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# OPTIMAL INCOME TAX IN A MONETARY ECONOMY

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ABSTRACT

This study examines the shape of an optimal income tax schedule in a monetary economy. In equilibrium, money's role is to allocate resources across generations, while a tax-transfer scheme serves as a form of social insurance. It is found that the optimal real income tax with money can be progressive.



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Optimal Income Tax in a Monetary Economy

This study examines the shape of an optimal income tax schedule in a monetary economy. Previous tax structure studies, which have not considered monetary economies, commonly find that an optimal income tax is regressive: the marginal tax rate falls as income rises.[1/] In the present study money is valued endogenously, and it is found, in contrast, that the optimal real income tax need not be regressive.[2/] There exist reasonable parameter values of utility and production functions for which it is progressive.

These conflicting findings have an intuitive explanation, but before it can be given some concepts and terms must be defined. Let the expected utility of a representative consumer-laborer be separated into a part consisting of the means of consumption and leisure and a part consisting of the variances of these variables.[3/] For any tax scheme define "distortion" as the difference between the mean part of expected utility under that tax scheme and the mean part under an optimal lump-sum tax.[4/] Distortion is then a measure of the loss in individual welfare caused by a tax's effects on average labor-leisure and consumption-savings allocations. Similarly, for any tax scheme define "instability" as the difference between the variance part of expected utility under that tax scheme and the variance part under an optimal lump-sum tax.[5/] Instability is then a measure of the loss in individual welfare caused by a tax's effects on the responses of labor and consumption to underlying shocks.

The optimal tax in any given class of taxes is the one that maximizes the expected utility of the representative consumer-laborer: the one in that class that minimizes the sum of distortion and instability. The shape of an optimal income tax is then determined by the trade-off that can be effected between distortion and instability. In general, a proportional income tax causes distortion. That distortion can be lessened by making the tax regressive, because then less income is taxed away with additional work effort. However, a regressive tax increases instability, since labor responds more to underlying shocks. Analogously, a progressive income tax adds distortion relative to a proportional tax but reduces instability. Individual preferences, the production technology, and the exchange process determine how much weight the tax system should give to reducing distortion as opposed to reducing instability.

Drawing from these notions, it is straightforward to explain the different shapes of optimal tax structures which result in this model with and without money. Taxes are used in this model solely to transfer goods to retirees. When there is no money, all intertemporal reallocations of consumption are done through the tax-transfer scheme. With a high rate of taxation required, even a proportional income tax causes considerable distortion. And with distortion being the primary loss in welfare, a regressive real income tax can effect a favorable trade of less distortion for more instability.

When there is money in the economy, however, intertemporal reallocations of consumption can be accomplished with little distortion through monetary exchange. The primary role of the tax-transfer scheme in this case is to provide insurance: to reduce instability.[6/] It is not surprising then that the optimal real income tax with money can be progressive, since that shape most effectively reduces instability.

Several other interesting results emerge from the model in this study:

1. A nominal income tax dominates a real income tax, *since the former allows different real income tax* schedules in different states of the economy whenever the price level depends on the state. In all the cases examined the optimal nominal income tax is regressive.
2. Prices and taxes are positively correlated, because the more reallocation which is done through the tax-transfer scheme, the less the role for money and the less its value.
3. The income velocity of money is procyclical.
4. One tax can imply a higher mean and lower variance of real income than another tax and still be less desirable.

The model used to generate these results can be considered a special case of the Enders and Lapan model with quadratic income tax schedules or it can be considered an extension of the Miller model to allow for a constant money stock. In addition to providing a role for money, the model extends the work of Diamond et al by making the rate of return on savings endogenous, and it extends the work of Varian by making labor supply endogenous.

The model is populated by overlapping generations of two-period lived agents. Individuals in each generation are identical, and the population is constant over time. Individuals work in the first period of life and retire in the second. A perishable consumption good is produced with a constant returns to labor technology, but the return is a random variable. In the first period of life individuals decide on how much labor to supply based on knowledge of that period's return to labor and then decide on how to divide their after-tax income between consumption of the produced good and savings in the form of money balances. In the second period of life individuals spend on consumption the sum of their money balances and transfers.

For given parameters of a tax system, a monetary equilibrium is determined in which the price level is a function of the state of technology and the price distribution is invariant over time. An optimal tax consists of tax parameters for which the resulting monetary equilibrium yields the maximum expected utility of the young and all future generations, where that expectation is taken prior to knowledge of the current state of technology.

The model is described formally in the next section. In the following section monetary equilibria are characterized under given tax parameters. Examples of optimal tax schemes and the equilibria they imply then are presented for numerical values of parameters from the utility and production functions. The paper concludes with an evaluation of the model's major strengths and weaknesses and a discussion of possible extensions of the model.

## The Model

1. Population: In each period N identical individuals are born. Each individual lives two periods. Thus, in each period, there are the "old" in the last period of their lives and the "young" in the first period of their lives.

2. Individual Welfare: Individual preferences are represented by a time-separable utility function in consumption and leisure.7/

$$W = U(c_1, L_1) + \beta U(c_2, L_2),$$

where for each individual of a given generation $c_i$ is consumption in the $i^{th}$ period of life; $\tilde{L}_i$ is leisure in the $i^{th}$ period of life, $0 < \tilde{L}_i < 1$; and the comtemporaneous utility function U is assumed to be concave.

A very special form of the contemporaneous utility function is assumed to simplify the analysis and the computations for the numerical examples.8/ In particular, it is asumed that

$$U(c,\tilde{L}) = -A(c-c^*)^2 - B(\tilde{L}-\tilde{L}^*)^2 ,$$

where $A > 0$, $B > 0$, the satiation level of consumption $c^*$ is at least as large as any feasible level, and the satiation level of leisure $\tilde{L}^*$ is one: $\tilde{L}^* \equiv 1$.

