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VINTAGE HUMAN CAPITAL AND THE DIFFUSION OF NEW TECHNOLOGY

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ABSTRACT.

We present a model of vintage human capital. The economy exhibits exogenous deterministic technological change. Technology requires skills that are specific to the vintage. A stationary competitive equilibrium is defined and shown to exist and be unique, as well as Pareto optimal. The stationary equilibrium is characterized by an endogenous distribution of skilled workers across vintages. The distribution is shown to be single peaked and there is diffusion of technology in the sense that there is a lag between the time when a technology appears and the peak of its usage. An increase in the rate of exogenous technological change shifts the distribution of human capital to more recent vintages and increases the relative wage of the unskilled workers in each vintage.

Section 1. Introduction.

Neoclassical models of growth have proved to be useful in both theoretical and empirical analysis. The key source of growth in per capita output in such models is exogenous technological change which is often assumed to be disembodied. Casual empiricism suggests that actual technological change is embodied in very specific types of skills as well as specific types of physical capital. The Schumpeterian notion of 'creative destruction' relies heavily on capital specificities. In a world characterized by such specificities we would expect that new, more productive technologies will advance more slowly than they would in a world where all capital is costlessly transferable.

In this paper, we construct a model in which we consider the polar extreme of costlessly transferable capital. Indeed we assume that decisions on investment are irreversible. The first model of this kind of 'putty clay' capital was by Solow (1960) who examined situations where the types of capital could be aggregated and economy wide output represented by a single production function.

Our interest is to develop a model in which we can analyze the interaction between capital specificities and the rate of advance of new technology. It is undeniable that dramatic advances in technology (the invention of computers and word processors comes to mind) do not achieve large scale soon after invention. Even today we find the abacus or the typewriter useful for some purposes. In addition, as Mansfield (1968) points out "it took 20 years or more for all of the major firms [in several industries] to install centralized traffic control, car retarders, byproduct coke ovens and continuous annealing." None of these inventions were patentable by their users.

Clearly, the fact that the capital goods have already been produced implies that, in general, it would not be rational simply to destroy them. We are interested, however, in going a step further. We wish to examine circumstances under which even though in a sense newer technologies are superior, resources are used to create capital which is specific to older technologies. Continuing with our earlier example we want to understand not only why typewriters are <u>used</u> but why they continue to be <u>produced</u>.

One possible explanation for this phenomena is that an important component of capital is in the form of human capial. The skills involved in a production process are sometimes transferable only to a limited extent to new production processes. In addition, it is often true that these skills are acquired only by partici-pation in the production process itself.

These considerations prompt us to develop a model of human capital which is acquired in the process of production and is specific to the particular technology of production. In order to focus our attention on this problem we abstract from physical capital entirely. In addition, the learning of new skills or the transfer of existing knowledge presumably occurs largely from older workers to younger workers.

We use an overlapping generations model in which a new generation is born in each period and lives for two periods. There is a single commodity produced in each period. At the beginning of each period, a new technology becomes available which is more productive than any of the preexisting technologies. We assume that this exogenous technical change is deterministic and that the new technology is more productive by a constant factor γ than the technology that became available one period earlier.

Young workers must decide which vintage of technology they should enter. Once they have entered a particular vintage, by working in that vintage they acquire skills that are specific to that technology at the end of the first period of their lives. All workers who work in a given vintage acquire the same skills. The state of the economy is described by the distribution of skills across vintages. The hiring decisions of firms and the entry decisions of young workers implies a distribution for succeeding periods. The existence of complementarities in production between skilled and unskilled workers will attract unskilled workers to vintages that have skilled workers. Thus in equilibrium old technologies continue to be used.

We establish that, for such a model, a unique stationary competitive equilibrium exists and is Pareto optimal. The stationary equilibrium is characterized by an endogenous distribution of skilled workers across vintages. This distribution is single-peaked. Under fairly general conditions we show that there is a lag between the time that a technology appears and the peak level of output from the use of that technology. The wage rates for unskilled workers increases monotonically with the age of the vintage while the wage rates for skilled workers declines monotonically with the age of the vintage.

We also examine the effect of a change in γ, the rate of exogenous technological change. In a stationary state, γ is alo the growth rate of the economy. We show that an increase in this growth rate shifts the associated stationary distribution to more recent vintages. Furthermore, it reduces the time lag between the introduction of new technologies and the peak of their usage. In other words, in faster growing economies new technologies diffuse more rapidly.

We also show that an increase in the growth rate causes wage profiles over time for any given generation to become flatter. In a sense, therefore, the return from investing in human capital by working in newer vintages where current wages are lower falls for each individual. However, since the distribution of skilled workers also shifts to newer vintages in which future income is larger the effect on overall investment in human capital is ambiguous.

Jungenfelt (1986) develops a related model in which capital specificities arise solely because of the fact that workers must be trained to produce new products. The key variable in his model is the length of training time. He shows that an increase in this variable leads to an increase in the number of old products which continue to be produced. Since there are no complementarities in his model it cannot generate the result that resources are invested to create capital which is specific to old technologies.

We present the model in Section 2. In Section 3 we prove that a stationary equilibrium exists and is unique. In section 4 we characterize the equilibrium and prove some comparative steady state results. In section 5 we show that the stationary competitive equilibrium is Pareto optimal. Some concluding remarks are contained in section 6.

Section 2. The Model

We consider an overlapping generations model of agents who live for two periods. The set of agents born in each period is given by the interval [0,1] with uniform distribution. Our structure has the following features: i) There is an exogenous technological change whereby new technologies appear each period. ii) Agents can make investments specific to a vintage so that the new technologies are diffused by the optimal decisions of agents. In our model, these investments take the form of human capital.

A new technology appears in every period. This technology is given by the production function

$$\gamma^t f(N,Z)$$

where t denotes to the period in which the technology appeared, N is the input of unskilled workers and Z is the input of experienced workers.

(A.1) The following assumptions are made on the production function.

i. f has constant returns to scale

ii. $f(N,0)=\omega_o N$ where $\omega_o \geq 0$

iii. $f(.,Z)$ is strictly concave for each $Z>0$.

In every period there are two generations of workers who live for two periods each, the experienced (old) and the unskilled (young). Young workers can choose to work in only one vintage. Experience is acquired by working in a firm in a particular vintage as an unskilled worker when young and is specific to the vintage corresponding to the firm's technology. The amount of expertise acquired by two young agents working in a firm of the same vintage is exactly the same. This will simplify the decision problem of young agents -as will be detailed later- who will just have to choose which vintage to enter on the basis of the wage offered and the valuation that the market will give to their specific expertise in the following period. We also assume that

(A.2) Old agents have zero productivity in the unskilled tasks.

This is an assumption of convenience. It simplifies the analysis and the most relevant results obtained hold also when old agents are allowed to perform the tasks of young, unskilled workers.

Agents have preferences defined over the two periods where they live given by utility function

$u(c_1, c_2) = c_1 + \beta c_2$ where $0 < \beta < 1$.

As usual in growth models some bound on the rate of technological change must be given. The following assumption plays a key role only in the issue of the optimality of the equilibrium:

(A.3) $\beta\gamma < 1$,

As we mentioned earlier, when young, agents can work in only one vintage. In the following period, they will have acquired experience in that vintage. Hence in each period there is a distribution of old agents' exper-ience across existing vintages. To make this precise, it will be convenient to introduce the following notation:

The letter 't' will index time and the letter 'τ' will index the vintage of the technology, with the following interpretation:

When referring in period t to technology of vintage τ, we will be referring to the technology that appeared τ periods before. For example $\tau=2$ indicates the vintage that appeared in t-2. Notice also that the same vintage in period t+1 will have $\tau=3$.

