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TRADE USING ASSETS WHICH ARE
DIVISIBLE AT A COST

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ABSTRACT _____

The consequences of costly divisibility of assets are studied using a model with the following features. The demand for assets is generated from an overlapping generations model with a continuum of agents in each generation and with intra-generation trade (intermediation) ruled out. There is a once-for-all supply of a stock of nonnegative-dividend assets in a large size, and there is a costly technology for dividing them into smaller sizes. Stationary equilibria are shown to exist. In contrast with similar models with costless divisibility of assets, competitive equilibria are not necessarily desirable; there can be Pareto-ordered equilibria.

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There is scarcely a Chinaman [one
European said (between 1733 and 1735)] how-
ever wretched he may be, who does not carry
scissors and a precision scale around with
him. The former is used to cut gold and
silver . . .; the latter . . . is used to
weigh the materials . . . (Cited in Braudel,
The Structures of Everyday Life, Vol. 1, p.
454).

Divisibility has often been noted to be one of the
properties of assets that importantly enhance their tradability.
Consistent with that notion, most financial intermediaries seem to
produce divisibility and some--money market mutual funds, for
example--seem to produce nothing but divisibility (see Klein
[1973]). Also consistent with that notion, some economies at some
times seem to have suffered significantly from shortages of divis-
ible or low value assets (see, for example, Hanson [1979] and
Timberlake [1978], Chapter 9). However, despite the conjectured
importance of divisibility of assets, we know of no general equi-
librium study of the provision of costly divisibility of assets.
Thus, no one has systematically studied whether private provision
of divisibility for assets differs significantly from such provi-
sion for other things, or whether private provision of divisibil-
ity for money-like assets differs significantly from such provi-
sion for other assets.

In this paper, we undertake an analysis that takes a
step toward answering such questions. To do that, we study a
model in which divisibility is scarce in the following sense. The
economy is endowed with assets in a large size and with a re-

source-using technology for dividing the large size into smaller sizes. As a way of generating a demand for divisible assets, we use an overlapping generations model of two-period lived agents in which members of the same generation cannot share assets or in any way intermediate among themselves. We assume that the environment is stationary and that each generation consists of a continuum of agents.

The model we study contains, essentially, two distinct objects. One object is a single consumption good (actually one per date) which is perfectly divisible and which cannot be produced. The economy's endowment of it at each date is exogenous and can either be consumed or used in a production process to produce limited divisibility of the other object in the model. The other object is an asset. (The consumption good cannot be stored and, hence, cannot itself be an asset.) The economy is endowed, once-for-all, with some number of units of the asset of a particular size with smaller sizes producible using the following technology. At any date, a unit of the asset of any size can be divided or cut in half any number of times at a given cost per division or cut in terms of the current consumption good. Constant costs are assumed to make it easy to price assets, while the cutting-in-half technology is adopted primarily because it limits in a simple way the variety of sizes that can be produced. We study two versions of this model which differ regarding whether assets physically depreciate.

The first version is one in which assets do not depreciate. In this version, the asset with which the economy is endowed ("trees" or "land") has an exogenous and constant dividend per period consisting of some nonnegative amount of that period's consumption good. Divided units of the asset have dividends proportional to their size. If the dividend is zero, then the asset is something like a fiat money in fixed supply.

The second version is our attempt to model situations in which coins or units of paper currency wear out so that there is a replacement problem. In this version, we assume that the dividend is zero and that units of the asset wear out in a particular way. For units of a particular size, there is a probability that units held from t to t + 1 "disintegrate." We assume that a disintegrated asset cannot be traded at t + 1, but can be converted at the cost implied by our technology into a non-disintegrated asset.

All our results are for stationary equilibria. We show that such equilibria exist and describe some of their characteristics. One characteristic of stationary equilibria in the positive dividend case is that different size assets have different rates of return. Moreover, as we show, it is possible to have equilibria in which assets with different rates of return are held. A second characteristic is that, because of the indivisibility, equilibrium marginal rates of substitution do not in general equal rates of returns. As noted below, this is consistent with one interpretation of shortages of low value assets.

As regards welfare properties, one general and obvious characteristic of equilibria in our model, whether stationary or nonstationary, is that they are nonoptimal if we regard as feasible any consumption allocation that does not more than exhaust the exogenous (and divisible) endowment of the consumption good. Since we severely limit trading opportunities by requiring that all trade be inter-generational and accomplished through purchases and subsequent sales of imperfectly divisible assets, very generally equilibrium allocations fail to satisfy two necessary conditions for optimality in that class of feasible allocations: efficiency--namely, that no resources be devoted to producing asset divisibility--and intra-generation equality of marginal rates of substitution. A less obvious characteristic--and, therefore, one we describe in some detail below--is that our model has multiple stationary equilibria, some of which can be ordered by Pareto-superiority. This multiplicity result and our analysis of taxes and subsidies--on the divisibility process in the first version and on asset replacement in the second version--imply that market provision of costly divisibility of assets can give rise to suboptimal outcomes even within the class of outcomes consistent with trade taking place according to the mechanism we have described.

The remainder of the paper is organized as follows. In Section I, we describe the nondepreciating asset version and present the stationary-equilibrium existence result. In Section II, we describe features of such stationary equilibria. Section

III contains the description and analysis of the version with disintegrating assets. Section IV contains all the existence proofs, while Section V contains concluding remarks.

I. Nondepreciating Assets: Structure and Equilibrium

We study a discrete time overlapping generations model of two-period lived agents defined over integer dates t, t ⩾ 1. Each generation consists of a continuum, the unit interval, of agents. Formally, each generation is represented by the atomless measure space $([0,1], B[0,1], \lambda)$, where $B([0,1])$ is the Borel σ-field on $[0,1]$ and λ is the Lebesgue measure. At each date, two kinds of objects exist: a single, nonproducible, nonstorable, and divisible consumption good; and assets which can be costlessly stored and can at a cost be divided.

1. Assets and the technology

The economy is endowed at t = 1 with some (average) amount of assets of a uniform size, which is normalized to be unity. Each unit of the asset throws off a dividend of d units of the consumption good at each date, where d ⩾ 0.

There is a technology, an irreversible Leontief technology, available to any agent at any time for dividing, possibly repeatedly, a unit of the asset of any size into two halves. The constant cost of a division or cut is α units of the current consumption good, where $\alpha > 0$. Formally, the economy's production set at any date t (of final or intermediate outputs at t) consists of all triplets (a_1, a_2, a_3) satisfying $(a_1, a_2, a_3) = (2m, -m, -m\alpha)$ for some nonnegative integer m, where a_1 is output of the asset in units of size $2^{-(n+1)}$, a_2 is output of the asset in units of size 2^{-n}, a_3 is output of the consumption good, and n is any nonnega-

tive integer. We assume that asset division gives rise to proportional dividend division so that one unit of size 2^{-n} has a dividend per period $d^n = 2^{-n}d$. Finally, in this version of the model, assets of any size are costlessly storable from one date to the next.

2. Agents

We separately describe all two-period lived agents and the agents who are in the second and last period of life at $t = 1$, generation 0. Members of generation t, $t \geqslant 1$, who are present at t and $t + 1$ have preferences, represented by utility functions, over their own consumption of time t and time $t + 1$ consumption good. They also have endowments of those goods. Members of generation 0 have preferences over time 1 consumption good only and are endowed with some of that good and with some nonnegative integer amount of the asset in size unity.

Formally, for all $t \geqslant 1$, generation t is described by a measurable mapping $G: [0,1] \to C(R_+^2) \times R_+^2$, where $C(R_+^2)$ is the space of continuous functions on R_+^2. (R^n is n dimensional Euclidean space, R_+^n is the nonnegative orthant, and R_{++}^n is the positive orthant.) Given $i \in [0,1]$, $G(i) = \{u_i(\cdot), w_i\}$, where $u_i: R_+^2 \to R$ is the utility function of i, $w_i = (w_i^1, w_i^2)$, and w_i^k is i's endowment of the consumption good in the kth period of life or at age k. It is assumed that mean endowments are finite; i.e., $\int_{[0,1]} w_i^k d\lambda(i) < +\infty$ for $k = 1, 2$. (From now on, when the context is clear, we write $\int_{[0,1]} f(i)d\lambda(i)$ as $\int f(i)d\lambda$.) Generation 0 is described by a measurable mapping $G_0: [0,1] \to C(R_+) \times R_+ \times Z_+$; i.e., $G_0(i) =$

$(u_{0i}(\cdot), w_{0i}^2, z_{0i}^0)$, where $u_{0i}: R_+ \to R$ is i's utility function which is assumed to be strictly increasing, w_{0i}^2 is i's endowment of date 1 consumption good, and z_{0i}^0 is i's endowment of assets of size unity. (Z_+ denotes the set of nonnegative integers.) Again, we assume finiteness; for mean endowments, $\int w_{0i}^2 d\lambda < +\infty$ and for the mean exogenous supply of the undivided asset, $\int z_{0i}^0 d\lambda \equiv \bar{x}^0 < +\infty$.

Later we will impose additional assumptions on G and $\bar{x}^0$. Note that we have imposed stationarity on the environment by supposing that the technology is unchanging and that the mapping G describes all generations t for t $\geqslant$ 1. Note also that only members of generation 0 are endowed with assets. We now turn to describing competitive equilibrium.

3. Prices and the consequences of profit maximization

We let the current consumption good serve as a numeraire and denote by p_t^n the period t price of one unit of the asset of size 2^{-n}. We also let $p_t = (p_t^0, p_t^1, \ldots)$ and $P = (p_1, p_2, \ldots)$.

Given our constant-returns-to-scale technology, the condition that profits not be positive, a necessary condition for competitive equilibrium, is that for all n and t, $p_t^n \leqslant 2^{-1}(p_t^{n-1} + \alpha)$. Moreover, if size 2^{-n} assets are produced at t, then $p_t^n = 2^{-1}(p_t^{n-1} + \alpha)$. Also, if at t, assets of 2^{-n} are produced starting with units of size one, then $p_t^r = 2^{-1}(p_t^{r-1} + \alpha)$ for r = 1, 2, ..., n so that $p_t^n = 2^{-n} p_t^0 + (1-2^{-n})\alpha$.

4. Competitive utility maximization

The budget set we specify for an agent i of generation t, t ⩾ 1 allows the agent to depart from his or her endowment only by purchasing and subsequently selling nonnegative integer amounts of the (outside) asset in the sizes that can be produced. In particular, agent i cannot borrow and cannot (therefore) jointly own or share assets with other agents. We also build perfect foresight into the budget set by having agent i at t face actual asset prices at t and at t + 1. Finally, we do not include production possibilities directly, because for prices satisfying the nonpositive profit condition, production possibilities are redundant given trading opportunities. Thus, given a system of prices P satisfying $p_t^n \leqslant 2^{-1}(p_t^{n-1}+\alpha)$, we write the budget set of agent i of generation t, t ⩾ 1, as

$$\beta(p_t, p_{t+1}; w_i) = \left\{ c \in R_+^2: \text{ there is } z \in Z_+^\infty \text{ such that} \right.$$

$$\left. c^1 \leqslant w_i^1 - \sum_{n=0}^\infty p_t^n z^n, \quad c^2 \leqslant w_i^2 + \sum_{n=0}^\infty (p_{t+1}^n + d^n) z^n \right\}$$

where Z_+^∞ is the set of sequences $(z^0, z^1, \ldots)$ with $z^n \in Z_+$. The vector z is the portfolio, consisting of integer amounts of assets of producible sizes, that supports the consumption bundle c.

In order that demands satisfy the continuity properties we need for establishing existence of an equilibrium, we define consumption demand of an agent i in generation t, t ⩾ 1, over the closure of $\beta(p_t, p_{t+1}; w_i)$, denoted $c\ell\beta(\ )$. That is, this demand is defined by $\phi_{ti}(P) = \text{argmax } \{u_i(c): c \in c\ell\beta(p_t, p_{t+1}; w_i)\}$. Note that the closure of $\beta(\ )$ contains the limit as n → ∞ of the con-

sumption supported by portfolios which contain assets of size 2^{-n}. For example, for stationary prices for which $p^n \to \alpha$ as $n \to \infty$, the closure of $\beta(\ )$ contains $(w_i^1 - \alpha m, w_i^2 + \alpha m)$, which is the limit as $n \to \infty$ of the consumption supported by the portfolio $z^n = m$, $z^k = 0$, $k \neq n$.

Having defined consumption demand over the closure of $\beta(\ )$, we cannot define the demand for assets in the obvious way as the set of portfolios that support a given consumption demand. Instead, we proceed as follows. Given $c_{ti} \in \phi_{ti}(P)$, we define the corresponding demand for assets by $\psi(c_{ti}; P, w_i) = \{z \in Z_+^\infty$: there exist sequences $\{c(q)\}$ and $\{z(q)\}$ with $c(q) \in R_+^2$ and $z(q) \in Z_+^\infty$ such that $c(q) \to c_{ti}$, $z(q) \to z$ and, for all q, $c^1(q) \leq w_i^1 - \sum_{n=0}^\infty p_t^n z^n(q)$, and $c^2(q) \leq w_i^2 + \sum_{n=0}^\infty (p_{t+1}^n + d^n) z^n(q)\}$, where $z(q) \to z$ means that $z(q)$ converges to z in the product topology (i.e., $z(q)$ converges to z pointwise (see Robertson and Robertson [1973])). Note that if $c_{ti} \in \beta(\ )$, then this definition picks out exactly those portfolios that support c_{ti} in the ordinary sense. For c_{ti} in the closure of $\beta(\ )$ but not in $\beta(\ )$, this definition does not require that the corresponding portfolio, call it z, support c_{ti} in the sense that $c_{ti}^2 \leq w_{ti}^2 + \sum_{n=0}^\infty (p_{t+1}^n + d^n) z^n$. Thus, for example, at stationary prices this definition assigns to $c_{ti} = (w_i^1 - \alpha m, w_i^2 + \alpha m)$, the portfolio $z^n = 0$ for all n even though $w_i^2 + \alpha m > w_i^2$ for $m > 0$. (Note that $z^n = 0$ for all n is the limit in the product topology as $j \to \infty$ of $z^j = m$, $z^k = 0$, $k \neq j$.) We denote elements of ψ by z_{ti}.