3. Endowments: Each individual is endowed with one unit of time each period. In the first period that time can be divided between leisure $\tilde{L}_1$ and labor L: $\tilde{L}_1 + L = 1$. In the second period all the time goes to leisure: $\tilde{L}_2 \equiv 1$.
Each individual in the current old generation is endowed with M/N units of fiat money.

4. Production: Production of a perishable consumption good y is governed by the linear process.

$y^\mu = \mu L$, where $\mu$ is distributed independently and identically over time and

$$\mu = \begin{cases} \mu_B \equiv 1; \text{ prob } (\mu = \mu_B) = 1/2; \text{ B is bad state} \\ \mu_G \equiv \theta > 1; \text{ prob } (\mu = \mu_G) = 1/2; \text{ G is good state.} \end{cases}$$

The young in any period are assumed to observe $\mu$ before they decide on how much labor to supply.

5. Exchange: In any period the young and old exchange goods and money. The old sell money, their savings, to purchase goods. The young sell goods to acquire money, their savings. Private insurance markets cannot operate due to the assumed timing of productivity shocks.[9] This assumed timing is intended to capture the notion that in the real world agents cannot fully insure againist business cycle risks.

6. Government: On a one-time basis, the government issues fiat money to the current old at t = 0. On a continuing basis, the government taxes the working young and transfers the proceeds to the retired old. The government's period budget constraint is then $NT(t)^\mu = NTr(t-1)^\mu$, where

$T(t)^{\mu}$ is the real tax collected from the young of generation t when state $\mu$ occurs, and

$Tr(t-1)^{\mu}$ is the real transfer paid to the old of generation t-1 when state $\mu$ occurs.

The following income taxes are considered:

a. Proportional: $T^{\mu} = t_1 y^{\mu}$

b. Nonproportional real: $T^{\mu} = t_1 y^{\mu} + t_2 (y^{\mu})^2$

c. Nonproportional nominal: $p^{\mu}\, T^{\mu} = t_1 p^{\mu} y^{\mu} + t_2 (p^{\mu} y^{\mu})^2$,

where $p^{\mu}$ is the money price of goods in state $\mu$.

An income tax is progressive, proportional, or regressive, respectively, as $t_2$ is greater than, equal to or less than zero.

In addition to these taxes, two other systems are examined for comparison. The first has M = 0 and contingent lump-sum taxes[10/]:

$$T^{\mu} = t_0^{\mu} .$$

The second has M > 0 and zero taxes:

$$t_0 = t_1 = t_2 = 0.$$

## Monetary Equilibria

A monetary equilibrium is a vector of state-contingent prices for which in each state all markets clear and all individuals maximize expected utility subject to their budget constraints. There are three markets: labor, goods, and money; and by Walras' law equilibrium in any two of them implies equilibrium in the third. In order to determine a monetary equilibrium, we need consider, then, only the labor and money markets. In the labor market, demand is identically equal to supply. In the money market supply is a constant, and demand in any period will take on one of two values depending on the state of the economy. Thus, only two prices need be determined: the price levels that equate the demand for money to the fixed supply in each of the two states.

A monetary equilibrium can be derived from the maximization problem of the representative consumer-laborer. First, a tax system is posited with $t_0$, $t_1$, and $t_2$ taken as parameters. Next, an individual's state-contingent supply of labor and demand for money are determined by maximizing expected utility conditional on the current period's productivity. Finally, equilibrium price levels are found by setting the state-contingent demand for money to the per capita money supply.

The maximization problems and equilibrium conditions for the various income taxes are described below. The analogous derivations for the lump-sum tax are described in Miller.

A. Proportional Income Tax

1. Individual choice problems

a. Young:

$$\max_{L,m} E[W|\mu(t)], \text{ where}$$

$$E[W|\mu(t)] = -A(c_1^{\mu(t)}(t)-c^*)^2 - BL^{\mu(t)}(t)^2$$
$$-\beta AE[(c_2^{\mu(t+1)}(t)-c^*)^2|\mu(t)] \quad ,$$

$$c_1^B(t) = y^B(t) - t_1 y^B(t) - S^B(t)m^B(t)$$

$$= (1-t_1)L^B(t) - S^B(t)m^B(t), \text{ for which}$$

$S \equiv 1/p$ is equivalently the inverse of the price level or the goods price of money, and $(\cdot)^B$ for any variable is its value conditional on $\mu(t) = \mu_B$,

$$c_1^G(t) = y^G(t) - t_1 y^G(t) - S^G(t)m^G(t)$$

$$= (1-t_1)\theta L^G(t) - S^G(t)m^G(t), \text{ for which}$$

$(\cdot)^G$ for any variable is its value conditional on $\mu(t) = \mu_G$.

For second-period consumption there are four contingencies:

$$c_2^{BB}(t) \equiv [c_2(t) | \mu(t) = \mu_B, \mu(t+1) = \mu_B]$$

$$= Tr^B(t) + S^B(t+1)m^B(t), \text{ where}$$

$Tr^B(t)$ is the real transfer to an individual of generation t in his or her second period of life conditional on $\mu(t+1) = \mu_B$. This transfer is assumed by the individual to be given and to be independent of his or her work effort in the first period of life.

$$c_2^{BG}(t) \equiv [c_2(t) | \mu(t) = \mu_B, \mu(t+1) = \mu_G]$$

$$= Tr^G(t) + S^G(t+1)m^B(t)$$

$$c_2^{GB}(t) \equiv [c_2(t) | \mu(t) = \mu_G, \mu(t+1) = \mu_B]$$

$$= Tr^B(t) + S^B(t+1)m^G(t)$$

$$c_2^{GG}(t) \equiv [c_2(t) | \mu(t) = \mu_G, \mu(t+1) = \mu_G]$$

$$= Tr^G(t) + S^G(t+1)m^G(t) .$$

We then have

$$\underset{\mu(t+1)}{E}[(c_2^{\mu(t+1)}(t)-c^*)^2 | \mu(t)] \equiv [\sigma_{c_2}^2 + (\bar{c}_2 - c^*)^2) | \mu(t)]$$

$$= \begin{cases} \frac{1}{4}(c_2^{BG}(t) - c_2^{BB}(t))^2 + (\overline{Tr}(t) + \bar{S}(t+1)m^B(t) - c^*)^2, \text{ if } \mu(t) = \mu_B \\ \\ \frac{1}{4}(c_2^{GG}(t) - c_2^{GB}(t))^2 + (\overline{Tr}(t) + \bar{S}(t+1)m^G(t) - c^*)^2, \text{ if } \mu(t) = \mu_G \end{cases}$$

where $\overline{(\cdot)} = \dfrac{(\cdot)^B + (\cdot)^G}{2}$ ,

$\sigma_{(\cdot)}^2 = \frac{1}{4}[(\cdot)^G - (\cdot)^B]^2$ for any given variable.

