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PROCEDURES FOR SOLVING AND ESTIMATING A CLASS OF HIERARCHICAL LINEAR RATIONAL EXPECTATIONS MODELS

William T. Roberds

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ABSTRACT

A method is presented for solving a certain class of hierarchical rational expectations models, principally models that arise from Stackelberg dynamic games. The method allows for numerical solution using spectral factorization algorithms, and estimation of these models using standard maximum likelihood techniques.

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The purpose of the present paper is to outline some procedures that are useful for solving and estimating certain types of rational expectations models. The models described below are rational expectations models in the sense that all agents' expectations of future variables are taken to be the conditional expectations of these variables on the agents' current information sets. These models are also hierarchical in the sense that the information that agents may use in making decisions is not the same for all agents. Instead, the information of one agent is restricted to be "smaller" than that of the others. To be more precise, one agent (perhaps a central monetary authority or a dominant firm) is assumed to be a Stackelberg leader. The other agents are assumed to know from the outset the strategies (sometimes called decision rules) of the leader for all time periods. The leader, however, does not take the strategies of the other agents as given, but exploits his knowledge of the other agents' reaction functions in maximizing his own objective.

The principal application of the techniques outlined below is in solving and estimating discrete time Stackelberg dynamic games of the linear-quadratic Gaussian (LQG) variety. These techniques are nonrecursive "open loop" and are intended to complement recursive "closed loop" or "feedback" procedures such as those described in Kydland and Prescott (1980) or Chow (1981).

The paper is presented in the following order: Section 1 lays out a two-player LQG Stackelberg game; Section 2 gives some examples of oligopoly models that fit into the framework of Section 1; Section 3 presents the solution procedure; extensions and some estimation strategies are discussed in Section 4.

1. Dynamic Game Models

An important class of hierarchical rational expectations models are models describing Stackelberg equilibria of dynamic games. In the present paper, I analyze a game with two infinitely lived players, each having a time-invariant, time-additive discounted quadratic objective functional. All stochastic variables enter into the players' objective functional in a linear fashion, and are assumed to be normally distributed. The two-player assumption can be relaxed, subject to computational constraints, but the other assumptions cannot. The purpose of the other assumptions is to facilitate econometric application by allowing linear least squares projections to be used in place of conditional means. In terms of notation, let

- u_{1t} be a column vector of decision variables of player 1 (the Stackelberg leader) at time t;
- u_{2t} be analogously defined for player 2 (the follower);
- f_{1t} be a column vector of uncontrollable forcing variables influencing player 1's payoff at time t;
- f_{2t} be analogously defined for player 2;
- b be the discount factor common to both players, where $0 < b < 1$.

Player one's objective is given by:

$$J_1 = E_1 \sum_{t=1}^{\infty} b^{t-1} \{f_{1t}'u_{1t} - [A(L)u_{1t}]' \tfrac{1}{2} M_1 [A(L)u_{1t}]$$

$$- [B(L)u_{2t}]' \tfrac{1}{2} M_2 [B(L)u_{2t}]$$

$$- [A(L)u_{1t}]' M_3 [B(L)u_{2t}]$$

$$- u_{1t}' \tfrac{1}{2} N_1 u_{1t} - u_{2t}' \tfrac{1}{2} N_2 u_{2t} - u_{1t}' N_3 u_{2t}\}$$

where A(L) and B(L) are matrix polynomials in the lag operator L, of finite order and degree; M_1, M_2, M_3, N_1, N_2 and N_3 are matrices of the appropriate dimension, M_1, M_2, N_1, and N_2 are symmetric, E_1 is the expectations operator, conditional on information available at time $t = 1$. Player two's objective is given by

$$J_2 = E_1 \sum_{t=1}^{\infty} b^{t-1} \{f_{2t}'u_{2t} - [C(L)u_{1t}]' \frac{1}{2} P_1[C(L)u_{1t}]$$

$$- [D(L)u_{2t}]' \frac{1}{2} P_2[D(L)u_{2t}]$$

$$- [D(L)u_{2t}]' P_3[C(L)u_{1t}]$$

$$- u_{1t}' \frac{1}{2} Q_1 u_{1t} - u_{2t}' \frac{1}{2} Q_2 u_{2t} - u_{2t}' Q_3 u_{1t}\}$$

where C(L) and D(L) are finite order, finite degree matrix polynomials in the lag operator L, P_1, P_2, P_3, Q_1, Q_2, Q_3 are matrices of the appropriate dimension, P_1, P_2, Q_1, Q_2, are symmetric, and the definiteness condition

$$D(b^{1/2}e^{-i\omega})' P_2 D(b^{1/2}e^{i\omega}) + Q_2 > 0$$

is satisfied for $\omega \in [-\pi, \pi]$. An example of a similar definiteness condition for the first player is given in the Appendix.

The uncontrollable forcing vector $f_t = [f_{1t}' f_{2t}']'$ is assumed to be Gaussian and to have time-invariant fundamental moving average representation

$$f_t = F(L)v_t + K$$

where v_t is vector white noise, and K is constant.

