L} X_t - \frac{\lambda_2 b}{\lambda_1-\lambda_2} \cdot \frac{1}{1-\lambda_2 L} X_t$$

(19) $$Y_t = a \sum_{i=0}^{\infty} \lambda_1^i \sum_{j=0}^{\infty} \lambda_2^j + \frac{\lambda_1 b}{\lambda_1-\lambda_2} \sum_{i=0}^{\infty} \lambda_1^i X_{t-i}$$

$$- \frac{\lambda_2 b}{\lambda_1-\lambda_2} \sum_{i=0}^{\infty} \lambda_2^i X_{t-i},$$

where we are making use of the fact that for a constant a

$$H(L)a = \sum_{i=0}^{\infty} h_i L^i a$$

$$= a \sum_{i=0}^{\infty} h_i = a\, H(1).$$

Notice that

$$\frac{1}{1-\lambda_1 L} \cdot \frac{1}{1-\lambda_2 L} = \sum_{i=0}^{\infty} \lambda_1^i L^i \sum_{j=0}^{\infty} \lambda_2^j L^j$$

so that the sum of the distributed lag weights

$\sum_{i=0}^{\infty} \lambda_1^i \sum_{i=0}^{\infty} \lambda_2^i$ is finite and equals $\frac{1}{(1-\lambda_1)(1-\lambda_2)}$

provided that both $|\lambda_1| < 1$, $|\lambda_2| < 1$. So in writing (19), we require either that both $|\lambda_1|$ and $|\lambda_2|$ be less than unity or that $a = 0$, so that $a \sum_{i=0}^{\infty} \lambda_1^i \sum_{j=0}^{\infty} \lambda_2^j$ is defined. Furthermore, we require that

$$(9') \qquad \lim_{n\to\infty} \sum_{i=n}^{\infty} \lambda_j^i X_{t-i} = 0, \text{ all } t,$$

hold for j = 1, 2, so that the geometric sums in (19) are both finite.

Suppose that a = 0. On this assumption write (19) as

$$(20) \qquad y_t = \frac{\lambda_1 b}{\lambda_1-\lambda_2} \sum_{i=0}^{t-1} \lambda_1^i X_{t-i} - \frac{\lambda_2 b}{\lambda_1-\lambda_2} \sum_{i=0}^{t-1} \lambda_2^i X_{t-i}, \ t \geq 1$$

$$+ \frac{\lambda_1^{t+1}}{\lambda_1-\lambda_2} \theta_0 + \frac{\lambda_2^{t+1}}{\lambda_1-\lambda_2} n_0$$

where $\theta_0 = b \sum_{i=0}^{\infty} \lambda^i X_{0-i}$

$$n_0 = -b \sum_{i=0}^{\infty} \lambda^i X_{0-i}.$$

The case in which $X_t = 0$ for $t \geq 1$ is often analyzed, as for the first-order case. On this assumption, (20) becomes

$$(21) \qquad Y_t = \frac{\lambda_1^{t+1}}{\lambda_1-\lambda_2} \theta_0 + \frac{\lambda_2^{t+1}}{\lambda_1-\lambda_2} n_0, \ t \geq 1.$$

If $\theta_0 = n_0 = 0$, $Y_t = 0$ for all $t \geq 1$, regardless of the values of λ_1 and λ_2. So Y = 0 is the stationary point or long-run equilibrium value of (21).

If λ_1 and λ_2 are real, then $\lim_{t\to\infty} Y_t$ will equal zero if and only if both $|\lambda_1| < 1$ and $|\lambda_2| < 1$, regardless of the values of the parameters θ_0 and n_0, so long as they are finite. If, however,

$|\lambda_1| > 1$, $|\lambda_2| < |\lambda_1|$, and $\theta_0 > 0$, then $\lim_{t\to\infty} Y_t = +\infty$. If $|\lambda_2| < |\lambda_1| > 1$, and $\theta_0 < 0$, then $\lim_{t\to\infty} Y_t = -\infty$. Thus, Y will tend toward the stationary point zero as time passes provided that both $|\lambda_1| < 1$ and $|\lambda_2| < 1$. If one or both of the λ's exceed one in absolute value, the behavior of Y will eventually be "dominated" by the term in (21) associated with the λ that is larger in absolute value; that is, eventually Y will grow approximately as λ_m^t, where λ_m is the λ_j with the larger absolute value.