Finally by restricting consideration to stationary equilibria the dependence of variables on t can be dropped. The young's choice problem can then be written:

if $\mu(t) = \mu_B$,

$$\max_{L^B, m^B} \{-A[(1-t_1)L^B - S^B m^B - c^*]^2 - BL^{B^2}$$
$$-\beta A[\tfrac{1}{4}((Tr^G - Tr^B) + (S^G - S^B)m^B)^2 + (\overline{Tr} + \bar{S}m^B - c^*)^2]\}$$

if $\mu(t) = \mu_G$,

$$\max_{L^G, m^G} \{-A[(1-t)\theta L^G - S^G m^G - c^*]^2 - BL^{G^2}$$
$$-\beta A[\tfrac{1}{4}((Tr^G - Tr^B) + (S^G - S^B)m^G)^2 + (\overline{Tr} + \bar{S}m^G - c^*)^2]\}$$

b. Old:

The old simply choose to consume the goods which can be purchased with their transfers and money holdings. They then are inelastic suppliers of money: in the aggregate they supply M units of money at any price S.

2. Stationary monetary equilibria

A stationary monetary equilibrium is completely described by the contingent labor supply functions of the young and by the goods prices of money $S^G$ and $S^B$, respectively, which equate the young's contingent demands for money $m^G$ and $m^B$, respectively, with the old's inelastic supply $M/N = m$. An equilibrium is found by computing the first-order conditions for a maximum, setting $Tr^\mu = t_1 y^\mu$ $\mu = B, G$ from the government's budget constraint, and imposing $m^B = m^G = M/N \equiv m$.

From the four first-order conditions we have,[11/]

$$\text{(i)} \qquad L^B = \frac{A(1-t_1)(S^B m^B + c^*)}{A(1-t_1)^2 + B}$$

$$\text{(ii)} \qquad L^G = \frac{A(1-t_1)\theta(S^G m^G + c^*)}{A(1-t_1)^2\theta^2 + B}$$

$$\text{(iii)} \qquad m^B = \frac{[\beta(S^G+S^B)-2(1-\Pi)S^B]c^* - \beta[S^G Tr^G + S^B Tr^B]}{\beta({S^G}^2+{S^B}^2)+2(1-\Pi){S^B}^2}, \text{ where}$$

$$0 < \Pi \equiv \frac{A(1-t_1)^2}{A(1-t_1)^2+B} < 1$$

$$\text{(iv)} \qquad m^G = \frac{[\beta(S^G+S^B)-2(1-\delta)S^G]c^* - \beta[S^G Tr^G + S^B Tr^B]}{\beta({S^G}^2+{S^B}^2)+2(1-\delta){S^G}^2}, \text{ where}$$

$$0 < \delta \equiv \frac{A(1-t_1)^2\theta^2}{A(1-t_1)^2\theta^2+B} < 1.$$

N w, imposing money market equilibrium $m^B = m^G = m$ and budget balance $Tr^\mu = t_1y^\mu$ and defining the relative price k by $S^G = kS^B$, (iii) and (iv) yield the following two equations in k and $S^B$:

(v) $$S^B = \left[\frac{\beta(k+1-\nu_1k-\nu_2)-2(1-\Pi)}{\beta(k^2+1+\nu_1k^2+\nu_2)+2(1-\Pi)}\right]\frac{c^*}{m} \equiv \phi_1(k)\frac{c^*}{m}, \text{ where}$$

$$\nu_1 \equiv \xi\delta$$

$$\nu_2 \equiv \xi\Pi$$

$$\xi = t_1/(1-t_1)$$

(vi) $$S^B = \left[\frac{\beta(k+1-\nu_1k-\nu_2)-2(1-\delta)k}{\beta(k^2+1+\nu_1k^2+\nu_2)+2(1-\delta)k^2}\right]\frac{c^*}{m} \equiv \phi_2(k)\frac{c^*}{m}$$

Thus, k can be found by equating $\phi_1(k)$ with $\phi_2(k)$ and then $S^B$ can be found from either (v) or (vi). It follows that k is the unique root greater than 1 to the cubic equation

(vii) $$W_3k^3 + W_2k^2 + W_1k + W_o = 0, \text{ where}$$

$$W_3 \equiv 2\beta(1-\delta)$$

$$W_2 \equiv \beta[-(1+\xi)\delta+(1-\xi)\Pi+2\xi\delta\Pi]-2(1-\delta)(1-\Pi)$$

$$W_1 \equiv -\beta[-(1-\xi)\delta+(1+\xi)\Pi-2\xi\delta\Pi]+2(1-\delta)(1-\Pi)$$

$$W_o \equiv -2\beta(1-\Pi).$$

B. <u>Nonproportional Real Income Tax</u>

1. Individual choice problems

a. Young:

Following the same steps as for the proportional income tax, the young's choice problem can be written:

if $\mu(t) = \mu_B$,

$$\max_{L^B, m^B} \quad -A[(1-t_1)L^B - t_2 {L^B}^2 - S^B m^B - c^*]^2 - B{L^B}^2$$
$$-\beta A[\tfrac{1}{4}((Tr^G - Tr^B) + (S^G - S^B)m^B)^2 + (\overline{Tr} + \bar{S}m^B - c^*)^2]$$

if $\mu(t) = \mu_G$,

$$\max_{L^G, m^G} \quad -A[(1-t_1)\theta L^G - t_2\theta^2 {L^G}^2 - S^G m^G - c^*]^2 - B{L^G}^2$$
$$-\beta A[\tfrac{1}{4}(Tr^G - Tr^B) + (S^G - S^B)m^G)^2 + (\overline{Tr} + \bar{S}m^B - c^*)^2]$$

Old:

The choice problem of the old does not depend on the income tax structure, so it as described under the proportional income tax.