The budget set of agent i of generation 0 is

$$\beta_0(p_1^0; w_{0i}, z_{0i}^0) = \{c \in R_+ : c \leq w_{0i}^2 + p_1^0 z_{0i}^0\} \text{ and } \phi_{0i}(P) = \text{argmax}$$

$\{u_{0i}(c) : c \in \beta_0(p_1^0; w_{0i}, z_{0i}^0)\}$ is the agent's demand. It follows

from monotonicity of u_{0i} that $\phi_{0i}(P) = w_{0i}^2 + p_1^0 z_{0i}^0$ and that agent

i supplies the initial asset endowment, z_{0i}^0, at any positive

price p_1^0, facts we use below.

5. Definition of competitive equilibrium

A sequence of integrable mappings $\{c_0^2, (c_t^1, c_t^2, z_t)_1^\infty\}$ where

$c_t^k : [0,1] \to R_+$ (the mapping describing age k consumption of gener-

ation t) and $z_t : [0,1] \to Z_+^\infty$ (the mapping describing the assets

held by generation t from t to t + 1, the integrability of which

is defined in the proof of Proposition 1), and a sequence of

prices P is a <u>competitive equilibrium</u> if: (a) for all $t \geq 1$,

$p_t^0 > 0$ and $p_t^n \leq 2^{-1}(p_t^{n-1} + \alpha)$ and the latter with equality if

$\int (z_t^n - z_{t-1}^n) d\lambda > 0$; (b) for all $t \geq 1$ and almost all $i \in [0,1]$,

$(c_{ti}^1, c_{ti}^2) \in \phi_{ti}(P)$ and $z_{ti} \in \psi(c_{ti}; P, w_i)$, and $c_{0i}^2 \in \phi_{0i}(P)$; (c)

for all $t \geq 1$

(i) $\qquad \int (c_t^2 - w_t^2) d\lambda + \int (c_{t+1}^1 - w_{t+1}^1) d\lambda +$

$\qquad \qquad \alpha \sum_{n=1}^\infty (\sum_{r=1}^n 2^{-r}) \int (z_{t+1}^n - z_t^n) d\lambda - \bar{x}^0 d = 0$

and

(ii) $\qquad \sum_{n=0}^\infty 2^{-n} \int z_t^n d\lambda = \bar{x}^0;$

and (d) the sequence $\{\{\int z_t^n d\lambda\}_{n=0}^\infty\}_{t=1}^\infty$ satisfies the irreversibility

of the production process.

A competitive equilibrium is <u>stationary</u> if for all $t >$ 1, $(c_t^1, c_t^2, z_t) = (c^1, c^2, z)$ a.s. and $p_t = p$. Alternatively, the collection of integrable mappings $\{c_0^2, c^1, c^2, z\}$ and the sequence $p = (p^0, p^1, p^2, \ldots)$ is a <u>stationary equilibrium</u> if: (a') $p^0 > 0$ and $p^n = 2^{-1}(p^{n-1} + \alpha)$; (b') $(c_i^1, c_i^2) \epsilon$ argmax $\{u_i(c): c \epsilon c\ell\beta(p, p, w_i)\}$, $z_i \epsilon \psi(c_i; p, w_i)$, and $c_{0i}^2 \epsilon$ argmax $\{u_{0i}(c): c \epsilon \beta_0(p; w_{0i}, z_{0i}^0)\}$ a.s.; and (c')

(i') $\quad \int(c_0^2 - w_0^2) d\lambda + \int(c^1 - w^1) d\lambda + \alpha \sum_{n=1}^{\infty} (\sum_{r=1}^{n} 2^{-r}) \int z^n d\lambda - \bar{x}^0 d = 0$

(ii') $\quad \int(c^2 - w^2) d\lambda + \int(c^1 - w^1) d\lambda - \bar{x}^0 d = 0$

(iii') $\quad \sum_{n=0}^{\infty} 2^{-n} \int z^n d\lambda = \bar{x}^0$.

Notice that (i) and (ii) and (i')-(iii') are market-clearing conditions for the consumption good and for the asset. The third term on the left side in (i) and (i') is the input of the consumption good into the division process. As implied by (i') and (ii'), in a stationary equilibrium all asset division occurs at $t = 1$. This, in turn, guarantees satisfaction of the irreversibility of the production process.

6. Existence of stationary equilibria

We establish the existence of a stationary equilibrium under the following additional assumptions.

A.1 (Monotonicity and continuity.) For almost all $i \epsilon [0,1]$, u_i is strictly monotone and is continuous.

A.2 (Desirability of consumption when young.) For almost all $i \in [0,1]$, there is no $c^2 \in R_+$ such that $u_i(0,c^2) > u_i(w_i)$.

A.3 (Ability to save.) For almost all $i \in [0,1]$, $\alpha < w_i^1$.

A.4 (Desire to save.) For almost all $i \in [0,1]$, $u_i(w_i^1-\alpha, w_i^2+\alpha) > u_i(w_i)$.

A.5 (Nontriviality.) $\overline{x}^0 \equiv \int z_{0i}^0 d\lambda \in (0,1]$.

Proposition 1. For an economy $\{G_0,G,\alpha,d\}$ satisfying $d \geqslant 0$, $\alpha > 0$, and A.1-A.5, there exists a stationary equilibrium with $p \in [\alpha,\overline{w}^1/\overline{x}^0]$, where $\overline{w}^1 = \int w_i^1 d\lambda$. (See Section IV for a proof.)

Although assumptions A.3-A.5 are fairly special, their role is straightforward. They guarantee that assets are divided in most cases. If the nontriviality assumption, A.5, is strengthened so that $\overline{x}^0 \in (0,1)$, then in equilibrium there is division of assets.

While the Section IV proof of Proposition 1 is standard, two special features of our model play a crucial role in the argument. First, our choice of technology allows us to parameterize the stationary prices of a portfolio by a scalar, rather than by an infinite dimensional vector. Second, even though there is an infinite (countable) number of available assets, for the purpose of equating demand and supply, only the total amount demanded by an individual (the derived or implied demand for units of size unity) matters, not its composition. (This follows from (iii') with the order of summation and integration reversed.) We use

this fact and our continuum-of-agents hypothesis to obtain the
necessary convexification effect; that is, to eliminate in the
aggregate the possible nonconvexity of individual demand implied
by the fact that individuals choose integer amounts of assets.

II. Nondepreciating Assets: Features of Stationary Equilibria

In this section, we describe some of the characteristics of stationary equilibria of the model described in the last section. Since some of the discussion involves the construction of examples which depend heavily on the features of rates of return and budget sets implied by stationary prices, we review those features first.

1. Stationary rates of return and budget sets

As noted above, purchase of an asset of size 2^{-n} at the (ex-dividend) price p^n entitles the purchaser to a dividend or coupon at the next date in the amount $(2^{-n})d$ and to the proceeds from selling the asset at the next date, which assuming stationary prices, is p^n. At stationary prices consistent with positive production of assets of size 2^{-n}, $p^n = (2^{-n})p^0 + (1-2^{-n})\alpha$. (Note that, for a given p^0, $p^n \to \alpha$ as $n \to \infty$.) It follows that the one period interest rate on an asset of size 2^{-n} is $d/[p^0+(2^n-1)\alpha]$. Thus, for a given $p^0 \geqslant \alpha$, the interest rate is decreasing in n and approaches zero as $n \to \infty$. [1/] (In what follows, we again let p denote p^0.)

For an individual i faced with stationary prices, there is a simple characterization of consumption bundles that are attainable and that satisfy the budget constraints without slack. It is easily shown that any consumption bundle that is supported (without slack) by $\{z_i^n\}$ with $\sum_0^\infty z_i^n = m$ is a weighted average of $(w_i^1-mp, w_i^2+m(p+d))$ and $(w_i^1-m\alpha, w_i^2+m\alpha)$--the former being

the bundle implied by the portfolio $\{z_i^n\}$ = $(m,0,0,\dots)$ at the price p, the latter being the limit as $k \to \infty$ of the bundle implied by $z_i^k = m$, $z_i^n = 0$ for $n \neq k$. In particular, at any price p, the consumption bundle implied by $\{z_i^n\}$ with $\sum_0^\infty z_i^n = m$ is $\mu(w_i^1 - mp, w_i^2 + m(p+d)) + (1-\mu)(w_i^1 - m\alpha, w_i^2 + m\alpha)$ where $\mu = \left(\sum_0^\infty 2^{-n} z_i^n\right)/m \; \varepsilon$ $(0,1]$. [2] Figure 1 shows an example of the closure of the budget set. It is drawn to scale for $p = (3/2)\alpha$, $d = \alpha = 1$, $(w_i^1, w_i^2) = (4,1)$. For these parameters, only portfolios with $m = 1$ and $m = 2$ are affordable.

2. An example in which different return assets are held

We now use the above characterization of budget sets and the particular example displayed in Figure 1 to show that it is possible to have an equilibrium in which different size assets (with different rates of return) are held.

The example is of an economy with identical agents in which each agent ends up holding the portfolio $\{z_i^n\} = (0,1,1,0,0,\dots)$. (Here and below the term "agents," when not qualified, refers to members of all generations other than generation 0.) We will work backwards to the specification of an economy that has such an equilibrium.

First, since $z_i^n = z^n$ for all i implies $\int z_i^n d\lambda = z^n$, market clearing requires $\sum 2^{-n} z^n = \overline{x}^0$ or $\overline{x}^0 = 3/4$.

Second, since the corresponding proposed consumption bundle is point B in Figure 1, we need to have preferences that imply that this point is preferred to all other points in the budget set. It is obvious from Figure 1 that there exist well-behaved indifference curve maps that make B most preferred. [3]

This example and equilibria, in general, are consistent with shortages of low value assets in the following sense. Very generally, equilibrium marginal rates of substitution of most two-period lived agents do not equal the returns on any of the assets that are traded in equilibrium. Since such agents could achieve preferred consumption streams if there existed lower value assets with the same returns as existing assets, the nonexistence of such assets can be regarded as a shortage.

3. Multiple equilibria ordered by Pareto superiority

We have such multiplicity results both for $d > 0$ and $d = 0$. For $d > 0$, we produce particular examples with multiple equilibria ordered by Pareto superiority. For $d = 0$, a more general argument can be given. Below we show that if $d = 0$, then there is always an equilibrium with $p = \alpha$ and that any $p > \alpha$ equilibrium is Pareto superior to the $p = \alpha$ equilibrium. We also show, in part by appealing to the proof of Proposition 1, that many, if not most, $d = 0$ economies have an equilibrium with $p > \alpha$.

(a) Multiple equilibria ordered by Pareto superiority: $d > 0$.

For $d = \alpha = 1$ and $(w^1, w^2) = (4,1)$, Figure 2a shows some of the attainable consumptions at two prices: $p = (3/2)\alpha$ and $p' = p + \alpha$. This relationship between p and p' implies that the portfolio $z = (1,0,0,\ldots)$ at the price p' supports the same consumption as does the portfolio $z = (0,2,0,0,\ldots)$ at the price p, point B in Figure 2a. If all agents are identical, if B is preferred under both prices, and if $\overline{x}^0 = 1$, then we have multiple equilibria

ordered by Pareto superiority. (All agents other than members of generation 0 are indifferent between the equilibria; generation 0 does better under p' because they receive more for what they own.)

There exist preferences so that B is preferred under both p and p'. For example, an indifference curve map with an indifference curve that is tangent at B to the line connecting B_1 and B and that is above point A_1 gives that result (see Figure 2a). Note that point C is the consumption implied by z = $(0,1,0,0,\ldots)$ at p' and that no consumption on the line segment connecting B and C is attainable at p'.

An economy of the sort just described generates a mean demand for units of size unity like that shown in Figure 2b. Thus, it has other equilibria, ones with prices in a neighborhood of p' and with an equilibrium portfolio for each person consisting of one unit of undivided asset. The equilibrium consumptions for these appear in Figure 2a on a line segment of slope -1 passing through the equilibrium consumption for the price p'. If an indifference curve is tangent at the p' consumption to a line with slope less than -1, it follows that there are equilibria at prices near to and lower than p' which give equilibria which all two period lived agents strictly prefer to the price p' equilibrium and, hence, to the price p equilibrium. Since prices in the neighborhood of p' exceed p, members of generation 0 also do better under such prices than under p.

Although the above example is very simple, it is not, unfortunately, robust to perturbations in the underlying space of

characteristics. In particular, as is clear from Figure 2b, any perturbation of $\bar{x}^0$ from unity eliminates the multiplicity. We next describe a robust example of Pareto ordered equilibria-- robust in the sense that such multiplicity occurs in an open set in the space of characteristics.[4/]

Figure 3a, which is drawn to scale for $(w^1, w^2) = (5,1)$, $\alpha = 3/2$, and $d = 1/2$, shows the consumptions supported by several portfolios at several prices. The points labelled $a(p)$, $b(p)$ and $c(p)$ are the consumptions supported at the price p by the port- folios $z = (1,0,0, \ldots)$, $z = (0,1,0,0, \ldots)$ and $z = (0,1,1,0,0, \ldots)$, respectively. As we now explain, for these parameters there exists a distribution of preferences that generates mean demand for units of size unity like that shown in Figure 3b. This, in turn, implies that Pareto-ordered equilibria exist for any $\bar{x}^0$ in an interval and for perturbations of all other characteristics.

Suppose the distribution of preferences is such that at $p = 2$, the consumption supported by $z = (0,1,1,0,0, \ldots)$--namely $c(2)$--is the unique preferred consumption for everyone. One such indifference curve is shown in Figure 3a and, since $c(2)$ is a corner, there can be a nondegenerate distribution of preferences satisfying that condition. Now consider prices slightly higher than $p = 2$. As p increases from 2, the portfolio $z = (1,0,0, \ldots)$ becomes preferred to $z = (0,1,1,0,0, \ldots)$ for more and more peo- ple. This implies mean demand that increases toward unity (see Figure 3b). As the price increases further toward $p = 2 + \alpha = 7/2$, the portfolio $z = (0,1,0,0, \ldots)$ becomes preferred for more

and more people. This implies a mean demand that decreases toward 1/2 (see Figure 3b).

As shown by Figure 3a, everyone is better off at any price in a neighborhood below p = 7/2 than at any price in a neighborhood above p = 2. Moreover, since there is an equilibrium in each of these neighborhoods for any $\bar{x}^0$ in a neighborhood above 3/4, this example gives rise to Pareto-ordered equilibria. Finally, since small perturbations in the distribution of endowments and preferences and in d and α generate small perturbations in the mean demand for units of size unity, such multiplicity occurs in an open set in the space of all characteristics.

The crucial feature of this example (and of the previous one) is that α is sufficiently large relative to d. If, instead, α is sufficiently small relative to d, then utility is decreasing in p for all savers.

