NONFUNDAMENTAL UNCERTAINTY
AND EXCHANGE RATES*

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ABSTRACT_____

This paper shows that there can be equilibria in which exchange rates display randomness unrelated to fundamentals. This is demonstrated in the context of a two currency, one good model, with three agent types and cash–in–advance constraints. A crucial feature is that the type i agents, for i=1, 2, must satisfy a cash–in–advance constraint by holding currency i, while type 3 agents can satisfy it by holding either currency. It is shown that real allocations vary across the multiple equilibria if markets for hedging exchange risk do not exist and that the randomness is innocuous if complete markets exist.

In their 1981 paper, Kareken and Wallace (1981) questioned whether markets can determine exchange rates among fiat monies. They displayed a model in which markets cannot in the following sense: in their two currency framework, any unchanging exchange rate between the two currencies is consistent with a perfect foresight equilibrium. Two features account for their result. One is that the real return on a fiat currency (on any nondividend paying asset) is simply the ratio of its price at two different dates and, hence, is homogeneous of degree zero in those prices. The second is that people in their model are indifferent between currencies whose real returns are equal. In this paper, we use versions of those features to show that the indeterminacy extends to a large class of random processes for exchange rates, where the randomness is nonfundamental in the sense of having nothing to with preferences, endowments, technologies, or government policies. While our contribution is theoretical, it is motivated largely by recent experience with floating exchange rates. For some currencies and time periods exchange rate movements appear largely unrelated to factors identified by standard economic models as determining exchange rates--national money stocks, incomes, nominal interest rates, etc. (see Meese and Rogoff (1983) and Adams and Boyer (1986)).

Although the Kareken and Wallace model can be shown to have (rational expectations) equilibria in which exchange rates display nonfundamental uncertainty,[1] their model has two implausible consequences. First, under certainty, a currency is held only if its return is as high as that of any other asset, including any other currency. Second, almost all equilibria are such that the faster growing currency becomes the world's money supply with world-wide inflation converging to its growth rate. For this reason, here we use a version of a cash-in-advance model in which only some people, a kind of speculative fringe, substitute among currencies on the basis of their return

distributions; other people must use particular currencies. The cash-in-advance feature allows all currencies to have lower returns than other assets. The existence of people who must use particular currencies limits the range over which exchange rates are indeterminate and completely reverses the Kareken and Wallace conclusion concerning currencies whose growth rates differ. It implies that a currency which grows relative to others is held only by those who must use it. Our model also differs from models of exploding "bubble" equilibria in foreign exchange markets (see Singleton (1987)). All our exchange rate paths are bounded, with bounds implied by the behavior of the people who must use particular currencies.

Although our model is motivated by features of the observed behavior of exchange rates, we do not attempt to explain puzzles related to real exchange rates or deviations from purchasing power parity. We use a one good model, or equivalently, a model in which all goods are freely traded, so purchasing power parity always holds exactly in our model. Although the model can be amended to include nontraded goods and deviations from purchasing power parity, we do not pursue that development here.[2]

The paper has two main sections. The first is concerned primarily with showing that there can be equilibria in which exchange rates display randomness unrelated to fundamentals. That being the case, we do not strive for generality. We set out a model of two currencies and three agent types, with one type being the speculative fringe and with each of the other types constrained to using a particular currency. After setting out the model, we first describe some of its equilibria with all fundamentals held fixed--in particular, with the stocks of both currencies held constant through time. We first show the sense in which Kareken and Wallace indeterminacy appears in this model. Then, we introduce a stationary two-state sunspot process and

show that there is a large class of stochastic processes for the exchange rate which are equilibria. We do this for two versions: one with complete securities markets so that exchange-rate risk can be insured against and the other with no such possibilities available to people. In the former case, expected returns on currencies are equalized and real allocations are unaffected by the uncertainty. In the latter case, neither of these implications hold. We then use two specific parametric examples to explore whether standard tests for currency substitution performed on equilibria from our model would uncover the true nature of currency demands in the model. We show that they would not. We end the first section by considering situations in which the two currencies grow at different rates. We show that in such situations nonfundamental uncertainty cannot play a role and that fundamentals like the stocks of the two currencies should explain the exchange rate. In other words and somewhat paradoxically, exchange rate indeterminacy appears in our model when countries pursue similar policies.

The second section is devoted to the normative implications of nonfundamental exchange-rate uncertainty. In particular, we consider whether such uncertainty is grounds for favoring some sort of fixed rate system rather than a floating rate system. We demonstrate that an "irrelevance result" holds for a quite general complete markets framework which includes the model of the first section as a special case. If everyone has unrestricted access to markets for sharing exchange-rate risk, then nonfundamental exchange-rate uncertainty is innocuous in that the corresponding real allocations could have arisen as equilibrium allocations in a comparable one currency world.

I. The Model

The example economy has 3 persons (really 3 trader-types), 2-currencies, 1-good per date, no production, and currency demands implied by a

variant of a cash-in-advance constraint. Two versions are considered: in one there is full participation in complete risk-sharing markets; in the other there are no such markets.

In both versions, trading at each date occurs in the following sequence. First, the stochastic shock is realized and the outcome made known to all three traders. Then, all traders meet: in the first version they settle up outstanding contingent claims, purchase contingent claims for the next period, and exchange currencies; in the second version they exchange currencies only. After the trading in securities and currencies is concluded, the individuals disperse to trade the single good for currencies. As in Lucas (1982), each person may be regarded as a partnership in which one partner purchases the good for currency while the other sells the endowment for currency. We assume that traders are unconstrained as to where or from whom they may buy the good, but that some persons are constrained as to which currency they may accept as payment for the endowment. Specifically, each of two people is constrained to accept only a particular currency, while the third person is free to demand payment in either currency or a combination of the two currencies. The currency composition which the third person demands can be viewed as known once the shock is realized.

This kind of cash-in-advance constraint differs from that in Helpman (1981) and Lucas. They constrain which currency agents may offer as payment for goods whereas we constrain which currency may be accepted as payment. Both formulations have the same timing of trading, have all sellers getting stuck holding currency, and have some sellers getting stuck holding particular currencies. We adopt our formulation because it yields an equilibrium in which there is a constant total of real balances even though there is substitution among currencies. A priori, it seems as attractive as the Helpman-Lucas formulation.

1. Assumptions and Equilibrium Conditions

There are three infinitely-lived people who have identical preferences and endowments of goods. Each person maximizes the expected value of $\sum_{t=0}^{\infty} \beta^t u(c^h(\theta_t))$, where $\beta \in (0,1)$ and $c^h(\theta_t)$ is consumption of time t good by person h in state θ_t. Here θ_t, which is assumed known at the beginning of period t, summarizes the relevant history of the economy through date t. We assume that u is increasing, strictly concave, and twice differentiable. We further assume that endowments of goods are constant over time with each person receiving y units of goods per period.

In the market setup that we study, within each period each person is subject to a sequence of constraints that corresponds to the order of transactions described above. All of these transactions are conditioned on the state θ_t. First, in the financial markets, individuals trade currencies and contingent claims competitively--$p_k(\theta_t)$ is the state θ_t price of the kth currency and $s(\theta_{t+1};\theta_t)$ is the state θ_t price of claims on state θ_{t+1} consumption. Agent h chooses nonnegative amounts of "pocket" currency, $\tilde{m}_k^h(\theta_t)$, and contingent claims, $q^h(\theta_{t+1};\theta_t)$, for those states θ_{t+1} that are possible given θ_t. The financial market constraint is

$$\sum_k p_k(\theta_t)\tilde{m}_k^h(\theta_t) + \sum_{\theta_{t+1}} s(\theta_{t+1};\theta_t)q^h(\theta_{t+1};\theta_t) \leq \sum_k p_k(\theta_t)m_k^h(\theta_{t-1})$$

$$+ q^h(\theta_t;\theta_{t-1}) + \tau^h(\theta_t).$$

The right side of this inequality is financial wealth carried over from t - 1 plus transfers in t, $\tau^h(\theta_t)$ (paid in currency). Second, agents proceed to the commodities markets, where their consumption expenditure is constrained by the requirement that currency is the means of payment,

$$c^h(\theta_t) \leq \sum_k p_k(\theta_t)\tilde{m}_k^h(\theta_t).$$

Then, the total value of currency carried out of the period by agent h is

$$\sum_k p_k(\theta_t) m_k^h(\theta_t) \leq y + \sum_k p_k(\theta_t) \tilde{m}_k^h(\theta_t) - c^h(\theta_t).$$

In addition, agent h for h = 1 and 2 is subject to selling y for currency h, or equivalently, is subject to $p_i(\theta_t) m_i^i(\theta_t) \geq y$, for i = 1, 2. For our purposes, it is convenient to express these constraints by eliminating the intermediate (pocket) currency holdings. This leads to the following equivalent constraints:

(1) $\qquad c^h(\theta_t) + \sum_k p_k(\theta_t) m_k^h(\theta_t) + \sum_{\theta_{t+1}} s(\theta_{t+1};\theta_t) q^h(\theta_{t+1};\theta_t)$

$$\leq y + \sum_k p_k(\theta_t) m_k^h(\theta_{t-1}) + q^h(\theta_t;\theta_{t-1}) + \tau^h(\theta_t)$$