Now suppose that the roots are complex. If the roots are complex, they will occur as a complex conjugate pair, as the quadratic formula (17) verifies. So assume that the roots are complex, and write them as

$$\lambda_1 = re^{iw} = r \cos w + i \sin w$$

$$\lambda_2 = re^{-iw} = r \cos w - i \sin w$$

where the real part is r cos w and the imaginary part is $\pm$ r sin w. Notice that

$$(22) \qquad \lambda_1 - \lambda_2 = r(e^{iw} - e^{-iw}) = 2r\, i \sin w.$$

Furthermore, notice that equation (21) can be written

$$(23) \qquad Y_t = \frac{b\,\lambda_1^{t+1}}{\lambda_1 - \lambda_2} \sum_{i=0}^{\infty} \lambda_1^i X_{0-i} - \frac{b\,\lambda_2^{t+1}}{\lambda_1 - \lambda_2} \sum_{i=0}^{\infty} \lambda_2^i X_{0-i}$$

$$= \frac{b}{\lambda_1 - \lambda_2} \sum_{j=0}^{\infty} (\lambda_1^{t+j+1} - \lambda_2^{t+j+1}) X_{0-j}.$$

Notice that

$$\lambda_1^{t+j+1} - \lambda_2^{t+j+1} = (re^{iw})^{t+j+1} - (re^{-iw})^{t+j+1}$$

$$= r^{t+j+1} (e^{iw(t+j+1)} - e^{-iw(t+j+1)})$$

$$= r^{t+j+1} (2i \sin (w(t+j+1)).$$

But from trigonemtric formulas $\sin(wt+w(j+1)) = \sin wt \cos w(j+1) + \cos wt \sin w(j+1)$.* Substituting this into the above formula gives

$$\lambda_1^{t+j+1} - \lambda_2^{t+j+1} = r^{t+j+1} (2i[\sin wt \cos w(j+1) + \cos wt \sin w(j+1)]).$$

Substituting the above equation and (22) into (23) gives

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*Notice that

$$e^{iw_1} = \cos w_i + i \sin w_1$$

$$e^{iw_1} e^{iw_2} = (\cos w_1 + i \sin w_1)(\cos w_2 + i \sin w_2)$$

$$= (\cos w_1 \cos w_2 - \sin w_1 \sin w_2)$$

$$+ i(\sin w_1 \cos w_3 + \sin w_2 \cos w_1).$$

Also notice that

$$e^{i(w_1+w_2)} = \cos (w_1+w_2) + i \sin (w_1+w_2).$$

Therefore

$$\sin (w_1+w_2) = \sin w_1 \cos w_2 + \sin w_2 \cos w_1$$

and

$$\cos (w_1+w_2) = \cos w_1 \cos w_2 - \sin w_1 \sin w_2.$$

$$Y_t = \frac{br^{t+1}}{2ri \sin w} \sum_{j=0}^{\infty} r^j \cdot (2i\,[\sin(wt) \cos w(j+1) + \cos wt \sin w(j+1))]\, X_{0-j}$$

$$Y_t = \frac{br^t}{\sin w} [\sin wt \sum_{j=0}^{\infty} r^j \cos w(j+1) + \cos wt \sum_{j=0}^{\infty} r^j \sin w(j+1)]\, X_{0-j}$$

$$t \geq 1$$

or

$$(24) \qquad Y_t = \frac{br^t}{\sin w} \cdot \sin wt \cdot Z_0 + \frac{br^t}{\sin w} \cos wt \cdot Z_1,$$

where $Z_0 = \sum_{j=0}^{\infty} r^j \cos w(j+1)\; X_{0-j}$

$$Z_1 = \sum_{j=0}^{\infty} r^j \sin w(j+1)\; X_{0-j}.$$

As before $Y = 0$ is the stationary point of the difference equation. For arbitrary initial conditions, i.e., for arbitrary values of the parameters Z_0 and Z_1, Y_t will approach zero as time passes provided that $r < 1$; for (24) describes the evolution of Y over time as the sum of "damped" sin and cosine functions, the damping factor being r^t. (Notice that for it to be possible to divide by sin w in (24), it is necessary that $\sin w \neq 0$, which means that w cannot equal zero, π, 2π,.... This will be satisfied so long as the roots are complex (remember $\lambda_1 = r \cos w + i \sin w$)). If $r > 1$, the oscillations are explosive, while if $r < 1$, the oscillations are damped, and Y_t approaches its stationary value of zero in an oscillatory fashion as time passes.

Notice that if λ_1 and λ_2 are complex, the distributed lag weights of (19) oscillate. Rewrite (19) as

(19') $$Y_t = a \sum_{i=0}^{\infty} \lambda_1^i \sum_{j=0}^{\infty} \lambda_2^j + \frac{b}{\lambda_1 - \lambda_2} \sum_{j=0}^{\infty} (\lambda_1^{j+1} - \lambda_2^{j+1})\ X_{t-j},$$

which using calculations similar to those above can be rewritten as

$$Y_t = a \sum_{i=0}^{\infty} \lambda_1^i \sum_{i=0}^{\infty} \lambda_2^i + \frac{b}{\sin w} \sum_{j=0}^{\infty} r^j \sin w\ (j+1)\ X_{t-j}$$

Notice that the damping factor multiplying the sin curve is r^j, so that the range of the weights decreases as the lag j increases, provided that $r < 1$.