(b) Multiple equilibria ordered by Pareto superiority: d = 0

We begin by showing constructively that when d = 0, there is always an equilibrium with p = α.

Proposition 2. Assume A.1, A.3-A.5. If d = 0, then p = α is an equilibrium.

Proof. Let m_i^* be the nonnegative integer value of m_i that maximizes $u_i(w_i^1 - \alpha m_i, w_i^2 + \alpha m_i)$. At p = α, all $\{z_i^n\}$ such that $\sum_0^\infty z_i^n = m_i^*$ support utility maximizing consumption. Therefore, to meet the demand of the young for assets at p = α, the only requirement is that $\sum_0^\infty \int z^n d\lambda = \int m_i^* d\lambda \equiv m^*$.

Let i be the integer satisfying $2^{i-1} \leqslant m^*/\bar{x}^0 \leqslant 2^i$. Since assumptions A.3-A.5 imply $m^* \geqslant \bar{x}^0$, we have $i > 0$. With i so determined, suppose we let the average number of units of size 2^{-i+1}, $\int z^{i-1} d\lambda$, be $2^i \bar{x}^0 - m^*$, the average number of units of size 2^{-i}, $\int z^i d\lambda$, be $2m^* - 2^i \bar{x}^0$, and $\int z^n d\lambda = 0$ for $n \notin \{i-1, i\}$.

By choice of i, these are nonnegative quantities. Also, the total average number of units is m^*. The only other equilibrium requirement is $\int_0^\infty 2^{-n} \int z^n d\lambda = \bar{x}^0$. This, too, is easily seen to be satisfied. Δ

It is tempting to interpret this $p = \alpha$ equilibrium as one in which the value of fiat money is its cost of production. At $p = \alpha$, the average quantity of fiat money supplied not distinguished by size is perfectly elastic in that any average quantity larger than unity can be produced while covering costs and meeting the constraint on $\bar{x}^0$ (see the proof). Since demanders do not distinguish among sizes at $p = \alpha$ and since, by assumption, demand is at least one unit on average, it is not surprising that there is an equilibrium at $p = \alpha$.

We now show that this $p = \alpha$ equilibrium is inferior to any $p > \alpha$ equilibrium.

<u>Proposition 3</u>. Assume A.1. If $d = 0$, then any $p > \alpha$ equilibrium is Pareto superior to any $p = \alpha$ equilibrium.

<u>Proof</u>. Let $c_i(\alpha)$ denote the equilibrium consumption of person i in generation t, $t \geqslant 1$, when $p = \alpha$. Since $c_i(\alpha) \in c\ell\beta(p, w_i)$ for any p, it follows that person i weakly prefers her or his consump-

tion under the $p > \alpha$ equilibrium. Since members of generation 0 strictly prefer $p > \alpha$ to $p = \alpha$, we have Pareto superiority. Δ

Given these propositions, multiplicity of equilibria ordered by Pareto superiority occurs when $d = 0$ for any economy for which there exists a $p > \alpha$ equilibrium. (Such a situation is illustrated in Figure 4.) Our general existence proof implies that such an equilibrium exists if there exists a $p > \alpha$ at which there is excess demand--namely, $\int_0^\infty \int z^n d\lambda > \bar{x}^0$. Although assumptions A.3-A.5 do not imply that such a p exists, one exists for many economies. 5/ There might also be Pareto-ordered equilibria within the set of equilibria with $p > \alpha$. In fact, the type of multiplicities studied for $d > 0$ are more likely to occur when $d = 0$.

Thus, whether $d > 0$ or $d = 0$, versions of our model have Pareto-ordered stationary equilibria with positive valuation of an outside asset. The presence of positive divisibility costs is necessary for such occurrence in the sense that if α is set at zero, then our model--at least versions in which everyone satisfies A.3 and A.4 and hence, is a "saver"--becomes a standard stationary, pure exchange overlapping generations model. In such models, it is well known that any stationary equilibrium with a constant nonnegative interest rate, as is implied by constant and positive valuation of an outside asset, is Pareto optimal.

Note, however, that setting $\alpha = 0$ does two things in our model: it effectively eliminates indivisibilities and it eliminates from the model a production technology. In the case of

assets without dividends, the mere presence of a production tech-
nology--one having nothing to do with divisibility-- can give rise
to Pareto-ordered stationary equilibria with positive valuation of
an outside asset. For example, suppose an overlapping generations
(or other infinitely lived economy) starts with divisible green
money and a constant-returns-to-scale technology for converting it
into divisible red money (by "painting" it), where α is the cost
in current consumption good per unit of money in making the con-
version. Then, if $(p_g, p_r) = (p^*, p^*+\alpha)$ are constant values of
green and red money in an (optimal) equilibrium in which no red
money is produced and held and if $p^* > \alpha$, it follows that there is
a Pareto-inferior equilibrium with $(p_g, p_r) = (p^*-\alpha, p^*)$ in which
only red money is held. The multiplicity we find in our model
when $d = 0$ may be related to this trivial kind of multiplicity but
the multiplicity we find when $d > 0$ is not. It is easy to see
that such trivial multiplicies cannot arise if $d > 0$; if $d > 0$,
then "painted" divisible assets are not held because the lower
priced "unpainted" divisible assets have a higher rate of re-
turn. Thus, at least for the case of positive dividend assets,
the indivisibility is playing a crucial role in generating Pareto-
ordered stationary equilibria. [6/]

4. A tax on the divisibility process

Suppose a tax τ (a subsidy if $\tau < 0$) is levied on the
divisibility process so that the cost to an individual of making a
division is $(1+\tau)\alpha$ instead of α. We can preserve stationarity
under such a tax by distributing the proceeds of the tax to mem-

bers of generation 0. If we do, then we can simply reinterpret the no-tax stationary equilibrium as a tax equilibrium by letting $\alpha = (1+\tau)\alpha'$, the gross-of-tax cost of dividing, and by appropriately adjusting consumption of generation 0.

Three points can be made about such a tax scheme. First, the scheme is nonneutral in the sense that $\tau \neq 0$ generates equilibrium allocations which, in general, are different from those implied by $\tau = 0$. Second, there is no obvious sense in which the presence of such a tax is distorting. Since there is no straightforward relationship between (the gross of tax) α and marginal rates of substitution in equilibrium, the usual "wedge-type" arguments for taxes on production do not apply to our model. Third, if one were to posit a social welfare function as a weighted average of utilities of the members of generation 0 and of the utilities of all two-period lived agents, then there would generally be some equilibrium for some $\tau \neq 0$ that implied a higher value for this function than is attained under any equilibrium with $\tau = 0$. This is an implication of the nonneutrality of the tax.

These implications of a tax or subsidy on the divisibility process, like the existence of Pareto-ordered equilibria, suggest that laissez-faire provision of costly divisibility of assets is not necessarily desirable.

III. Disintegrating Assets

Here we maintain the model of Section I except that we assume that assets depreciate in a particular way. We also assume that there are no dividends (d = 0), so that we are focussing on the consequences of currency wearing out. We use this model to compare what happens if agents face directly the implied replacement costs, a regime we label laissez-faire, with what happens if replacement is subsidized, financed by taxation. Historically, both kinds of situations have been experienced. A version of laissez-faire reigned during the nineteenth century, at least in the United States and in England, when holders of gold coins bore directly the consequences of wear of such coins. 7/ The subsidy situation is the common one in place today; in most countries, sufficiently worn units of paper currency are replaced by new units at government expense. 8/

We assume that depreciation of assets occurs as follows. A unit of an asset of size 2^{-n} held from t to t + 1 has a probability of "disintegrating" at the beginning of or just prior to t + 1. At t + 1, a disintegrated unit continues to be the fraction 2^{-n} of an undivided unit except that it takes a form that cannot be stored. It can only be combined with other disintegrated units to produce assets of size unity, which can then be divided into storable units of various sizes using the same costly division technology assumed in Section I. The combining of disintegrated units into units of size unity is assumed to be costless.

One can interpret the above assumptions in terms of gold coins. For a gold coin of size 2^{-n}, there is a probability that the coin turns into powdered gold (disintegrates) which cannot be used as a coin (stored). However, the powder can costlessly be combined with other powder to produce units of size unity. Alternatively, a disintegrated coin can only be "melted down" in combination with other such coins to produce units of size unity.

This way of modeling the depreciation process is simple in several respects. First, the process does not itself produce additional sizes of assets. Second, it is consistent under laissez-faire with the existence of stationary equilibria (in which, however, consumption in the second period of life is in general a random variable). Third, under some additional assumptions, it is consistent with an absence of aggregate risk. This, in turn, makes it relatively easy to describe the consequences of tax schemes that finance replacement of disintegrated units. [9]

We study two tax-scheme alternatives to laissez-faire. [10] The first scheme has lump-sum taxes payable in the second period of life which finance all replacement. The taxes are lump-sum in that they are not viewed by agents as dependent on their portfolio decisions. Since the tax finances all replacement, taxes aside, the situation facing agents is the same as that when assets do not disintegrate. The second scheme has insurance taxes. Under it, an agent pays a tax in the second period of life equal to the expected replacement cost of the agent's portfolio. (This scheme can also be thought of as the result of the operation

of a private insurance market.) We are interested primarily in comparing laissez-faire with lump-sum taxation, the latter being the system that resembles current policy in most countries.

1. The model and stationary equilibrium

The model is identical to that described in Section I except that here a unit of the asset of size 2^{-n} held from t to t + 1 disintegrates with constant probability θ_n. Thus, if z_{ti}^n denotes the integer number of units of assets of size 2^{-n} held by agent i from t to t + 1 and y_{ti}^n denotes the number of these that disintegrate, then y_{ti}^n is a binomial random variable with parameters z_{ti}^n and θ_n.

In this section, we define demands only for stationary prices that satisfy $p^n = 2^{-n}p + (1-2^{-n})\alpha$. Moreover, we simplify the description of the decision problem by assuming, as implied by strict monotonicity of preferences, that budget constraints are satisfied as equalities.

We begin by introducing some notation which allows us to describe the distribution of disintegration losses implied by a portfolio. Let $M(R_+)$ denote the space of probability measures on $(R_+, B(R_+))$ where $B(R_+)$ is the Borel sigma field on R_+. Then we define $\pi(\cdot; p, w, z) \in M(R_+)$ by $\pi(x; p, w, z) = \text{prob} \{x = w^2 + \Lambda_p z - \alpha(I-\Lambda)y: y^n$ has the binomial distribution with parameters (z^n, θ_n) for all $n \geqslant 0\}$, where $I = (1,1,\ldots)$, $\Lambda = (2^{-0}, 2^{-1}, 2^{-2}, \ldots)$ and $\Lambda_p = p\Lambda + \alpha(I-\Lambda)$, p being the price of a unit of size unity. Note that $\pi(\cdot; p, w, z)$ is the distribution of second-period income (consumption under monotonicity) under laissez-faire.

Given a measure $\pi \in M(R_+)$ with countable support and a utility function $u: R_+^2 \to R$, we can define the following utility functions on $R_+ \times M(R_+)$: $v(c,\pi) = \sum_x \pi(x)u(c,x)$ and $\bar{v}(c,\pi) = u(c, \sum_x \pi(x)x)$, the summations being over the support of π. The function v is expected utility under laissez-faire while the function $\bar{v}$ is utility under insurance taxation. Note that for given (p,w,z), second-period consumption under insurance taxation is simply the expected value of second-period consumption under laissez-faire. We also define the budget constraint $\hat{\beta}(p,w) = \{(c,\pi) \in R_+ \times M(R_+): \text{ there exists } z \in Z_+^\infty \text{ such that } c = w^1 - \Lambda_p z \text{ and } \pi = \pi(\cdot;p,w,z)\}$.

Under <u>laissez-faire</u>, the demand for consumption of agent i of generation t, $t \geq 1$ (facing stationary prices satisfying $p^n = 2^{-n}p + (1-2^{-n})\alpha$) is $\tilde{\phi}_i(p) = \text{argmax } \{v_i(c,\pi): (c,\pi) \in c\ell\hat{\beta}(p,w_i)\}$, where the closure of $\hat{\beta}(\cdot)$ is taken with respect to the product topology induced by the Euclidean topology and the topology of convergence in measure. Similarly, under insurance taxation, the demand of agent i is given by $\bar{\phi}_i(p) = \text{argmax } \{\bar{v}_i(c,\pi): (c,\pi) \in c\ell\hat{\beta}(p,w_i)\}$.

The set of portfolios supporting a choice $(c,\pi) \in c\ell\hat{\beta}(p,w_i)$ is $\psi((c,\pi);p,w_i) = \{z \in Z_+^\infty: \text{ there exist sequences } \{c(q)\} \text{ and } \{z(q)\} \text{ such that, for all } q, c(q) \in R_+, z(q) \in Z_+^\infty, c(q) = w_i^1 - \Lambda_p z(q), \pi(\ ;p,w_i,z(q)) \to \pi(\ ), c(q) \to c, \text{ and } z(q) \to z\}$, where convergence for π is convergence in measure. Therefore, asset demand is given by $\tilde{\psi}_i(p) = \{\psi((c,\pi);p,w_1): (c,\pi) \in \tilde{\phi}_i(p)\}$ and $\bar{\psi}_i(p) = \{\psi((c,\pi);p,w_i): (c,\pi) \in \bar{\phi}_i(p)\}$ under laissez-faire and insurance taxation, respectively.

With <u>lump-sum</u> taxation, demands differ from those of the nondepreciating asset model only in that they depend on nonrandom lump-sum taxes, τ, payable in the second period of life. Thus, the budget constraint of agent i of generation t, t $\geqslant$ 1, is $\beta(p,\tau,w_i) = \{c \in R_+^2:$ there exists $z \in Z_+^\infty$ such that $c^1 = w_i^1 - \Lambda_p z$ and $c^2 = w_i^2 + \Lambda_p z - \tau\}$. The demand for consumption is then $\phi_i(p,\tau) = \text{argmax} \{u_i(c): c \in c\ell\beta(p,\tau,w_i)\}$ and the demand for assets is $\psi_i(p,\tau) = \{\psi(c;p,\tau,w_i): c \in \phi_i(p,\tau)\}$, where $\psi(c;p,\tau,w_i)$ = $\{z \in Z_+^\infty:$ there exist sequences $\{c(q)\}$ and $\{z(q)\}$ such that for all q $c(q) \in R_+^2$, $z(q) \in Z_+^\infty$, $c^1(q) = w_i^1 - \Lambda_p z(q)$, $c^2(q) = w_i^2 + \Lambda_p z(q) - \tau$, $c(q) \rightarrow c$, and $z(q) \rightarrow z\}$.