Each player $\underline{i}$ seeks to maximize his objective by choosing a sequence of strategies $\{g_{it}\}_{t=1}^{\infty}$. Each strategy maps the player's information (I_{it})

into a decision taken at time t, i.e., $u_{it} = g_{it}(I_{it})$. [1/] Because of the asymmetric nature of Stackelberg equilibria $I_{1t} \neq I_{2t}$ for any time t. For player one (the leader)

$$(1.1) \qquad I_{1t} = \{v_t, v_{t-1}, \ldots\} \cup \{\text{initial conditions for all state variables}\}$$

while for player two (the follower)

$$(1.2) \qquad I_{2t} = I_{1t} \cup \{g_{1t}\}_{t=1}^{\infty}.$$

An equilibrium $(\{g_{1t}^*\}, \{g_{2t}^*\})$ occurs when $\{g_{1t}^*\}$ maximizes J_1 and $\{g_{2t}^*\}$ maximizes J_2, where $\{g_{1t}\} = \{g_{1t}^*\}$ in equation (1.2). The strategies $\{g_{1t}\}$ are restricted to be affine functions of the shocks $\{v_t\}$, and to be of mean exponential order less than $b^{-1/2}$. Note that player two's information does not include knowledge of player one's future decisions, but instead knowledge of player one's future strategies. Note also that the equilibrium strategies must be optimal for almost every sequence of realizations of $\{v_t\}$.

The distinction between decisions and strategies is an important one: the leader is a "dominant player" in this game because he is the first to announce a sequence of strategies. Decisions, on the other hand, are taken simultaneously by both players in every period.

It is also important to note that the restriction to affine strategies is a nontrivial one. With this assumption, certainty equivalence can be exploited in the solution of the model considered above, even though the leader's problem is not a classical LQG problem. [2/] To see why this restriction is important, suppose the leader were allowed to play nonaffine strategies; then the follower's conditional distribution on the future decisions of the leader would no longer be Gaussian, resulting in nonaffine conditional expectations of the leader's future decisions. Since the leader must take the

follower's (again, possibly nonaffine) strategies into account when formulating his own optimal strategy, certainty equivalence would no longer hold for the leader's problem. That affine strategies can be suboptimal for nonclassical LQG problems, and that certainty equivalence can fail to hold in such problems, has been demonstrated by Witsenhausen (1968) in the context of a two-period model. Hence, one would ideally allow for nonaffine strategies in the equilibrium considered above. However, since an extension to nonaffine strategies could complicate the analysis to the point of intractability,[3] only affine strategies will be considered.

Because the information sets I_{it} contain no state variables other than uncontrollable shocks, this sort of game is described by dynamic game theorists as "open loop".[4] Games in which controllable state variables appear in players' information sets fall into the "closed loop" and "feedback" categories. As emphasized by Kydland (1977) and others, the equilibria of open loop dynamic dominant player games will in general be different from the feedback or closed loop equilibria. The open-loop approach taken in this paper is justified largely by computational considerations. Particularly for econometric applications, the nonrecursive procedures discussed below may offer considerable gains in computational convenience over the recursive procedures used to obtain closed loop and feedback equilibria.

2. Examples

This section contains three examples of dynamic game models that fall into the class of models discussed above. Each of these models can, therefore, be solved using the techniques outlined in the next section.

Example 1. Exhaustible Resource Depletion.

Hansen, Epple, and Roberds (1984) consider the case of an extractive industry consisting of two producers, where the first producer is a Stackelberg leader. Resource extraction cost at time t for producer $\underline{i}$ is given by

$$\phi_i \Delta r_{it} + s_{it} \Delta r_{it} + \left(\frac{\theta_i + \pi_i}{2}\right) \Delta r_{it}^2 + \pi_i \Delta r_{it} r_{it-1}$$

where r_{it} denotes the cumulated amount of the resource extracted by firm i as of time t, ϕ_i, π_i, and θ_i are positive constants, and s_{it} is a random shock to the costs of firm i at time t. The exhaustive nature of the resource is represented by the presence of the terms $\pi_i \Delta r_{it} r_{it-1}$. The inverse demand function for the resource is given by

$$p_t = d_t - \delta[\Delta r_{1t} + \Delta r_{2t}]$$

where p_t is the real price of the resource at time t, d_t is a random shock to demand, and δ is a positive constant. Each producer $\underline{i}$ is assumed to maximize

$$E_1 \sum_{t=1}^{\infty} b^{t-1} \left\{p_t \Delta r_{it} - \phi_i \Delta r_{it} - s_{it} \Delta r_{it} - \left(\frac{\theta_i + \pi_i}{2}\right) \Delta r_{it}^2 - \pi_i \Delta r_{it-1}\right\}$$

The Hansen, Epple, and Roberds model can be fit into the current setup via the substitutions

$$u_{it} = r_{it}, \quad i = 1, 2$$

$$f_{it} = (1 - bL^{-1})(d_t - \phi_i - s_{it}), \quad i = 1, 2$$

$$A(L) = B(L) = C(L) = D(L) = (1 - L)$$

$$M_1 = \theta_1 + 2\delta$$

$$P_2 = \theta_2 + 2\delta$$

$$M_2 = N_2 = N_3 = P_1 = Q_1 = Q_3 = 0$$

$$M_3 = P_3 = \delta$$

$$N_1 = \pi_1(1-b)$$

$$Q_2 = \pi_2(1-b)$$

Example 2. Adjustment Cost Model

Kydland (1979) uses the following oligopoly model to illustrate the feedback solution in dynamic dominant player games. Suppose an industry consists of two firms, each with output y_{it} at time t. Investment by firm i over period t, x_{it}, is taken as

$$x_{it} = y_{i,t+1} - (1-\delta)y_{it}$$

where δ = the depreciation rate. The cost of investment to firm i at time t is given by

$$qx_{it} + c(x_{it}-\delta y_{it})^2$$

where q is the unit cost of capital and the term $c(x_i-\delta y_{it})^2$, $c > 0$, represents the adjustment cost associated with changing the firm's capital stock. Each firm i seeks to maximize