As noted above, the roots λ_1 and λ_2 are the reciprocals of the roots of the polynomial

(61) $$1-t_1\ L - t_2\ L^2 = 0.$$

For we know that $1-t_1\ L-t_2\ L^2 = (1-\lambda_1 L)(1-\lambda_2 L)$, with roots $1/\lambda_1$ and $1/\lambda_2$. Alternatively, multiply the above equation by L^{-2} to obtain

$$L^{-2} - L^{-1}\ t_1 - t_2 = 0 = (L^{-1}-\lambda_1)(L^{-1}-\lambda_2)$$

or

(62) $$X^2 - t_1\ X - t_2 = 0$$

where $X = L^{-1}$. Notice that the roots of (62) are the reciprocals of the roots of (61). Thus, λ_1 and λ_2 are the roots of (62).

It is interesting to know what values of t_1 and t_2 yield complex roots. Using the quadratic formula we have that the roots of (62) are

$$\lambda_i = X = \frac{t_1 \pm \sqrt{t_1^2 + 4t_2}}{2} .$$

For the roots to be complex, the term whose square root is taken must be negative, i.e.,

(63) $$t_1^2 + 4t_2 < 0,$$

which implies that $t_2 < 0$. In case (63) is satisfied, the roots are

$$\lambda_1 = \frac{t_1}{2} + \frac{i\sqrt{-(t_1^2+4t_2)}}{2} = a + bi$$

$$\lambda_2 = \frac{t_1}{2} - \frac{i\sqrt{-(t_1^2+4t_2)}}{2} = a - bi$$

To write $a + bi$ in polar form we recall that

$$a + bi = r \cos w + ri \sin w = re^{iw}$$

where $r = \; a^2 + b^2$ and where $\cos w = a/r$. Thus we have that

$$r = \sqrt{\left(\frac{t_1}{2}\right)^2 - \frac{(t_1^2+4t_2)}{4}}$$

$$= \sqrt{-t_2} .$$

We also have that

$$\cos w = \frac{t_1}{\sqrt{2 \quad -t_2}} \qquad \text{or } w = \cos^{-1}\left(\frac{t_1}{2\sqrt{-t_2}}\right)$$

For the oscillations to be damped we require that $r = \sqrt{-t_2} < 1$, which requires that $-t_2 < 1$.

The periodicity of the oscillations is $2\pi/\cos^{-1}(t_1/\sqrt{-t_2})$; i.e., this is the number of periods from peak to peak in the oscillations.

If the roots are real, movements will be damped if both roots are less than one in absolute value. That requires

$$-1 < \frac{t_1 + \sqrt{t_1^2 + 4t_2}}{2} < 1$$

and

$$-1 < \frac{t_1 - \sqrt{t_1^2 + 4t_2}}{2} < 1.$$

The condition

$$\frac{t_1 + \sqrt{t_1^2 + 4t_2}}{2} < 1$$

implies

$$\sqrt{t_1^2 + 4t_2} < 2 - t_1$$

$$t_1^2 + 4t_2 < 4 + t_1^2 - 4t_1$$

(64) $$t_1 + t_2 < 1$$

The condition

$$\frac{t_1 - \sqrt{t_1^2 + 4t_2}}{2} > - 1$$

implies that

$$-\sqrt{t_1^2 + 4t_2} > -2 - t_1$$

$$\sqrt{t_1^2 + 4t_2} < 2 + t_1$$

$$t_1^2 + 4t_2 < t_1^2 + 4 + 4t_1$$

(65) $$t_2 < 1 + t_1.$$

Conditions (64) and (65) must be satisfied for the roots, if real, to be less than unity in absolute value.

Notice that both roots are negative and real if $t_1^2 + 4t_2 > 0$ and

$$\frac{t_1 + \sqrt{t_1^2 + 4t_2}}{2} < 0 \text{ which implies}$$

$$t_1 < -\sqrt{t_1^2 + 4t_2}$$

$$t_1^2 > t_1^2 + 4t_2$$

$$0 > t_2.$$

Figure 1 depicts regions of the t_1, t_2 plane for which conditions (63), (64), or (65) are or are not satisfied. The graph shows combinations of t_1 and t_2 that give rise to damped oscillations, explosive oscillations, etc.