Let μ_t be the distribution of experience of old agents across vintages $\tau \epsilon \{0, 1, 2, \ldots\}$. Thus $\mu_t(\tau)$ indicates the number (more precisely mass) of old agents with experience in vintage τ. These are the old people who when young worked in the vintage that appeared in $t-\tau$. Since there are no experienced workers in the 'just born' vintage, $\mu_t(0)=0$ for all t. We will often refer to μ_t as the <u>state of the economy</u>.

The existence of constant returns to scale makes irrelevant the number and distribution of property rights of firms in each vintage. For simplicity, and without loss of generality, we will assume that each old agent 'runs' a firm and competitively hires young agents.

Let $w(t,\tau,\mu_t)$ indicate the minimum wage needed to attract unskilled workers to vintage τ at period t when the state of the economy is μ_t. Old agents with skills τ solve the following problem:

$$(1) \quad v(t,\tau,\mu_t) = \max_{n\geq 0} \gamma^{t-\tau} f(n,1) - w(t,\tau,\mu_t)\, n$$

As a consequence of (A.1) there is a unique solution to the above problem which will be given by $n(t,\tau,\mu_t)$.

We will now analyze the decision problem faced by young agents born in period t. If they decide to enter vintage τ, their earnings in the following period will be

$$(2) \quad v(t+1,\tau+1,\mu_{t+1}),$$

since in the following period they will be skilled in vintage $\tau+1$.. Young agents will be assumed to have perfect foresight on the returns to experience in each vintage. Since they maximize discounted earnings, for them to be indifferent as to which vintage to enter the following must be satisfied:

$$w(t,1,\mu_t)+\beta v(t+1,2,\mu_{t+1})=w(t,0,\mu_t)+\beta v(t+1,1,\mu_{t+1})$$

$$w(t,2,\mu_t)+\beta v(t+1,3,\mu_{t+1})=w(t,1,\mu_t)+\beta v(t+1,2,\mu_{t+1})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

(3) $$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$w(t,\tau,\mu_t)+\beta v(t+1,\tau+1,\mu_{t+1})=w(t,\tau-1,\mu_t)+\beta v(t+1,\tau,\mu_{t+1})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Let $y_o(t,\mu_t)$ denote the mass of entrants in the new vintage at period t when the state of the economy is μ_t. Since $f(0,1)=\omega_o$, output and consumption for a young agent that enters this vintage is given by $\gamma^t\omega_o$. Thus there will be no incentives to enter the new vintage unless

$$\gamma^t\omega_o\geq w(t,0,\mu_t)$$

where $w(t,0,\mu_t)$ is given by equation (3).

In order to complete the description of the environment, we will assume that at period 0 there is a set of old agents indexed by [0,1] with uniform distribution. We assume also that they have experience on a set of existing technologies and that the corresponding distribution of expertise is given by μ_o. Thus $\mu_o(\tau)$ is the mass of those workers experienced in vintage $-\tau$, i.e. the vintage with production function $\gamma^{-\tau}f(n,z)$.

We can now define an equilibrium for this economy.

<u>Definition</u>: A <u>competitive equilibrium</u> for this economy is:

a. a wage function $w(t,\tau,\mu_t)$

b. an employment function $n(t,\tau,\mu_t)$ and new vintage entry function $y_o(t,\mu_t)$.

c. a sequence of distribution functions $\{\mu_t\}$

such that:

i. $n(t,\tau,\mu_t)$ is an optimal response to $w(t,\tau,\mu_t)$, i.e.it satisfies (1).

ii. $w(t,\tau,\mu_t)$ makes young agents indifferent as to which vintage to enter, i.e. $w(t,\tau,\mu_t)$ satisfies equation (3) and $w(t,0,\mu_t)\geq\gamma^t\omega_0$.

iii. $\sum_{t=1}^{\infty} n(t,\tau,\mu_t)\mu_t(\tau) = 1-y_0(t,\mu_t)$.

where $\gamma^t\omega_0<w(t,0,\mu_t)$ implies $y_0(t,\mu_t)=0$.

iv. $\mu_{t+1}(\tau)=n(t,\tau-1,\mu_t)\mu_t(\tau-1)$ for $\tau\geq2$ and $\mu_{t+1}(1)=y_0(t,\mu_t)$.

Conditions i. and ii. state that agents make their decisions optimally. Condition iii. is the labor market clearing condition. Condition iv. states that the law of motion for μ_t is precisely the one generated by the optimal rules described.

In the rest of the paper we will concentrate our attention on the <u>stationary equilibrium</u>, i.e. a competitive equilibrium with the additional condition:

v. $\mu_t=\mu_{t+1}=\mu$ for all t.

We will establish that a stationary distribution exists and is unique. Then we will analyze the properties the economy has if it were at a stationary equilibrium.

Section 3. Existence of a Stationary Equilibrium.

We will first establish some necessary and sufficient conditions for the existence of a stationary equilibrium. Then we will show that under the assumptions made, these conditions are satisfied by a unique set of equilibrium values. Since in the stationary equilibrium μ_t is constant, we will supress μ_t as an argument to the functions defined above.

Proposition 1 Suppose $w^*(.)$, $n^*(.)$, $y_o^*(.)$ and μ^* is a stationary equilibrium. Then

1. $n^*(t,\tau)$ is independent of t.
2. $n^*(t,\tau)>0$ implies $n^*(t,\tau')>0$ for all $\tau'\leq\tau$.
3. $y_o^*(t)$ is independent of t and $y_o>0$.
4. $w^*(t,\tau)=\gamma^t w^*(0,\tau)$ for all t and $w^*(0,0)=\omega_o$.
5. $v^*(t,\tau)=\gamma^t v^*(0,\tau)$ all t.

Proof. 1. Since $\mu(\tau+1)=\mu(\tau)n^*(t,\tau)$ for any t, $n^*(.)$ must be independent of t.

2. If $n(t,\tau')=0$, then $\mu(\tau)=0$ for all $\tau>\tau'$. But then $\gamma^t\omega_o>\gamma^{t-\tau'}\omega_o$ and so young people will not enter vintage τ'. Hence $n^*(t,\tau')=0$.

3. Since $\mu(1)=y_o(t)$, y_o must be constant. If $y_o=0$, then $\mu(\tau)=0$ for all τ. But if that is the case, $y_o=1$.

4. Note that

$$n(t,\tau) = \text{argmax } \gamma^t f(n,1)-w(t,\tau)n$$
$$= \text{argmax } f(n,1)-[w(t,\tau)/\gamma^t]\ n$$

Since $f(.,1)$ is strictly concave, if $n>0$ then it is strictly decreasing in w. But then for $n(t,\tau)$ to be independent of t it must be the case that $n(t,\tau)=0$ or that $w(t,\tau)/\gamma^t$ is independent of t. Hence $w(t,\tau)=\gamma^t w(0,\tau)$ for all t. If $w_o>\omega_o$ then $y_o=0$ which we just proved cannot be true.

5. $$v(t,\tau) = \max \gamma^t f(n,1)-\gamma^t w(0,\tau)n$$
$$= \gamma^t \max f(n,1)-w(0,\tau)n$$
$$= \gamma^t v(0,\tau).$$

As a consequence of the above proposition, we can simplify considerably the notation employed from now on. We have supressed the μ_t arguments and we can now supress the t arguments from the functions used. This leaves us only with 'τ' as the only argument. We will thus define:

$n_\tau = n(t,\tau,\mu)$

$w_\tau = w(t,\tau,\mu)$ and

$v_\tau = v(t,\tau,\mu)$.