(2) $\qquad p_i(\theta_t) m_i^i(\theta_t) \geq y;\ i = 1,\ 2;\ \sum_k p_k(\theta_t) m_k^3(\theta_t) \geq y.$

By describing these constraints as equivalent, we mean the following. For any consumption level, $c^h(\theta_t)$, contingent claims position, $q^h(\theta_{t+1};\theta_t)$, and end of period cash balance position, $m_k^h(\theta_t)$, that satisfies (1) and (2), we can find pocket currency quantities, $\tilde{m}_k^h(\theta_t) \geq 0$, such that the original constraints hold. Conversely, if a configuration of $c^h(\theta_t)$, $q^h(\theta_{t+1};\theta_t)$, $m_k^h(\theta_t)$ and $\tilde{m}_k^h(\theta_t)$ satisfies the original market constraints, then $c^h(\theta_t)$, $q^h(\theta_{t+1};\theta_t)$ and $m_k^h(\theta_t)$ satisfy (1) and (2).[3]

Thus, at t, taking current prices and the processes for future prices as given, person h chooses date t consumption, $c^h(\theta_t)$, holdings of the kth currency, $m_k^h(\theta_t)$, and contingent claims, $q^h(\theta_{t+1};\theta_t)$, to maximize lifetime expected utility subject to (1) and (2). (We require that the $q^h(\theta_{t+1};\theta_t)$ be bounded in absolute value. This bound, which will turn out not to be binding, is needed to make the constraint set compact.) As initial conditions, we assume that all three agents enter period 0 with identical portfolios of

positive currency holdings and without contingent claims or transfers.[4] Of course, in the version without risk-sharing, we also impose $q^h(\theta_{t+1};\theta_t) \equiv 0$. An equilibrium consists of stochastic processes for quantities and prices such that the quantities are optimal for the individuals at those prices and are market clearing.

Since person h's choice problem involves maximizing a concave function over a nonempty, linear, and compact constraint set, subject to nonbindingness of the bound on the q^h's (to be verified later), the following standard first-order conditions are necessary and sufficient for a maximum to that problem: (1) and (2);

(3) $\qquad c^h(\theta_t): \quad u'\left(c^h(\theta_t)\right) \leq \lambda_1^h(\theta_t)$

and with equality if $c^h(\theta_t) > 0$;

(4) $\qquad m_k^h(\theta_t): \quad \beta \sum_{\theta_{t+1}} \left[\pi(\theta_{t+1};\theta_t)p_k(\theta_{t+1})u'\left(c^h(\theta_{t+1})\right)\right]/p_k(\theta_t) \leq \lambda_1^h(\theta_t) - \lambda_{2k}^h(\theta_t)$

for k = 1, 2 and with equality if $m_k^h(\theta_t) > 0$ and where $\lambda_{22}^1(\theta_t) \equiv \lambda_{21}^2(\theta_t) \equiv 0$ and $\lambda_{21}^3(\theta_t) \equiv \lambda_{22}^3(\theta_t)$; and

(5) $\qquad q^h(\theta_{t+1};\theta_t): \quad \beta\pi(\theta_{t+1};\theta_t)u'\left(c^h(\theta_{t+1})\right) = \lambda_1^h(\theta_t)s(\theta_{t+1};\theta_t)$

for each θ_{t+1}. In the no risk-sharing version, (5) is replaced by $q^h(\theta_{t+1};\theta_t) \equiv 0$. Here the λ's are nonnegative multipliers--$\lambda_1^h(\theta_t)$ that associated with (1) and $\lambda_{2k}^h(\theta_t)$ that associated with (2)--and $\pi(\theta_{t+1};\theta_t)$ is the probability that the state at t + 1 is θ_{t+1}.

When there are complete markets, general equilibrium requires that commodity demand equal commodity supply; that the excess demand for contingent claims be zero; and that the demand for each national money equal its outstanding stock. The market clearing conditions, which must hold for each history θ_t, are thus:

(6) $\sum_h c^h(\theta_t) = 3y$

(7) $\sum_h q^h(\theta_{t+1};\theta_t) = 0$, for each θ_{t+1}

(8) $\sum_h m_k^h(\theta_t) = M_k(\theta_t)$, k = 1, 2

where $M_k(\theta_t)$ is the supply of currency k in state θ_t. When securities trading is ruled out, only (6) and (8) are relevant.

We will say that an equilibrium is binding if (2) holds at equality and if the money holdings that satisfy (2) exhaust the supplies; or, equivalently, if (2) holds at equality and if person 1 does not hold currency 2 and person 2 does not hold currency 1. It follows immediately that in a binding equilibrium there are lower and upper bounds on the price of each money; namely, $p_k(\theta_t)M_k(\theta_t)$ ε [y,2y]. Given that we define the exchange rate, denoted $e(\theta_t)$, by $e(\theta_t) = p_1(\theta_t)/p_2(\theta)$, it follows that in a binding equilibrium $e(\theta_t)$ ε $(M_2(\theta_t)/M_1(\theta_t))[1/2,2]$. All our results for this model concern binding equilibria.

2. A Class of Equilibria Under Constant Currency Supplies

Here we are concerned with constructing a class of binding equilibria for our model economy under the assumption that the currency supplies are constant through time--and, for convenience, fixed at unity--and that there are no transfers. We assume that there is an exogenous, nonfundamental process ξ_t with the following simple probability structure: $pr(\xi_{t+1}=\xi^1;\theta_t) = pr(\xi_{t+1}=\xi^2;\theta_t) = 1/2$. This is an independently and identically distributed "sunspot." It follows that the state θ_t is the history of sunspots, $\theta_t = \{\xi_{t-s}\}$ for all $s \geq 0$, and that the probabilities of state evolutions in the preceding expressions, the $\pi(\theta_{t+1};\theta_t)$, are constant and equal to 1/2.

We consider as a candidate equilibrium process for the price of currency 1 a stochastic process of the form:

$$(9) \qquad p_1(\theta_{t+1}) = \begin{cases} p_1(\theta_t) + \alpha\big(p_1(\theta_t)\big) & \text{if } \xi_{t+1} = \xi^1 \\ p_1(\theta_t) - \delta\big(p_1(\theta_t)\big) & \text{if } \xi_{t+1} = \xi^2 \end{cases}$$

where α and δ are functions we will describe. Our candidate is also to be a binding equilibrium, one that satisfies $p_1(\theta_{t+1}) + p_2(\theta_{t+1}) = 3y$ and $p_1(\theta_{t+1})$ ε [y,2y]. Obviously, this bound constrains the functions α and δ.

Nonstochastic equilibria. Our example economy displays a multiplicity of binding equilibria in which price levels and exchange rates are nonstochastic. In terms of (9), this corresponds to equilibria with $\alpha = \delta \equiv 0$. This multiplicity originates in the fact that agents of type 3 regard the two currencies as perfect substitutes when their returns are equal. In particular, with constant fundamentals, the following constant and nonstochastic processes are a binding equilibrium for any $p_1 \varepsilon$ [y,2y]: $p_1(\theta_t) = p_1$, $p_2(\theta_t)$ $= p_2 = 3y-p_1$, $e(\theta_t) = e = p_1/p_2$, $s(\theta_{t+1};\theta_t) = \beta/2$, and $c^h(\theta_t) = y$. (This is a consequence of proposition 1, set out and proved below.) In such an equilibrium, the gross rate of return on each currency is constant over time at unity and there is a positive nominal and real gross interest rate equal to $1/\beta$. However, as noted, any price levels can prevail which are consistent with the requirement that agents of types h, for h = 1 and 2, hold a real value of currency equal to y. Since the price levels are indeterminate, the exchange rate is as well.

Complete contingent claims markets. When complete contingent claims can be traded in the securities market, there are equilibria with random values of the currencies, but with consumption nonrandom and the same as in the nonstochastic equilibria just described. These are equilibria in which

the _expected_ gross returns on the two currencies are unity, which in terms of (9) is the restriction $\alpha = \delta$. In fact, as the following proposition asserts, there is an equilibrium for any such pair, (α, δ), that satisfies some boundary conditions.

Proposition 1. Let $\alpha: [y, 2y] \rightarrow R_+$ and $\alpha(x) \leq \min(x-y, 2y-x)$. For any such α and $\delta = \alpha$ and any $p_1(\theta_0) \in [y, 2y]$, there is a binding equilibrium with $p_1(\theta_{t+1})$ given by (9).

The proof is given in the appendix. An example of a permissible function is shown in Figure 1. Bindingness implies that the graph of any such function be in the isosceles triangle with base $[y, 2y]$ and height $y/2$.

To summarize under complete markets, the equilibrium values of national monies may evolve according to many different stochastic processes, even within the simple class posited above. The common features of the equilibrium price process are (i) that expected gross returns to currency holding are equal to each unity for each currency; and (ii) that values of the monies are bounded by the demands of the restricted agents.

Prohibition on security trading. When there is no security trading, there are also binding equilibria with random values of currencies. However, these equilibria are quite different from the equilibria under complete contingent claims markets. We describe some of their features here, leaving the detailed construction to the proof of the proposition set out below.