An Example

Maybe the most famous second-order difference equation in economics is the one associated with Samuelson's multiplier accelerator model. Samuelson posited the model

$$C_t = c\, Y_{t-1} + \alpha \qquad 1 > c > 0 \text{ (consumption function)}$$

FIGURE 1

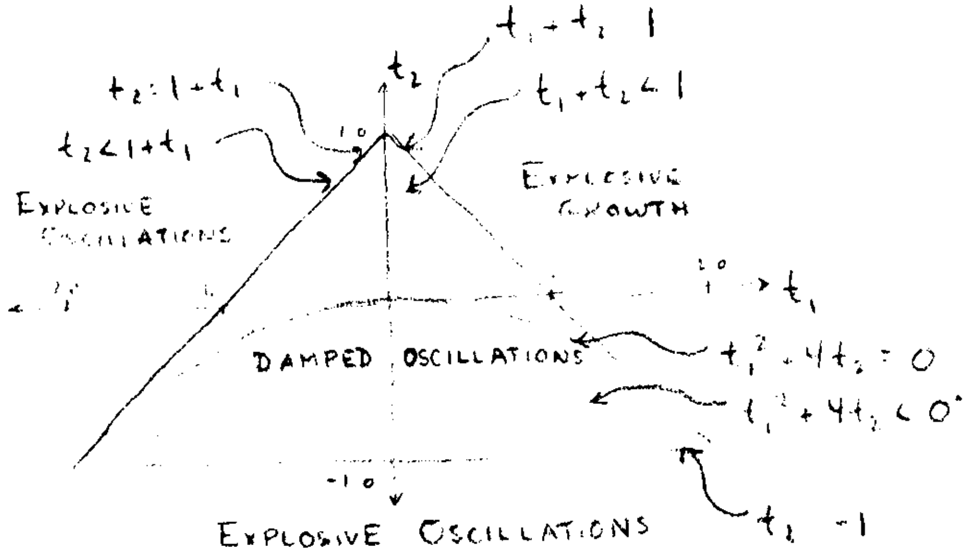
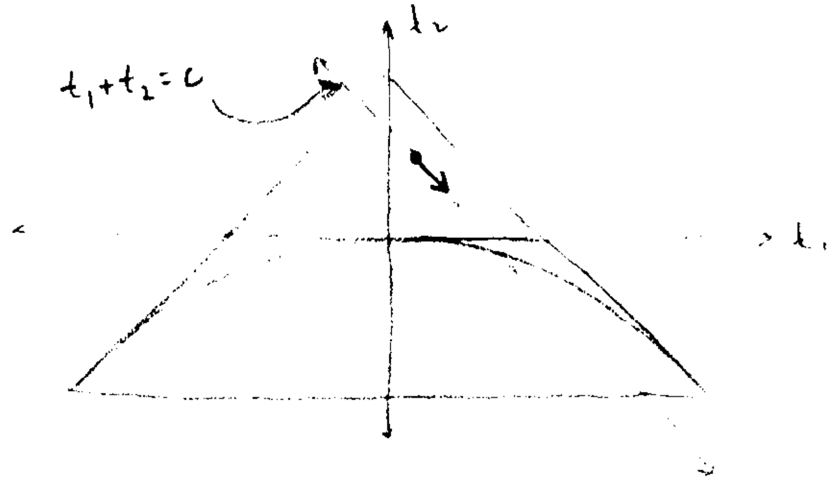


FIGURE 2



$$I_t = \gamma(Y_{t-1}-Y_{t-2}) \qquad \gamma > 0 \text{ (accelerator)}$$

$$C_t + I_t = Y_t$$

where C_t is consumption and I_t is investment. Substituting the first two equations into the third gives

$$Y_t = (c+\gamma)\, Y_{t-1} + \gamma Y_{t-2} + \alpha$$

or

$$Y_t = t_1\, Y_{t-1} + t_2\, Y_{t-2},$$

where $t_1 = c + \gamma$, $t_2 = -\gamma$. Notice that $t_1 + t_2 = c$. So variations in the parameter γ move the parameters t_1 and t_2 downward and to the right along the line $t_1 + t_2 = c$ in figure 2. Using figure 2, the values of c and γ compatible with damped oscillations, explosive oscillations, and so on, can easily be determined.

Figure 3 shows the path of Y over time for various values of c and γ, and for the initial conditions $Y_0 = Y_1 = 10$.