$$E_1 \sum_{t=1}^{\infty} b^{t-1} \{p_t y_t - qx_{it} - c(x_{it}-\delta y_{it})^2\}$$

where p_t is the real price of the firm's output at time t, net of any constant unit production cost. As in the first example, p_t is determined by a linear inverse demand function

$$p_t = a_t - \alpha[y_1+y_{2t}]$$

where a_t is a random shock to demand and α is a positive constant. To map Kydland's model into the present setup, set

$$u_{it} = y_{it+1}, \quad i = 1, 2$$

$$A(L) = B(L) = C(L) = D(L) = (1-L)$$

$$f_{it} = ba_{t+1} - (1-b(1-\delta)q), \quad i = 1, 2$$

$$M_1 = P_2 = 2c$$

$$M_2 = M_3 = P_1 = P_3 = 0$$

$$N_3 = Q_3 = b\alpha$$

$$N_1 = Q_2 = 2b\alpha$$

$$N_2 = Q_1 = 0$$

Example 3. A Model of Learning by Doing

Dennis Epple has suggested a linear-quadratic model of learning by doing, in which each of the two producers in an industry seeks to maximize

$$J_i = \sum_{t=1}^{\infty} b^{t-1} \{p_t q_{it} - \mu_i q_{it}^2 + \pi_i q_{it} k_{it}\}$$

where p_t is the real price of the industry's product, q_{it} is the current output of firm i, and k_{it} represents the "experience" of firm i in producing the product. The parameters μ_i and π_i are assumed to be positive. Real price is given by the demand equation

$$p_t = d_t - \delta[q_{1t} + q_{2t}]$$

where d_t is a random shock to demand and δ is a positive constant. Experience, k_{it}, evolves according to the law of motion

$$k_{it} = \gamma k_{it-1} + q_{it}$$

where γ is a constant greater than zero, and the condition $(\mu_i+\delta)\ (1-\gamma b^{-1/2}) > \pi_i$ is satisfied.

In this model, the cost-reducing effects of learning are represented by the term $\pi_i q_{it} k_{it}$, i.e., one-period marginal costs are linearly decreasing in experience. Experience must grow at a rate less than $b^{-t/2}$, however, in order that one-period costs cannot be driven to negative infinity by the learning effect.

The learning model is mapped into the present setup by the substitutions

$$u_{it} = k_{it}, \qquad i = 1, 2$$

$$f_{it} = (1-\gamma b L^{-1})d_t, \quad i = 1, 2$$

$$A(L) = B(L) = C(L) = D(L) = (1-\gamma L)$$

$$M_1 = 2\mu_1 + 2\delta + \gamma^{-1}\pi_1$$

$$P_2 = 2\mu_2 + 2\delta + \gamma^{-1}\pi_2$$

$$M_2 = N_2 = P_1 = Q_1 = Q_3 = 0$$

$$M_3 = P_3 = \delta$$

$$N_1 = \pi_1(\gamma^{-1}-b\gamma)$$

$$N_2 = \pi_2(\gamma^{-1}-b\gamma)$$

3. The Solution Procedure

By "solving" the class of models described in Section 1, I mean to derive a method of mapping the parameters of the two players' objective functionals into the parameters of the equilibrium law of motion in the variables u_{1t} and u_{2t}. Explicit formulas for $\{g_{1t}^*\}$ and $\{g_{2t}^*\}$ are not derived. The solution procedure makes heavy use of techniques developed by Hansen and Sargent in (1981). Especially useful are the following differentiation rules: let $\{x_t\}$ and $\{y_t\}$ be vector sequences such that

$$S_1 = \sum_{t=1}^{\infty} b^{t-1} [a(L)y_t]' B[c(L)x_t]$$

and

$$S_2 = \sum_{t=1}^{\infty} b^{t-1} [d(L)y_t]' \frac{1}{2} F[d(L)y_t]$$

are bounded, where $0 < b < 1$, $a(L)$, $c(L)$, and $d(L)$ are finite order matrix polynomials in the lag operator, and B and F are appropriately dimensioned matrices. Then

(D1) $$\partial S_1/\partial y_t = b^{t-1} a(bL^{-1})' B\, c(L)x_t$$

(D2) $$\partial S_2/\partial y_t = b^{t-1} d(bL^{-1})' F\, d(L)y_t$$

The certainty equivalence properties of LQG optimization are also exploited, in that the model is first solved for conditional means. Terms involving expectations are then evaluated using the standard Wiener-Kolmogorov prediction formulas.

To initiate the solution procedure suppose that player two knows the sequence of equilibrium strategies $\{g_{1t}^*\}$ of player one. Since player two knows $\{g_{1t}^*\}$, and at every time t, $I_{2t} \supset I_{1t}$, player one's decision $u_{1t}^* = g_{1t}^*(I_{1t})$ will be known to player two as of time t. The necessary

first-order conditions for player two will thus be the following expectational Euler equations:[5]

$$(3.1) \qquad [D(bL^{-1})'\, P_2 D(L) + Q_2]\, E_t u^*_{2t} =$$

$$- [Q_3 + D(bL^{-1})'\, P_3 C(L)]\, E_t u^*_{1t} + f_{2t} \quad t = 1, 2, \ldots$$

where again E_t represents the conditional expectations operator. The operators L and L^{-1} are defined as follows for the sequence of conditional means $E_t u_{1t}$: $LE_t u_{1t} \equiv E_{t-1} u_{1t-1}$, and $L^{-1} E_t u_{1t} \equiv E_t u_{1t+1}$, i.e., negative powers of L do not operate on agents information sets.[6] Now, the characteristic polynomial of equation (3.1) has factorization[7]