$\mu_\tau = \mu(\tau)$.

Given these facts it is now possible to rewrite equation (3). It will be convenient to write the profit v_τ as a function of the wage w_τ. Then using equation (3) and Proposition 1 we have that a necessary condition for a stationary equilibrium is that there exist sequences $\{w_\tau\}$ which satisfy:

$$
(4)\quad
\begin{aligned}
&w_1+\beta\gamma v_2(w_2) = w_o+\beta\gamma v_1(w_1)\\
&w_2+\beta\gamma v_3(w_3) = w_1+\beta\gamma v_2(w_2)\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
&w_\tau+\beta\gamma v_{\tau+1}(w_{\tau+1}) = w_{\tau-1}+\beta\gamma v_\tau(w_\tau)\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots
\end{aligned}
$$

subject to $w_\tau \geq 0$.

where $v_\tau(w) = \max_n \gamma^{-\tau} f(n,1)-wn$

Suppose for the moment we could find a sequence $\{w_\tau\}_{\tau=1}^{\infty}$ that satisfied equation (4) with $w_o=\omega_o$. This would imply a sequence $\{n_\tau\}_{\tau=1}^{\infty}$ given by $n_\tau = \text{argmax}\ \gamma^{-\tau} f(n,1)-w_\tau n$, the optimal input decision if there are specialized workers in vintage τ. Market clearing requires that

$$(5)\quad \Sigma n_\tau \mu_\tau = 1-y_o$$

Using the fact that $\mu_{\tau+1}=\mu_\tau n_\tau$ and $\mu_1=y_o$, given y_o (or μ_1) we obtain by induction the whole sequence $\{\mu_\tau\}$ by setting $\mu(\tau) = \mu_1 \prod_{\tau'=1}^{\tau-1} n_{\tau'}$.

Then

$$(6)\quad \mu_\tau n_\tau = (\mu_1 \prod_{\tau'=1}^{\tau-1} n_{\tau'}) n_\tau = \mu_1 \prod_{\tau'=1}^{\tau} n_{\tau'}$$

and thus equation (5) can be rewritten as

$$(7)\quad \mu_1 \sum_{\tau=1}^{\infty} \prod_{\tau'=1}^{\tau} n_{\tau'} = 1-\mu_1$$

If $\mu_1=0$ the left side is equal to zero and the right side equals one. If μ_t equals one then the right side equals zero. The left side is continuous and nondecreasing in μ_1 and the right side is strictly decreasing. Hence as long as equation (7) is well defined, then given $\{n_\tau\}$ there exists a unique μ_1 consistent with this equation. As will be shown in the appendix, n_τ decreases monotonically to zero. Hence there is a T such that $n_t<1$ and so

$$\begin{aligned}\mu_1 \sum_{\tau=1}^{\infty} \prod_{\tau'=1}^{\tau} n_\tau &= \mu_1 \sum_{\tau=1}^{T} \prod_{\tau'=1}^{\tau} n_\tau + \mu_T \prod_{\tau'=1}^{T} n_\tau \sum_{\tau=T+1}^{\infty} \prod_{\tau'=T+1}^{\tau} n_\tau \\ &\le \mu_1 \sum_{\tau=1}^{T} \prod_{\tau'=1}^{\tau} n_\tau + \mu_T \prod_{\tau'=1}^{T} n_\tau \sum_{\tau=0}^{\infty} n_T^\tau \\ &< \infty .\end{aligned}$$

so that equation (7) is well defined.

This suggests the following procedure for finding an equilibrium:

<u>Step 1</u>. Obtain a solution to equation (4).

<u>Step 2</u>. Find the corresponding input demands $\{n_\tau\}$

<u>Step 3</u>. Find μ_1 from equation (7).

This is summarized in the following proposition.

<u>Propositon 2</u> $w^*(.)$, $n^*(.)$, $y_o^*(.)$ and μ^* is a stationary equilibrium if and only if there exist $\{w_\tau\}$, $\{n_\tau\}$ and μ_1 such that

$\mu^*(\tau)$ is given by equation (6)

$n^*(t,\tau,\mu)=n_\tau$ for all t

$y^*(t)=\mu_1$

$w^*(t,\tau,\mu)=\gamma^t w_\tau$ for all t

$\{w_\tau\}$ satisfies equation (4) and $w_o=\omega_o$, n_τ are optimal input decisions given w_τ and μ_1 and $\{n_\tau\}$ satisfy equation (7).

Proof. That these conditions are sufficient can easily be checked. By Proposition 1, $w^*(t,\tau,\mu)=\gamma^t w^*(0,\tau,\mu)$. Hence we can set w_τ = $w^*(t,\tau,\mu)$ and also by Proposition 1 $w_o=\omega_o$. The construction above shows that the rest of the conditions follow from this one □.

The next proposition states the existence and uniqueness of a stationary equilibrium.

Theorem 1. There exists a unique stationary competitive equilibrium.

Proof. Follows immediately from Proposition A.1 in the appendix and Proposition 2.

Section 4. Properties of the equilibrium:

The first question to be asked is about the distribution of skilled workers across vintages. We show in the appendix that the wage rate of unskilled workers, w_τ, is increasing in the age of the vintage and that the wage paid to skilled workers, v_τ, is decreasing in the age of the vintage. Since productivity is decreasing with the age of the vintage it follows that n_τ is decreasing in τ. We show in Proposition 3 below that if an Inada condition is satisfied then all vintages are used in a stationary equilibrium.

Proposition 3. If $f_1(0,1)=\infty$ then $\mu_\tau>0$ for all τ. Otherwise there exists some T such that $\mu_\tau>0$ if and only if $\tau\leq T$.

Proof. Let τ be the smallest number such that $\mu_\tau=0$. Recall that in a stationary equilibrium $\mu_\tau=\mu_1 \prod_{\tau'=1}^{\tau-1} n_{\tau'}$, and $\mu_1=y_o$. We have already established that $y_o>0$. Consequently if $\mu_\tau=0$ then $\mu_{\tau'}=0$ for all $\tau'\geq\tau$ and $n_{\tau-1}=0$. If $f_1(0,1)=\infty$ then for any finite $w_{\tau-1}$, $n_{\tau-1}>0$.

On the other hand, suppose that $f_1(0,1)<\infty$. A necessary condition for an equilibrium is that if $n_\tau>0$ then $w_\tau=\gamma^{-\tau}f_1(n_\tau,1)\leq \gamma^{-\tau}f_1(0,1)$. Consequently w_τ must converge to zero. However since w_τ is an increasing sequence it must be bounded away from zero □.

Clearly, the first part of Proposition 3 depends critically on assumption A.2. If old agents could work as unskilled workers in any vintage then the wages of unskilled workers cannot exceed the wages of skilled workers. Therefore we would need to impose the condition that $v_\tau\geq w_{\tau'}$ for all τ,τ'. In this case the number of vintages will be finite. Other than that, none of our results change.

It is of interest to examine the shape of the distribution of skilled workers as well as the distribution of output. We establish below that employment of skilled workers will rise and then fall with the age of the vintage.

Proposition 4 (Single peakedness). There exists T such that for all $\tau \leq T$ $\mu_\tau \geq \mu_{\tau-1}$ and for $\tau > T$ $\mu_\tau \leq \mu_{\tau-1}$. Furthermore, if $\omega_0 = 0$ and $\beta\gamma \leq \frac{f_1(1,1)}{f_2(1,1)}$ then $T \geq 2$.