All of the above pertain to agents of generation t, t $\geqslant$ 1, two-period lived agents. Agents of generation 0 behave exactly as in the no-depreciation model; they supply all their assets at any positive price and consume the proceeds and their endowment.

Therefore, a collection $(c_0^2,(c^1,\pi),z,p)$ --where (c_0^2,c^1,z) are integrable maps; for all i, π_i is a probability measure, and p is a scalar--is a <u>stationary equilibrium under laissez-faire</u> if: (a) p > 0; (b1) $c_{0i}^2 = w_i^2 + pz_i^0$ a.s.; (b2) $(c_i,\pi_i) \in \tilde{\phi}_i(p)$ and $z_i \in \tilde{\psi}_i((c_i^1,\pi_i);p,w_i)$ a.s.; (c1) $\int(c_0^2-w_0^2)d\lambda + \int(c^1-w^1)d\lambda + \int\alpha(I-\Lambda)zd\lambda = 0$; (c2) $\int(c^2-w^2)d\lambda + \int(c^1-w^1)d\lambda + \int\alpha(I-\Lambda)yd\lambda = 0$; where c_i^2 and y_i are the random variables induced by z_i; and (c3) $\int\Lambda zd\lambda = \bar{x}^0$.

Similarly, a collection $(c_0^2,(c^1,\pi),z,p)$, with the above properties, is a <u>stationary equilibrium under insurance taxation</u> if it satisfies (a), (b1), (b2') $(c_i^1,\pi_i) \in \bar{\phi}_i(p)$ and $z_i \in \bar{\psi}_i((c_i^1,\pi_i);p,w_i)$ a.s., and (c1)-(c3).

Finally, a collection $(c_0^2,(c^1,c^2),z,\tau,p)$, where $(c_0^2,(c^1,c^2),z,\tau)$ are integrable maps and p is a scalar, is a <u>stationary equilibrium under lump-sum taxation</u> if (a), (b1), (b2") $(c_i^1,c_i^2) \in \phi_i(p,\tau_i)$ and $z \in \psi_i((c_i^1,c_i^2);p,\tau_i,w_i)$ a.s., (c1)-(c3), and (d) $\int\alpha(I-\Lambda)yd\lambda = \int\tau d\lambda$, where y_i is the random variable induced by z_i.

Under either taxation scheme, these definitions can be satisfied only if there is no aggregate risk. (If there were aggregate risk, condition (c2) would imply random consumption, which violates the conditions underlying demands in the taxation cases.)

In order to assure no aggregate risk, the taxation schemes are studied only under the assumption of homogeneous consumers, although the results are easily extended to a model with a finite number of distinct consumer types. (The technicalities of assuming no-aggregate-risk in our model are discussed at the end of Section IV.) Under laissez-faire, aggregate risk is not troublesome because any aggregate variation in the total cost of replacing disintegrated assets is "financed" by exactly offsetting randomness in aggregate consumption in the second period of life. A formal statement of homogeneity of consumers is as follows.

A.6. (Homogeneous consumers.) $G: [0,1] \to C(R_+^2) \times R_+^2$ is constant almost surely.

We then have the following existence proposition for laissez-faire and insurance taxation.

Proposition 4.

(a) Assume A.1-A.5. Under laissez-faire, there exists a stationary equilibrium with $p \in [\alpha, \bar{w}^1/\bar{x}^0]$.

(b) Assume A.1-A.6. Under insurance taxation, there exists a stationary equilibrium with $p \in [\alpha, \bar{w}^1/\bar{x}^0]$.

(The proof is given in Section IV.)

We have two existence results under lump-sum taxation, which, however, are less general than the Proposition 4 results. The first result gives sufficient conditions for existence of a stationary equilibrium with $p > \alpha$; the second gives sufficient conditions for one with $p = \alpha$.

Both results depend on the following strengthened versions of A.2 and A.3, respectively:

A.2'. (Desirability of consumption.) For almost all $i \in [0,1]$, for all $c \in R_{++}^2$ and $c^* \in R_+$, $u_i(c^1, c^2,) > u_i(c^*, 0)$ and $u(c^1, c^2) > u_i(0, c^*)$.

A.3'. (Bounds on endowment.) There exists a positive integer m such that for almost all $i \in [0,1]$, $w_i^1 > m\alpha$ and $w_i^2 - w_i^1 + m\alpha > 0$.

The first result also assumes that the following condition holds.

<u>Condition B.</u> There exists a continuous function $\tilde{p}$: $[0,\bar{w}^1] \rightarrow (\alpha,\infty)$ such that if $x \; \varepsilon \; \int \psi(\tilde{p}(\tau),\tau)d\lambda$, then $x \geqslant \bar{x}^0$.

This condition asserts that there is a continuous lower bound on equilibrium prices. Unfortunately, it is difficult to describe other than quite special economies which satisfy it. [11]/

    The second result does not rely on Condition B, but uses as additional assumptions the following.

A.7. (Common probabilities of disintegration.) For all n, $\theta_n = \theta$.

A.8. (A version of normal goods.) For almost all i, if $u_i(c^1,c^2) \geqslant u_i(c^1+\delta,c^2-\delta)$ for some $\delta > 0$, then $u_i(c^1,c^2-\gamma) \geqslant u_i(c^1+\delta,c^2-\delta-\gamma)$ for all $\gamma > 0$.

Assumption A.8 is satisfied if u is twice differentiable and its second derivatives satisfy $u_{22} - u_{12} \leqslant 0$.

    The propositions, proofs of which are given in Section IV, are as follows.

<u>Proposition 5.</u> Assume A.1, A.2', A.3', A.6, and Condition B. Under lump-sum taxation, there exists a stationary equilibrium with $p \; \varepsilon \; (\alpha,\bar{w}^1/\bar{x}^0)$.

<u>Propostion 6.</u> Assume A.1, A.2', A.3', and A.4-A.8. Under lump-sum taxation, there exists an equilibrium with $p = \alpha$.

The proof of Proposition 5 uses a fixed point argument, while that of Proposition 6 is, in part, constructive along the lines of the proof of Proposition 2.

If the assumptions of both Propositions 5 and 6 hold, then there are multiple equilibria under lump-sum taxation, at least one with $p > \alpha$ and at least one with $p = \alpha$. In contrast to the situation in the no-depreciation version (Proposition 3), we cannot immediately conclude that a $p > \alpha$ equilibrium is Pareto superior to one with $p = \alpha$. Such a conclusion follows, according to an argument exactly like that used for Proposition 3, if taxes in the $p > \alpha$ equilibrium do not exceed taxes in the $p = \alpha$ equilibrium.

2. Asset returns and stationary equilibrium budget sets

Before we compare equilibria under different policies, we describe features of the stationary return distributions on assets of various sizes under the different policies and the related features of budget sets.

Table 1 lists the expected value of the interest rate on an asset of size 2^{-n}, $E(r^n)$, and its variance $V(r^n)$. Notice that under laissez-faire, the mean is decreasing and the variance is increasing in n if $\theta_n = \theta$ for all n. Under insurance taxation, the (mean) return is the mean return under laissez-faire, while under lump-sum taxation all returns are zero. This gives some indication that in a stationary equilibrium, the incentive to avoid small-sized assets is greatest under laissez-faire and weakest under lump-sum taxation (provided that the sequence $\{\theta_n\}$ is nondecreasing).

Figure 5 displays some features of the budget sets under the different policies. It is drawn to scale for $(w^1, w^2) = (15,3)$, $\alpha = 5$, $\theta_n = \theta = 1/4$ and $p = 7$.

Under insurance taxation, if $\theta_n = \theta$, all consumptions supportable by a portfolio with $Iz = m$ are convex combinations of the consumption implied by the portfolio $(m,0,0,\ldots)$ and the limit as $k \to \infty$ of the consumption implied by $z^k = m$ and $z^n = 0$ for $n \neq k$. The former is $(w^1 - mp, w^2 + mp)$ while the latter is $(w^1 - m\alpha, w^2 + m(1-\theta)\alpha)$. These consumptions are shown in Figure 3 for $m = 1$ and $m = 2$. As above, a convex combination of these with weights μ and $1 - \mu$, respectively, is attainable if and only if $\mu = (\Lambda z)/(Iz)$ for some $z \in Z_+^\infty$ and $(Iz) = m$. Point A in Figure 5 is the consumption implied by the portfolio $z = (0,2,0,0,\ldots)$ under insurance taxation.

Under laissez-faire, different distributions of consumption correspond to different portfolios. For portfolios with $m = 2$, these are in general 4-point distributions, but the distribution collapses for some $m = 2$ portfolios. Thus, for the portfolio $z = (2,0,0,\ldots)$, the distribution collapses to $(w^1 - 2p, w^2 + 2p)$ because disintegration of undivided assets does not imply a loss. For the portfolio $z = (0,2,0,0,\ldots)$, the distribution collapses to 3 points: a_0 (corresponding to $y^1 = 0$) with probability $(3/4)^2$, a_1 (corresponding to $y^1 = 1$) with probability $2(3/4)(1/4)$, and a_2 (corresponding to $y^1 = 2$) with probability $(1/4)^2$.

Under lump-sum taxation, the set of attainable consumption depends on the tax τ. If $\tau = 0$, then the attainable consumptions are the same as those in the nondepreciating asset model with $d = 0$. They are on the line with slope -1 passing through the endowment. If $\tau > 0$, they are similar except that the line with slope -1 passes through the after-tax endowment $(w^2, w^2 - \tau)$.

3. Laissez-faire can Pareto dominate lump-sum taxation and vice versa

    We now discuss examples that show that laissez-faire can be either Pareto superior or Pareto inferior to lump-sum taxation.

(a) Laissez-faire can be Pareto superior to lump-sum taxation

    This example is one of homogeneous consumers with $(w^1, w^2) = (15, 3)$, $\theta_n = \theta = 1/4$, $\alpha = 5$, and $\bar{x}^0 = 1$. We will show, using Figure 6, that there are preferences such that: (i) under lump-sum taxation, there is an equilibrium with $(p, \tau) = (7, 5/4)$ and $z_i = (0, 2, 0, 0, \ldots)$ for all i; and (ii) under laissez-faire, there is a Pareto superior equilibrium with $p = 7$ and $z_i = (1, 0, 0, \ldots)$ for all i. Note that both portfolios satisfy market clearing.

    Since $\tau = 5/4 = \alpha/4$ is the amount that finances replacement for a common portfolio $z = (0, 2, 0, 0, \ldots)$, it follows that the consumption implied by that portfolio and tax under lump-sum taxation is the one implied by the same portfolio under insurance taxation--point A in Figure 6. Let preferences be such that the indifference curve through point A has slope -1 at A. This guarantees that claim (i) holds.

Assume also that the indifference curve through point B is higher than that through point A and is also above all bundles implied by portfolios with m = 2 under insurance taxation. Then p = 7 and the portfolio z_i = (1,0,0,...) for all i is an equilibrium under insurance taxation. It follows that this is also an equilibrium under laissez-faire, because point B is affordable under laissez-faire and is preferred to all other affordable consumption distributions at p = 7 so long as the agent is risk averse or neutral. This last fact is true because for any affordable laissez-faire consumption distribution at a given price p, there is an affordable consumption under insurance taxation at that price which is weakly preferred. Since B is by construction preferred to A, claim (ii) is established. (Note that (p,τ) = (7,0) is not an equilibrium under laissez-faire taxation because there are affordable consumptions under (p,τ) = (7,0) that are preferred to B and because the portfolio supporting any such consumption is not consistent with market clearing. This follows because $(w^1-2\alpha, w^2+2\alpha)$ is preferred to B, which, in turn, implies that there are m = 2 portfolios that support consumption preferred to B.)

(b) Lump-sum taxation can be Pareto-superior to laissez-faire

In this example, consumers are homogeneous with (w^1, w^2) = (20,3) and $u(c^1, c^2) = (3/4)\log(c^1+1) + (1/4)\log(c^2+1) - \log((c^1-c^2)^2+1)$, $\theta_n = \theta = 1/4$ for all n, $\alpha = 4$ and $\bar{x}^0 = 3/2$. There is a lump-sum taxation equilibrium with (p,τ) = (9/2,1/2) and z_i = (1,1,0,0,...) which supports consumption (c^1,c^2) = (45/4,45/4)

with $u(45/4,45/4) = 1.088$. In particular, at $(p,\tau) = (9/2,1/2)$, the portfolio $z = (2,0,0,\ldots)$ supports $c = (11,11.5)$, while the portfolio $z = (1,0,1,0,0,\ldots)$ supports $(91/8,89/8)$, both of which yield lower utility. Notice that $p = 9/2$ and $z = (1,1,0,0,\ldots)$ is also an equilibrium under insurance taxation. In particular, $u(c) = u(45/4,45/4)$ is tangent at $(45/4,45/4)$ to the line segment containing all $m = 2$ attainable consumptions under insurance taxation.

Under laissez-faire, the portfolio $z = (1,1,0,\ldots)$ supports $c = (45/4,47/4)$ with probability $3/4$ and $c = (45/4,39/4)$ with probability $1/4$ and, hence, implies expected utility equal to $.887$. It is easily shown that this portfolio is preferred to any other portfolio with $m = 2$. In particular, $z = (2,0,0,\ldots)$ supports $c = (11,12)$ and $u(11,12) = .787$. It is also the case that no $m = 1$ or $m > 2$ portfolio is preferred under laissez-faire. In particular, $z = (1,0,0,\ldots)$ supports $c = (31/2,15/2)$ and $u(31/2,15/2) = -.667$.

An example of this sort produces the intended result in part because demands are sufficiently unresponsive that the "distortion" implied by agents not facing replacement costs under lump-sum taxation does not operate. What remains is the gain from eliminating risk.

The above examples show that any general case for either laissez-faire or lump-sum taxation must somehow cope with counter examples.

IV. Existence Proofs

1. Proof of Proposition 1

We prove Proposition 1 by applying a simple generalization of the Intermediate Value Theorem. However, this final argument is based on properties of the budget constraint and demand correspondences that we establish first in a series of lemmas.

We simplify our notation by using the following symbols: $\Lambda \equiv (1, 2^{-1}, 2^{-2}, \ldots, 2^{-n}, \ldots)$, $I \equiv (1, 1, \ldots, 1, \ldots)$, $\Lambda_p \equiv p\Lambda + \alpha(I-\Lambda)$ and $\Lambda_{(p+d)} \equiv (p+d)\Lambda + \alpha(I-\Lambda)$, where p, α, and d are real numbers. Finally, if $z = (z^0, z^1, \ldots)$ then $\Lambda z = \sum_{n=0}^{\infty} 2^{-n} z^n$ and similarly for $\Lambda_p z$ and $\Lambda_{(p+d)} z$.