$$D(bz^{-1})'\, P_2 D(z) + Q_2 = G(bz^{-1})'\, G(z)$$

where G(z) has only nonnegative powers of z, the roots of det G(z) exceed $b^{-1/2}$ in modulus, and the moduli of the roots of det $G(bz^{-1})$ are less than $b^{-1/2}$. The homogeneous solution to equation (3.1) may thus be written

$$u^h_{2t} = \sum_i (h_i r_i^{-t} + k_i (b^{-1} r_i)^t)$$

where the r_i are the roots of det G(z), and the vectors $\{h_i\}$ and $\{k_i\}$ are of the same dimension as u_{2t}. Since the $\{u^*_{it}\}$ are restricted to be "stable", i.e., of mean exponential order less than $b^{-1/2}$, the coefficients of the unstable portion of the homogeneous solution, the k_i's, must be equal to zero. Euler equations (3.1) may thus be solved forward to yield

$$(3.2) \qquad G(L) u^*_{2t} = G(bL^{-1})'^{-1} \{-[Q_3 + D(bL^{-1})'\, P_3 C(L)] E_t u^*_{1t} + f_{2t}\}.$$

Equation (3.2) is a "closed loop" representation of the sequence of optimal decisions $\{u^*_{2t}\}$, i.e., the current optimal decision for player two, u^*_{2t}, is expressed as a function of lagged values of itself, and current and lagged

values of u_{1t}^* and f_{2t} (after making the appropriate substitution for terms involving expectations of future variables). Using this representation, one could go one step further and derive the sequence of optimal open-loop strategies $\{g_{2t}^*\}$ by operating on (3.2) with $g(L)^{-1}$. However, for the present purpose of deriving the equilibrium law of motion for u_{1t} and u_{2t}, this extra step is not necessary.[8/]

The next step in solving the model is to formulate the leader's problem as a constrained maximization problem:

$$\max_{\{g_{1t}\}\{g_{2t}\}} J_2$$

$$\text{s.t.} \quad u_{1t} = g_{1t}(I_{1t}),\ u_{2t} = g_{2t}(I_{2t}) \text{ and equation (3.2).}$$

To solve the leader's problem, form the Lagrangian expression

$$\begin{aligned} L_1 = E_1 \sum_{t=1}^{\infty} b^{t-1} \{ & f_{1t}'u_{1t} - [A(L)u_{1t}]' \tfrac{1}{2} M_1 [A(L)u_{1t}] \\ & - [B(L)u_{2t}]' \tfrac{1}{2} M_2 [B(L)u_{2t}] \\ & - [A(L)u_{1t}]' M_3 [B(L)u_{2t}] \\ & - u_{1t}' \tfrac{1}{2} N_1 u_{1t} - u_{2t}' \tfrac{1}{2} N_2 u_{2t} - u_{1t}' N_3 u_{2t} \} \\ & + E_1 \sum_{t=1}^{\infty} b^{t-1} \{ \lambda_t' [G(bL^{-1})'^{-1} f_{2t} - G(L)u_{2t} \\ & - G(bL^{-1})'^{-1} [Q_3 + D(bL^{-1})' P_3 C(L)] E_t u_{1t}] \} \end{aligned}$$

where λ_t is a vector of Lagrange multipliers. Necessary first-order conditions for the leader's maximization problem are

$$(3.3)\qquad [A(bL^{-1})'M_1A(L)+N_1]E_tu_{1t}^* + [A(bL^{-1})'M_3B(L)+N_3]E_tu_{2t}^* + [Q_3'+C(bL^{-1})'P_3'D(L)]\,G(L)^{-1}E_t\lambda_t^* = f_{1t}$$

$$(3.4)\qquad [N_3'+B(bL^{-1})'M_3'A(L)]E_tu_{1t}^* + [B(bL^{-1})'M_2'B(L)+N_2]E_tu_{1t}^* + G(bL^{-1})'E_t\lambda_t^* = 0$$

$$(3.5)\qquad G(bL^{-1})'^{-1}[Q_3+D(bL^{-1})'P_3C(L)]E_tu_{1t}^* + G(L)E_tu_{2t}^* = G(bL^{-1})'^{-1}f_{2t}$$

$$t = 1, 2, 3, \ldots$$

Now substitute $\lambda_t^0 \equiv G(L)^{-1}\lambda_t^*$, operate on (3.5) with $G(bL^{-1})'$, and stack the above equations to obtain the system

$$(3.6)\qquad H(L)E_tu_t^* = z_t$$

where

$$u_t^* = [u_{1t}^{*\prime}\ u_{2t}^{*\prime}\ \lambda_t^{0\prime}]',$$

$$H(L) = \begin{bmatrix} N_1 + A(bL^{-1})'M_1A(L) & N_3 + A(bL^{-1})'M_3B(L) & Q_3' + C(bL^{-1})'P_3'D(L) \\ N_3' + B(bL^{-1})'M_3'A(L) & N_2 + B(bL^{-1})'M_2B(L) & Q_2 + D(bL^{-1})'P_2D(L) \\ Q_3 + D(bL^{-1})'P_3C(L) & Q_2 + D(bL^{-1})'P_2D(L) & 0 \end{bmatrix}$$

and

$$z_t = [f_{1t}'\ 0\ f_{2t}']'$$

Alternatively, equations (3.6) may be derived by taking the constraint in the leader's optimization problem to be the follower's Euler equations.[9/] It is convenient to rewrite H(L) in partitioned form as

$$H(L) = \begin{vmatrix} H_{11}(L) & H_{12}(L) & H_{13}(L) \\ H_{12}(L)' & H_{22}(L) & H_{23}(L) \\ H_{13}(L)' & H_{23}(L) & 0 \end{vmatrix}$$