Proof. We have established that n_τ is decreasing in τ. Let T be the smallest τ such that $n_\tau < 1$. Recall that $\mu_\tau = \mu_{\tau-1} n_{\tau-1}$. Consequently, for $\tau \leq T$ $\mu_\tau \geq \mu_{\tau-1}$ and for $\tau > T$ $\mu_\tau \leq \mu_{\tau-1}$.

We have from equation (4) and $\omega_0 = 0$ that

$$\beta\gamma v_1(w_1) = w_1 + \beta\gamma v_2(w_2).$$

By the definition of v(.) and the fact that $w_1 = \gamma^{-1} f_1(n_1, 1)$ we have that

$$\beta[f(n_1,1) - f_1(n_1,1)n_1] \geq \frac{1}{\gamma} f_1(n_1,1).$$

Therefore

$$(8) \qquad \frac{\beta\gamma f_2(n_1,1)}{f_1(n_1,1)} \geq 1.$$

The numerator of this inequality is increasing in n_1 and the denominator is decreasing. Hence, if

$$(9) \qquad \beta\gamma \leq \frac{f_1(1,1)}{f_2(1,1)}$$

then $n_1 > 1$ □.

We have established that under mild conditions the peak of the distributions of skilled workers will occur for some $T \geq 2$. In order to obtain sharper results about the peak of this distribution as well as results about the peak of the distribution of output we consider a particular production function. Assume that the production function is Cobb-Douglas; $f(n,z) = n^\alpha z^{1-\alpha}$. It is plausible to assume that $\alpha \leq 1/2$. Inequality (9) which guarantees that $T \geq 2$ can then be written as $\frac{\beta\gamma(1-\alpha)}{\alpha} \leq 1$. In Proposition 5 below we strengthen this condition to ensure that $T \geq 3$ and the peak of the distribution of output occurs at vintage $\tau \geq 2$.

Proposition 5. If $\frac{\beta\gamma(1-\alpha)}{\alpha} + \gamma\left[\frac{\beta\gamma(1-\alpha)}{\alpha}\right]^{1-\alpha} \leq 1$ then $T\geq 3$ and the peak of the output distribution occurs at a vintage $\tau\geq 2$.

Proof. It follows from inequality (8) that $n_1\geq\alpha/\beta\gamma(1-\alpha)$. Since $\alpha<1$, $n_1^{\alpha-1}\leq[\alpha/\beta\gamma(1-\alpha)]^{\alpha-1}$. From equation (4) we have that $w_2\leq w_1+\beta\gamma v_2(w_2)$. Hence using the fact that $w_\tau=\gamma^{-\tau}f_1(n_\tau,1)$ we have that

$$n_2^{\alpha-1}\leq\gamma n_1^{\alpha-1}+\frac{\beta\gamma(1-\alpha)}{\alpha}n_2^{\alpha}.$$

It follows that

$$(10)\quad n_2^{\alpha-1}\leq\gamma\left[\frac{\beta\gamma(1-\alpha)}{\alpha}\right]^{1-\alpha}+\frac{\beta\gamma(1-\alpha)}{\alpha}n_2^{\alpha}.$$

Suppose that $n_2<1$. Then the right side of inequality (10) is at most 1. Hence $n_2^{\alpha-1}\leq 1$ so $n_2>1$.

Note that output at vintage τ is given by $\gamma^{-\tau}f(\mu_\tau n_\tau,\mu_\tau)$. Therefore output at vintage 2 is greater than output at vintage 1 if and only if

$$(11)\quad \gamma^{-1}f(\mu_2 n_2,\mu_2)>f(\mu_1 n_1,\mu_1).$$

Since $\mu_2=\mu_1 n_1$, inequality (11) is satisfied iff $\gamma^{-1}n_2^{\alpha}>n_1^{\alpha-1}$. Recall however that $w_2\geq w_1$. Hence $\alpha\gamma^{-1}n_2^{\alpha-1}\geq\alpha n_1^{\alpha-1}$. But $n_2>1$. Therefore $\gamma^{-1}n_2^{\alpha}>n_1^{\alpha-1}$ □.

It is of interest to examine the effect of a change in the rate of technological change on the stationary distribution. Our main result is that when $\gamma'>\gamma$ then the distribution corresponding to the higher growth rate, say μ', will be dominated in the sense of stochastic dominance by the original distribution. In other words, when the growth rate increases the distribution of skilled workers is concentrated among more recent vintages. This also implies that the rate of diffusion of new technologies change is higher if the economy grows more rapidly.

<u>Proposition 6</u>. Consider two economies with $\gamma'>\gamma$. Let μ',μ denote the respective stationary distributions. Then μ stochastically dominates μ', i.e.

$$\sum_{\tau=1}^{t} \mu_\tau \leq \sum_{\tau=1}^{t} \mu'_\tau \text{ for all } t.$$

<u>Proof</u>.

In Proposition A.2 in the appendix we prove that $n(\tau,\mu')<n(\tau,\mu)$. Hence if $\mu'(1)\leq\mu(1)$ then $\mu'(\tau)<\mu(\tau)$ for all $\tau>1$ and hence $\sum_{\tau=1}^{\infty} n_\tau \mu'_\tau<1-\mu_1$ and thus $\mu'_1>\mu_1$. Let $T=\{\min t:\mu'(t)<\mu(t)\}$. Then $T\geq2$. For $t<T$, $\sum_{\tau=1}^{t} \mu'(\tau)> \sum_{\tau=1}^{t} \mu(t)$. Since $n(t,\mu')<n(t,\mu)$ for all t and $\mu'(T)<\mu(T)$, by construction $\mu'(t)<\mu(t)$ for all $t\geq T$. But then for any $t\geq T$,

$$\sum_{\tau=1}^{t} \mu'(\tau) = 1 - \sum_{\tau>t} \mu'(\tau) > 1 - \sum_{\tau>t} \mu(\tau) = \sum_{\tau=1}^{t} \mu(\tau),$$

which establishes the result □.

Our next result shows that the earnings profile becomes flatter as the growth rate of the economy increases. As shown in Proposition A.2 of the appendix

$$w(t,\mu')\geq \left[\frac{\gamma}{\gamma'}\right]^t w(t,\mu).$$

But

$$v'_t(w)=\max_n \gamma'^{-t}f(n)-wn \leq \left[\frac{\gamma}{\gamma'}\right]^t \left[\gamma^{-t}f(n)-wn\right]\leq \left[\frac{\gamma}{\gamma'}\right]^t v_t(w)$$

Since $\gamma' > \gamma$,

$$(12) \quad \frac{\gamma' v'_{\tau+1}(w'_{\tau+1})}{w'_{\tau}} \leq \frac{\gamma v_{\tau+1}(w_{\tau+1})}{w_{\tau}}$$

This establishes that the earnings profile becomes flatter with a higher growth rate. One measure of investment in human capital is foregone earnings. If an individual joins a sufficiently old vintage we have shown that future earnings will be close to zero and current wages will be high. This individual would then be making very small investments in human capital. The present value of income is equated across vin-tages. Hence individuals who join new vintages will be making large investments in human capital. These can be measured by the ratio of future earnings to the wage that implies no investment. Of course the latter equals the present value of earnings. Inequality (12) implies that this measure of investment in human capital declines. However, note that aggregate investment in human capital does not necessarily fall since the distribution of skill levels shifts to vintages with higher rates of investment.

Section 5. Optimality of the Competitive Equilibrium.