Second-Order Difference Equations (Equal Roots)

The preceding treatment assumed that $\lambda_1 \neq \lambda_2$. (Notice that we divided by $\lambda_1 - \lambda_2$ to obtain (19).) If $\lambda_1 = \lambda_2$, then the polynomial we must study is

$$\frac{1}{(1-\lambda L)(1-\lambda L)} = \frac{1}{1-\lambda L}\,(1+\lambda L+\lambda^2L^2+\ldots)$$

$$= (1+\lambda L+\lambda^2L^2+\ldots) + \lambda L(1+\lambda L+\lambda^2L^2+\ldots)$$

$$+ \lambda^2L^2(1+\lambda L+\lambda^2L^2+\ldots) +\ldots$$

$$= 1+2\lambda L+3\lambda^2L^2+\ldots$$

$$(25) \qquad \frac{1}{(1-\lambda L)^2} = \sum_{i=0}^{\infty} (i+1)\lambda^i L^i.$$

The polynomial in (25) is called a second-order Pascal lag distribution. It is the product of two geometric lag distributions with the same decay parameter λ.

With the aid of (25) we can study the solution to difference equations of the form

$$(1-\lambda L)^2 Y_t = a + b X_t .$$

The solution is

$$(26) \qquad Y_t = a \sum_{i=0}^{\infty} (i+1)\lambda^i + b \sum_{i=0}^{\infty} (i+1)\lambda^i X_{t-i} .$$

For $a \sum_{i=0}^{\infty} (i+1)\lambda^i$ to be finite, either $|\lambda| < 1$ or $a = 0$ must be satisfied.

To aid in studying difference equations with arbitrary initial conditions, we assume that $a = 0$ and rewrite (26) as

$$(27) \qquad Y_t = b \sum_{i=0}^{t-1} (i+1)\lambda^i X_{t-i} + b \sum_{i=t}^{\infty} (i+1)\lambda^i X_{t-i} .$$

The second sum can be written as

$$b \sum_{j=0}^{\infty} (j+1+t)\lambda^{t+j} X_{0-j}$$

$$= b \sum_{j=0}^{\infty} (j+1)\lambda^{t+j} X_{0-j} + b \sum_{j=0}^{\infty} t\lambda^{t+j} X_{0-j}$$

$$= b\lambda^t \sum_{j=0}^{\infty} (j+1)\lambda^j X_{0-j} + b\, t\lambda^t \sum_{j=0}^{\infty} \lambda^j X_{0-j} .$$

So for the special case $X_t = 0$ for $t > 0$, (27) becomes

$$(28) \qquad Y_t = \lambda^t \theta_0 + t\lambda^t n_0$$

$$\text{where } \theta_0 = b \sum_{j=0}^{\infty} (j+1)\lambda^j X_{0-j}$$

and

$$\eta_0 = b \sum_{j=0}^{\infty} \lambda^j X_{0-j}.$$

The stationary point of the equation is zero. For arbitrary initial conditions θ_0 and η_0, Y_t will approach the stationary point as time passes if $|\lambda| < 1$. If $|\lambda| > 1$, the value of Y_t will diverge from the stationary point zero as time passes, unless $\theta_0 = \eta_0$. Thus, if $|\lambda| > 1$, the stationary point is a "razor's edge" equilibrium.

N^{th}-Order Difference Equations (distinct roots)

Consider a rational polynomial with n^{th} order denominator:

$$A(L) = \frac{F(L)}{G(L)} = \frac{F(L)}{(1-\lambda_1 L)(1-\lambda_2 L)\ldots(1-\lambda_n L)}$$

The zeroes of G(L) are $L_1 = 1/\lambda_1$, $L_2 = 1/\lambda_2, \ldots, L_n, \ldots, L_n = 1/\lambda_n$, since each of these values for L satisfies the equation

$$G(L) = (1-\lambda_1 L)(1-\lambda_2 L)\ldots(1-\lambda_n L) = 0.$$

Suppose the n roots are distinct. Now the method of partial fractions enables us to express A(L) as

$$\text{(29)} \qquad \frac{F(L)}{G(L)} = \sum_{r=1}^{n} \frac{F(Lr)}{G'(Lr)} \cdot \frac{-1}{1 - \frac{1}{Lr} \cdot L}$$

where $Lr = \frac{1}{\lambda r}$ is the r^{th} zero of G(L),

$G'(Lr)$ is the derivative of G(L) with respect to L evaluated at Lr, and

F(Lr) is F evaluated at Lr. Letting

$$G(L) = \sum_{j=0}^{\infty} g_j L^j,$$

we have $G'(L) = \sum_{j=1}^{\infty} j\ g_j L^{j-1}$.