We now proceed to study the properties of the budget constraint and demand correspondences for sequences of stationary prices of the form $P = (\Lambda_p, \Lambda_p, \ldots)$ (that is, stationary prices satisfying $p^{n+1} = 2^{-1}(p^n + \alpha)$ for all $n \geq 0$, where $p \equiv p^0$).

Let $\beta(p, w) \equiv \beta(\Lambda_p, \Lambda_p, w) = \{c \in R_+^2: \text{there exists } z \in Z_+^\infty$ such that $c^1 \leq w^1 - \Lambda_z p$ and $c^2 \leq w^2 + \Lambda_{(p+d)} z\}$, and let $\chi(p, w) = \{z \in Z_+^\infty: w^1 - \Lambda_p z \geq 0 \text{ and } w^2 + \Lambda_{(p+d)} z \geq 0\}$, so that $\chi(p, w)$ is the set of portfolios that support $\beta(p, w)$.

<u>Lemma 1</u>: If $p \in [\alpha, \infty)$ and $z \in \chi(p, w)$, then $Iz \leq w^1/\alpha$.

<u>Proof.</u> If $z \in \chi(p, w)$, then $w^1 \geq \Lambda_p z = [p\Lambda + \alpha(I-\Lambda)]z \geq [\alpha\Lambda + \alpha(I-\Lambda)]z = \alpha Iz$. Δ

Let $\beta_+(p,w) = \{c \ \varepsilon \ \beta(p,w): c^1 >> 0\}$, where $x >> 0$ means that there exists an $\varepsilon > 0$ such that $x > \varepsilon$. ($\beta_+(p,w)$ is the set of consumptions in the budget set with consumption at age 1 bounded away from 0.)

<u>Lemma 2</u>: The correspondence $\beta_+(\cdot,w)$ is continuous at every $p \ \varepsilon \ [\alpha,\infty)$.

<u>Proof</u>. We first show that $\beta_+(\cdot,w)$ is upper hemi-continuous. Fix $\bar{p} \ \varepsilon \ [\alpha,\infty)$. By assumption A.3, $\beta_+(\bar{p},w)$ is not empty. We show that for every open subset U of R^2 such that $\beta_+(\bar{p},w)$ is contained in U, there is a relatively open subset V of R such that for all $p \ \varepsilon \ V$, $\beta_+(p,w)$ is contained in U.

Without loss of generality, consider open sets of the form $U_\varepsilon = \{c \ \varepsilon \ R^2: |\bar{c}-c| < \varepsilon$ and $\bar{c} \ \varepsilon \ \beta_+(\bar{p},w)\}$ and let $V_\varepsilon = (\max(\alpha,\bar{p}-\varepsilon\alpha/2w^1),\bar{p}+\varepsilon\alpha/2w^1)$. Also note that $\beta_+(p,w) = \{c \ \varepsilon \ R_+^2:$ there exists $z \ \varepsilon \ Z_+^\infty$ such that $0 << c^1 \leqslant w^1 - (p\Lambda z+\alpha(I-\Lambda)z) = w^1 - (\bar{p}\Lambda z+\alpha(I-\Lambda)z) + (\bar{p}-p)\Lambda z$ and $0 \leqslant c^2 \leqslant w^2 + ((p+d)\Lambda z + \alpha(I-\Lambda)z) = w^2 + ((\bar{p}+d)\Lambda d + \alpha(I-\Lambda)z - (\bar{p}-p)\Lambda z\}$.

Now given $c \ \varepsilon \ \beta_+(p,w)$, let $\bar{c}^1 = \max \ (\varepsilon/2,c^1-(\bar{p}-p)\Lambda z)$ and $\bar{c}^2 = \max \ (0,c^2+(\bar{p}-p)\Lambda z)$. Then $\bar{c} = (\bar{c}^1,\bar{c}^2) \ \varepsilon \ \beta_+(\bar{p},w)$ and $|c-\bar{c}| \leqslant \max \ \{\varepsilon/2,|\bar{p}-p|\Lambda z\}$. But $|\bar{p}-p|\Lambda z \leqslant |\bar{p}-p|w^1/\alpha$ by lemma 1. Therefore, if $p \ \varepsilon \ V_\varepsilon$, then $|\bar{p}-p|\Lambda z \leqslant \varepsilon/2$. Thus, if $p \ \varepsilon \ V_\varepsilon$ and $c \ \varepsilon \ \beta_+(p,w)$, then $|c-\bar{c}| \leqslant \varepsilon/2$ so that $\beta_+(p,w)$ is contained in U_ε.

We now show that the correspondence $\beta_+(\cdot,w)$ is lower hemi-continuous. Fix $p \ \varepsilon \ [\alpha,\infty)$ and let $\{p(q)\}$ be an arbitrary sequence converging to p. For any $c \ \varepsilon \ \beta_+(p,w)$, there is a $z \ \varepsilon \ Z_+^\infty$

such that $0 \ll c^1 + \gamma^1 = w^1 - \Lambda_p z$ and $0 < c^2 + \gamma^2 = w^2 + \Lambda_{(p+d)} z$, where $\gamma^1 > 0$ and $\gamma^2 > 0$ are slack variables. Define the sequence $\{c(q)\}$ by $c^1(q) = w^1 - \Lambda_{p(q)} z - \gamma^1$ and $c^2(q) = w^2 + \Lambda_{(p(q)+d)} z - \gamma^2$. Then for $\bar{q}$ large enough, $c(q) \; \varepsilon \; \beta_+(p(q),w)$ for all $q > \bar{q}$, which establishes the lower hemi-continuity of $\beta_+(\cdot,w)$ at p. Δ

Recall that the individual demand for consumption for a two-period lived agent i is $\phi_i(p) = \text{argmax} \{u_i(c): c \; \varepsilon \; c\ell\beta(p,w_i)\}$.

Lemma 3: Assume A.1-A.3. For almost all $i \; \varepsilon \; [0,1]$, ϕ_i is non-empty and upper hemi-continuous at every $p \; \varepsilon \; [\alpha,\infty)$.

Proof. Let $\phi_i^+(p) = \text{argmax} \{u_i(c): c \; \varepsilon \; c\ell\beta_+(p,w_i)\}$, where $c\ell\beta_+(p,w_i)$ is the closure of $\beta_+(p,w_i)$. We first note that $\phi_i^+(p)$ is nonempty. This follows from continuity of preferences (assumption A.1), boundedness of $\beta(p,w_i)$ and nonemptiness of $\beta_+(p,w_i)$ (assumption A.3). We next note that $\phi_i^+(p) = \phi_i(p)$. This follows from assumptions A.1 and A.2, which imply that if $c \; \varepsilon \; \phi_i(p)$, then $c^1 \gg 0$ and, hence, $c \; \varepsilon \; c\ell\beta_+(p,w_i)$.

By lemma 2, $c\ell\beta_+(\cdot,w_i)$ is a continuous correspondence. (The closure of a continuous correspondence is continuous.) It is also compact valued. The conclusion of the lemma is, then, a consequence of the Maximum Principle (see, for example, Hildenbrand [1974], p. 29). Δ

Next, recall that $\psi_i(p) = \{\psi(c;p,w_i): c \; \varepsilon \; \phi_i(p)\}$. By monotonicity, assumption A.1, it follows that $\psi_i(p) = \{z \; \varepsilon \; Z_+^\infty:$

there exists $c \in \phi_i(p)$ satisfying $\Lambda_p z = w_i^1 - c^1$ and $\Lambda_{(p+d)} z = c^2 - w_1^2$ }.

Lemma 4: Assume A.1-A.3. For almost all $i \in [0,1]$, ψ_i (which maps R_+ to subsets of Z_+^∞) is upper hemi-continuous at every $p \in [\alpha, \infty)$.

Proof. By lemma 1, if a sequence $\{z(q)\}$ is in $\chi(p,w)$, then $Iz = ||z(q)||_1 \le w^1/\alpha$. Therefore, there is a subsequence $\{z(q_i)\}$ for which $\lim ||z(q_i)||_1$ exists. Given this fact and the monotonicity assumption, A.1, it is easy to see that $\psi_i(p) = \{z \in Z_+^\infty$: there exists $c \in \phi_i(p)$ and a sequence $\{z(q)\}$ satisfying $z(q) \in Z_+^\infty$, $z(q) \to z$, $c^1 = w^1 - (p-\alpha)\Lambda z - \alpha \lim ||z(q)||_1$, and $c^2 = w^2 + -(p+d-\alpha)\Lambda z + \alpha \lim ||z(q)||_1$ }.

Fix $p \in [\alpha, \infty)$ and let $p(q) \to p$, $z(q) \to z$ and $z(q) \in \psi_i(p(q))$ for all q. Given $z(q) \in \psi_i(p(q))$, there is a $c(q) \in \phi_i(p(q))$ and a sequence $\{z(q)_r\}$ such that $z(q)_r \to z(q)$ and $c^1(q) = w^1 = (p(q)-\alpha)\Lambda z(q) - \alpha \lim ||z(q)_r||_1$ and analogously for $c^2(q)$. Let $\{z_q\}$ be a subsequence of the sequence $\{z(q)_q\}$ for which $\lim ||z_q||_1$ exists. Then $c^1(q) \to w^1 - (p-\alpha)\Lambda z - \alpha \lim ||z_q||_1 = c^1$ and analogously for $c^2(q)$. By upper hemi-continuity of $\phi_i(\ )$ at p, it follows that $c \in \phi_i(p)$. Finally, since Z_+^∞ is a closed subset of R^∞ in the product topology, we can conclude that $z \in \psi_i(p)$. Δ

Now let $\Psi(p) = \int \Lambda \psi_i(p) d\lambda$ where $\int \Lambda \psi_i(p) d\lambda = \{ \int \Lambda z_i d\lambda$: $z_i \in \psi_i(p)$ a.s. }. That is, $\Psi(p)$ is the implied or derived mean demand for units of size unity at p.

Lemma 5: Assume A.1-A.3. The correspondence Ψ is upper hemi-continuous and convex valued at every $p \in [\alpha, \infty)$.

Proof. For almost all $i \in [0,1]$, it follows from lemma 4 and the fact that Λ is a continuous linear map that $\Lambda \psi_i$ is an upper hemi-continuous correspondence. Furthermore, by lemma 1, for any $p \in [\alpha, \infty)$, $\Lambda z_i \leqslant w_i^1/\alpha$ for all $z_i \in \psi_i(p)$. Then, since $\int (w_i^1/\alpha) d\lambda < \infty$, the upper hemi-continuity of $\Psi(\cdot)$ follows from Proposition 8 in Hildenbrand [1974], page 73. The convexity of $\Psi(\cdot)$ follows from Theorem 3 in Hildenbrand [1974], page 62. Δ

The next lemma simply shows that over the range of prices $[\alpha, \bar{w}^1/\bar{x}^0]$, there is positive and nonnegative excess supply of assets.

Lemma 6. Assume A.1-A.5. (i) (Positive excess supply.) For all $p > \bar{w}^1/\bar{x}^0$, where $\bar{w}^1 = \int w_i^1 d\lambda$, $x < \bar{x}^0$ whenever $x \in \Psi(p)$. (ii) (Nonnegative excess supply.) (a) if $d > 0$ and if $x \in \Psi(\alpha)$, then $x \geqslant \bar{x}^0$. (b) If $d = 0$, then there exists $x \in \Psi(\alpha)$ such that $x \geqslant \bar{x}^0$.

Proof. (i) Fix $p > \bar{w}^1/\bar{x}^0$. If $x \in \Psi(p)$, then there exists $z_i \in \psi_i(p)$ a.s. such that $x = \int \Lambda z_i d\lambda$ and $\Lambda_p z_i = p\Lambda z_i + \alpha(I-\Lambda)z_i = w_i^1 - c_i^1$ for some $c_i \in \phi_i(p)$. Since for almost all i, $c_i^1 > 0$ and $\alpha(I-\Lambda)z_i \geqslant 0$, $p\Lambda z_i < w_i^1$. Therefore, $x = \int \Lambda z d\lambda < \bar{w}^1/p \leqslant \bar{w}^1/(\bar{w}^1/\bar{x}^0) = \bar{x}^0$.

(iia) If $p = \alpha$, then $p^n = \alpha$ for all n. It follows by monotonicity of preference and $d > 0$, that if z_i is any portfolio with $Iz_i = m$ and $z_i^n > 0$ for some $n > 0$, then $z_i \notin \psi_i(\alpha)$. That is, only portfolios of the form $z_i = (m_i, 0, 0, \ldots)$ are demanded at $p =$

α. Since assumption A.4 implies that $m_i > 1$ for almost all i, $\int \Lambda z_i d\lambda = \int m_i d\lambda > 1 > \bar{x}^0$.

(iib) If $p = \alpha$, so that $p^n = \alpha$, and $d = 0$, it follows that if some portfolio $z_i \varepsilon \psi_i(\alpha)$ and $Iz_i = m_i$, then $z_i = (m_i, 0, 0, \ldots) \varepsilon \psi_i(\alpha)$. And since A.4 implies that $m_i > 1$ for almost i, it follows as in (iia) that there exists $x \varepsilon \Psi(\alpha)$ such that $x > 1 > \bar{x}^0$. Δ

Proof of the Proposition

By monotonicity of preferences, all budget constraints are satisfied without slack. Then (iii') and the integrals of these budget constraints (at equality) imply (i') and (ii') (Walras' Law), where (i')-(iii') are the market clearing conditions in the definition of a stationary equilibrium. In other words, in order to prove existence of a stationary equilibrium, it is enough to show that there exists $p^* \varepsilon [\alpha, \bar{w}^1/\bar{x}^0]$ such that $\bar{x}^0 \varepsilon \Psi(p^*)$.