In this section, following a standard approach in overlapping generations(OLG) models, we will construct an Arrow-Debreu economy that corresponds to the OLG environment. We show that the stationary equilibrium of the OLG economy is a competitive equilibrium for the Arrow-Debreu economy and that it is Pareto optimal. In our specific problem there is an additional complication to the standard case. There is not only an infinite number of goods and agents but also infinite technology sets, one for each vintage.

The commodity space will be

$$L = \{(\{c_t\}_{t=o}^{\infty}, \{n_t\}_{t=o}^{\infty}, \{z_{-k}\}_{k=1}^{\infty}) \text{ where } (c_t, n_t, z_{-k}) \in R^3\}$$

The commodity space is the linear space of sequences in R^3.

There will be one agent for each generation indexed by $i \in [-1, 0, 1, \ldots \infty)$. For $i \geq 0$, i indexes the generation born at $t=i$ and $i=-1$ corresponds to the old at $t=0$. For each i, the con-sumption set will be a subset of L of the following form:

$$X_i = \{(\{c_t\}_{t=o}^{\infty}, \{-n_t\}_{t=o}^{\infty}, \{-z_{-k}\}_{k=1}^{\infty}) \text{ with } c_t \geq 0,\ n_t = 0 \text{ for } t \neq i \text{ and } -1 \leq -n_i \leq 0\}$$

with the following endowments :

$z^i_{-k} = \mu_k$ for all k if $i=-1$ and $z^i_{-k} = 0$ otherwise.

Note that we follow the usual convention of denoting consumers' inputs by positive numbers and their outputs by negative numbers while the reverse criterion is used for firms.

Preferences must be assigned which are consistent with the ones agents have over consumption in the two periods they live in the OLG economy. This is done by setting

$$U_i(\{c_t\}) = c_i + \beta c_{i+1} \ .$$

We will assume that one firm operates each vintage. Naturally we have to restrict firms operations to those periods where its vintage has already appeared. With this proviso, the technology sets will be:

$$Y_j = \{(\{q_t^j\},\{-n_t^j\},\{-z_{-k}^j\}) \text{ with } -n_t^j \leq 0 \text{ for } j \leq t \text{ and } n_t^j = 0 \text{ if } j > t.$$

$$q_t^j = \gamma^j f(n_t^j, n_{t-1}^j)\} \qquad \text{if } j \leq t \text{ and } t > 0$$

$$q_t^j = 0 \text{ if } j > t$$

$$q_o^o = f(n_o^o, 0)$$

$$q_o^j = \gamma^j f(n_o^j, z_{-j}^j) \text{ if } j < 0.$$

where j ranges over the set of negative and positive integers.

Note that this environment corresponds to the case where firms pay a wage in period t to the young people hired at t and use the labor in periods t and t+1; heuristically this is a 'slave economy'.

In the OLG economy each generation consists of a measure space of agents. Technology is given by a function mapping measures of skilled and unskilled workers into output of the consumption good, and workers can only be assigned to one vintage. To establish the correct connection with the OLG environment, we can think of the time a person born in period i allocates to vintage j, $n^{ij} \in [0,1]$ in the Arrow-Debreu economy, as the fraction of people assigned to that vintage in the OLG setting. By the same token, the consumption allocation of this agent can be interpreted as the sum of the consumption of members of the corresponding generation..

We will now define feasibility for this economy and a competitive equilibrium. We will slightly modify the consumption allocatons for the OLG economy considered above, while leaving labor supply allocations unchanged . The modified allocations will give each young worker the average consumption of his cohort in the first period of their lives. Each worker will receive the average consumption of his generation when old. Obviously these allocations are feasible and yield the same utilities as those of the stationary equilibrium.

We will show that these modified allocations together with a price system are a competitive equibrium for the Arrow-Debreu economy. Furthermore, we will show that the first welfare theorem applies to this economy and hence that the competitive equilibrium thus constructed is a Pareto optimal allocation. Since these modified allocations yield the same utilities as the ones corresponding to the OLG economy, the equilibrium of the OLG economy is also Pareto optimal.

A <u>feasible allocation</u> for this economy is a pair $(\{x^i\}_{i=-1}^{\infty}, \{y^j\}_{j=-\infty}^{\infty})$ with the following properties:

i) $x^i \epsilon X_i$ for all i

ii) $y^j \epsilon Y_j$ for all j

iii) $\sum_{j=-\infty}^{t} y_t^j - \sum_{i=-1}^{\infty} x_t^i = 0$ if $t>0$

and $\sum_{j=0}^{\infty} y_o^{-j} - \sum_{i=-1}^{\infty} x_o^i = (0,0,\{\mu_k\}_{k=o}^{\infty})$

It can easily be verified that the technology described has constant returns to scale. Hence given a set of prices any profit maximizing vector y^j has to yield zero profits. As a consequence, there is no need to take into account the distribution of profits of firms, thus justifying the following definition.

A <u>competitive equilibrium</u> for this economy relative to a price system $(\{p_t\}_{t=0}^{\infty}, \{\omega_t\}_{t=0}^{\infty}, \{\omega_{-k}^o\}_{k=1}^{\infty})$ is a feasible allocation which satisfies:

iv) <u>profit maximization</u>:

$$y^j = \operatorname{argmax} \sum_{t=0}^{\infty} (p_t q_t^j - \omega_t n_t^j) - \sum_{k=1}^{\infty} \omega_{-k}^o z_{-k}^j$$

for any $(\{q_t^j\}, \{-n_t^j\}, \{-z_{-k}^j\}) \epsilon Y_j$.

Note that the technology sets described imply that $z_{-k}^j=0$ if $j \neq -k$, and that $n_t^j=0$ if $t<j$. Thus for $j \geq 0$ the above objective can be written as $\sum_{t=j}^{\infty} p_t q_t^j - \omega_t n_t^j$ and for $j<t$ it becomes $\sum_{t=0}^{\infty} (p_t q_t^j - \omega_t n_t^j) - w_{-j}^o z_{-j}^j$.

v) <u>utility maximization</u>:

$$x^i = \operatorname{argmax} U(c^i)$$

$$\text{subject to } \sum_{t=0}^{\infty} p_t c_t^i \leq \omega_i n_i + \sum_{k=1}^{\infty} \omega_{-k}^o z_{-k}^i$$

Note that since $z_{-k}^i=0$ for $i \neq -1$ the above constraint specializes to $\sum_{t=0}^{\infty} p_t c_t^i \leq \omega_i n_i$ if $i \neq -1$ and to $\sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{k=1}^{\infty} \omega_{-k}^o z_{-k}^i$ for $i=-1$.

We will now construct a competitive equilibrium by assigning allocations to agents similar to those of the OLG economy. The first step is to assign consumption allocations. Recall that in the OLG economy, the consumption allocations to members of generation t who worked when young in the vintage that appeared in period j is given by

$$\gamma^t(c^1_\tau, c^2_\tau) = \gamma^t(w_\tau, \gamma v_{\tau+1}).$$

where $\tau = t-j$.

Consider the consumption allocation

$(c^i_t, c^i_{t+1}) = \gamma^t(\sum_{\tau=0}^{\infty} c^1_\tau \mu_\tau n_\tau, \sum_{\tau=0}^{\infty} c^2_\tau \mu_{\tau+1})$ -which is the sum of consumption allocations corresponding to members of generation t, for $0 \leq i = t$ and $c^i_t = 0$ for $t < i$ and $t > i+1$. For the agent born at $t=-1$, the consumption allocation will be $c^{-1}_o = \sum_{\tau=1}^{\infty} v_\tau z_{-\tau} = \sum_{\tau=1}^{\infty} v_\tau \mu_\tau$.