Using equation (3.4), the Lagrange multiplier vector $E_t\lambda_t^0$ may be eliminated from the system (3.6), yielding[10/]

$$(3.7) \qquad \begin{vmatrix} H_{11}(L)-H_{13}(L)H_{23}(L)^{-1}H_{12}(L)' & H_{12}(L)-H_{13}(L)H_{23}(L)^{-1}H_{22}(L) \\ H_{13}(L)' & H_{23}(L) \end{vmatrix} \begin{vmatrix} E_t u_{1t}^* \\ E_t u_{2t}^* \end{vmatrix} = \begin{vmatrix} f_{1t} \\ f_{2t} \end{vmatrix}$$

It is of interest to compare the equations (3.7) with the Euler equations that would be obtained in an open loop Nash game. For the Nash game, one obtains by stacking the first-order conditions for the first and second players

$$(3.8) \qquad \begin{vmatrix} H_{11}(L) & H_{12}(L) \\ H_{13}(L)' & H_{23}(L) \end{vmatrix} \begin{vmatrix} E_t u_{1t}^* \\ E_t u_{2t}^* \end{vmatrix} = \begin{vmatrix} f_{1t} \\ f_{2t} \end{vmatrix}$$

By comparing (3.8) and (3.7), it is clear that the Euler equations of the dominant player and Nash games differ only in the leader's equation; the follower's equation is the same for both games. Also note that, since the polynomials $H_{ij}(L)$ are of finite order, the leader's Euler equation will be of finite order in the Nash game, but in general of infinite order in the Stackelberg game.

For reasons of brevity, general sufficiency conditions for the leader's problem are not given here. Rather, an example of how sufficiency conditions can be obtained is given in the Appendix, for the third example of Section 2.

The last step in the solution procedure is to use equations (3.6) to obtain a vector ARMA or MA representation for u_t^*. To obtain an MA representation for z_t, first factor $H(L)$ as[11]

$$H(L) = K(bL^{-1})K(L)$$

where $K(L)$ involves only positive powers of L, and the roots of $\det K(s)$ are outside the circle $|s| = b^{-1/2}$. One algorithm for obtaining this factorization numerically is suggested by Whittle (1983). An alternative method[12] for obtaining the necessary factorization would be to interpret equations (3.6) as the first-order conditions for an unconstrained LQG problem, then to solve the matrix algebraic Riccati equation for that problem.

Assuming the roots of $\det K(s)$ to be distinct, write $K(bL^{-1})^{-1}$ in matrix partial fractions form as

$$K(bL^{-1})^{-1} = \sum_j \frac{N_j}{L-s_j}$$

where the N_j are matrices of the appropriate dimension, and the s_j are the roots of $\det K(bs^{-1})$. Requiring the homogeneous solution of (3.6) to be equal to zero, one obtains by a slight modification of a result by Whiteman (1983)[13] the unique equilibrium moving average representation for u_t^*:

$$(3.9) \qquad u_t^* = \sum_{j=0}^{\infty} C_j v_{t-j}^* \equiv C(L)v_t^*$$

where

v_t^* is the innovation in $[f_{1t}' \; 0 \; f_{2t}']$

$$C(L) = K(L)^{-1} \sum_j \frac{N_j}{L-s_j} [L^n F(L) - s_j^n F(s_j)]$$

where n is the highest order of the polynomials $A(L)$, $B(L)$, $C(L)$, and $D(L)$. Again the summation is over the roots of $\det K(bs^{-1})$.

A difficulty with the above solution is that while the sequence $\{tr[b^j C_j C_j']\}_{j=0}^{\infty}$ is guaranteed to be summable, the sequence $\{tr\,[C_j C_j']\}$ is not, i.e., u_t^* may not be stationary. In the case that det K(s) has no roots for $|s| \leq 1$, however, the latter sequence will be summable and the solution u_t^* will be stationary. This will in fact be the case for the examples of Section 2.

An ARMA representation for u_t^* can be obtained by operating on both sides of (3.9) with $K_0^{-1}K(L)$ to obtain

$$(3.10) \qquad K_0^{-1}K(L)\, u_t^* = K_0^{-1} \sum_j \frac{N_j}{L-s_j} [L^n F(L) - s_j^n F(s_j)] v_t^*$$

where K_0 is the constant term of K(L). A researcher might prefer to work with the ARMA representation in a models such as Example 1 and where it is important to allow for nonzero stable homogeneous solutions to the first order conditions (3.6).

The ARMA representation (3.10) is also interesting in that it reveals the time inconsistent nature of the open-loop equilibrium. To see this, first note that the correct initial conditions for the vector of Lagrange multipliers λ_t^0 are $\lambda_t^0 = 0$ for $t \leq 0$. As the game evolves according to equation (3.10), however, $\lambda_1^0, \lambda_2^0, \ldots$, will in general be nonzero. Now consider a dynamic subgame starting in period $\tau > 1$. For such a subgame, the correct initial conditions for λ_t^0 are $\lambda_t^0 = 0$ for $t < \tau$, implying that $\lambda_1^0, \ldots, \lambda_{\tau-1}^0$ must be zero. Hence, the equilibrium for the subgame will be different from the original equilibrium, and the optimal strategy for the leader will not be time consistent.[14/]

4. Extensions and Empirical Applications

There are two immediate extensions to the model presented in Sections 2 and 3.