Let $P^+ = \{p \varepsilon [\alpha, \bar{w}^1/\bar{x}_0]:$ there exists $x \varepsilon \Psi(p)$ satisfying $x > \bar{x}^0\}$ and $P^- = \{p \varepsilon [\alpha, \bar{w}^1/\bar{x}^0]:$ there exists $x \varepsilon \Psi(p)$ satisfying $x < \bar{x}^0\}$. By the upper hemi-continuity of $\Psi(\cdot)$, lemma 5, P^- and P^+ are closed subsets of $[\alpha, \bar{w}^1/\bar{x}^0]$. By definition, their union is $[\alpha, \bar{w}^1/\bar{x}^0]$ and, by lemma 6, both subsets are nonempty. It follows that the intersection of P^+ and P^- is nonempty, because emptiness would imply that $[x, \bar{w}^1/\bar{x}^0]$ is the union of two nonempty disjoint closed sets, a contradiction. Finally, if $p^* \varepsilon P^+$ and $p^* \varepsilon P^-$, then by the convexity of $\Psi(p)$, lemma 5, $\bar{x}^0 \varepsilon \Psi(p^*)$. Δ

2. Proof of Proposition 4

We prove this proposition by suitably modifying the argument used to prove Proposition 1. Basically, we have to show that individual demands satisfy the required continuity properties. The rest of the argument is the same and will not be repeated.

Lemma 7. The correspondence $c\ell\hat{\beta}_+(\cdot,w)$ is continuous and compact valued at every $p \in [\alpha,\infty)$.

Proof. Fix $\bar{p} \in [\alpha,\infty)$. Since for any $(c,\pi) \in c\ell\hat{\beta}_+(\bar{p},w)$, $c \in [0,w^1]$ and $\pi(x) = 0$ for any $x \ge w^1 + w^2$, $c\ell\hat{\beta}_+(\bar{p},w)$ is compact. (Formally, the set $\{\pi \in M(R_+):$ there exists $c \in R_+$ and $(c,\pi) \in \hat{\beta}_+(\bar{p},w)\}$ is "tight" and therefore compact (theorem 6, p. 240 in Billingsley [1968]).)

Now let $U_\epsilon = \{(c,\pi) \in R_+ \times M(R_+): \max(|c-\bar{c}|,\rho(\pi,\bar{\pi})) < \epsilon$ and $(\bar{c},\pi) \in \hat{\beta}_+(\bar{p},w)\}$ and let $V_\epsilon = (\max(\alpha,\bar{p}-\epsilon\alpha/2w^1), \bar{p} + \epsilon\alpha/2w^1)$, where ρ is the Prohorov metric.[12] Then, proceeding as in lemma 2, it follows that if $p \in V_\epsilon$, then $\hat{\beta}_+(p,w)$ is contained in U_ϵ. This establishes upper hemi-continuity of $\hat{\beta}_+(\cdot,w)$ at $\bar{p}$.

Finally, lower hemi-continuity of $\hat{\beta}(\cdot,w)$ is proved as in lemma 2. Δ

Lemma 8. The functions $v_i(\ ,\ )$ and $\bar{v}_i(\cdot,\cdot)$ are continuous on $R_+ \times M(R_+)$.

Proof. The result is a direct application of Theorem 5.5 in Billingsley [1968].

The rest of the proof proceeds using arguments similar to those used in the proof of Proposition 1. Δ

3. Proof of Proposition 5.

This proof differs from that of Proposition 1 mainly in that it uses a fixed point argument on prices and taxes. Again, some lemmas are required to support the final argument.

Recall that $\beta_{++}(p,\tau,w) = \{c \in R^2_{++}:$ there exists $z \in Z^\infty_+$ such that $c^1 \leq w^1 - \Lambda_p z, c^2 \leq w^2 + \Lambda_p z - \tau\}$. An argument similar to that in the proof of lemma 2 establishes the following results.

<u>Lemma 9</u>. The correspondence $\beta_{++}(\cdot,\cdot,w)$ is continuous at every $(p,\tau) \in [\alpha,\infty) \times [0,w^1]$.

The individual demand for consumption for a two-period lived agent i is defined by $\phi_i(p,\tau) = \text{argmax} \{u_i(c): c \in c\ell\beta(p,\tau,w_i)\}$. A straightforward modification of the proof of lemma 3 gives the following result.

<u>Lemma 10</u>. Assume A.1, A.2', and A.3'. For almost all $i \in [0,1]$, ϕ_i is nonempty and upper hemi-continuous at every $(p,\tau) \in [\alpha,\infty) \times [0,w^1]$.

Instead of proving upper hemi-continuity of the demand correspondence for assets, we study the continuity properties of the correspondence $\Delta_i(\ ,\ )$ defined by $\Delta_i(p,\tau) = \{(x,y) \in R^2_+:$ there exists $c \in \phi_i(p,\tau)$ and a sequence $\{z(q)\}$ such that, for all q, $z(q) \in Z^\infty_+$, $w^1 - \Lambda_p z(q) \gg 0$ and $w^2 + \Lambda_p z(q) - \tau \gg 0$, $z(q) \to z$, $c^1 = w^1 - (p-\alpha)\Lambda z - \alpha \lim ||z(q)||_1$, $c^2 = w^2 + (p-\alpha)\Lambda z + \alpha \lim$

<u>Proof.</u> See the proof of lemma 6, part (i).

Proof of the Proposition

As in the proof of Proposition 1, by Walras' law, it is enough to show that there exists $(p^*, \tau^*) \ \varepsilon \ (\alpha, \infty) \ \times \ [0, w^1]$ such that $(\bar{x}^0, \tau^*) \ \varepsilon \ \int \Delta_i(p^*, \tau^*) d\lambda$.

Define the correspondence $\mu(\cdot, \cdot)$ by $\mu(x, \tau) = \text{argmax} \ \{p(x-\bar{x}^0): p \ \varepsilon \ [\tilde{p}(\tau), w^1/\bar{x}^0]\}$, where $\tilde{p}(\cdot)$, by condition B, is a continuous function such that $\tilde{p}(\tau) > \alpha$ and $x \geqslant \bar{x}^0$ if $x \ \varepsilon \ \Psi(\tilde{p}(\tau), \tau)$. Next, define the correspondence $\gamma(p, x, \tau) = \mu(x, \tau) \ \times \ \Delta(p, \tau)$, where $\Delta(p, \tau) = \int \Delta_i(p, \tau) d\lambda$. By lemma 11 and the argument used in lemma 5, $\Delta(\cdot, \cdot)$ is upper hemi-continuous and convex valued. Furthermore, the same is true for $\mu(\cdot, \cdot)$. Then, since $\gamma(p, x, \tau)$ maps $[\alpha, w^1/\bar{x}^0] \ \times \ [0, w^1/\alpha] \ \times \ [0, w^1]$ into subsets of itself, a standard generalization of Kakutani's fixed point theorem shows that γ has a fixed point. That is, there exists (p^*, x^*, τ^*) such that $(p^*, x^*, \tau^*) = \gamma(p^*, x^*, \tau^*)$.

Notice that since $\tau^* \ \varepsilon \ \text{proj}_2 \ \Delta(p^*, \tau^*)$, taxes finance replacement as required by condition (d) in the definition of equilibrium under lump-sum taxes. Notice also that since $p^* = \mu(x^*, \tau^*)$, $p^* > \alpha$. It only remains to show that $x^* = \bar{x}^0$.

Suppose that $x^* > \bar{x}^0$. Then, by the definition of μ, $p^* = w^1/\bar{x}^0$. However, by lemma 12, this implies $x^* < \bar{x}^0$, a contradiction. Suppose, alternatively, that $x^* < \bar{x}^0$. Then μ implies that $p^* = \tilde{p}(\tau^*)$, which, by hypothesis (condition B), implies $x^* \geqslant \bar{x}^0$, a contradiction. Δ

4. Proof of Proposition 6

We show that there exists a monotone function g that maps $[0,w^1]$ into $[0,w^1]$ and is such that a fixed point of it is the required equilibrium. (We take for granted the obvious fact that such a monotone function has a fixed point.)

The function g is defined as follows. Fix $p = \alpha$. Then, for each $\tau \varepsilon [0,w^1]$, we use the argument in the proof of Proposition 2 to find an integrable function z, a portfolio, such that $z_i \varepsilon \psi_i(\alpha,\tau)$ almost surely and such that $\int \Lambda z d\lambda = \bar{x}^0$. Then we let $g(\tau) = \theta\alpha\int(I-\Lambda)zd\lambda$, the cost of expected disintegration of the portfolio z. It follows that if τ^* is a fixed point of g, then $(p,\tau) = (\alpha,\tau^*)$ is a stationary equilibrium under lump-sum taxation.

We now establish the existence and required properties of g. For a fixed τ, let $m_i^*(\tau)$ be the largest nonnegative integer m_i that maximizes $u_i(w_i^1-\alpha m_i, w_i^2+\alpha m_i-\tau)$. By assumptions A.2', A.3', A.4, and A.8, $m_i^*(\tau) \geqslant 1$. (Note that A.3' and A.4 imply that $m_i^*(0) \geqslant 1$. Then assumption A.8, along with A.3', implies that $m_i^*(\tau) \geqslant 1$ for all τ.)

At $p = \alpha$, all z_i such that $Iz_i = m_i^*(\tau)$ support utility maximizing consumption. Therefore, to support such consumption, we require only that z satisfy $\int Iz d\lambda = \int m_i^*(\tau)d\lambda \equiv m^*(\tau)$.

Let i be the integer satisfying $2^{i-1} \leqslant m^*(\tau)/\bar{x}^0 \leqslant 2^i$. It follows from A.5 that $i > 0$. As in the proof of Proposition 2, we let $\int z^{i-1}d\lambda = 2^i\bar{x}^0 - m^*(\tau)$, $\int z^i d\lambda = 2m^*(\tau) - 2^i\bar{x}^0$, and let $\int z^n d\lambda = 0$ for $n \notin \{i-1,i\}$. It follows that $\int Iz^n d\lambda = m^*(\tau)$, that

$\int \Lambda z^n d\lambda = \bar{x}^0$ and that $g(\tau) = \theta\alpha \int (I-\Lambda)zd\lambda = \theta\alpha[\int Izd\lambda - \int \Lambda zd\lambda] = \theta\alpha[m^*(\tau)-\bar{x}^0]$. Note that although the integer i depends on τ, $g(\tau)$ depends on τ only by way of $m^*(\tau)$, a result that uses assumption A.7.

With $g(\tau) = \theta\alpha[m^*(\tau)-\bar{x}^0]$, it follows that $g(\tau) \in [0,w^1]$, since $\alpha m^*(\tau) < w^1$. Finally, assumption A.8 implies that $m^*(\tau)$ is nondecreasing in τ, which implies that g is nondecreasing. (Obviously, $g(\tau)$ is a nondecreasing step function.) Δ

5. The absence of aggregate risk.

As we have said, equilibria with taxation schemes are well defined provided there is no aggregate risk. It is well known that in models where individuals bear some risk and those risks are independent, the existence of a continuum of agents is not sufficient to rule out aggregate risk. In our model there is an additional complication in that the distributions of the individual random variables are endogenous. However, as we now briefly explain, there is no aggregate risk at equilibrium if the numbers of types is finite (in fact, we assume only one type).

For any $p \in [\alpha,\infty)$, $\bar{\Psi}(p) \equiv \int \bar{\psi}(p)d\lambda$ (alternatively, for any $(p,\tau) \in [\alpha,\infty] \times [0,w^1]$, $\Delta(p,t))$ is a closed convex subset of R (of R^2). By the Krein-Milman Theorem (Rudin [1973] p. 70), $\Psi(p)$ $(\Delta(p,t))$ is the convex hull of the set of its extreme points. Due to the structure of the budget constraints and the finiteness of the number of types, the set of extreme points is a finite set and corresponding to each extreme point there is a, possibly limiting, portfolio supporting such a point.

Therefore, if $x \in \bar{\Psi}(p)$ $((x,y) \in \Delta(p,\tau))$, then there exist $(\mu_1, \mu_2, \ldots, \mu_m)$ such that $\mu_k \geqslant 0, \sum \mu_k = 1$ and $x = \sum_k \mu_k (\Lambda z_k)$ $((x,y) = \sum_k \mu_k (\Lambda z_k, \alpha \lim_q ||\theta z_k(q)||_1 - \Lambda\theta z_k)$, where $z_k(q) \rightarrow z_k$). Since each μ_k can be identified with a segment on $[0,1]$ on which there is a single distribution of y, that induced by the supporting portfolio z_k, we can apply proposition 2 of Feldman and Gilles [1985] to obtain the absence of aggregate risk.

The Feldman-Gilles construction orders agents and realizations so that the distribution of realizations over agents is the population distribution of the random variable. (This assures that the expectation over agents is the population expectation.) Strictly speaking, such an ordering of agents and realizations contradicts independence of realizations across agents. That, however, is of no concern in our context because we do not permit intra-generation trade of any sort.

V. Concluding Remarks

We began by posing two questions. Is private provision of divisibility for assets different from such provision for other things? And is such provision for money-like assets different from its provision for other assets?

We approached these questions using a model with the following crucial ingredients. We modelled the supply of asset divisibility by assuming that the costs of producing divisibility for assets resembles the costs of splitting logs or candy bars. In other words, we modelled the costly provision of divisibility for assets in much the same way as one would model the costly provision of divisibility for consumption goods. We modelled the demand for divisible assets by assuming that the only way to provide for future consumption in excess of future income is through spot purchases and subsequent spot sales of outside assets. (In the context of the overlapping generations model we used, we did not permit members of a generation to form coalitions to share assets.)

These assumptions imply that the feasible trades in our model depend on both the extent to which assets are made divisible and on prices. In contrast, in models where the consumption goods themselves are costly to divide, feasible trades depend only on the extent to which consumption is made divisible. This difference between indivisibilities for consumption goods and for assets accounts for a corresponding difference in welfare consequences. In models with costly divisibility of consumption goods, any

competitive equilibrium is Pareto optimal.[13] In that sense, there is nothing special about private provision of divisibility for consumption goods. In our model, there is something special about private provision of divisibility for assets in that the welfare properties of competitive equilibria depend significantly on whether asset divisibility is costly or costless. With costly divisibility, in the non-depreciating asset version, there are multiple equilibria ordered by Pareto superiority. Also, both for that version and for the depreciating asset version, equilibria with taxes or subsidies on the divisibility process or on replacement are not in any obvious way worse than laissez-faire equilibria.

In our model, market provision of divisibility for money-like assets and for other assets have similar consequences. We did, however, find that the occurrence of multiple equilibria ordered by Pareto superiority is more general for money-like assets.

Since the conclusion that no obviously desirable properties follow from laissez-faire provision of costly divisibility for assets is rather startling, one would like to know how robust it is. In particular, one would like to know how robust it is with regard to the way we have modelled the demand for assets. Unfortunately, we cannot say. We settled on our way of modelling asset demand because we surmised that relatively simple stationary equilibria would exist for that specification.