Note that bindingness, $p_1(\theta_t) + p_2(\theta_t) = 3y$, implies that the speculator can always guarantee a sure return of unity on a portfolio by holding equal amounts of the two currencies. It also implies that all other choices imply a random return and, hence, by (1), random consumption. Since the speculator is risk averse, any such choice must give a higher-than-unity expected return on the portfolio. Bindingness also implies that the equilibrium

value of a currency varies directly with the speculator's holdings of that currency, and that $p_1 = p_2 = 3y/2$ when the speculator holds equal amounts of the two currencies. Together these imply that the speculator holds equal amounts of the two currencies if and only if the expected returns on the two currencies are equal; and that the speculator tilts holdings toward currency k if and only if currency k has a higher expected return than the other currency. From the implied relationship between currency values and expected returns it follows that a currency must appreciate on average whenever its value is high or that α and δ functions consistent with bindingness satisfy $\alpha(x) >(=)(<) \delta(x)$ as $x >(=)(<) 3y/2$ for $x \in (y,2y)$ (see Figure 2). Note, by the way, that these inequalities on α and δ suggest that most realizations of the process for $p_k(\theta_t)$ tend either toward y or 2y, one of the bounds.

In the appendix, we prove the following proposition:

Proposition 2. Absent security trading, there exist functions α and δ, each defined on $[y,2y]$ and positive except at the endpoints, such that for any $p_1(\theta_0) \in [y,2y]$ there is a binding equilibrium with $p_1(\theta_{t+1})$ given by (9).

Although this proposition says only that there is at least one nontrivial pair (α,δ), the proof makes clear that there is an uncountable set of them, but a set more tightly bounded and not as simply described as that in Proposition 1.

3. Tests for Currency Substitution.

We view standard tests for currency substitution as proceeding by regressing the real value of a currency on discrepancies in expected returns between this currency and others. Currency substitution is rejected if the regression coefficient is not positive and large. Here we comment on how these tests would fare on equilibrium observations generated by the two ver-

sions of our example economy. We first examine such regressions using the true difference in expected returns. We then examine them assuming that proxies for expected returns, functions of exchange rates, are used.[5]

Suppose first that the true model is the complete risk-sharing model with an equilibrium with nonrandom consumption and with random currency values satisfying (9). Since expected returns are equalized for all currency values in this equilibrium, no regression would be necessary (or could be computed). The currencies would appear as perfect substitutes.

Suppose alternatively that the true model is the incomplete risk-sharing version. We compute the expected returns for a simple example. It starts with the following α and δ functions in (9) over part of their domains; namely,

$$\delta(x) = (x-y)/2 \text{ for } x \in [y, 3y/2]$$

$$\alpha(x) = (2y-x)/2 \text{ for } x \in [3y/2, 2y].$$

The extensions of $\alpha(x)$ for $x \in [y, 3y/2]$ and $\delta(x)$ for $x \in [3y/2, 2y]$ are not arbitrary, but must satisfy equation (v) in the proof of Proposition 2 in the Appendix, which in turn makes them dependents on the type 3 person's marginal utility of consumption. We set $u'[c^3(\theta_t)] = c^3(\theta_t)^{-1}$ (logarithmic utility) and $y = 1$. The implied functions are shown in Figure 3. Note that they satisfy $\alpha(x) > (=)(<) \delta(x)$ as $x > (=)(<) 3y/2$ for $x \in [y, 2y]$.

Figure 4 shows the implied difference in expected returns, $E_t p_1(t+1)/p_1(t) - E_t p_2(t+1)/p_2(t)$, as a function of the real value of currency 1 at t, $p_1(t)$. The regression coefficient in a regression of $p_1(t)$ on the difference in expected returns will be very dependent on the sample. It is as easy to find a negative relationship as a positive relationship, and it seems

impossible to find no relationship, a zero regression coefficient. Needless to say, the regression coefficient does not reveal the true structure under- lying currency demands in the incomplete risk-sharing version.

The behavior of the difference in expected returns in Figure 4 is easy to interpret. It is negative when $p_1(t)$ is less than $3y/2$ (which happens when the speculator has a portfolio tilted toward currency 2) and is positive when $p_1(t)$ exceeds $3y/2$ (which happens when the speculator has a portfolio tilted toward currency 1). The absolute magnitude is explained by the risk aversion of the speculator and the amount of risk borne. The current real value of currency 1 is $3y/2$ when the speculator holds equal amounts of the two currencies and, as a consequence, bears no risk. That explains why the dif- ference is zero when $p_1(t) = 3y/2$. The speculator also bears no risk at each endpoint. At all other $p_1(t)$s, the speculator does bear risk and this and continuity more or less explain the shape of the function. Note, by the way, that the features noted in this discussion will hold for any binding equilib- rium with random exchange rates in the incomplete risk-sharing version.

Presumably because it is unclear which price indices to use to measure expected returns, it is common to measure differences between them using proxies that are functions only of exchange rate data. Note that the condition for equality of expected returns can be written as

(10) $E_t p_1(t+1)/E_t p_2(t+1) = p_1(t)/p_2(t)$

where E_t denotes mathematical expectation conditional on the state at t. Two procedures using exchange rate data seem to be common. One amounts to replac- ing the left side of (10) by the expectation of the exchange rate and the right side by the exchange rate. The other amounts to taking the logarithm of both sides of (10) and replacing the logarithm of the left side by the expec-

tation of the logarithm of the exchange rate. We now examine how regressions of the real value of a currency on such exchange rate proxies for expected return differences between currencies would fare on equilibrium observations generated by our example economy.

First, suppose that the true model is the complete risk-sharing model with an equilibrium with nonrandom consumption and with random currency values satisfying (9) for $\delta = \alpha$ and the following α function:

$$\alpha(x) = \begin{cases} (x-y)/2 \text{ for } x \ \epsilon \ [y,3y/2] \\ (2y-x)/2 \text{ for } x \ \epsilon \ [3y/2,2y] \end{cases}.$$

The results are reported in Figure 5 (exchange rate) and Figure 6 (log of exchange rate). (We have reversed the axes and plotted the exchange rate rather than the value of currency 1 on the horizontal axis for reasons to be discussed later.) If the exchange rate proxies were correct, the regression coefficient would always be infinite (the function would be identically zero). The Figures indicate that such an outcome is not likely. Depending on the sample, small positive or negative regression coefficients are possible, although a zero regression coefficient is not. Further, even if a large positive regression coefficient is obtained, the Figures show that the fit of the regression will be poor. In any case, the exchange rate proxies do not perform well as proxies for the true differences in expected returns in this model. Finally, the Figures also show that samples of observations from an equilibrium of this model would lead to rejection of the null hypothesis that the exchange rate (or the logarithm of the exchange rate) is a martingale, even though the true difference in returns is a martingale.

If the true model is the incomplete risk-sharing model described above, then the expected value of the exchange rate again performs poorly as a

proxy for the left-side of (10). The expected value of the logarithm of exchange rates seems to perform well as a proxy for the logarithm of the left-side of (10). That is, the true difference in expected returns and the difference between the expectation of the logarithm of the future exchange rate and the logarithm of the current exchange rate have the same general shapes as functions of the current exchange rate. Thus, regressions with this proxy would resemble those implied by Figure 4.

From these experiments we draw the following conclusions. First, if the true model is the complete risk sharing version, then correctly measured expected returns will imply that such returns are equalized. However, the use of exchange-rate proxies will not. Second, if the true model is the incomplete risk-sharing version, then the standard test for currency substitution, either using correctly measured expected returns or exchange-rate proxies, will not uncover anything like the true structure of currency demand.

4. The Role of Changing Currency Supplies

We now indicate how binding equilibria are affected by changing currency supplies given that the currency supplies follow deterministic paths. As noted above, if bindingness holds at t, then $e(\theta_t) \in \left(M_2(t)/M_1(t)\right)[1/2,2] \equiv [\underline{e}_t, \bar{e}_t] \equiv I_t$, where $M_k(t)$ is the supply of the kth currency at t. The idea behind the next proposition is that a necessary condition for diversification between two currencies at some date t is that the one-period return distribution at t be such that one currency does not dominate the other. This, in turn, implies that a necessary condition for bindingness and diversification at t is that $e_t \in I_{t+1}$; otherwise, one currency would necessarily appreciate or necessarily depreciate relative to the other between t and $t + 1$. Since e_t must also be in I_t, it follows that e_t is in the intersection of I_t and I_{t+1}. Working backwards, then, as shown below,

one gets the result that a necessary condition for bindingness and for diversification for all dates $t = 1, 2, \ldots, T$ is nonemptiness of the intersection of $I_1, I_2, \ldots, I_{T+1}$. If this is violated, then in a binding equilibrium, there is no diversification at some date t. This implies that e_t is either at $\underline{e}_t$ with probability one or at $\bar{e}_t$ with probability one, which, in turn, implies that the exchange rate is unique at all dates prior to t.

Proposition 3. Let V_T denote the intersection of $I_0, I_1, I_2, \ldots, I_T$.

(a) If for any $T \geq 0$, V_T is not empty, then $e(\theta_0) \in V_T$ in any binding equilibrium.

(b) If for any $T \geq 0$, V_T is empty, then $e(\theta_0)$ is unique in any binding equilibrium.