2. Stationary monetary equilibria

A stationary monetary equilibrium is found by computing the first-order conditions for a maximum, setting $Tr^\mu = t_1 y^\mu + t_2 (y^\mu)^2$ $\mu = B,G$ from the government's budget constraint, and imposing $m^B = m^G = M/N \equiv m$. This yields the following four equations in $L^B$, $L^G$, $S^B$, and $S^G$:

(i) $\quad a_{11}{L^B}^3 + a_{12}{L^B}^2 + a_{13}L^B + a_{14} = 0$, where

$$a_{11} \equiv 2At_2^2$$

$$a_{12} \equiv -3A(1-t_1)t_2$$

$$a_{13} \equiv A[(1-t_1)^2 + 2t_2(S^B m + c^*)] + B$$

$$a_{14} \equiv -A(1-t_1)(S^B m+c^*)$$

(ii) $$a_{21}L^{G3} + a_{22}L^{G2} + a_{23}L^G + a_{24} = 0, \text{ where}$$

$$a_{21} \equiv 2At_2^2\theta^4$$

$$a_{22} \equiv -3A(1-t_1)t_2\theta^3$$

$$a_{23} \equiv A\theta^2[(1-t_1)^2+2t_2(S^G m+c^*)]+B$$

$$a_{24} \equiv -A(1-t_1)\theta(S^G m+c^*)$$

(iii) $$S^B = \frac{a_{33}+ (a_{33}^2-4a_{32}a_{34})^{1/2}}{2a_{32}}, \text{ where}$$

$$a_{32} \equiv (2+\beta)m$$

$$a_{33} \equiv 2[c^*-(1-t_1)L^B+t_2L^{B^2}]-\beta(c^*-t_1L^B-t_2L^{B^2})$$

$$a_{34} \equiv -(c^*-S^G m-t_1\theta L^G-t_2\theta^2L^{G^2})\beta S^G$$

(iv) $$S^G = \frac{a_{43}+ (a_{43}^2-4a_{42}a_{44})^{1/2}}{2a_{42}}, \text{ where}$$

$$a_{42} \equiv (2+\beta)m$$

$$a_{43} \equiv 2[c^*-(1-t_1)\theta L^G+t_2\theta^2L^{G^2}]-\beta(c^*-t_1\theta L^G-t_2\theta^2L^{G^2})$$

$$a_{44} \equiv -(c^*-S^B m-t_1L^B-t_2L^{B^2})\beta S^B .$$

These conditions reveal that there always exists a nonmonetary equilibrium with $S^B = S^G = 0$. There also will exist a monetary equilibrium with $S^G > S^B > 0$ if $c^* - t_1\theta-t_2\theta^2 > 0$. For each pair $(S^B,S^G)$, however, there will be single roots $L^B$ and $L^G$ included in the unit interval.

C. Nonproportional Nominal Income Tax:

The choice problems and the derivation of the monetary equilibria are precisely the same as for the preceding taxes except that now the real tax payment is given by

$$T^{\mu} = t_1 y^{\mu} + t_2 (y^{\mu})^2 / S^{\mu}.$$

A stationary monetary equilibrium in this case is given by the following four equations in $L^B$, $L^G$, $S^B$, and $S^G$:

(i) $$b_{11} L^{B^3} + b_{12} L^{B^2} + b_{13} L^B + b_{14} = 0, \text{ where}$$

$$b_{11} \equiv 2At_2^2 / S^{B^2}$$

$$b_{12} \equiv -3A(1-t_1)t_2 / S^B$$

$$b_{13} \equiv A[(1-t_1)^2 + (2t_2/S^B)(S^B m + c^*)] + B$$

$$b_{14} \equiv -A(1-t_1)(S^B m + c^*)$$

(ii) $$b_{21} L^{G^3} + b_{22} L^{G^2} + b_{23} L^G + b_{24} = 0, \text{ where}$$

$$b_{21} \equiv 2At_2^2 \theta^4 / S^{G^2}$$

$$b_{22} \equiv -3A(1-t_1)t_2 \theta^3 / S^G$$

$$b_{23} \equiv A[(1-t_1)^2 \theta^2 + (2t_2 \theta^2 / S^G)(S^G m + c^*)] + B$$

$$b_{24} \equiv -A(1-t_1)\theta(S^G m + c^*)$$

(iii) $$S^B = \frac{b_{33} + (b_{33}^2 - 4b_{32}b_{34})^{1/2}}{2b_{32}}, \text{ where}$$

$$b_{32} \equiv (2+\beta)m$$

$$b_{33} \equiv 2[c^* - (1-t_1)L^B] - \beta(c^* - t_1 L^B)$$

$$b_{34} \equiv t_2[(2+\beta)L^{B^2}] - \beta S^G(c^* - S^G m - t_1\theta L^G - t_2\theta^2 L^{G^2})$$

(iv) $$S^G = \frac{b_{43} + (b_{43}^2 - 4b_{42}b_{44})^{1/2}}{2b_{42}}, \text{ where}$$

$$b_{42} \equiv (2+\beta)m$$

$$b_{43} \equiv 2[c^* - (1-t_1)\theta L^G] - \beta(c^* - t_1\theta L^G)$$

$$b_{44} \equiv t_2[(2+\beta)\theta^2 L^{G^2}] - \beta S^B(c^* - S^B m - t_1 L^B - t_2 L^{B^2}).$$

Again, there are two equilibria: a nonmonetary equilibrium with $S^B = S^G = 0$ and a monetary equilibrium with $S^G > S^B > 0$ when $t_1$ and $t_2$ are not too large.