First, the model can be extended to a setup where there is more than one follower. Computational considerations will be a limiting factor for this extension, as each additional follower will cause the dimension of the $H(L)$ matrix to be augmented by $2k$, where k is the dimension of the follower's vector of control variables. An alternative technique for introducing more players into the model would be to treat the follower as a representative player whose objective function represents the aggregation of the objectives of a number of competitive agents.

The second extension results from the fact it is not really necessary to specify the follower's objective function in order to solve for the MA or ARMA representation of u_t^*. If the researcher is not interested in the parameters of the follower's objective function, then he need only posit a set of Euler equations such as (3.1) to obtain the solution to the model.[15/] This might be the case in a macroeconomic application, where the researcher was primarily interested in estimating the parameters of the objective function of a governmental agency.

In the case that the solution for u_t^* is stationary, two quasi-maximum likelihood estimation procedures may be used to estimate the model. These procedures are discussed in some detail in Hansen and Sargent (1980); a brief outline of their application to the present model is given below.

To apply these procedures, first partition f_t into observable and unobservable components, i.e., $f_t = [f_t^0 f_t^n]'$. The MA representation for f_t must have form

$$\begin{vmatrix} f_t^0 \\ f_t^n \end{vmatrix} = \begin{vmatrix} F^0(L) & 0 \\ 0 & F^n(L) \end{vmatrix} \begin{vmatrix} v_t^0 \\ v_t^n \end{vmatrix}$$

where v_t^0 and v_t^n are uncorrelated at all leads and lags. Denoting by πv_t^0 the linear least squares projection of v_t on v_t^0, the model to be estimated is

$$\begin{vmatrix} u_{1t} \\ u_{2t} \end{vmatrix} = UC(L)(v_t^* - \pi v_t^0) + UC(L)\pi v_t^0$$

$$f_t^0 = F^0(L)v_t^0$$

together with any auxiliary equations (e.g., demand equations) of the model, where the matrix $U = [I:0]$. Defining $a_t = v_t^* - \pi v_t^0$, $\sum_a = E(a_t a_t')$, and $\sum_v = E(v_t^0 v_t^{0\prime})$, the theoretical spectral density matrix of the process $[u_{1t}' \; u_{2t}' \; f_t^{0\prime}]'$ is given by $S(\omega) =$

$$\begin{vmatrix} UC(e^{-i\omega}) & UC(e^{-i\omega})\pi \\ 0 & F^0(e^{-i\omega}) \end{vmatrix} \begin{vmatrix} \sum_a & 0 \\ 0 & \sum_v \end{vmatrix} \begin{vmatrix} C(e^{i\omega})'U' & F^0(e^{i\omega})' \\ \pi' C(e^{i\omega})'U' & 0 \end{vmatrix}$$

The first estimation procedure is to maximize a spectral approximation of the log likelihood function. Let $I(\omega_j)$ be the periodogram of $[u_{1t}' u_{2t}' f_t^{0\prime}]'$, where $\omega_j = 2\pi j/T$ for $j = 0, \ldots, T-1$ and T is the sample size. The spectral approximation of the log likelihood of the sample is given by

$$L_T^1 = \text{constant} - \frac{1}{2}\sum_{j=0}^{T-1} \log [\det S(\omega_j)] - \frac{1}{2}\sum_{j=0}^{T-1} \text{tr}\,[S(\omega_j)^{-1} I(\omega_j)].$$

The above procedure may appear more complicated than it actually is. To calculate the value of L_T^1 for a given set of model parameter values, one first needs the periodgram of the observable series u_{1t}, u_{2t}, and f_t^0. One

would then obtain the moving average polynomial C(L) by application of a spectral factorization algorithm to H(L). Having obtained C(L), one can then construct $S(\omega_j)$ at the harmonic frequencies ω_j. Having obtained $S(\omega_j)$, L_T^1 can then be obtained by straightforward matrix manipulations. An interesting aspect of this estimation procedure is that one need not construct the unobservable Lagrange multiplier series λ_t^0 for purposes of estimation.

The second estimation procedure is to maximize a time domain approximation to the log likelihood function, i.e., the log likelihood conditioned on some initial observations. This procedure makes use of the ARMA representation for u_t^*:

$$K_0^{-1}K(L)u_t^* = K_0^{-1} \sum_j \frac{N_j}{L-s_j} [L^n F(L) - s_j^n F(s_j)]v_t^*$$

which can be rewritten in invertible form as 16/

$$K_0^{-1}K(L)u_t^* = M(L)w_t$$

where w_t is white noise and M(L) is invertible. It is also assumed that f_t^0 has invertible ARMA representation

$$N(L)\ f_t^0 = P(L)x_t$$

where x_t is white noise and P(L) is invertible. Let ϕx_t be the projection of w_t on x_t. The time domain approximation of the log likelihood function is given by

$$L_T^2 = \text{constant} - \frac{T}{2} \log\ [\det\ V_1] - \frac{T}{2} \log\ [\det\ V_2]$$

where $$V_1 = \frac{1}{T} \sum_{t=1}^{T} \hat{x}_t \hat{x}_t'$$

$$\hat{x}_t = [N(L)\ f_t^0 - L(P(L)/L)_+\ \hat{x}_t]$$

$$V_2 = \frac{1}{T} \sum_{t=1}^{T} U(\hat{w}_t - \phi\hat{x}_t)(\hat{w}_t - \phi\hat{x}_t)'U'$$

$$\hat{w}_t = [K_0^{-1}K(L)u_t^* - L(M(L)/L)_+\hat{w}_t]$$

and the notation $(\ )_+$ means to ignore negative powers of L. The time domain approximation assumes that sufficient initial conditions exist for f_t^0, u_{1t}, and u_{2t} so that $\hat{x}_1$ and $\hat{w}_1$ and are well-defined. Presample values for $\hat{x}_t$ and $\hat{w}_t$ are taken to be zero. Initial values for the Lagrange multiplier λ_t^0 will also be equal to zero.