The proof makes use of the following lemma:

Lemma. Suppose it is known at t that $e(\theta_{t+1}) \in I_t \cap A_{t+1} = [\underline{a}, \bar{a}]$, $\underline{a} \leq \bar{a}$. Then in any binding equilibrium (i) if $I_t \cap A_{t+1}$ ($\cap$ denotes intersection) is not empty, then $e(\theta_t) \in I_t \cap A_{t+1}$; (ii) if $\bar{e}_t < \underline{a}$, then $e(\theta_t) = \bar{e}_t$; (iii) if $\underline{e}_t > \bar{a}$, then $e(\theta_t) = \underline{e}_t$.

Proof. To prove part (i), assume the contrary. Then $e(\theta_t) \notin A_{t+1}$. There are then two possibilities: either $e(\theta_t) < \underline{a}$ or $e(\theta_t) > \bar{a}$. In the former, it is certain that currency 1 will appreciate relative to currency 2 from t to $t + 1$. This implies that currency 2 is held only to meet constraints (2) or that $e(\theta_t) = \bar{e}_t$, which contradicts nonemptiness of $I_t \cap A_{t+1}$. The second possibility is ruled out by an analogous argument. Parts (ii) and (iii) are obvious. ◊

Proof of Proposition 3. We first define a set A_0 recursively as follows: Let $A_T = I_T$ and for $k \geq 1$, let

$$
A_{T-k} = \begin{cases} I_{T-k} \wedge A_{T-k+1} & \text{if this is not empty} \\ \bar{e}_{T-k} & \text{if (ii) of the lemma applies} \\ \underline{e}_{T-k} & \text{if (iii) of the lemma applies.} \end{cases}
$$

Given A_0, A_1, ..., A_T defined in this way, the lemma implies that $e(\theta_t) \in A_t$ for $t = 0, 1, 2, ..., T$.

If V_T is not empty, then $A_0 = V_T$. This establishes part (a). If V_T is empty, then A_t is a singleton for some t. This implies that A_0 is a singleton and establishes part (b). ◊

To illustrate the implications of Proposition 3 and the lemma, consider the following currency supply paths: $M_1(t) = M_1$ for all t, and $M_2(t) = M_2$ for $t = 1, 2, ..., K$ and $M_2(t+1) = \alpha M_2(t)$ for $t \geq K$ with $\alpha > 1$. It follows that for any t, there exists h such that the intersection of I_t, I_{t+1}, ..., I_{t+h} is empty. Thus, there is a unique binding equilibrium and the exchange rate path is deterministic in that equilibrium. Moreover, using the reasoning of the lemma, it is immediate that $e_t = \bar{e}_t$ for all t in that equilibrium so that the path of $M_2(t)/M_1(t)$ "explains" the path of e_t.

Thus, the model we have set out is consistent with exchange rates being well explained by fundamentals when those fundamentals are different enough in the sense that the I_t intervals vary enough over time. Put differently, in this model multiplicity of equilibria arises only if the two currencies have fundamentals that are similar in the sense that they imply that the I_t intervals do not vary too much over time.

In light of Proposition 3, it is tempting to propose generalizations of Propositions 1 and 2 to settings with deterministic currency supply paths. The main difficulty in establishing such results is insuring that the transfers (the $\tau^h(\theta_t)$ of (1)) that produce the currency supply changes are consistent with bindingness.

Proposition 3--and in particular, part (b)--must be amended if there are more than two currencies--for example, if there are K currencies and K + 1 people, with person h constrained to hold a minimum real value of currency h, h = 1, 2, ..., K and with person K + 1 constrained only to hold a minimum real value of total currency holdings. If V_T for a pair of currencies is empty, then it can be shown that the real value of one of the currencies is minimal; namely that $p_k(\theta_0)M_k(0) = y$ for one of the currencies. This, though does not imply that the exchange rate between such a currency and any other currency is unique; it does imply that some weighted average exchange rate for such a currency is unique in a binding equilibrium.

II. Normative Consequences of Nonfundamental Uncertainty

Propositions 1 and 2 show that a world in which some agents choose among currencies solely on the basis of prospective returns is subject to an extreme multiplicity of equilibria under floating exchange rates. However, those results only offer hints about whether the multiplicity is significant, or is, in some sense, innocuous. Here, we show that if there is full participation in complete risk sharing markets, then the multiplicity is innocuous in the sense of the following proposition.

Proposition 4. If an economy (a discrete-time, finite-number-of-states-at-each-date economy) with currencies 1, 2, ..., K has an equilibrium in which (a) anyone holding any of these currencies has access to complete contingent claims markets, and (b) $p_k(\theta_t) > 0$ and $p_j(\theta_t) > 0$ imply $r_k(\theta_t) = r_j(\theta_t)$, where $r_k(\theta_t) = \sum_{\theta_{t+1}} [s(\theta_{t+1};\theta_t)p_k(\theta_{t+1})]/p_k(\theta_t)$ (the interest factor on the k-th currency in state θ_t), then that economy with one currency (and hence, with no currency-specific constraints) has an equilibrium with the same consumption allocation.

We prove this in the appendix by constructing the one-currency equilibrium in two steps. In the first step, we start with any equilibrium satisfying the hypotheses, and, by adjusting individual portfolios only, construct an equilibrium in which everyone holds all the currencies in the same proportion at every date and state. In the second step, we take that equilibrium and, by adjusting prices of currencies and currency supplies only, construct an equilibrium with only the first of the K currencies having value.

The first step shows that the consumption allocation of the initial equilibrium could arise as an equilibrium in an economy in which there are no currency-specific constraints. In terms of the economy of Section I, the first step shows that any consumption allocation that is part of an equilibrium that satisfies hypotheses (a) and (b) could also be an equilibrium in an economy that is identical except that constraint (2) is replaced by $\sum_k p_k(\theta_t) m_k^h(\theta_t) \geq y$ for all h--an economy with no currency-specific constraints. The second step shows that the initial consumption allocation could be an equilibrium in a world economy with one currency. The first step implies that nothing about the initial equilibrium consumption allocation should be attributed to currency-specific constraints, while the second step shows that nothing about it should be attributed to the existence of several currencies and fluctuations in their relative values. The role of hypotheses (a) and (b) is to allow us to construct the one-currency equilibrium without adjusting any taxes (person-specific lump-sum taxes and direct taxes).[6]

Proposition 4 goes beyond what was established in Section I for the example economy, because we did not show there that any risk-sharing equilibrium has constant consumption. It establishes a feature of the entire set of equilibria for that and other economies in which some agents choose among currencies solely on the basis of prospective returns.[7] It says that if there

is full participation in risk-sharing markets then, despite the multiplicity of equilibria under a floating-rate system, that system cannot be indicted on welfare grounds because any equilibrium consumption allocation is also an equilibrium in a one-currency or fixed rate world.

The result depends, of course, on it being costless to participate in such markets. If it is not, then, as illustrated in an extreme way by Proposition 2, a floating rate system subjects people to risks that they would not have to bear under a fixed rate system. We should note, however, that our analysis implies that a fixed-rate system is subject to multiplicities for the equilibrium quantities of the different currencies linked by fixed exchange rates, and, hence, multiplicities in equilibrium reserves. If a fixed rate system is to work well, different outcomes for reserves must not be interpreted as requiring other adjustments.

III. Concluding Remarks

We have studied the positive and normative consequences of a somewhat unusual view of currency demands, the view that some people choose among currencies solely on the basis of their return distributions. There are two kinds of casual evidence that favor this speculative view of currency demands. One is the behavior of exchange rates; the volatility of exchange rates is difficult to explain if currency demands do not satisfy the speculative view. The other is the large amounts of the currencies of several countries held outside the respective countries of issue; the speculative view can be expected to apply to the choice among such foreign currencies.

The consequences of the speculative view for a floating rate system are not surprising. They follow from the fact that the component of demand that satisfies the speculative view is the "marginal" component that determines exchange rates if the fundamentals do not dictate corner solutions. If

they do not, then the "marginal" component implies the existence of a rich class of stochastic processes for exchange rates any member of which is a rational expectations equilibrium. The welfare consequences of such equilibria depend on whether there is complete participation in markets for hedging exchange-rate risk. If there is, then the randomness of exchange rates is innocuous. Otherwise, the randomness carries over to real wealth and consumption.

Footnotes

[1]Quite independently, Manuelli and Peck (1990) have generalized the Kareken–Wallace results to stochastic equilibria. They work with a model in which all agents are, in our terminology, part of the speculative fringe and show that the consumption allocation that is an equilibrium in a one–currency world can be supported in a multiple currency world by a large class of stochastic processes for exchange rates. Among our results, only Proposition 1 is closely related to theirs.

[2]This has been done by Barnett (1988).

[3]The latter is trivial. To establish the former note that by (1) and (2), $c^h(\theta_t) + \sum_{\theta_{t+1}} s(\theta_{t+1}; \theta_t) q^h(\theta_{t+1}, \theta_t)$ does not exceed the right–side of the securities market constraint. Since $c^h(\theta_t) \geq 0$, we can choose nonnegative $\tilde{m}_1^h(\theta_t)$ and $\tilde{m}_2^h(\theta_t)$ so the securities market constraint holds at equality; and, for such choices, $c^h(\theta_t) \leq \sum p_k(\theta_t) \tilde{m}_k^h(\theta_t)$, as required. Finally, it follows from (1) that such $\tilde{m}(\theta_t)$ choices satisfy the third constraint.