## Optimal Income Taxation

In the previous section it was shown that a monetary equilibrium exists for each of a number of parameter settings for a given income tax structure. Each equilibrium implies a state-contingent allocation of goods and leisure for each individual in each generation and, hence, a distribution of state-contingent utilities. An optimal tax in a given tax structure is one that implies an allocation which maximizes some weighting of state-contingent utilities. The weighting chosen in this study is average unconditional expected utility of the young and all future generations.12/

The optimal tax parameters for a given tax structure generates the state-contingent allocation of good and leisure that maximizes:

$$EW = E_{\mu(t)}U(c_1,1-L) + \beta E_{\mu(t+1)}U(c_2,1).$$

Under the special assumptions of quadratic utility and serially independent, identically distributed disturbances, the objective function can be written

$$EW = -A(\bar{c}_1-c^*)^2-A\sigma_{c_1}^{\;2} - B\bar{L}^2 - B\sigma_L^{\;2} - \beta A(\bar{c}_2-c^*)^2 - \beta A\sigma_{c_2}^{\;2},$$

where $(\bar{\cdot})$ and $\sigma_{(\cdot)}^2$ are unconditional means and variances, respectively.

Different tax structures can be ranked by comparing the max EW implied by each. The resulting EWs also can be decomposed in different ways. For any tax the resulting EW can be divided into a part composed of the means: $EW_1 = -A(\bar{c}_1- c^*)^2 - B\bar{L}^2 - \beta A(\bar{c}_2-c^*)^2$ and a part composed of the variances: $EW_2 = -A\sigma_{c_1}^{\;2} - B\sigma_L^2 - \beta A\sigma_{c_2}^{\;2}$; so that $EW = EW_1 + EW_2$. Let $EW_1{*}$ amd $EW_2{*}$ be the values that are acheived by an optimal contingent lump-sum tax. For any parameters of any other tax structure, then define distortion and instability by:

$$\text{Distortion} \equiv EW_1^* - EW_1$$

$$\text{Instability} \equiv EW_2^* - EW_2$$

where $EW_1$ and $EW_2$ are the values implied by that tax. Distortion and instability can be either positive or negative depending on whether the tax being considered does worse or better than the optimal contingent lump-sum tax for these components of EW.

It is also instructive to decompose EW into the part related to first-period consumption:

$EW[c_1] \equiv -A[(\bar{c}_1-c^*)^2 + \sigma_{c_1}^2]$; the part related to labor or leisure: $EW[L] \equiv -B[\bar{L}^2+\sigma_L^2]$; and the part related to second-period consumption: $EW[c_2] \equiv -\beta A[(\bar{c}_2-c^*)^2 + \sigma_{c_2}^2]$. Clearly, $EW = EW[c_1] + EW[L] + EW[c_2]$.

## Numerical Examples

Even with the many simplifying assumptions made in constructing the model, there seems to be little hope of deriving results analytically about the structure of optimal income taxes. Computation of an equilibrium for arbitrary tax parameters requires the simultaneous solution of four nonlinear equations in four unknowns. Yet, computation of such equilibria are necessary in order to determine the link between the tax parameters $t_0$, $t_1$, and $t_2$ and the objective function EW.

Instead of attempting an analytical approach, this study proceeds by computing optimal tax structures and associated equilibria for numerical values of utility, production, and money parameters. The examples are used in two ways. First, they are used as existence proofs. They illustrate well-specified general equilibrium economies with well-behaved utility and production functions which have some interesting properties. Second, they are used to infer some general results when general principles seem to be operating. This use is admittedly somewhat precarious given the many special assumptions made and the numerical solutions examined.

The initial parameter set examined is $< A, B, c^*, \beta, \theta > = < 2, 10, 4, .9, 1.5 >$, the same as in Miller. It was chosen to imply labor supply between 0 and 1 and upward sloping with respect to $\mu$ (e.g., the real wage) under a proportional tax system. Characteristics of equilibria for this parameter set under optimal tax parameters for given tax structure are displayed in Table 1.

Outcomes under five different tax structures are reported across the columns. The first column lists characteristics of an equilibrium under an optimal contingent lump-sum tax in a nonmonetary economy. The remaining columns list characteristics of equilibria under various income taxes in a monetary economy with the per capita money stock $M/N \equiv m = 1$. The taxes are zero, proportional, nonproportional real, and nonproportional nominal (as defined on page 7).[13/]

The rows list various characteristics of the equilibria under optimal settings of the various taxes. Rows 1-3 list the optimal tax parameters for a given tax structure. Only the lump-sum tax is contingent on the state of productivity. Rows 4-5 list the equilibrium prices of money in each state. Rows 6-9 list the means of consumption, labor, and real income; while rows 10-13 report coefficients of variation of these variables.[14/] Rows 14-17 list expected welfare, broken down into the parts attributable to first-period consumption, labor, and second-period consumption.[15/] The final rows list the differences in expected utility due to distortion and instability between a given tax and the optimal contingent lump-sum tax.

Probably the most interesting result in Table 1 is that the optimal real income tax is progressive, $\hat{t}_2 > 0$. In previous studies, such as Miller's which had not included money, it was found generally that the optimal income tax is regressive. Apparently, the introduction of money is what accounts for the turnaround in results. In order to gain insights into the reasons for this new finding, it's useful to compare the outcomes across columns.

The first column characterizes the best feasible allocation, while the second column characterizes the equilibrium under a constant money stock and no taxes or transfers. Comparing the two columns reveals that with only a constant money stock to effect intergenerational trades, the economy in equilibrium comes close to the optimal allocation. (In fact, its implied EW of -52.7641 is significantly higher than the EW of -52.9987 implied by a proportional tax and transfer scheme with no money--see Miller).