As an example, consider applying (29) to the second-order denominator polynomial

$$\frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)} = \frac{F(L)}{G(L)}$$

Since $G(L) = 1-(\lambda_1+\lambda_2)L + \lambda_1\lambda_2 L^2$, we have

$$G'(L) = -(\lambda_1+\lambda_2) + 2\lambda_1\lambda_2 L.$$

The zeroes of G(L) are $1/\lambda_1$ and $1/\lambda_2$, so that

$$G'(\frac{1}{\lambda_1}) = -(\lambda_1+\lambda_2) + 2\lambda_1\lambda_2 \frac{1}{\lambda_1} = \lambda_2-\lambda_1$$

$$G'(\frac{1}{\lambda_2}) = -\ (\lambda_1+\lambda_2) + 2\lambda_1\lambda_2 \frac{1}{\lambda_2} = \lambda_1-\lambda_2$$

So applying (29) we have

$$\frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)} = \frac{1}{(\lambda_2-\lambda_1)} \cdot \frac{-1}{(1-\lambda_1 L)} + \frac{1}{(\lambda_1-\lambda_2)}\ \frac{-1}{(1-\lambda_2 L)}$$

$$= \frac{1}{(\lambda_1-\lambda_2)}\ [\frac{1}{(1-\lambda_1 L)} - \frac{1}{(1-\lambda_2 L)}],$$

which can be verified directly, and agrees with the calculations used above to obtain (19).

Notice that F(Lr) and G'(Lr) in (29) are particular numbers, possibly complex ones, since they are F(L) and G'(L) evaluated at particular values of L.

Suppose we have an n^{th} order difference equation

(30) $$(1-\lambda_1 L)(1-\lambda_2 L)\ldots(1-\lambda_n L)Y_t = b\ X_t.$$

The solution to (30) is obtained by dividing by $(1-\lambda_1 L)\ldots(1-\lambda_n L)$ to obtain

$$Y_t = \frac{b}{(1-\lambda_1 L)\ldots(1-\lambda_n L)} X_t.$$

We suppose that the λ_j's are all distinct. Then application of (29) to the above equation gives

$$Y_t = b \sum_{r=1}^{n} \frac{1}{G'(\frac{1}{\lambda r})} \cdot \left(\frac{-1}{1-\lambda_r L}\right) X_t$$

(31) $$Y_t = b \sum_{r=1}^{n} \frac{-1}{G'(\frac{1}{\lambda r})} \sum_{i=0}^{\infty} \lambda_r^i X_{t-i},$$

which shows that Y_t can be expressed as the weighted sum of n geometric distributed lags with decay coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Given n initial value of Y, and assuming $X_t = 0$ always, it is possible to start up difference equation (30) finitely far back in the past, and to obtain a solution of the form

$$Y_t = \lambda_1^t n_1 + \lambda_2^t n_2 + \ldots + \lambda_n^t n_n$$

where $n_1, \ldots, n_n$ are constants chosen to satisfy the n initial values. The above equation can be derived from (31) by applying calculations analogous to those applied above in the first and second order cases.

N^{th}-Order Difference Equations (N equal roots)

Consider the n^{th} order difference equation

$$(1-\lambda L)^n \, Y_t = b \, X_t \tag{32}$$

which has the solution

$$Y_t = \frac{b}{(1-\lambda L)^n} X_t .$$

The polynomial $\frac{1}{(1-\lambda L)^n}$ is the one associated with an n^{th} order Pascal lag distribution, which is formed by multiplying (convolving) n geometric lag distributions with the same decay parameter λ. We have already studied the second-order Pascal distribution. By induction, it is possible to show that

$$\frac{1}{(1-\lambda L)^n} = \sum_{i=0}^{\infty} (i+1)^{n-1} \, \lambda^i L^i , \tag{33}$$

which agrees with our earlier formulas for the special cases $n = 1$ and $n = 2$.

With the aid of (33), the solution to (32) can be written

$$Y_t = b \sum_{i=0}^{\infty} (i+1)^{n-1} \, \lambda^i \, X_{t-i} .$$

Using calculations like those for the first and second order cases for the special case in which $X_t = 0$ for all t, and in which n arbitrary initial values are supplied to start up the process, it is straightforward to show that the solution obeys

$$Y_t = \lambda^t \, n_1 + t \, \lambda^t \, n_2 + \ldots + t^{n-1} \, \lambda^t \, n_n$$

where $n_1, \ldots, n_n$ are constants chosen to satisfy the n initial values.

An Example of a First Order System

Consider the following model studied by Cagan.* Let m_t be the log of the money supply, p_t the log of the price level and p^e_{t+1} the log of the price expected to prevail at time t+1 given information available at time t. The model is

$$(34) \qquad m_t - p_t = \alpha(p^e_{t+1} - p_t) \qquad\qquad \alpha < 0,$$

which is a portfolio equilibrium condition. The demand for real balances varies inversely with expected inflation $p^e_{t+1} - p_t$. The variable m_t is exogenous.