Again, the time domain approximation procedure may appear rather complicated at first glance. A verbal description of the precedure is as follows. Given a set of model parameters, factor H(L) to obtain an ARMA representation for u_t^*. Also obtain an ARMA representation for f_t^0. Sequentially construct the estimated innovations in f_t^0, i.e., the $\hat{x}_t$'s. One would then sequentially construct the estimated innovations in u_t^*, i.e., the $\hat{w}_t$'s. This last step necessarily involves construction of an estimated series of Lagrange multipliers. This can be done because at any time $t > 0$, λ_t^0 is an affine function of current and past observables, whose values are known, and their innovations, which can be approximated by setting presample innovations to zero.

An inconvenient feature of the time domain approximation is that a new Lagrange multiplier series must be constructed for each function evaluation. Because this added step will not be necessary in evaluating the spectral approximation of the log likelihood function, the frequency domain procedure may offer computational advantages over the time domain procedure.

5. Conclusion

By using a Lagrangian to solve the leader's maximization problem, the methods of Hansen and Sargent (1981) and Whiteman (1983) for solving linear rational expectations models can be extended to a class of hierarchical models. Although it will not generally be possible to obtain analytic solutions for these models, numerical solutions can be obtained using the algorithm suggested above. There will be some extra computational burden for hierarchical models because Lagrange multipliers must be added to the vector of decision variables, but the same computational techniques may be used as with nonhierarchical models.

Footnotes

1/The term "information" is used here in the game theoretic sense, i.e., a player's information as of time t is the domain of his strategy function as of time t.

2/The applicability of the certainty equivalence principle in such cases follows from the result IV.c., p. 1561 of Witsenhausen (1971).

3/The skeptical reader is referred to Witsenhausen (1968).

4/The term "open loop" is usually applied to games under certainty, i.e., the case for which $v_t = 0$ for all t in the game described above. The definition of open-loop equilibrium for the stochastic case corresponds to that of Kydland (1977), except for the restriction that strategies be affine. The reader is referred to Kydland's article for a comparison of open loop, feedback, and closed loop dominant player dynamic games.

5/The term "expectational Euler equation" was coined by Whiteman (1983).

6/Readers of Sargent's textbook (1980), should note that Sargent's operator "B" is identical to "L" as defined above.

7/This follows from well-known results for spectral density matrices. See Hansen and Sargent (1981) for a discussion of such factorizations.

8/This follows from the fact that adding current and lagged state variables to the follower's information set will not change the leader's optimal strategy. In the words of Basar and Olsder (1982) (Remark 1, page 309):

> "If the follower has, instead [of open-loop information], access to closed loop perfect state information, his optimal response will be any closed loop representation

> of the open-loop policy $[\{g^*_{2t}\}]$; however, since the constraints [placed on the leader, in maximizing his objective] are basically open-loop relations, these different representations do not lead to different optimization problems for the leader."

9/The reader is invited to verify this statement by taking equations (3.1) as the leader's constraint, then applying differentation rules D1 and D2.

10/Since the first-order conditions for the leader's problem only hold for positive t, some care must be exercised in interpreting the term $H_{23}(L)^{-1}$, which involves positive powers of L. In order for these inverses to be well-defined, I adopt the convention that both players' objective functions have been normalized so that all variables take on a value of zero for nonpositive time.

11/Again see Hansen and Sargent (1981) for a discussion of such factorizations.

12/Danny Quah suggested this method.

13/I.e., the corollary to Theorem 1, Chapter 4 of Whiteman (1983). The necessary modification is to restrict the region of existence for $K(s)^{-1}$ to be $\{s| \ |s|<b^{1/2}\}$ instead of the open unit disk. The result of this modification is that the $\{C_j\}$ are not necessarily square summable unless they are "deflated" by $\{b^j\}$, as explained above. The reader is cautioned that Theorem 1, which determines whether a unique solution to a given model exists, is false. However, the corollary, which gives the form of the unique solution when it does exist, remains true.

14/This point has been noted by a number of authors, perhaps most forcefully by Kydland and Prescott (1977). The reader should note that the term "optimal" in the sentence "optimal strategies are time inconsistent" only

means "optimal for the dynamic game under consideration" and not (in general) "globally optimal for the leader."

15/ In this type of analysis there would technically be no need for the LHS of Euler equations (3.1) to be symmetric in z and bz^{-1}. Proving sufficiency for the dominant player's problem and existence of the necessary factorizations would be more difficult than under the symmetry assumption.

16/ Hansen and Sargent (1980) describe procedures for obtaining such representations. As long as the MA component of the representation for u_t^* has no roots on the unit circle, such representations can be obtained in a straightforward fashion.