Comparing the second and third columns reveals the value of adding a proportional tax and transfer scheme to a monetary economy (as in Enders and Lapan). Although the tax causes increased distortion, it more than makes up for it in terms of reducing instability. The tax-transfer scheme, thus, seems to be providing a beneficial insurance role. Most of the intergenerational transfer of goods is handled by monetary exchange, however: without money the optimal tax rate is .1277 (see Miller), but with it the optimal tax rate drops to .0171.

Since the primary role of the tax-transfer scheme in this model is to reduce instability, it is not surprising that the optimal real income tax can be progressive. Progressivity reduces instability at the cost of higher distortion.

Several other interesting results merge from Table 1. One is that the nonproportional nominal income tax yields a higher level of welfare than does the real income tax, even though there is no form of money illusion in the economy. This occurs, apparently, because the nominal income tax offers an extra degree of flexibility: it allows the tax schedule to be a function of the state of the economy. With the real income tax $T = t_1y + t_2y^2$ the tax parameters do not depend on the state. With the nominal income tax $pT = t_1py + t_2(py)^2$ it follows $T = t_1y + (pt_2)y^2$ so that the coefficient on $y^2$ will vary with the state of the economy whenever p does.

A second interesting result is that the optimal nominal income tax is regressive while the optimal real income tax is progressive. This result reveals a weakness in studies of income tax indexing which commonly compare a given nonindexed tax and the same tax with indexing.[16/] The result suggests that a different tax structure may be desired with indexing. What should be compared is the "best" attainable nonindexed tax with the "best" attainable indexed tax, where "best" is determined by some welfare criterion.

A third interesting result is that if we compare the second and third columns, we observe the price level $p \equiv 1/S$ rises with the level of taxes. One explanation is that as more resources are moved through the tax-transfer system, money has less of a role to play and so its value falls. A second explanation is that higher taxes curtail work effort and so with a constant income velocity of money, $py/m$, the price level must rise.

We see from Table 1 that average real income does fall with the higher tax so the second explanation is at least part of the story. But for the second explanation to be the whole story, a constant income velocity of money would be required. In Table 2 it is shown that the average income velocity of money is higher under the proportional income tax than under zero taxes, so that the first explanation is also part of the story.

We also see in Table 2 that the income velocity of money under any tax scheme is higher in the good state than in the bad. If we interpret the two states as the peak and trough of the business cycle, respectively, then the model implies that the income velocity of money is procyclical.

Table 3 reports the same information as Table 1 except for a change in one parameter assumption: $\beta$ is increased from .9 to 1.0. With this change in $\beta$ both the optimal real income tax and the optimal nominal income tax are regressive. And while the nominal income tax still is better than the real income tax in terms of welfare, it implies a lower mean and higher variance of real income. Thus, this result demonstrates that the mean and variance of real income are not sufficient statistics in judging the desirability of alternative taxes.

Comparing the first and third tables reveals that as $\beta$ rises, more second-period consumption is desired and, hence, more goods must be moved from the young to the old. Almost all of this additional movement is handled through the money system as evidenced by the increased value of money and essentially unchanged rate of taxation. Again, this seems to be evidence that the primary role of the tax-transfer system in this model is to provide insurance and not to move goods across generations.

Table 4 has the same information as Table 1 with the single parameter change: $\theta$ is reduced from 1.5 to 1.25. Comparing the two tables reveals that with an increase in $\theta$ individuals work more (by assumption) and there is more real income to be divided between first- and second-period consumption. With more goods being moved across generations the value of money rises. But with an increase in $\theta$ instability also rises. There is then a greater need for insurance, so that the desired rate of taxation also rises.

In Tables 1, 3, and 4 the nominal income tax is better than the real income tax, and it is regressive. These results also hold for three other sets of parameters which were examined but not reported. It could well be that this is a general result in this model. The dominance of the nominal income tax was explained in terms of added flexibility. By inserting the value of the price level in each state, we can express the nominal tax in Table 1, for example, as

$$T = \begin{cases} (.0800)y - (.0343)y^2 & \text{if } \mu = \mu_B \\ (.0800)y - (.0286)y^2 & \text{if } \mu = \mu_G \end{cases}$$

This tax limits distortion by having a regressive schedule in each state. It also limits instability by having a less regressive tax in the good state.

Evaluation and Extensions

Although the model used in this paper is quite simple, it does have some nice features for tax structure analysis. First, it is a general equilibrium model which allows alternative tax structures to be ranked in terms of individual utilities. Second, it has endogenously valued money, and as this study shows, the existence of money can have important implications for the structure of taxes. Third, it includes an irreducible uncertainty. If there were no uncertainty, the optimal income tax seems almost certain to be regressive when economic efficiency is the criterion (as opposed to income distribution). Finally, it has an endogenous labor supply which permits distorting taxes to affect the distribution of real income.

With these nice features the model also has some obvious faults. Correcting these faults might provide fruitful avenues for future research. The model is very specialized. The utility function, production function, and productivity distribution are of very special forms, and on top of this only numerical solutions are found. Because of the complexity of the problem even with the simplifying assumptions, it appears that analytical solutions cannot be found. However, if only numerical solutions are examined, it should be possible to assume more general forms of utility functions, production functions, and productivity distributions. This would allow the results to be generalized somewhat.

There is no physical capital in the model. While money provides some intertemporal connectedness, the inclusion of capital would in addition allow analysis of the differences between a wage tax and an income tax.

The model has a very limited role for government: after distributing a fixed stock of money to the current old, it just manages a balanced tax and transfer system. It is straightforward to add consumption to the government's activities. As the government needs to raise more revenues to finance its expenditures, the distortion caused by taxes is likely to become relatively more important than instability. As expenditures increase, the optimal income tax is then likely to become increasingly regressive to give more weight to reducing distortion. It is also straightforward to allow the government to follow a constant growth rate rule for the money supply, so that it can tax implicitly by inflation as well as explicitly by income taxes. This extension would allow the optimal inflation tax and optimal income tax structure to be determined simultaneously for a given rate of government consumption.