Suppose first that

$$(35) \qquad p^e_{t+1} - p_t = \gamma(p_t - p_{t-1})$$

so that the public expects inflation next period to be the current rate of inflation, $p_t - p_{t-1}$ multiplied by the constant γ. Then (34) becomes

$$m_t - p_t = \alpha\gamma\ p_t - \alpha\gamma\ p_{t-1}.$$

Using lag operators, this can be written as

$$[(\alpha\gamma+1) - \alpha\gamma L]\ p_t = m_t$$

or

$[1 - \frac{\alpha\gamma}{1+\alpha\gamma} L]\ p_t = \frac{1}{1+\alpha\gamma} m_t$. The solution can be written

*"The Monetary Dynamics of Hyperinflation"

$$p_t = \frac{1}{1+\alpha\gamma} \sum_{i=0}^{\infty} \left(\frac{\alpha\gamma}{1+\alpha\gamma}\right)^i m_{t-i},$$

which will be finite for the time path $m_t = \bar{m}$ for all t, provided that

$$\left|\frac{\alpha\gamma}{1+\alpha\gamma}\right| < 1.$$

The above inequality is in the spirit of the "stability condition" developed by Cagan in his paper. It is a condition that delivers a finite p_t for all t for a certain time path of m. Notice that

$$\frac{1}{1+\alpha\gamma} \sum_{i=0}^{\infty} \left(\frac{\alpha\gamma}{1+\alpha\gamma}\right)^i = \frac{\frac{1}{1+\alpha\gamma}}{1 - \frac{\alpha\gamma}{1+\alpha\gamma}} = 1.$$

Thus, the long-run effect of a once-and-for-all jump in m is to drive p up by an equal amount (provided the above "stability condition" is met).

Returning to (34), let us abandon (35) and now assume perfect foresight:

(36) $$p^e_{t+1} = p_{t+1}.$$

Substituting (36) into (34) gives

$$m_t - p_t = \alpha p_{t+1} - \alpha p_t$$

or $$\alpha p_{t+1} + (1-\alpha)\, p_t = m_t.$$

Write this as

$$(L^{-1} + \frac{1-\alpha}{\alpha})\, p_t = \frac{1}{\alpha} m_t$$

or

(37) $$(1 - \frac{\alpha-1}{\alpha} L)\, p_t = \frac{1}{\alpha} m_{t-1}$$

Notice that since $\alpha < 0$, it follows that $\frac{\alpha-1}{\alpha} > 1$. This fact is an invitation to solve (37) in the "forward" direction, that is, to use (6). Dividing both sides of (37) by $(1 - \frac{\alpha-1}{\alpha} L)$ gives

$$p_t = \left(\frac{\frac{1}{\alpha}}{1-\frac{\alpha-1}{\alpha}L}\right) m_{t-1},$$

which using (6) becomes

$$p_t = \left(\frac{\frac{1}{\alpha}\cdot -\left(\frac{\alpha}{\alpha-1}\right)L^{-1}}{1-\frac{\alpha}{\alpha-1}L^{-1}}\right) m_{t-1} = -\frac{1}{\alpha-1}\left(\sum_{i=0}^{\infty}\left(\frac{\alpha}{\alpha-1}\right)^i L^i\right) m_t .$$

(38) $$p_t = \frac{1}{1-\alpha}\cdot\sum_{i=0}^{\infty}\left(\frac{\alpha}{\alpha-1}\right)^i m_{t+i}$$

Notice that since $\alpha < 0$, $0 < \frac{\alpha}{\alpha-1} < 1$, so that the sum of the lag weights is finite. Equation (38) expresses the log of the current price as a moving sum of current and <u>future</u> values of the log of the money supply. Notice that

$$\frac{1}{1-\alpha}\sum_{i=0}^{\infty}\left(\frac{\alpha}{\alpha-1}\right)^i = \frac{\frac{1}{1-\alpha}}{1-\frac{\alpha}{\alpha-1}} = 1,$$

so that p is a weighted <u>average</u> of current and future values of m.

An Example of a Second Order System

Consider the following model studied by Muth.* Let p_t be the price of a commodity at t, C_t the demand for current consumption, I_t the stock of inventories of the commodity, Y_t the output of the commodity, and p_t^e the price previously expected to prevail at time t;

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*"Rational Expectations and the Theory of Price Movements."