[4]This assumption could be weakened somewhat without affecting the results. Among other things, it insures that the constraint set includes positive consumption.

[5]Among the studies that test for currency substitution by using exchange rates to proxy for expected returns differences are those by Bordo and Choudhri (1982), Cuddington (1983), Brissimis and Leventakis (1985). Most of the evidence from these studies argues against currency substitution. For Canada, Bordo and Choudhri obtained estimated coefficients that have the correct sign but are statistically insignificant. For Canada, the U.K. and the U.S., Cuddington obtains estimated coefficients that either have the correct sign but are statistically insignificant or have the wrong sign and are statistically significant. Only for Germany does he obtain a statistically significant coefficient with the correct sign. For Greece, Brissimis and Leventakis obtained estimated coefficients that have the correct sign and are statistically significant. However, when interest rates are added to these equations the coefficients become insignificant. While Marquez (1987) also includes expected exchange rate changes in his theoretical model, they do not appear in his empirical investigation since he assumes expected changes to be zero.

[6]Without condition (b), we can duplicate a multiple currency equilibrium with a single currency equilibrium in general settings only if we can tax the money holdings of different people differently. In particular, if there are two currencies with different returns in the multiple currency equilibrium, then in the single currency analogue we would have to tax the money holdings of those who in the multiple currency equilibrium are forced to hold the lower return money. Our proof does not use any such taxes. Moreover, condition (b) is a consequence of the way we construct equilibria with nonfundamental exchange rate uncertainty. In that sense, it is not restrictive.

[7]Both the proposition and the proof are somewhat complicated because we permit nonfundamental uncertainty, as represented by the state θ_t, to affect real quantities like consumption. It is known that single currency versions of cash-in-advance models (see Woodford 1987) and single currency versions of overlapping generations models (see Azariadis 1981) can have equilibria in which

quantities like consumption. It is known that single currency versions of cash-in-advance models (see Woodford 1987) and single currency versions of overlapping generations models (see Azariadis 1981) can have equilibria in which nonfundamental uncertainty can affect allocations. Proposition 4 is applicable to multiple currency versions of such models.

Appendix

1. Proof of Proposition 1

The proof is constructive in that a candidate equilibrium is proposed and is shown to satisfy the equilibrium conditions.

The candidate for $p_1(\theta_t)$ is given in the statement of the proposition. For $p_2(\theta_t)$, we propose $p_2(\theta_t) = 3y - p_1(\theta_t)$ (as implied by bindingness). Given these prices, let nominal money holdings be those implied by (2) at equality and (8). The bound on α implies that such holdings are nonnegative.

We also propose $c^h(\theta_t) \equiv y$ and $\lambda_1^h(\theta_t) \equiv u'(y)$ so that (3) and (6) hold. Note that this proposal for consumption and that for prices implies that the left side of (4) is $\beta u'(y)$. Since $\beta < 1$, we can choose nonnegative multipliers for constraint (2) so that (4) holds.

We let the contingent claims prices be given by (5)--namely, $s(\theta_{t+1};\theta_t) = \beta/2$. Then we let $q^h(\theta_{t+1};\theta_t) = y - \sum_k p_k(\theta_{t+1})m_k^h(\theta_t)$ for each θ_{t+1}. This is bounded below by $-2y$ and above by y and makes the right-side of (1) equal to $2y$ for $t \geq 1$. (Our assumption about prices and initial conditions imply that the right-side of (1) equals $2y$ at $t=0$.) It remains only to show that the proposal for $q^h(\theta_{t+1};\theta_t)$ satisfies (7) and is such that (1) holds at equality, which requires that $\sum_{\theta_{t+1}} s(\theta_{t+1};\theta_t)q^h(\theta_{t+1};\theta_t) = 0$.

From our proposal for $q^h(\theta_{t+1};\theta_t)$,

$$\sum_h q^h(\theta_{t+1};\theta_t) = 3y - \sum_h [\sum_k p_k(\theta_{t+1})m_k^h(\theta_t)]$$

$$= 3y - \sum_k [\sum_h p_k(\theta_{t+1})m_k^h(\theta_t)]$$

$$= 3y - \sum_k p_k(\theta_{t+1})[\sum_h m_k^h(\theta_t)]$$

$$= 3y - \sum_k p_k(\theta_{t+1}) = 0.$$

Also from the proposal,

$$\sum_{\theta_{t+1}} s(\theta_{t+1};\theta_t) q^h(\theta_{t+1};\theta_t) = \beta\{y - \sum_{\theta_{t+1}} \pi(\theta_{t+1};\theta_t)[\sum_k p_k(\theta_{t+1}) m_k^h(\theta_t)]\}.$$

But,

$$\sum_{\theta_{t+1}} \pi(\theta_{t+1};\theta_t)[\sum_k p_k(\theta_{t+1}) m_k^h(\theta_t)] = \sum_k m_k^h(\theta_t)[\sum_{\theta_{t+1}} \pi(\theta_{t+1};\theta_t) p_k(\theta_{t+1})]$$

$$= \sum_k p_k(\theta_t) m_k^h(\theta_t) = y,$$

where the last equality follows from bindingness and the next to last from each expected return being unity.

Finally, as an aside, note that the gross nominal interest rate at t on a bond which pays one unit of currency k in each state at $t + 1$ is $p_k(\theta_t)/\sum_{\theta_{t+1}} s(\theta_{t+1};\theta_t) p_k(\theta_{t+1})$ and equals β^{-1} in the proposed equilibrium. $\Diamond$

2. Proof of Proposition 2

This proof is also constructive. Let K be the unique and positive solution to $\beta u'(y-K)(1+K/y) = u'(y+K)$. We start with a function δ defined on $[y,3y/2]$ and a function α defined on $[3y/2,2y]$, where these are nonnegative and bounded as in Proposition 1, are positive except at y and $2y$, respectively, are bounded above by K, satisfy $\alpha(3y/2) = \delta(3y/2)$, but are otherwise arbitrary. The bounding by K limits the return realizations on all possible portfolios so that every person in every period wants to spend all of financial wealth on consumption. We show that corresponding to any such pair (α,δ) is an equilibrium.

As above, our candidate equilibrium is a binding equilibrium--one with $p_1(\theta_t) + p_2(\theta_t) \equiv 3y$ and with nominal money holdings implied by such prices, (2) at equality, and (8). Letting $p_1(\theta_t) = x$, we begin by expressing

the distribution of consumption implied by (1) at equality and bindingness in terms of x and the functions α and δ. From (1) at equality,

(i) $\quad c^h(\theta_{t+1}) = \sum_k p_k(\theta_{t+1}) m_k^h(\theta_t) = \sum_k [p_k(\theta_t) m_k^h(\theta_t)][p_k(\theta_{t+1})/p_k(\theta_t)].$

Bindingness and (9) imply the following distributions for returns on monies, $p_k(\theta_{t+1})/p_k(\theta_t)$:

(ii)

k ξ	ξ^1	ξ^2
1	$1 + \alpha(x)/x$	$1 - \delta(x)/x$
2	$1 - \alpha(x)/(3y-x)$	$1 + \delta(x)/(3y-x)$

Also, for k = 1, 2

(iii) $\quad p_k(\theta_t) m_k^3(\theta_t) = p_k(\theta_t) M_k - p_k(\theta_t) m_k^k(\theta_t) = p_k(\theta_t) - y$

where the first equality follows from bindingness and (8), and the second from $M_k \equiv 1$ and (2) at equality.

Then (i)-(iii), bindingness and (1) at equality imply the following distributions for the ratio of consumption to y $\left(c^h(\theta_{t+1})/y\right)$:

(iv)

h ξ	ξ^1	ξ^2
1	$1+\alpha(x)/x$	$1-\delta(x)/x$
2	$1-\alpha(x)/(3y-x)$	$1+\delta(x)/(3y-x)$
3	$1+\alpha(x)(x-3y/2)/(3y-x)(x/2)$	$1-\delta(x)(x-3y/2)/(3y-x)(x/2)$

We now extend α and δ so that h = 3 diversifies--so that the left-side of (4) does not depend on k for h = 3. From (ii) and $\pi(\theta_{t+1};\theta_t) \equiv 1/2$, this condition can be written

(v) $\quad \alpha(x)u'\left[c^3(\theta_{t+1}; \xi_{t+1}= \xi^1)\right] = \delta(x)u'\left[c^3(\theta_{t+1}; \xi_{t+1}= \xi^2)\right].$

We first let $x \in (y, 3y/2)$ and extend α. For a fixed x, the right-side is determined by the given δ and is positive. Letting $\alpha(x) \equiv z$, for fixed x and $z \in R_+$, the left-side of (v), call it $H(z)$, is a continuous function of z (since u' is continuous). Moreover, $H(0) = 0$ and $H(\delta(x)) > \delta(x)u'[c^3(\theta_{t+1}; \xi_{t+1} = \xi^2)]$, where the inequality follows from (iv) and $u'' < 0$. By the intermediate value theorem, there exists $z \in (0, \delta(x))$ satisfying (v). The extension of δ to $(3y/2, 2y)$ so as to satisfy (v) is established in the same way and implies $\delta(x) \in (0, \alpha(x))$ for $x \in (3y/2, 2y)$. Note that this construction implies that the extended functions satisfy all the bounds imposed initially, a fact we use next.