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Appendix

Sufficiency Conditions for the Learning by Doing Example

In this Appendix, I investigate sufficiency conditions for a certainty version of Example 3 of Section 2. First, it is convenient to rewrite producer i's objective as

$$J_i = \sum_{t=1}^{\infty} b^{t-1}\{[d_t - \delta(1-\gamma L)(k_{1t}+k_{2t})](1-\gamma L)k_{it} - \mu_i[(1-\gamma L)k_{it}]^2 + \pi_i k_{it}[(1-\gamma L)k_{it}]\}.$$

Using the differentation rules given above, it can easily be shown that for this example, the leader's (producer 1's) first-order conditions are

$$\begin{bmatrix} 2\alpha(L)+\beta_1(L) & \alpha(L) & \alpha(L) \\ \alpha(L) & 0 & 2\alpha(L)+\beta_2(L) \\ \alpha(L) & 2\alpha(L)+\beta_2(L) & 0 \end{bmatrix} \begin{bmatrix} k_{1t} \\ k_{2t} \\ \lambda_t \end{bmatrix} = \begin{bmatrix} (1-\gamma b L^{-1})d_t \\ 0 \\ (1-\gamma b L^{-1})d_t \end{bmatrix}$$

where

$$\alpha(L) = \delta(1-\gamma L)(1-\gamma b L^{-1})$$

$$\beta_i(L) = 2\mu_i(1-\gamma L)(1-\gamma b L^{-1}) - \pi_i(2-\gamma L-\gamma b L^{-1}).$$

The leader's first-order conditions correspond to equations (3.6) above.

The discussion of sufficiency conditions below will follow that given in Telser and Graves (1971), pp. 58-81. In their notation, we have

$$x_t = [k_{1t} k_{2t}]';$$

$$m = 1; \quad n = 2; \quad \delta = b;$$

$$B(\delta^{1/2} e^{i\theta}) + B'(\delta^{1/2} e^{-i\theta}) =$$

$$- \begin{bmatrix} 0 & \alpha(b^{1/2} e^{i\theta}) \\ \alpha(b^{1/2} e^{i\theta}) & 2\alpha(b^{1/2} e^{i\theta}) + \beta_1(b^{1/2} e^{i\theta}) \end{bmatrix};$$

$$A(b^{1/2} e^{i\theta}) =$$

$$- [\alpha(b^{1/2} e^{i\theta}) \quad 2\alpha(b^{1/2} e^{i\theta}) + \beta_2(b^{1/2} e^{i\theta})];$$

$$B_0 + B_0' =$$

$$- \begin{bmatrix} 0 & \delta(1+b\gamma^2) \\ \delta(1+b\gamma^2) & 2(\delta+\mu_1)(1+b\gamma^2) - \pi_1(2-(1+b\gamma^2) \end{bmatrix};$$

$$A_0 =$$

$$- [\delta(1+b\gamma^2) \quad 2(\delta+\mu_2)(1+b\gamma^2) - \pi_2(2-(1+b\gamma)].$$

Telser and Graves have derived second-order conditions sufficient for a constrained maximum. These are, for the case under consideration:

(1) The determinant of the matrix $H_0 \equiv$

$$\begin{bmatrix} 0 & A_0 \\ A_0' & B_0 + B_0' \end{bmatrix}$$

must be positive. (See Theorem 6.1, page 79);

(2) (a) $B(\delta^{1/2} S) + B'(\delta^{1/2} \overline{S})$ must be nonsingular for all complex S on the unit circle;

(b) $A(\delta^{1/2} S)$ must have rank one on the unit circle;

(c) The determinant of the bordered Hermitian matrix $H(s) \equiv$

$$\begin{bmatrix} 0 & A(\delta^{1/2}\overline{S}) \\ A'(\delta^{1/2}S) & B(\delta^{1/2}S)+B'(\delta^{1/2}\overline{S}) \end{bmatrix}$$

is positive on the unit circle. (See Theorem 5.3 on page 76; note that Telser and Graves mistakenly give sufficient conditions for a constrained minimum.)

After some algebra, condition (1) reduces to

$$2(\delta+\mu_1)/\pi_1 > (1-b\gamma^2)/(1+b\gamma^2)$$

which must always be satisfied if

$$\delta + \mu_1 > \pi_1/2.$$

Turning to condition (2a), note that $\det (B(\delta^{1/2}e^{i\theta}) + B'(\delta^{1/2}e^{-i\theta})) = (\alpha(b^{1/2}e^{i\theta}))^2$. It is relatively easy to show that $\alpha(b^{1/2}S)$ is real and positive on the unit circle, hence (2a) is satisfied. Similarly, $A(\delta^{1/2}S)$ can never vanish for S on the unit circle, hence (2b) is satisfied.

To investigate condition (2c), first note that

$$\det H(S) = \left|\alpha(b^{1/2}S)\right|^2 \left(2\alpha(b^{1/2}S) + \beta_1(b^{1/2}S)\right)$$

which implies that

$$\operatorname{sgn} (\det H(S)) = \operatorname{sgn} \left(2\alpha(b^{1/2}S) + \beta_1(b^{1/2})\right).$$

Now, for $S = e^{i\theta}$, $2\alpha(b^{1/2}S) + \beta_1(b^{1/2}S) = 2(\mu_1+\delta)\ (1+b\gamma^2-2b^{1/2}\gamma \cos\theta)$

$$- \pi_1(1-2b^{1/2}\gamma \cos\theta),$$

which has minimum at $\theta = 0$ or $S = 1$, assuming $\mu_1 + \delta > \pi_1/2$. Now,

$$\left[2\alpha(b^{1/2}S) + \beta_1(b^{1/2}S)\right]\Big|_{S=1} =$$

$$2(\mu_1+\delta)(1-b^{1/2}\gamma)^2 - 2\pi_1(1-b^{1/2}\gamma)$$

which is positive so long as

$$(*) \qquad (\mu_1+\delta)(1-b^{1/2}\gamma) > \pi_1.$$

Hence, (2c) will be satisfied when condition (*) is met. Since the previous condition $\mu_1 + \delta > \pi_1/2$ is implied by (*), sufficient conditions for a maximum will hold as long as (*) is satisfied.

The sufficiency conditions suggested by Telser and Graves could be applied in a similar fashion to other models. The reader is referred to Telser and Graves for the details.