X_t represents the effects of the weather on supply. The model is

$$C_t = -\beta\, p_t, \qquad \beta > 0 \qquad \text{demand curve}$$

$$Y_t = \gamma p_t^e + X_t, \qquad \gamma > 0 \qquad \text{supply curve}$$

$$I_t = \alpha(p_{t+1}^e - p_t) \qquad \alpha > 0 \qquad \text{inventory demand}$$

$$Y_t = C_t + (I_t - I_{t-1}) \qquad \text{market clearing}$$

Let us suppose that there is perfect foresight so that $p_t^e = p_t$ for all t. Making this assumption and substituting the first three equations into the fourth gives

$$\gamma\, p_t + X_t = \alpha(p_{t+1} - p_t) - \alpha(p_t - p_{t-1}) - \beta\, p_t$$

or

$$\gamma\, p_{t+1} - (2\alpha + \beta + \gamma)\, p_t + \alpha p_{t-1} = X_t.$$

Dividing by α gives

$$(p_{t+1} - \frac{(2\alpha+\beta+\gamma)}{\alpha}\, p_t - \alpha p_{t-1}) = \frac{1}{\alpha}\, X_t$$

or

$$(L^{-1} - \phi + L)\, p_t = \frac{1}{\alpha}\, X_t$$

where $\phi = \frac{\beta+\gamma}{\alpha} + 2 > 0$. Multiplying by L gives

(39) $$(1 - \phi L + L^2)\, p_t = \frac{1}{\alpha}\, X_{t-1}.$$

We need to factor the polynomial $(1-\phi L + L^2)$ as

$$(1-\phi L + L^2) = (1-\lambda_1 L)(1-\lambda_2 L)$$

$$= (1-(\lambda_1+\lambda_2)L + \lambda_1\lambda_2 L^2)$$

so that we require that

$$\lambda_1 + \lambda_2 = \phi$$

$$\lambda_1 \lambda_2 = 1.$$

The second equality establishes that $\lambda_1 = 1/\lambda_2$,

so that the two roots appear as a reciprocal pair. So we can write

$$(1-\phi L + L^2) = (1-\lambda L)(1-\frac{1}{\lambda}L)$$

where λ is chosen to satisfy $\lambda + \frac{1}{\lambda} = \phi$.

So (39) can be written

(39') $$(1-\lambda L)(1-\frac{1}{\lambda}L)\ p_t = \frac{1}{\alpha}\ X_{t-1}.$$

Since $\frac{B+\gamma}{\alpha} > 0$, it follows that $\lambda = \frac{B+\gamma}{\alpha} + 2 > 2$. That implies that λ does not equal 1, since $\lambda + 1/\lambda = \phi$. Notice that if $\lambda > 1$, $1/\lambda < 1$. So one of our roots necessarily exceeds 1, the other necessarily is less than 1.

We divide both sides of (39') by $(1-\lambda L)(1-(1/\lambda)L)$ to obtain

$$p_t = \frac{1}{\alpha}\frac{1}{(1-\lambda L)(1-\frac{1}{\lambda}L)}\ X_{t-1}.$$

Without loss of generality, suppose $\lambda < 1$ and let $\lambda_2 = 1/\lambda$. Use (6) and (19) to write the solution as

$$p_t = \frac{\frac{1}{\alpha}\lambda}{\lambda-\lambda_2}\left(\frac{1}{1-\lambda L}\right) X_{t-1} - \frac{\frac{1}{\alpha}\lambda_2}{(\lambda-\lambda_2)}\left(\frac{-\frac{1}{\lambda_2 L}}{1-\frac{1}{\lambda_2}L^{-1}}\right) X_{t-1}$$

$$= \frac{\frac{1}{\alpha}\lambda}{\lambda-\frac{1}{\lambda}}\left(\frac{1}{1-\lambda L}\right) X_{t-1} + \frac{\frac{1}{\alpha}\cdot\frac{1}{\lambda}}{\lambda-\frac{1}{\lambda}}\left(\frac{\lambda L^{-1}}{1-\lambda L^{-1}}\right) X_{t-1}$$

$$= \frac{\frac{1}{\alpha}\lambda}{\lambda-\frac{1}{\lambda}}\left(\frac{1}{1-\lambda L}\right) X_{t-1} + \frac{\frac{1}{\alpha}}{\lambda-\frac{1}{\lambda}}\left(\frac{1}{1-\lambda L^{-1}}\right) X_t$$

$$p_t = \frac{\frac{1}{\alpha}}{\lambda-\frac{1}{\lambda}}\sum_{i=1}^{\infty}\lambda^i X_{t-i} + \frac{\frac{1}{\alpha}}{\lambda-\frac{1}{\lambda}}\sum_{i=0}^{\infty}\lambda^i X_{t+i}$$

(40) $$p_t = \frac{\frac{1}{\alpha}}{\lambda-\frac{1}{\lambda}}\sum_{i=-\infty}^{\infty}\lambda^{|i|} X_{t-i}$$

The solution (40) expresses p_t as a "two-sided" distributed lag of X, that is, as a weighted sum of past, present, and future values of X. In this model, the current price depends on the entire path of the exogenous shock X over the entire past <u>and</u> the entire future.