We have now produced stochastic processes for prices, nominal money holdings, and consumptions that by construction satisfy constraints (1) and (2) at equality and the market clearing conditions, (6) and (8), and are such that the left-side of (4) does not depend on k for $h = 3$. To show that these constitute an equilibrium, it remains only to show that (3) and (4) hold-- namely, that we can choose nonnegative multipliers so that (3) and (4) hold.

We let $\lambda_1^h(\theta_t)$ be given by (3) at equality. Then (4) holds for nonnegative multipliers for constraints (2) if and only if

$$\text{(vi)} \qquad (\beta/2) \sum_j u'[c^h(\theta_{t+1}; \xi_{t+1} = \xi^j)] p_k(\theta_{t+1}; \xi_{t+1} = \xi^j)/p_k(\theta_t) \leq u'[c^h(\theta_t)]$$

for all h, k and θ_t. From (ii), $p_k(\theta_{t+1})/p_k(\theta_t) \in [1-K/y, 1+K/y]$, while from (iv) and our initial conditions it follows that $c^h(\theta_t) \in [y-K, y+K]$ for all h, k, and θ_t (or x). These bounds and $u'' < 0$ imply that the left-side of (vi) is bounded above by $\beta u'(y-K)(1+K/y)$ and that the right-side is bounded below by $u'(y+K)$. Since K is such that $\beta u'(y-K)(1+K/y) = u'(y+K)$, (vi) holds. ◊

3. Proof of Proposition 4

The proof consists of the following two lemmas. We use the notation of Section I except that we now use a symbol without subscripts or arguments to denote the entire corresponding vector across states, time, and, if relevant, monies and people. For example, p denotes the entire vector of prices of currencies across currencies, states, and time.

Lemma 1. If an economy has an equilibrium $(\hat{p}, \hat{m}, \hat{q}, \hat{M}; \hat{X})$ that satisfies the hypotheses of Proposition 4, then the same economy but without currency-specific constraints has an equilibrium $(\bar{p}, \bar{m}, \bar{q}, \bar{M}; \bar{X}) = (\hat{p}, \bar{m}, \bar{q}, \hat{M}; \hat{X})$, where $\bar{m}$ is given by

(i) $\qquad \bar{m}_1^h(\theta_t)/\hat{M}_1(\theta_t) = \sum_k \hat{p}_k(\theta_t)\hat{m}_k^h(\theta_t)/\sum_k \hat{p}_k(\theta_t)\hat{M}_k(\theta_t)$

and

(ii) $\qquad \bar{m}_k^h(\theta_t)/\bar{m}_1^h(\theta_t) = \hat{M}_k(\theta_t)/\hat{M}_1(\theta_t)$

and $\bar{q}$ is given by

(iii) $\qquad \sum_k \hat{p}_k(\theta_{t+1})\hat{m}_k^h(\theta_t)+\hat{q}^h(\theta_{t+1};\theta_t) = \sum_k \hat{p}_k(\theta_{t+1})\bar{m}_k^h(\theta_t) + \bar{q}^h(\theta_{t+1};\theta_t).$

Here X denotes all variables other than those that appear prior to the semi-colon. Thus, X includes consumption of all goods, holdings of all assets other than currencies and contingent claims (all of which are assumed to have payoffs denominated in terms of a numeraire good not in terms of currencies), and all prices including the contingent claims prices, but not including the prices of the currencies.

Proof. First we show that for each h, "-" satisfies any constraints on total currency holdings and is affordable. From (i) and (ii), it is immediate that

(iv) $\quad \sum_k \hat{p}_k(\theta_t) \bar{m}_k^h(\theta_t) = \sum_k \hat{p}_k(\theta_t) \hat{m}_k^h(\theta_t)$

so that any constraints on an individual's total currency holdings are met. To establish affordability, we show that (a possibly generalized version of) (1) is satisfied by "-". Given that "-" satisfies (iv) and is constructed to satisfy (iii), "-" will satisfy (1) if it satisfies

(v) $\quad \sum_{\theta_{t+1}} \hat{s}(\theta_{t+1};\theta_t) [\hat{q}^h(\theta_{t+1};\theta_t) - \bar{q}^h(\theta_{t+1};\theta_t)] = 0.$

To verify that (v) holds, multiply (iii), by $\hat{s}(\theta_{t+1};\theta_t)$ and sum over θ_{t+1}. Upon rearranging the order of summation, the terms involving $\hat{m}_k^h(\theta_t)$ (or $\bar{m}_k^h(\theta_t)$) can be written as $\sum_k m_k^h(\theta_t) p_k(\theta_t) r_k(\theta_t) = r(\theta_t) \sum_k m_k^h(\theta) p_k(\theta_t)$, where the equality uses hypothesis (b) of the Proposition. Then, using (iv), we get (v).

Next we note that "-" is utility maximizing. Since all prices are the same under "*" and "-", the set of affordable bundles is the same. It follows that, "-" is (remains) utility maximizing. (This is true even if h's utility depends on "total real balances" held by h.)

Finally, we have to show that $\bar{m}$ and $\bar{q}$ satisfy market clearing. Summing (i) over h implies $\sum_h \bar{m}_1^h(\theta_t) = \hat{M}_1(\theta_t)$. This and (ii) imply $\sum_h \bar{m}_k^h(\theta_t) = \hat{M}_k(\theta_t)$ for all k. Market clearing for $\bar{q}$ follows then from (iii) summed over h and from market-clearing for $\bar{m}$. ◊

Lemma 2. If $(\bar{p}, \bar{m}, \bar{q}, \bar{M}; \bar{X})$ is the equilibrium in the conclusion of Lemma 1, then $(\tilde{p}, \tilde{m}, \tilde{q}, \tilde{M}; \tilde{X}) = (\tilde{p}, \tilde{m}, \bar{q}, \tilde{M}; \bar{X})$ is also an equilibrium, where $\tilde{p}_k(\theta_t) = 0$ for k = 2, 3, ..., K and all θ_t, and

(vi) $\quad \tilde{p}_1(\theta_t) \tilde{M}_1(\theta_t) = \sum_k \bar{p}_k(\theta_t) \bar{M}_k(\theta_t); \quad t \geq 1$

(vii) $\quad \tilde{p}_1(\theta_t) [\tilde{M}_1(\theta_t) - \tilde{M}_1(\theta_{t-1})] = \sum_k \bar{p}_k(\theta_t) [\bar{M}_k(\theta_t) - \bar{M}_k(\theta_{t-1})], \quad t \geq 1$

(viii) $\tilde{M}_1(\theta_0) = \bar{M}_1(\theta_0)$

(ix) $\tilde{m}_1^h(\theta_t) = \sum_k \bar{m}_k^h(\theta_t)\bar{p}_k(\theta_t)/\tilde{p}_1(\theta_t)$.

(Note that (vi) and (vii) and the initial condition (viii) determine a unique $(\tilde{p}_1, \tilde{M}_1)$. It follows trivially, then, that $(\tilde{p}, \tilde{m}, \tilde{q}, \tilde{M}; \tilde{X})$ is uniquely determined by "-".)

<u>Proof</u>. Since condition (ix) holds individual real balances unchanged between "~" and "-", any constraint on an individual's total currency holdings are satisfied by "~".

As regards affordability of "~", since (ix) implies unchanged individual real balances, "~" satisfies (1) if we can show that "~" and "-" give rise to the same value for the right-side of (1). Since we hold taxes unchanged and $\tilde{q} = \bar{q}$, we have only to show that $\tilde{p}_1(\theta_{t+1})\tilde{m}_1^h(\theta_t) = \sum_k \bar{p}_k(\theta_{t+1})\bar{m}_k^h(\theta_t)$.

By (ii) (of Lemma 1),

$$\sum_k \bar{p}_k(\theta_{t+1})\bar{m}_k(\theta_t) = [\bar{m}_1^k(\theta_t)/\bar{M}_1(\theta_t)] \sum_k \bar{p}_k(\theta_{t+1})\bar{M}_k(\theta_t).$$

Since (vi) and (vii) imply

$$\tilde{p}_1(\theta_{t+1})\tilde{M}_1(\theta_t) = \sum_k \bar{p}_k(\theta_{t+1})\bar{M}_k(\theta_t),$$

we get

$$\sum_k \bar{p}_k(\theta_{t+1})\bar{m}_k^h(\theta_t) = \tilde{p}_1(\theta_{t+1})\bar{m}_1^k(\theta_t)\tilde{M}_1(\theta_t)/\bar{M}_1(\theta_t).$$

The last step is to verify that $\bar{m}^k(\theta_t)\tilde{M}_1(\theta_t)/\bar{M}_1(\theta_t) = \tilde{m}_1^k(\theta_t)$. This follows from (ix) upon first substituting into its right-side for $\bar{m}_k^h(\theta_t)$ the right side of (ii) and then using (vi).