Footnotes

[1]/Of course, if $d = 0$, then the stationary rate of interest is zero for all sizes. However, for nonstationary prices satisfying $p_t^{n+1} = 2^{-1}(p_t^n + \alpha)$ for all n and t, assets have returns that vary with size even if $d = 0$.

[2]/The boundary of the stationary budget set can be written $c_i^1 = w_i^1 - [p\Lambda + \alpha(I-\Lambda)]z$, $c_i^2 = w_i^2 + [(p+d)\Lambda + \alpha(I-\Lambda)]z$, where $\Lambda = (2^0, 2^{-1}, \ldots, 2^{-n}, \ldots)$ and $I = (1, 1, \ldots)$. It follows that $c_i^1 = w_i^1 - [(\Lambda z)/(Iz)](Iz)p - (1 - \Lambda z/Iz)(Iz)\alpha = w_i^1 - [(\Lambda z)/m]mp - (1 - \Lambda z/m)m\alpha = \mu w_i^1 - \mu mp + (1-\mu)w_i^1 - (1-\mu)m\alpha = \mu(w_i^1 - mp) + (1-\mu)(w_i^1 - m\alpha)$, where $\mu = \Lambda z/m = \Lambda z/Iz \, \epsilon \, (0,1]$; and similarly for c_i^2.

[3]/Our model and, in particular, examples like this one are consistent with the assertion made in Wallace [1983] that if the government and the private sector have access to the same constant-returns-to-scale technology for producing a property like divisibility, then government asset exchanges accomplished at market prices leave unchanged the set of equilibria.

[4]/Notice that it is enough to consider small perturbations of $(\alpha, d, \overline{x}^0)$ and of the distribution of preferences and endowments among agents of generation t, $t \geq 1$. Each distribution is given by a measure ν on $C(R_+^2) \times R_+^2$ defined by $\nu(A) = \lambda\{i\epsilon[0,1]: (u_i, w_i)\epsilon A\}$ where $\lambda\{ \}$ denotes the Lebesque measure of $\{ \}$. Two sets of characteristics, $(x, d, \overline{x}^0, \nu)$ and $(\alpha', d', \overline{x}^0{}', \nu')$, are close if $(\alpha, d, \overline{x}^0)$ and $(\alpha', d', \overline{x}^0{}')$ are close in Euclidean distance, if the supports of ν and ν' are close, and if ν and ν' are close in the topology of weak convergence of measures.

$\underline{5/}$For example, one exists for all identical agent economies satisfying $\overline{x}^0 < 1$ and $v_i(w_i^1-\alpha, w_i^2+\alpha) < 1$, where $v_i \equiv u_{i1}/u_{i2}$ is the marginal rate of substitution of agent i. (The inequality on v_i implies that there exists $p > \alpha$ such that $v_i(w_i^1-p, w_i^2+p) = 1$. At this price, demand is $\{z_i^n\} = (1,0,0,\ldots)$, which implies excess demand.)

$\underline{6/}$The existence of competitive allocations which are dominated within the class of equilibrium allocations is a standard result in models with incomplete markets (see Hart [1975] for an example and Geanakopolos and Pelemarchakis [1985] for generic results) and in models of overlapping generations. As we have indicated, at least when $d > 0$, the costly provision of divisible assets is a critical ingredient for the result in our model.

$\underline{7/}$See Jevons ([1875], Chapter 10) for a description and analysis of England's gold coinage system. In the preface, Jevons listed among "currency questions which press for solution" the following: "How long shall we in England allow our gold coinage to degenerate in weight? Shall we recoin it at the expense of the state or of the unlucky individuals who happen to hold light sovereigns?" (p. viii). His answer was that "the only thorough remedy is for the government to bear the loss occasioned by the wear of the gold, ..." (p. 111).

$\underline{8/}$See Supel and Todd [1984] for a discussion of the problem facing the U.S. government under such a replacement policy.

9/There are other depreciation schemes which are simple in these respects and which we could also analyze. For example, one could assume that disintegration implies total loss of the asset, but that the government is able to costlessly produce new units of size unity. A policy under which the government sells enough assets to replace disintegrated units and transfers its profits in a lump-sum fashion to the old at each date is consistent with stationarity.

10/In the case of paper currency, one may want to think of laissez-faire as a government policy under which the government stands ready to exchange new units of currency for disintegrated units plus the cost of producing new units.

11/For homogeneous consumers, the following situation gives a $\tilde{p}$ function. Let $\tilde{s}(\tau)$ maximize $u(c)$ subject to $c^1 = w^1 - s$ and $c^2 = w^2 - \tau + s$ by choice of $s \ \varepsilon \ R_+$. If (i) $\tilde{s}(\tau)$ is continuous on $[0,w^1]$ and (ii) there exists an integer $m > 0$ such that $(m+1)\alpha > \tilde{s}(\tau) > m\alpha$ for all $\tau \ \varepsilon \ [0,w^1]$, then $\tilde{p}(\tau) = \tilde{s}(\tau)/m$ satisfies condition B. This $\tilde{p}(\tau)$ implies excess demand because at $p = \tilde{p}(\tau)$, only the portfolio $z = (m,0,0,...)$ supports $\tilde{s}(\tau)$ and this portfolio implies excess demand. The stringent assumption is (ii), an almost vertical Engel curve assumption. We need it for this construction because if $\tilde{s}(\tau) = m\alpha$ for some integer m, then at any price p, $z = (0,0,...) \ \varepsilon \ \psi(p,\tau)$ and this z implies excess supply.

12/Here we use two facts. First, convergence in measure is equivalent to convergence in the weak-star topology on $M(R)$

(also called the topology of weak convergence in probability theory). Second, since R_+ is separable, the weak-star topology is metrizable. This metric, known as the Prohorov metric, is defined as follows. If P and Q are elements of $M(R_+)$, then $\rho(P,Q) = \inf_{\varepsilon>0} \{P(A) \leqslant Q(A_\varepsilon) + \varepsilon$ and $Q(A) \leqslant P(A_\varepsilon) + \varepsilon$, for all $A \varepsilon B(R_+)\}$, where $A_\varepsilon = \{x \varepsilon R_+: \text{dist} (x,A) < \varepsilon\}$ (Billingsley [1968], pp. 236-238).

[13]/For an analysis of models with indivisible consumption and a continuum of agents, see Mas-Colell [1977].

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Table 1. Expected Returns, $E(r^n)$,

and Variances, $V(r^n)$, of Size 2^{-n} Assets

$$(\gamma_n = (1-2^{-n})\alpha/[2^{-n}p+(1-2^{-n})\alpha])$$

	$E(r^n)$	$V(r^n)$
Laissez-Faire	$-\theta_n\gamma_n$	$\theta_n(1-\theta_n)(\gamma_n)^2$
Insurance Taxation	$-\theta_n\gamma_n$	0
Lump-Sum Taxation	0	0

Figure 1

An Example of a Budget Set
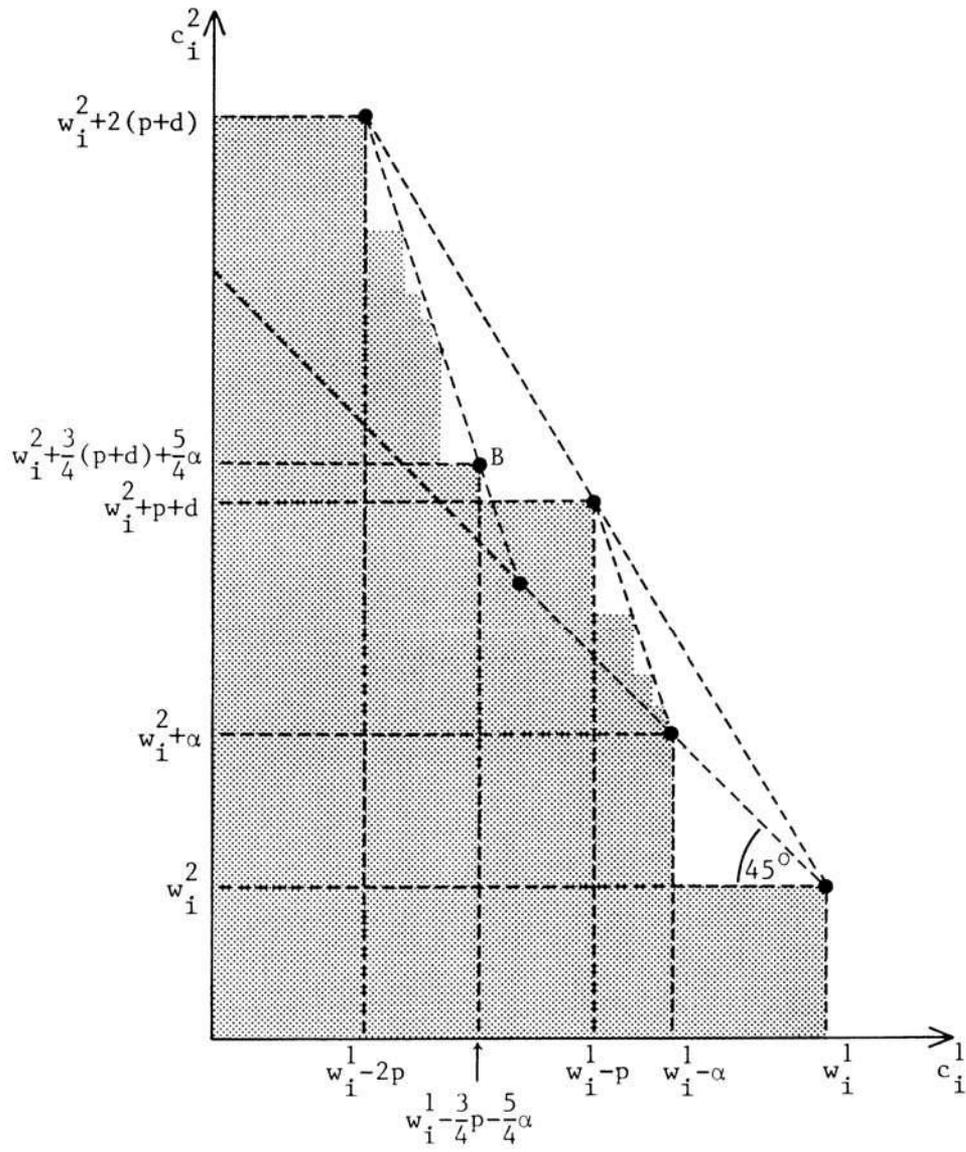
Figure 2a

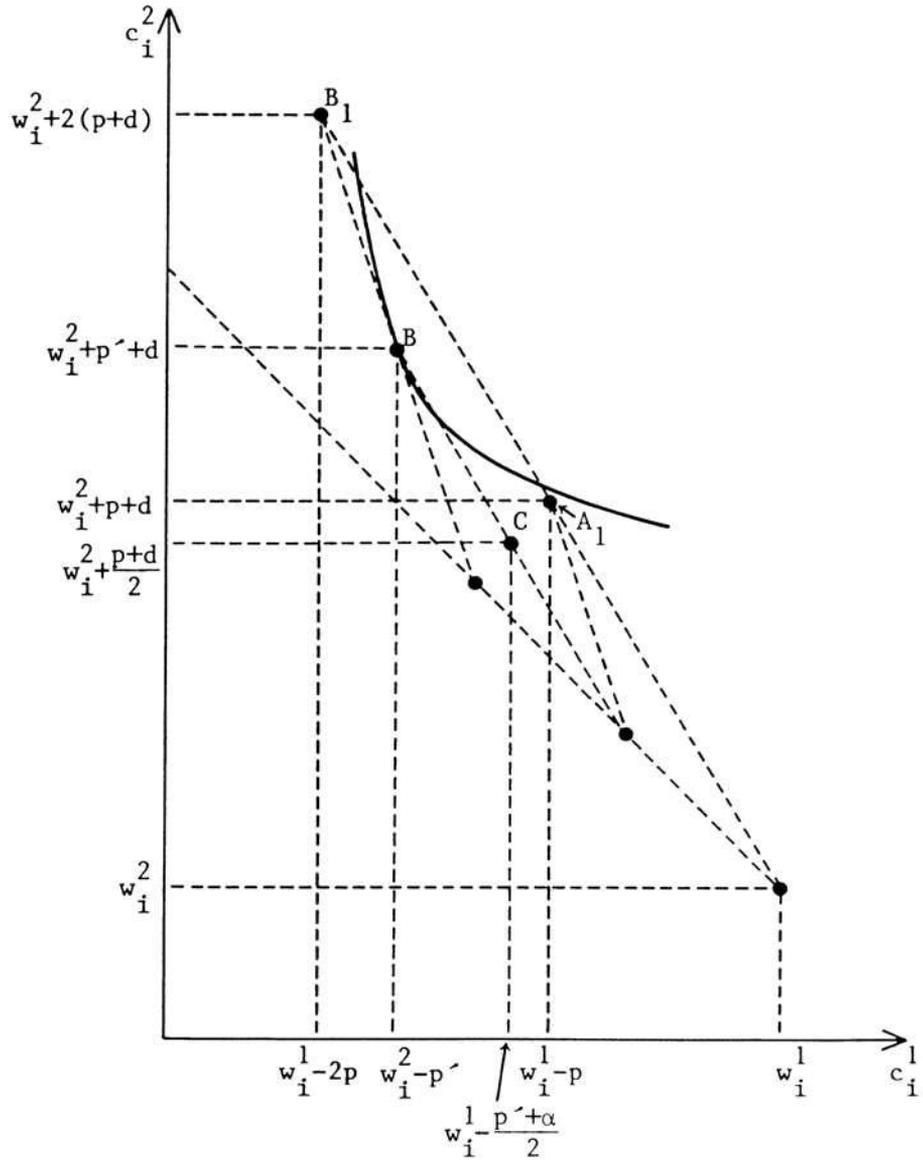Budget Sets Consistent With Multiple Equilibria
Ordered by Pareto Superiority $(p'=p+\alpha)$