Finally, the model has an inadequate information structure, but this problem is not easily remedied. Two related assumptions were simply imposed in constructing the model:

1. There is no lump-sum taxation.
2. Individual choice problems are conditional on knowledge of $\mu(t)$, but the government's choice problem is not.

These assumptions are related, because an explanation for either of them would be based on private information and moral hazard. While it is a weakness of the model to have to impose the assumptions, to include an information structure adequate to endogenously generate them would likely make the model unmanageable.

Footnotes

[1] See, for example, Miller, Mirrlees, Phelps, Seade, Stern and Stiglitz. This result, however, did not follow necessarily in Varian.

[2] When there is money in the model, the income tax schedule can relate real tax payments to real income (as in previous studies without money) or it can relate nominal tax payments to nominal income. The first is termed a real income tax and the second is termed a nominal income tax. There is a difference between the two if nonproportional taxation is allowed.

[3] This separation can be done globally and exactly with a quadratic utility function. It can be done locally and approximately with other utility functions by taking their second-order Taylor series expansions.

[4] This definition requires a unique optimum under lump-sum taxation, which is satisfied in this study.

[5] This definition also requires a unique optimum under lump-sum taxation.

[6] Enders and Lapan proved that in an economy similar to the one I posit, some amount of a proportional income tax and transfer scheme increases welfare in a monetary economy. My results give insights into why it does, and the paper extends their results by solving for the optimal income tax without requiring it to be proportional.

[7] The dependence of variables on time or on generations is suppressed whenever the meaning is clear without it. The welfare of a representative agent in generation t would be written in full as: $W(t) = U(c_1(t),\tilde{L}_1(t)) + \beta U(c_2(t),\tilde{L}_2(t))$.

8/A defense of the assumptions used can be found in Miller.

9/The current young cannot take out insurance on their first-period income, because they appear after the shock to technology has occurred. They would like to take out insurance on their income when they are old. They can't insure among themselves, however, since they are identical. The current old can't insure them, because they won't be around in the next period. If there were insurance, it would have to be between the young and the unborn, which, of course, is not feasible. See also Enders and Lapan, footnote 9, page 656 on this point.

10/Since the contingent lump-sum taxes deliver the optimal allocation of goods, money has no role in this economy. There is nothing lost by setting $M = 0$ then, because if $M > 0$, it would have zero value in equilibrium.

11/In general, there are two equilibria: a nonmonetary equilibrium with $S^B = S^G = 0$ and a monetary equilibrium with $S^G > S^B > 0$. The expressions (iii) and (iv) assume a monetary equilibrium.

12/The choice of this welfare criterion is defended in Miller. The author realizes, however, that this choice is somewhat arbitrary and others could be defended as well. This matter is discussed further in the concluding section.

13/In addition, the outcomes for these taxes in a monetary economy are directly comparable to those for taxes in a nonmonetary economy reported in Miller.

[14] Although the variances of these variables are the relevant measures of variation according to the utility function, the coefficients of variation are independent of the units of measurement and are easier to interpret.

[15] Only four decimal places are reported in the table. When it is reported in the text that one tax is better than another and their EW values in the table are the same, it should be understood that there is a difference beyond four decimal places.

[16] See CBO.

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Table 1

Equilibria Under Different Tax Schemes

| Outcome \ Tax Scheme | m = 0: Contingent Lump Sum | m = 1: Zero | m = 1: Proportional | m = 1: Non-Proportional Real | m = 1: Non-Proportional Nominal |
|---|---|---|---|---|---|
| $\hat{t}_0$ | B: .1538 G: .5293 | 0.0 | 0.0 | 0.0 | 0.0 |
| $\hat{t}_1$ | 0.0 | 0.0 | 0.0171 | 0.0157 | 0.0800 |
| $\hat{t}_2$ | 0.0 | 0.0 | 0.0 | 0.0006 | -0.0090 |
| $s^B$ | 0.0 | .3181 | .2950 | .2956 | .2621 |
| $s^G$ | 0.0 | .3790 | .3502 | .3508 | .3143 |
| $\bar{c}_1$ | .7075 | .6908 | .6843 | .6843 | .6897 |
| $\bar{L}$ | .8147 | .8128 | .8008 | .8009 | .7999 |
| $\bar{c}_2$ | .3416 | .3485 | .3400 | .3400 | .3358 |
| $\bar{y}$ | 1.0490 | 1.0393 | 1.0243 | 1.0243 | 1.0255 |
| $\sigma_{c_1}/\bar{c}_1$ | 23.9% | 41.9% | 41.5% | 41.4% | 42.6% |
| $\sigma_L/\bar{L}$ | 15.0% | 11.5% | 11.6% | 11.6% | 12.8% |
| $\sigma_{c_2}/\bar{c}_2$ | 55.0% | 8.7% | 9.7% | 9.7% | 10.3% |
| $\sigma_y/\bar{y}$ | 34.0% | 30.8% | 30.9% | 30.9% | 32.0% |
| $EW[c_1]$ | -21.7388 | -22.0688 | -22.1487 | -22.1482 | -22.0880 |
| $EW[L]$ | -6.7870 | -6.6939 | -6.5002 | -6.5004 | -6.5039 |
| $EW[c_2]$ | -24.1553 | -24.0014 | -24.1135 | -24.1139 | -24.1697 |
| EW | -52.6811 | -52.7641 | -52.7624 | -52.7624 | -52.7615 |
| Distortion | -52.4105 1/ | +.0979 | +.1020 | +.1028 | +.0716 |
| Instability | -.2705 1/ | -.0148 | -.0206 | -.0214 | +.0090 |

Parameters

| A | B | c* | β | θ |
|---|---|---|---|---|
| 2 | 10 | 4 | .9 | 1.5 |

1/ Figures for distortion and instability under contingent lump-sum tax are $EW_1^*$ and $EW_2^*$, respectively.