Let $c^i = \{c^i_t\}_{t=0}^{\infty}$.

Let $n^i_t = -1$ if $t=i$ and 0 otherwise.

Let $z^{-1} = -\mu$ and $z^i = 0$ otherwise.

Let $x^i = (c^i, n^i, z^i)$

We will now construct the input allocations consistent with those of the OLG economy. For $t>j$ let $n^j_t = \mu_\tau n_\tau$ where $\tau = t-j$ and let $n^j_j = \mu_o$. Let $z^j_j = \mu_j$ when $j<0$. Recall that $\mu_\tau = \mu_{\tau-1} n_{\tau-1} = n^j_{t-1}$ when $\tau > 1$. Hence

$$q^j_t = \gamma^j f(n^j_t, n^j_{t-1}) = \gamma^j f(\mu_\tau n_\tau, \mu_\tau) = \gamma^j \mu_\tau f(n_\tau, 1)$$

whenever $j<t$ and

$q_t^j=f(n_t^j,0)=\gamma^j\omega_o n_t^j$ when $j=t$.

Set $y^j=(\{q_t^j\},\{n_t^j\},\{z_{-k}^j\})$

Finally let $x=\{x^i\}_{i=-1}^{\infty}$ and $y=\{y^j\}_{j=-\infty}^{\infty}$.

We now state the main proposition of this section.

Proposition 6. There exist prices such that the allocation (x,y) is a competitive equilibrium for the Arrow-Debreu economy.

Proof. The allocation considered is clearly feasible. Consider the following price system:

$$p_o=1$$
$$p_t=\beta^t$$
$$\omega_t=(\gamma\beta)^t[w_0+\beta\gamma v_1(w_1)]$$
$$\omega_{-k}^o=v_k(w_k)).$$

We must show that conditions (iv) and (v) of the definition of competitive equilibrium are satisfied. We will first establish profit maximization.

profit maximization:

The profits of a firm of vintage j can be written as

$$(13)\quad \pi^j(n^j)=\sum_{t=0}^{\infty}p_t\gamma^j f(n_t^j,n_{t-1}^j)-w_t n_t^j.$$

where $n_{-1}^j=0$ if $j\geq 0$ and $n_{-1}^j=z_{-j}^j$ if $j<0$.

Since f is continuously differentiable, the Euler equations

$$(14)\quad p_t\gamma^j f_1(n_t^j, n_{t-1}^j) + p_{t+1}\gamma^j f_2(n_{t+1}^j, n_t^j) - w_t \le 0 \qquad t=0,1,\ldots$$

where $n_{-1}^j=0$ for $j\geq 0$ and $n_{-1}^j = z_{-j}^j$ for $j<0$.

$$(15)\quad p_o\gamma^j f_2(n_o^j, z_j^j) - w_j^o \le 0 \text{ if } j<0.$$

are necessary for the sequence n^j to give a maximum. As shown in Lemma 4, these conditions are also sufficient. We will show that when these wages are the ones defined above, the labor inputs given by the allocation constructed above are a solution to equations (14) and (15) so that they maximize profits.

Equation (14) can be conveniently written as

$$(16)\quad p_t\gamma^t\gamma^{-\tau} f_1(n_t^j, n_{t-1}^j) + p_{t+1}\gamma^t\gamma\gamma^{-(\tau+1)} f_2(n_{t+1}^j, n_t^j) - w_t \le 0$$

$t=j, j+1, \ldots,$ where $\tau = t-j$

and given the definition of prices we obtain

$$(17)\quad \gamma^{-\tau} f_1(n_t^j, n_{t-1}^j) + \beta\gamma\gamma^{-(\tau+1)} f_2(n_{t+1}^j, n_t^j) \le [w_0 + \beta\gamma v_1(w_1)]$$

Recall that

$$\gamma^{-\tau} f_1(\mu_\tau n_\tau, \mu_\tau) = \gamma^{-\tau} f_1(n_\tau, 1) = w_\tau \text{ for } \tau\geq 0$$

and

$$\gamma^{-\tau} f_2(\mu_\tau n_\tau, \mu_\tau) = \gamma^{-\tau}\frac{1}{\mu_\tau} f(\mu_\tau n_\tau, \mu_\tau) - f_1(\mu_\tau n_\tau, \mu_\tau) n_\tau$$

$$= \gamma^{-\tau} f(n_\tau, 1) - w_\tau = v_\tau \text{ for } \tau\geq 1 .$$

Thus, replacing n_t^j in the left side of equation (17) as defined in the constructed allocation, we obtain for $t\geq j$

$$(18)\quad \gamma^{-\tau} f_1(\mu_\tau n_\tau, \mu_\tau) + \beta\gamma\gamma^{-(\tau+1)} f_2(\mu_{\tau+1} n_{\tau+1}, \mu_{\tau+1}) = w_\tau + \beta\gamma v_{\tau+1}$$

But $w_\tau+\beta\gamma v_{\tau+1} = \omega_o+\beta\gamma v_1$, so that the defined allocations satisfy equation (18) and thus also satisfy equation (14).

For $t<j$, $f(n_t,n_{t-1})=0$ so $f_1=0$ and $f_2=0$, so equation (14) is satisfied. As for equation (15), for $j<0$ let $k=-j$ and then

$$p_o\gamma^j f_2(n_o^j,z_j^j)=\gamma^{-k}f_2(\mu_k n_k,\mu_k)$$
$$=\gamma^{-k}f_2(n_k,1)=v_k=w_{-k}^o=w_j^o.$$

Utility maximization

For any wage, the supply of labor will always equal 1.

We must show that the consumption allocations maximize utility subject to the budget constraint.

Consider first the agent born at $t\geq 0$. The rate at which he can substitute consumption between the only two perioods of his life he cares about, namely t and t+1, is equal to the rate of time preference, i.e. $\frac{p_t}{p_{t+1}} = \frac{1}{\beta}$. Hence the consumer is indifferent between spending his budget between period t or t+1.

Hence we only need to check that budget constraints are satisfied with equality. The consumption allocations of the agent born at t are

(19) $c_t^i=\gamma^t \sum_{\tau=0}^{\infty} w_\tau\mu_\tau n_\tau$ and $c_{t+1}^i=\gamma^t \sum_{\tau=0}^{\infty} \gamma v_{\tau+1}\mu_{\tau+1}$

Then

$$(20)\qquad p_t c_t + p_{t+1} c_{t+1} = p_t \gamma^t \sum_{\tau=0}^{\infty} w_\tau \mu_\tau n_\tau + p_{t+1}\gamma^t \sum_{\tau=0}^{\infty} \gamma v_{\tau+1}\mu_{\tau+1}$$

$$= \beta^t \gamma^t \sum_{\tau=0}^{\infty} (w_\tau + \beta\gamma v_\tau)\mu_{\tau+1} \quad \text{since } \mu_\tau n_\tau = \mu_{\tau+1}$$

$$= \omega_t \sum_{\tau=0}^{\infty} \mu_{\tau+1} = \omega_t$$

The above follows because $w_\tau + \beta\gamma v_\tau = \omega_o + \beta\gamma v_1$, $\omega_t = (\gamma\beta)^t [\omega_o + \beta\gamma v_1]$ and $\sum_{\tau=0}^{\infty} \mu_{\tau+1} = 1$.

But then equation (20) states that the consumption allocated to this agent strictly satisfies his budget constraint.

The agent i=-1 supplies inelastically all his human capital z^i. Since

$$c_o^{-1} = \sum_{\tau=1}^{\infty} v_\tau (w_\tau) z_{-\tau} = \sum_{k=1}^{\infty} w_k^o z_{-k}$$

his budget constraint is also satisfied with equality □.