Figure 2b

Mean Demand for Units of Size Unity
Implied by the Figure 2a Economy
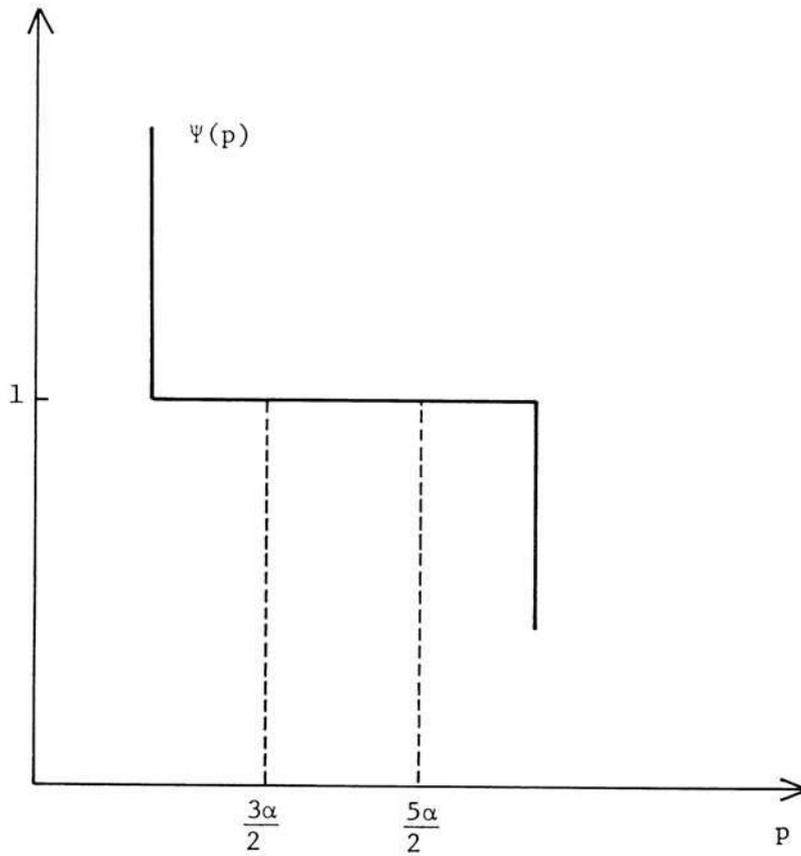
Figure 3a

A Robust Example of Pareto-Ordered Equilibria
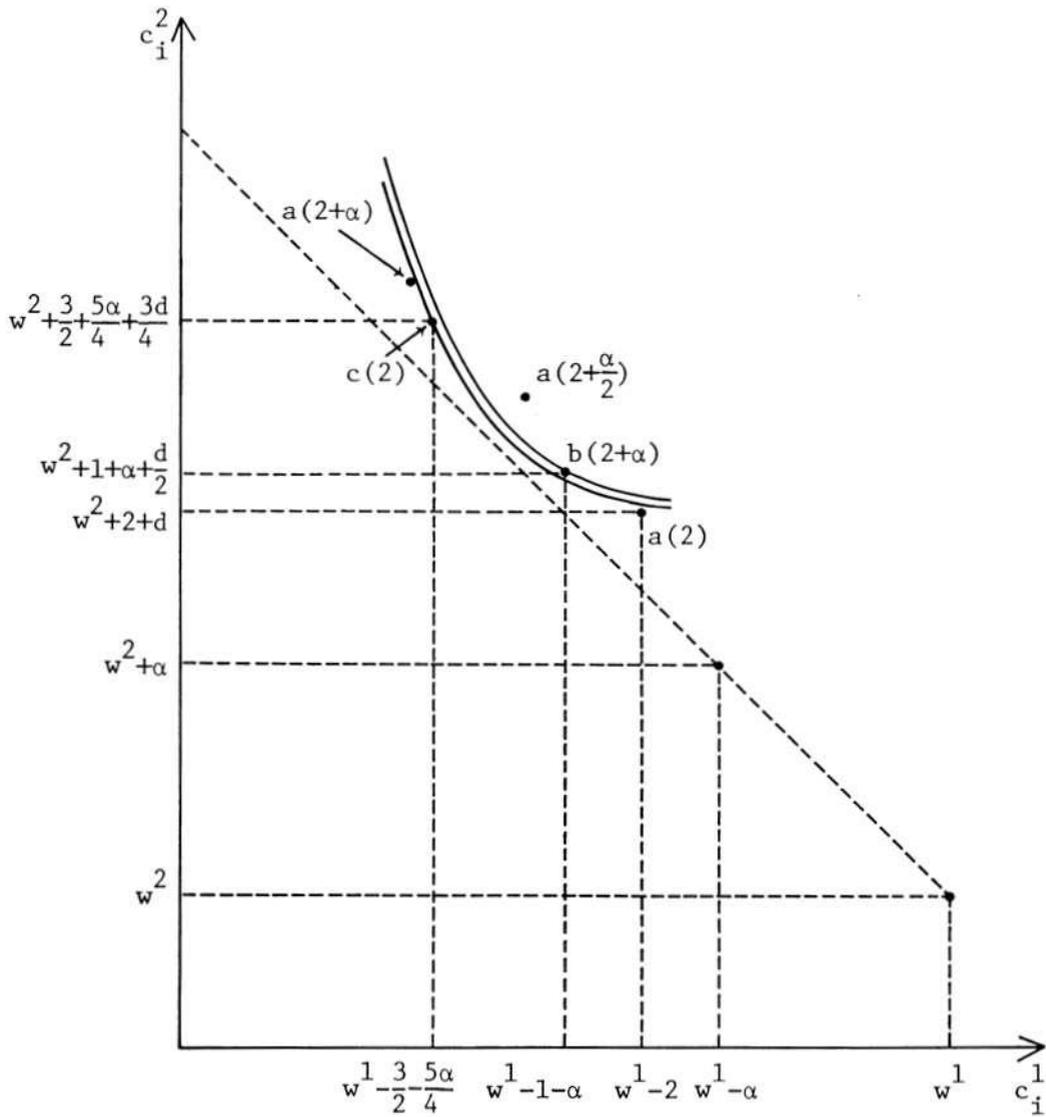
$$[(w^1, w^2) = (5,1), \quad \alpha = 3/2, \quad d = 1/2]$$

Figure 3b

Mean Demand for Units of Size Unity
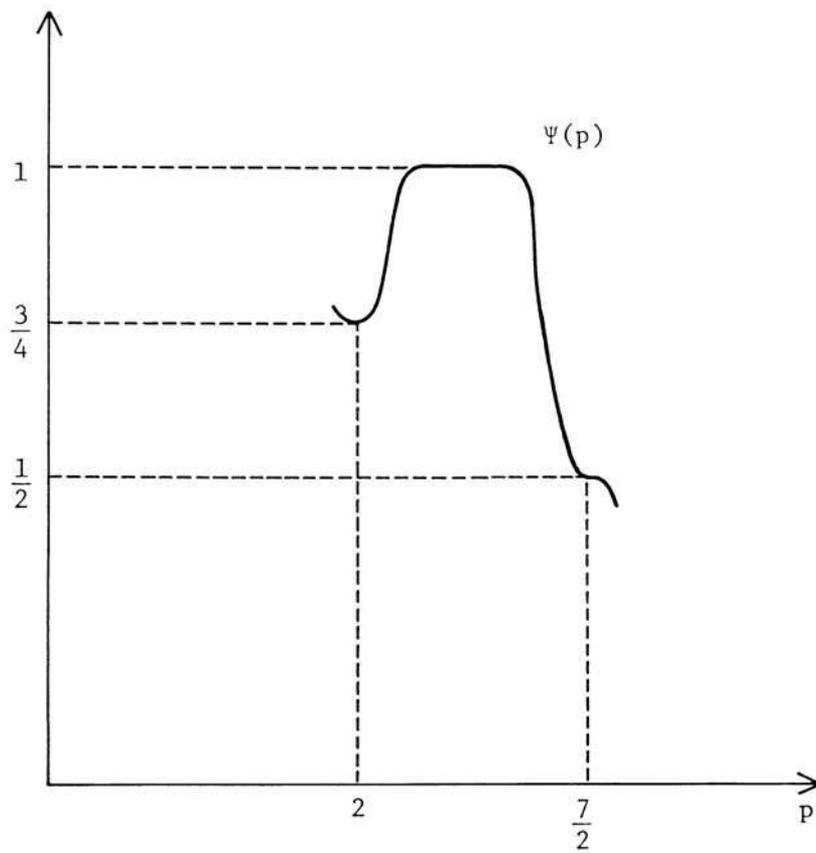Implied by the Figure 3a Economy

Figure 4

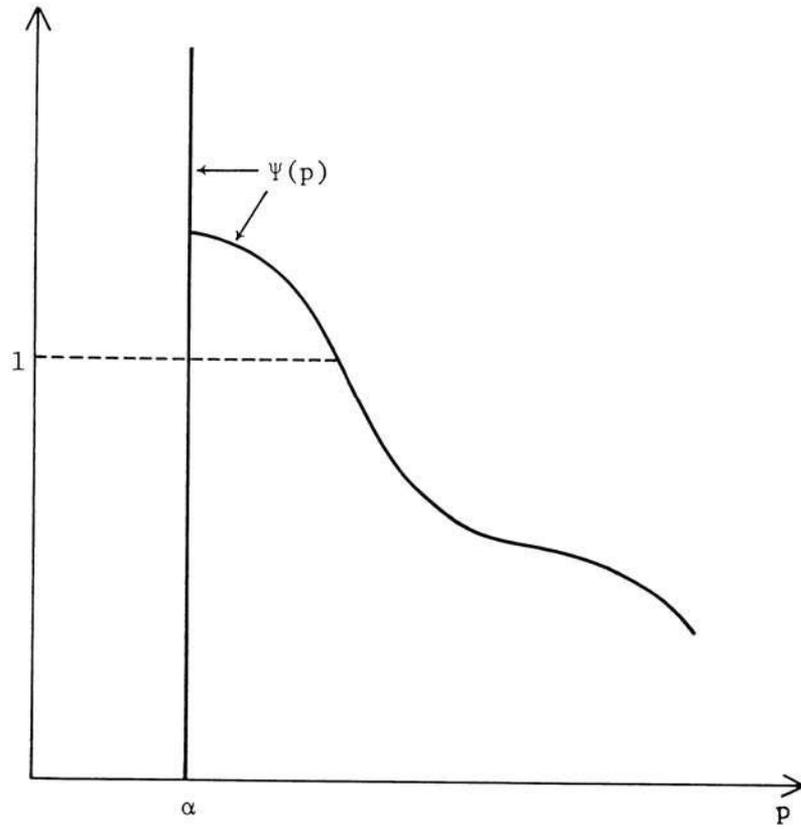Illustrative Mean Demand for Units of Size Unity

[d = 0]

Ψ(p)

1

α

p

Figure 5

Some Affordable Consumption Realizations
in the Disintegrating Asset Model

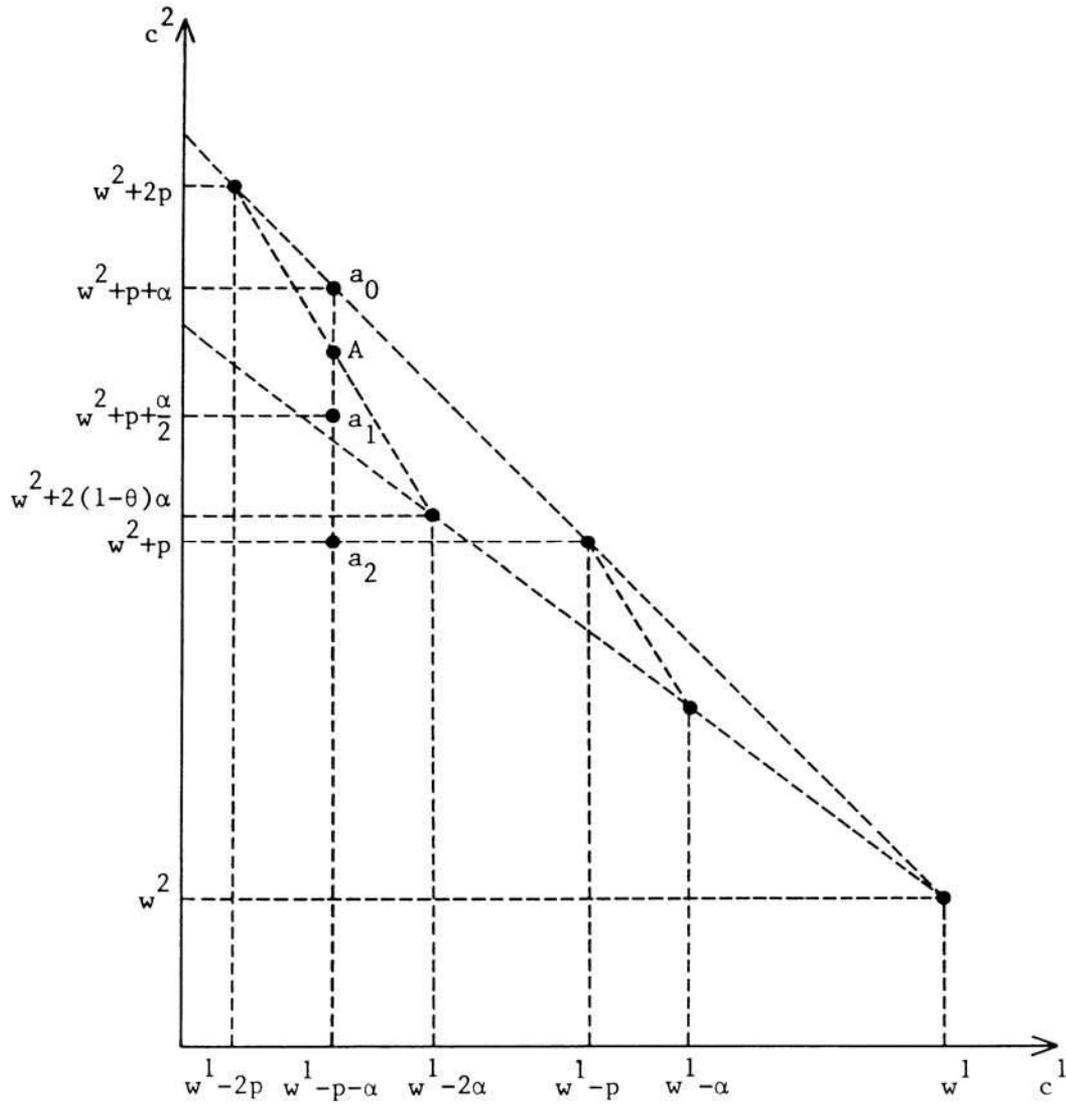$$[(w^1, w^2) = (15,3), \; \alpha = 5, \; \theta_n = \theta = 1/4, \; p = 7]$$

Figure 6

A Laissez-Faire Equilibrium (Point B) Pareto Superior to
a Lump-Sum Taxation Equilibrium (Point A)

$[(w^1, w^2) = (15, 3), \alpha = 5, \theta_n = \theta = 1/4, p = 7]$