We next show that $(\tilde{m}, \tilde{q}, \tilde{X})$ is utility maximizing among affordable bundles. Since $(\tilde{m}, \tilde{q}, \tilde{X})$ at $\tilde{p}$ gives h the same utility as does $(\bar{m}, \bar{q}, \bar{X})$ at $\bar{p}$, if there is an alternative, say (m^*, q^*, X^*), strictly preferred to $(\tilde{m}, \tilde{q}, \tilde{X})$ and affordable at $\tilde{p}$, then it gives higher utility than $(\bar{m}, \bar{q}, \bar{X})$ at $\bar{p}$. We show that this implies the existence of (m', q^*, X^*) affordable at $\bar{p}$ and strictly preferred to $(\bar{m}, \bar{q}, \bar{X})$, a contradiction.

Let m' be given by $m'^h_k(\theta_t) = m^{*h}_1(\theta_t)\bar{M}_k(\theta_t)/\bar{M}_1(\theta_t)$. Then (m', q^*, X^*) at $\bar{p}$ gives the same utility as (m^*, q^*, X^*) at $\tilde{p}$ if

(x) $\qquad \sum_k \bar{p}_k(\theta_t) m'^h_k(\theta_t) = \tilde{p}_1(\theta_t) m^{*h}_1(\theta_t)$ for all h and θ_t.

Affordability of (m', q^*, X^*) at $\bar{p}$ follows if (x) holds and if

(xi) $\qquad \sum_k \bar{p}_k(\theta_{t+1}) m'^h_k(\theta_t) \geq \tilde{p}_1(\theta_{t+1}) m^{*h}_1(\theta_t)$.

Equality (x) follows from the definition of m' and from (vi). Inequality (xi) follows from the definition of m', (vi), and (vii). Thus we have the desired contradiction.

Since the only quantities for which "~" differs from "-" are currency holdings, the only market clearing condition to verify is $\sum_h \tilde{m}^h_1(\theta_t) = \tilde{M}_1(\theta_t)$. Summing (ix) over h we get

$$\sum_h \tilde{m}^h_1(\theta_t) = \sum_k \bar{p}_k(\theta_t)\bar{M}_k(\theta_t)/\tilde{p}_1(\theta_t) = \tilde{M}_1(\theta_t),$$

where the first equality uses the fact that "-" is an equilibrium and the second uses (vi). Note that since (vii) holds total revenue from currency creation equal under "~" and "-", holding taxes and governments' real borrowing the same under "~" and "-" is consistent with satisfaction of an aggregate government cash-flow constraint. ◊

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Figure 1

Equilibrium change-in-price of currency 1 functions: complete markets

Upper bound

$y/2$

$\alpha = \delta$

0

y

$3y/2$

$2y$
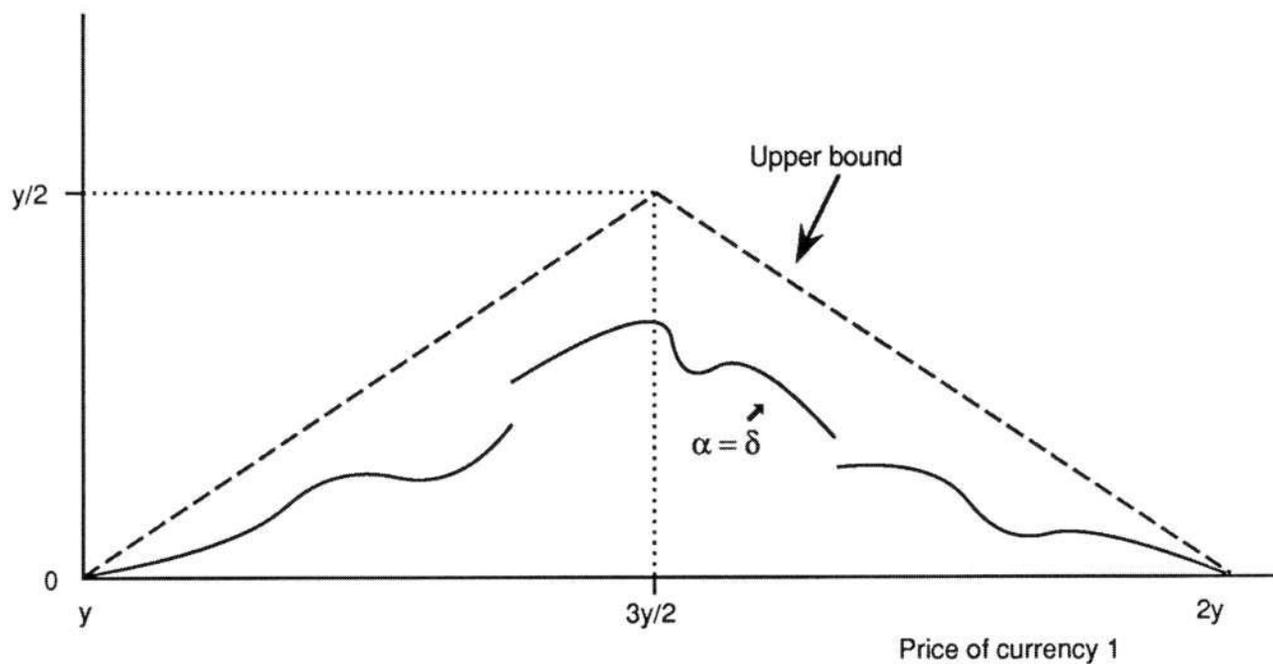
Price of currency 1

Figure 2

Equilibrium change-in-price of currency 1 functions: incomplete markets
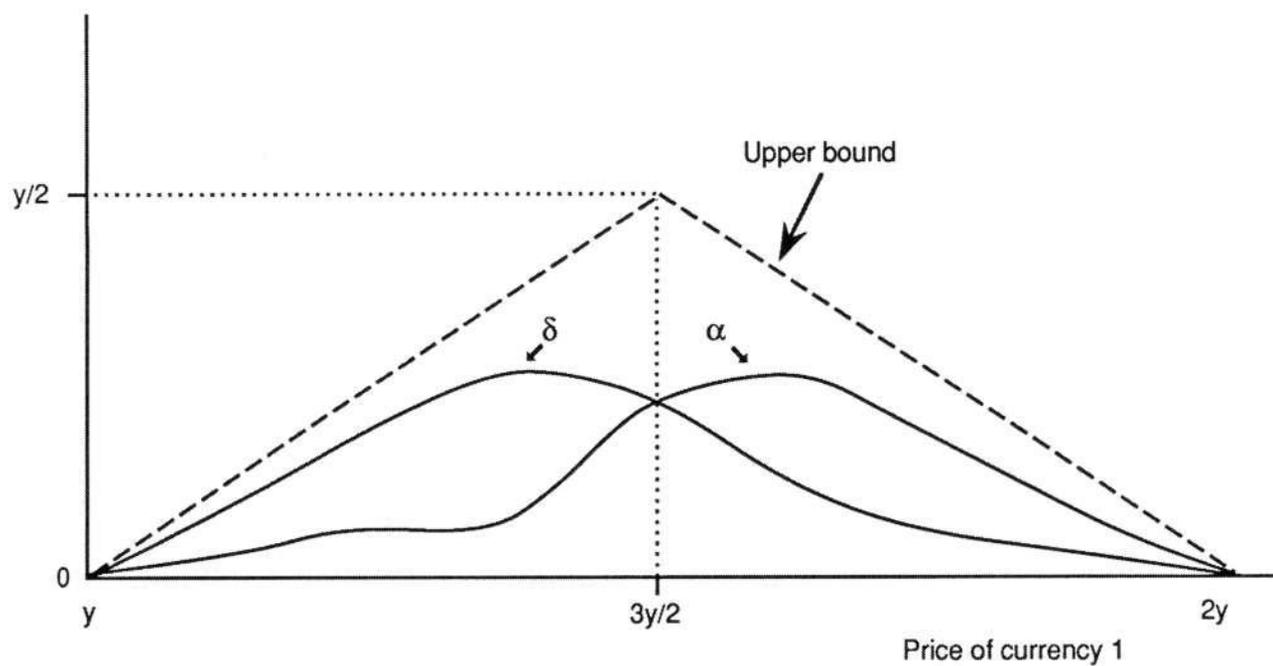