Table 2

Income Velocity of Money

| State \ Tax Scheme | Zero | Proportional | Non-Proportional Real | Non-Proportional Nominal |
|---|---|---|---|---|
| $\mu = \mu_B$ | 2.2626 | 2.3991 | 2.3950 | 2.6613 |
| $\mu = \mu_G$ | 3.5858 | 3.8299 | 3.8218 | 4.3072 |
| Average | 2.9242 | 3.1145 | 3.1084 | 3.4843 |

Parameters

| $A$ | $B$ | $c^*$ | $\beta$ | $\theta$ |
|---|---|---|---|---|
| 2 | 10 | 4 | .9 | 1.5 |

Table 3

Equilibria Under Different Tax Schemes

| Outcome \ Tax Scheme | m = 0 | m = 1 | | | |
|---|---|---|---|---|---|
| | Contingent Lump Sum | Zero | Proportional | Non-Proportional Real | Non-Proportional Nominal |
| $\hat{t}_0$ | B: .3636 G: .7347 | 0.0 | 0.0 | 0.0 | 0.0 |
| $\hat{t}_1$ | 0.0 | 0.0 | 0.0144 | 0.0266 | 0.0811 |
| $\hat{t}_2$ | 0.0 | 0.0 | 0.0 | -0.0053 | -0.0145 |
| $s^B$ | 0.0 | .5085 | .4886 | .4826 | .4494 |
| $s^G$ | 0.0 | .6019 | .5767 | .5704 | .5345 |
| $\bar{c}_1$ | .5492 | .5346 | .5285 | .5283 | .5312 |
| $\bar{L}$ | .8535 | .8518 | .8412 | .8408 | .8375 |
| $\bar{c}_2$ | .5492 | .5552 | .5482 | .5484 | .5433 |
| $\bar{y}$ | 1.0984 | 1.0898 | 1.0766 | 1.0767 | 1.0745 |
| $\sigma_{c_1}/\bar{c}_1$ | 33.8% | 54.6% | 54.3% | 54.8% | 55.6% |
| $\sigma_L/\bar{L}$ | 14.8% | 11.8% | 11.9% | 12.2% | 13.2% |
| $\sigma_{c_2}/\bar{c}_2$ | 33.8% | 8.4% | 8.9% | 9.0% | 9.6% |
| $\sigma_y/\bar{y}$ | 33.8% | 31.1% | 31.2% | 31.4% | 32.3% |
| $EW[c_1]$ | -23.8848 | -24.1882 | -24.2675 | -24.2727 | -24.2395 |
| $EW[L]$ | -7.4430 | -7.3558 | -7.1776 | -7.1753 | -7.1360 |
| $EW[c_2]$ | -23.8848 | -23.7376 | 23.8352 | -23.8322 | -23.9035 |
| $EW$ | -55.2126 | -55.2816 | -55.2803 | -55.2802 | -55.2789 |
| Distortion | -54.9160 1/ | +.0904 | +.0941 | +.0862 | +.0608 |
| Instability | -.2966 1/ | -.0214 | -.0264 | -.0186 | +.0055 |

Parameters

| A | B | c* | β | θ |
|---|---|---|---|---|
| 2 | 10 | 4 | 1.0 | 1.5 |

1/ Figures for distortion and instability under contingent lump-sum tax are $EW_1^*$ and $EW_2^*$, respectively.

Table 4

Equilibria Under Different Tax Schemes

| Outcome \ Tax Scheme | m = 0: Contingent Lump Sum | | m = 1: Zero | Proportional | Non-Proportional Real | Non-Proportional Nominal |
|---|---|---|---|---|---|---|
| | B | G | | | | |
| $\hat{t}_0$ | .1538 | .3324 | 0.0 | 0.0 | 0.0 | 0.0 |
| $\hat{t}_1$ | 0.0 | | 0.0 | 0.0054 | -0.0056 | 0.0280 |
| $\hat{t}_2$ | 0.0 | | 0.0 | 0.0 | 0.0062 | -0.0030 |
| $s^B$ | 0.0 | | .2344 | .2280 | .2325 | .2172 |
| $s^G$ | 0.0 | | .2552 | .2480 | .2527 | .2367 |
| $\bar{c}_1$ | .6188 | | .6146 | .6128 | .6128 | .6145 |
| $\bar{L}$ | .7588 | | .7581 | .7545 | .7546 | .7544 |
| $\bar{c}_2$ | .2431 | | .2448 | .2426 | .2425 | .2409 |
| $\bar{y}$ | .8619 | | .8594 | .8554 | .8554 | .8554 |
| $\sigma_{c_1}/\bar{c}_1$ | 12.9% | | 23.3% | 23.2% | 23.0% | 23.5% |
| $\sigma_L/\bar{L}$ | 8.7% | | 6.9% | 6.9% | 6.8% | 7.2% |
| $\sigma_{c_2}/\bar{c}_2$ | 36.8% | | 4.3% | 4.5% | 4.5% | 4.5% |
| $\sigma_y/\bar{y}$ | 19.7% | | 17.9% | 17.9% | 17.8% | 18.2% |
| $EW[c_1]$ | -22.8778 | | -22.9624 | -22.9873 | -22.9858 | -22.9645 |
| $EW[L]$ | -5.8010 | | -5.7749 | -5.7202 | -5.7209 | -5.7199 |
| $EW[c_2]$ | -25.4201 | | -25.3830 | -25.4127 | -25.4135 | -25.4358 |
| EW | -54.0989 | | -54.1204 | -54.1202 | -54.1202 | -54.1201 |
| Distortion | -54.0277 1/ | | +.0239 | +.0243 | +.0262 | +.0210 |
| Instability | -.0712 1/ | | -.0025 | -.0030 | +.0049 | +.0002 |

Parameters

| A | B | c* | β | θ |
|---|---|---|---|---|
| 2 | 10 | 4 | 0.9 | 1.25 |

1/ Figures for distortion and instability under contingent lump-sum tax are $EW_1^*$ and $EW_2^*$, respectively.