As a final step, we need to show that this competitive equilibrium is Pareto optimal. This is established in the following proposition, with the proof following the First Welfare Theorem.

<u>Proposition</u>. The competitive equilibrium is Pareto optimal.

<u>Proof</u>. The proof mimics Debreu(1959). Suppose there exists another allocation (x'^i, y'^j) such that $x'^i \geq x^i$ for all i with strict preference for some generation $\underline{i}$. Then $py' = \sum_{j=-\infty}^{\infty} \pi'^j \leq py = \sum_{j=-\infty}^{\infty} \pi^j$ and $px'^i \geq px^i$, with strict inequality for agent $\underline{i}$. Let $x = \sum_{i=0}^{\infty} x^i$ and

$y=\sum_{j=-\infty}^{\infty} y^j$, and define x' and y' similarly. Note that since output each period is bounded by $\gamma^t f(1,1)$, the value of total output $py \leq \sum_{t=0}^{\infty} \beta^t \gamma^t f(1,1)$. From the above it is the case that $px' > px$ and $py' \leq py$, and thus $px' > py'$. But then $p(x'-y') > p(x-y)$ and since $x-y=\zeta$ it cannot be the case that $x'-y' \leq \zeta$ so (x'^i, y'^j) is not a feasible allocation.

Section 6. Concluding Remarks.

We have presented a model of investment in technology specific human capital. The central result is that such specificities lead to a lag between the time that a new technology becomes available and the peak of its usage. In that sense, this model is consistent with the slow diffusion of new technologies. It is certainly true that slow diffusion can be a consequence of the fact that consumers must learn how to use new products. Our focus, however, is on the fact that producers must acquire the skills necessary to produce the new product cheaply. In our model the marginal product of investment in such human capital is high when older workers already possess the required level of skill.

Our main result is that an increase in the rate of change of technology implies an increase in the rate of diffusion. We also show that the wage profiles over time are flatter in older technologies than in newer ones. In that sense, the value of investing in a newer technology is higher and our model is one of human capital accumulation. The equilibrium we describe is Pareto optimal.

An obvious extension of our model would be to allow for uncertainty in the rate of technological innovation. We conjecture that in such a case, a technological innovation which is substantially better than average will attract a large number of young workers and lead to larger than average investment in the newest technology. Since this capital is specific to the technology, in subsequent periods relatively few young workers will be attracted to even newer technologies. These technologies will then be adopted and diffused at a slower rate than average.

The assumption of exogenous technical change obviously does not do justice to the reality of the process of innovation which requires the use of resources. In addition, it would be of interest to examine a model where technological innovation as well as adoption are jointly and endogenously determined. One possible modification of our model would be to let the productivity of the newest vintage relative to the previous one, γ, be determined by the number of workers who enter the newest industry. In such a case workers in the newest vintages can be thought of as engaging in innovative activity.

The existence of this externality may well cause the equilibrium not to be Pareto optimal. The effects of various policies to remedy this externality could then be examined. In any case, we conjecture that an exogenous improvement in the technology of innovation will lead, as in this paper, to an increase in the rate of diffusion. The earning profiles will also likely get flatter with such an improvement.

APPENDIX.

Proposition A.1. There exists a unique sequence $\{w_t\}$ that solves equation (4).

Proof. To establish this result we will first show that if we truncate the system at any T and impose $v_T=0$, then there exists a unique solution to the truncated problem. Then we will show that the sequence of solutions to the truncated problem converges and that the limit is a solution to the original problem. Finally we establish that there is no other solution to equation (4).

Step 1: Truncated problem.

Consider the problem

$$w_1+\beta\gamma v_2(w_2)=w_0+\beta\gamma v_1(w_1)$$

$$(4') \quad w_2+\beta\gamma v_3(w_3)=w_1+\beta\gamma v_2(w_2)$$

$$\cdots\cdots\cdots$$

$$w_T=w_{T-1}+\beta\gamma v_T(w_T).$$

subject to $w_t \geq 0$.

Alternatively, this is equivalent to

$$w_T=w_0+\beta\gamma v_1(w_1)$$

$$w_T=w_1+\beta\gamma v_2(w_2)$$

$$\cdots\cdots\cdots$$

$$w_T=w_{T-1}+\beta\gamma v_T(w_T).$$

subject to $w_t \geq 0$.

We will now show that the set of w_T such that the w_τ induced by backward induction in equation (4') which are all nonnegative is nonempty. We will proceed by induction. Notice that $w_{T-1}=w_T-\beta\gamma v_T(w_T)$ and the right side is increasing in w_T and goes to infinity as $w_T\to\infty$. Hence there exists some w_T that makes $w_{T-1}\geq 0$. Suppose that for a given w_T and for all $\tau\geq\tau'$ the induced w_τ is nonnegative. If w_T increases, then w_{T-1} increases and inductively w_τ increases for all $\tau\geq\tau'$. Thus, as $w_T\to\infty$, $w_{\tau'-1}$ given by

$$w_{\tau'-1}=w_T-\beta\gamma v_{\tau'}(w_{\tau'})$$

also goes to ∞ and thus there exists some w_T such that $w_{\tau'-1}$ is nonnegative. Hence there exists some w_T such that the induced sequence $\{w_\tau\}_{\tau=0}^{\infty}$ is nonnegative.

Suppose w_T is such that all w_τ are nonnegative. Then $w_{T-1}=w_T-\beta\gamma v_T(w_T)\leq w_T$. By induction we will show that w_τ is increasing in τ. Assuming $w_\tau\leq w_{\tau+1}$ we have that $w_{\tau-1}=w_T-\beta\gamma v_\tau(w_\tau)\leq w_T-\beta\gamma v_\tau(w_{\tau+1})\leq w_T-\beta\gamma v_{\tau+1}(w_{\tau+1})=w_\tau$. Hence w_τ is increasing in τ.

Suppose that in the above case, $w_0(w_T)>0$. Start decreasing w_T For any $0<\epsilon<w_0(w_T)$ at some point some $w_\tau=\epsilon$. But in that case the corresponding ω_0 will be no greater than ϵ. Hence there exists some $\underline{w}$ such that if $w_T=\underline{w}$ then $w_\tau(\underline{w})\geq 0$ and $\omega_0(w_T)\leq w_0$. As $w_\tau\to\infty$ we already have shown that $w_0(w_\tau)\to\infty$.

We just need to establish that the mapping $w_0(w_T)$ is continuous. We will proceed by induction. For $\tau=T-1$, $w_{T-1}=w_T-\beta\gamma v_T(w_T)$. Since v_T is continuous, $w_{T-1}(w_T)$ is a continuous function. Suppose $w_{\tau+1}$ is a continuous function of w_T. Then $w_\tau=w_T-\beta\gamma v_{\tau+1}(w_{\tau+1})$ which is a continuous function of $w_{\tau+1}$ and by composition of w_T.

The above implies that for any $\omega_o>0$ there exists a solution to problem equation (4'). Furthermore, since $\omega_o(w_T)$ is strictly increasing, there is a unique solution to this problem.

Step 2: Convergence of the truncated solutions

We proceed to show that these solutions converge. More precisely, letting $\bar{w}_t$ be the unique solution to the truncated problem we establish that $\bar{w}_t \rightarrow \bar{w} < \infty$.