Upper bound

$y/2$

δ

α

0

y

$3y/2$

$2y$

Price of currency 1

Figure 3

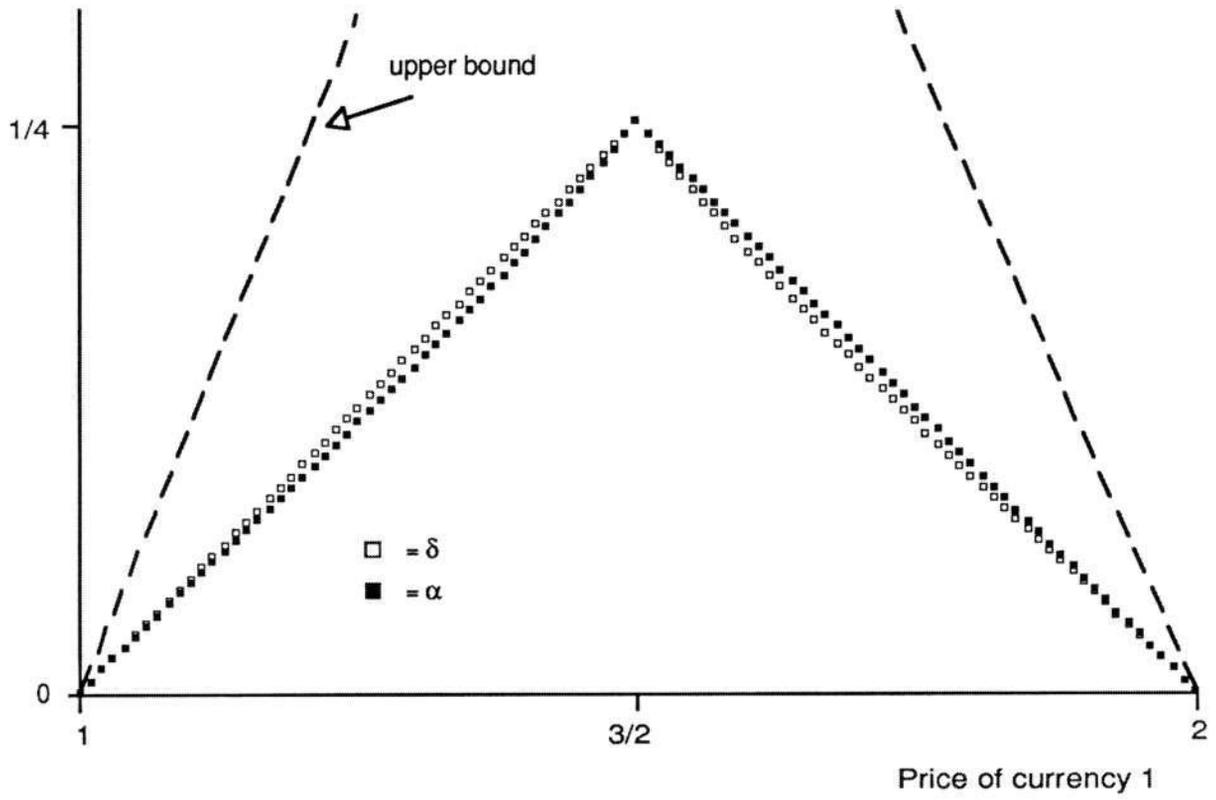## Equilibrium change-in-price of currency 1 functions: incomplete markets, logarithmic utility, and y=1

Figure 4

Equilibrium difference in expected return as a function of the price of currency1: incomplete markets, logarithmic utility, and y=1

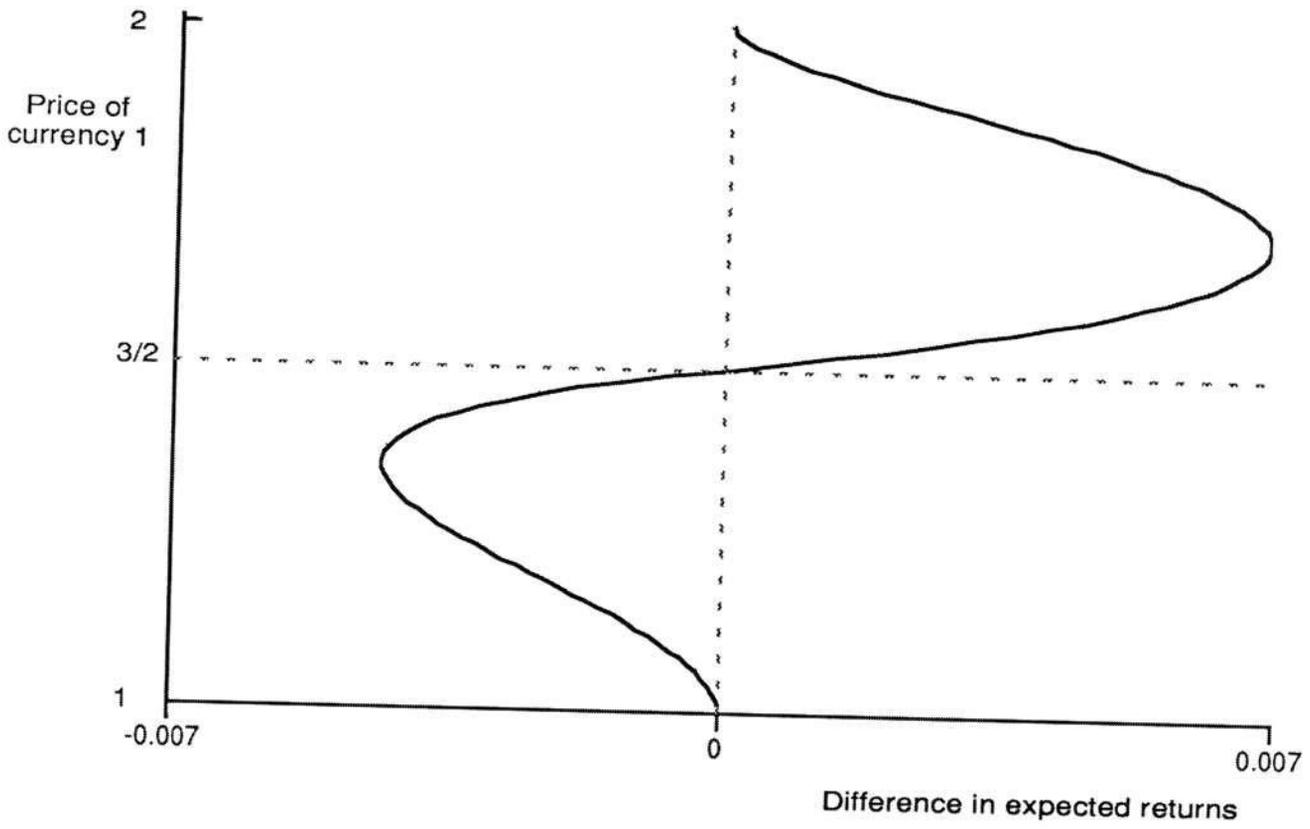# Figure 5

Expectation of change in the exchange rate: complete markets example
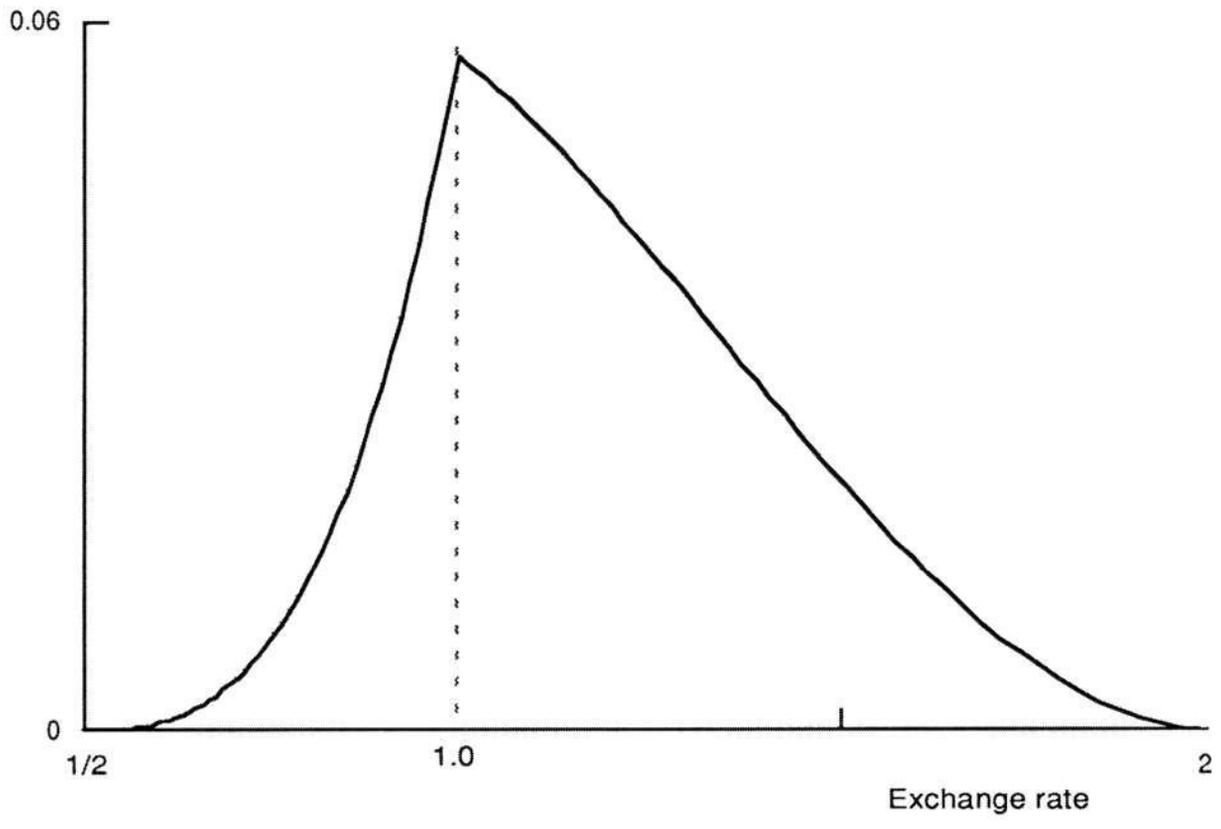
Figure 6

Expectation of change in the logarithm of the exchange rate: complete markets example