We will first establish that $\bar{w}_t \leq \bar{w}_{t+1}$.. Suppose, to the contrary, that $\bar{w}_t > \bar{w}_{t+1}$. We will first show that this implies that $w_1(\bar{w}_t) < w_1(\bar{w}_{t+1})$. If this were not the case, then

$$\bar{w}_t = \omega_o + \beta\gamma v_1(w_1(\bar{w}_t)) \leq \omega_o + \beta\gamma v_1(w_1(\bar{w}_{t+1})) = \bar{w}_{t+1}$$

so that $w_1(\bar{w}_t) < w_1(\bar{w}_{t+1})$

We now show that the contradiction hypothesis implies that $w_\tau(\bar{w}_t) < w_\tau(\bar{w}_{t+1})$ for all $\tau \leq t$. Suppose $w_\tau(\bar{w}_t) < w_\tau(\bar{w}_{t+1})$ and that $w_{\tau+1}(\bar{w}_t) \geq w_{\tau+1}(\bar{w}_{t+1})$. Then repeating the argument used above for w_1 we obtain a contradiction. Hence for all $\tau \leq t$ $w_\tau(\bar{w}_t) < w_\tau(\bar{w}_{t+1})$ and in particular, $\bar{w}_t < w_t(\bar{w}_{t+1}) < w_t(\bar{w}_{t+1}) + \beta\gamma v_{t+1}(\bar{w}_{t+1}) = \bar{w}_{t+1}$. This proves that $\bar{w}_{t+1} \geq \bar{w}_t$, as desired.

We now turn to the other side of the inequality. Suppose, to the contrary, that $\bar{w}_t < w_t(\bar{w}_{t+1})$. Then

$$w_{t-1}(\bar{w}_{t+1}) = \bar{w}_{t+1} - \beta\gamma v_t(w_t(\bar{w}_{t+1})) > \bar{w}_t - \beta\gamma v_t(\bar{w}_t) = w_{t-1}(\bar{w}_t).$$

By the same argument if $w_\tau(\bar{w}_{t+1}) > w_\tau(\bar{w}_t)$ the same will be true for all $\tau' \leq \tau$. Hence by induction, $\omega_o(\bar{w}_{t+1}) > \omega_o(\bar{w}_t) > \omega_o$.

We have thus established that $w_t(\bar{w}_{t+1})<\bar{w}_t<\bar{w}_{t+1}$. As a consequence the following inequality holds

$$|\bar{w}_{t+1}-\bar{w}_t|\leq|\bar{w}_{t+1}-w_t(\bar{w}_{t+1})|=\beta\gamma v_{t+1}(\bar{w}_{t+1}).$$

But as can easily be checked, $v_{t+1}(\bar{w}_{t+1})\leq\gamma^{-1}v_t(\bar{w}_{t+1})$ so that

$$|\bar{w}_t-\bar{w}_1|\leq\gamma^{-t}[\beta\gamma v_1(\bar{w}_t)]\leq\gamma^{-t}\beta\gamma v_1(\omega_o).$$

This implies that $\{\bar{w}_t\}$ is a Cauchy sequence so it converges.

Denoting the limit of $\{\bar{w}_t\}$ by $\bar{w}$, we now proceed to show that the solution to equation (4) induced by $\bar{w}$ from equation (4') is well defined. We will proceed again by induction.

$\bar{w} = \omega_o+\beta\gamma v_1(w_1)$ gives w_1.

$\bar{w} = w_1+\beta\gamma v_2(w_2)$ gives w_2. In order for this to be well defined, we need that $\bar{w}>w_1$. Suppose, to the contrary, that $\bar{w}\leq w_1$. Since $\bar{w}_t\leq\bar{w}$ for all t $w_1(\bar{w}_t)\geq w_1(\bar{w})$ and so the above would imply that $\bar{w}_t\leq w_1(\bar{w}_t)$, a contradiction. Suppose $w_\tau(\bar{w})$ is well defined. Then $\bar{w}=w_\tau(\bar{w})+\beta\gamma v_{\tau+1}(w_{\tau+1})$ and we thus require that $\bar{w}>w_\tau(\bar{w})$. Suppose to the contrary that $\bar{w}\leq w_\tau(\bar{w})$. Since $\bar{w}_\tau$ was assumed well defined it is easy to see that $w_\tau(\bar{w}_t)\geq w_\tau(\bar{w})$ and so the above would imply that $\bar{w}_t\leq w_\tau(\bar{w}_t)+\beta\gamma v_{\tau+1}(w_{\tau+1}(\bar{w}_t))$, and hence $w_{\tau+1}(\bar{w}_t)$ would not be well defined. This proves that the sequence $w_\tau(\bar{w})$ is a well defined solution to equation (4).

Step 3: This is the only solution.

Suppose $\bar{w}'$ is another equilibrium. We will denote by w_τ' the wage induced for vintage τ by $\bar{w}'$. We will first show that $\bar{w}'\geq\bar{w}$.

Suppose to the contrary that $\bar{w}' < \bar{w}$. Then there exists some t such that $w_t' \leq w'_t + \beta\gamma v_{t+1}(w_{t+1}') = \bar{w}' < \bar{w}_t$, where $\bar{w}_t$ corresponds to the truncated solution at t. By the inductive argument used in step 2, $w_1' < w_1(\bar{w}_t)$. But this implies that
$\bar{w}' = \omega_o + \beta\gamma v_1(w_1') \geq \omega_o + \beta\gamma v_1(w_1(\bar{w}_t)) = \bar{w}_t$, which yields a contradiction. This establishes that $\bar{w}' \geq \bar{w}$.

We now show that $\bar{w}' \leq \bar{w}$, which will complete the proof. Suppose to the contrary that $\bar{w}' > \bar{w}$. Then there is some t such that $\bar{w}' \geq w_t' > \bar{w} \geq \bar{w}_t$. This implies that $w_1' > w_1(\bar{w}_t)$ and hence that $\bar{w}' = \omega_o + \beta\gamma v_1(w_1') \leq \omega_o + \beta\gamma v_1(w_1(\bar{w}_t)) = \bar{w}$, which yields a contradiction □.

We will now prove some results which are used in Proposition 6. Let $\gamma' \geq \gamma$ and let $\bar{w}_t$ and $\bar{w}_t'$ correspond to the wage for the problem truncated at t for γ and γ', respectively. For simplicity we will denote by w_τ' and w_τ the wages for period τ corresponding to the problem truncated at t for γ' and γ, respectively.

<u>Lemma 1</u>. $\bar{w}_t' \geq \left[\frac{\gamma}{\gamma'}\right]^t \bar{w}_t$.

<u>Proof</u>. Suppose to the contrary that $\bar{w}_t' < \left[\frac{\gamma}{\gamma'}\right]^t \bar{w}_t$. If $w_{t-1}' \geq \left[\frac{\gamma}{\gamma'}\right]^{t-1} w_{t-1}$ then

$$\bar{w}'_t = w_{t-1}' + \beta\gamma v_t'(w_{t-1}')$$
$$> \left[\frac{\gamma}{\gamma'}\right]^{t-1} [w_{t-1} + \beta\gamma v_t(w_t)] = \left[\frac{\gamma}{\gamma'}\right]^{t-1} \bar{w}_t \geq \left[\frac{\gamma}{\gamma'}\right]^t \bar{w}_t$$

which is a contradiction. We will now show by backward induction that the contradiction hypothesis implies $w_1' < \frac{\gamma}{\gamma'} w_1$. Suppose that $w_\tau' < \left[\frac{\gamma}{\gamma'}\right]^\tau w_\tau$. Then if $w_{\tau-1}' \geq \left[\frac{\gamma}{\gamma'}\right]^{\tau-1} w_{\tau-1}$,

$$\bar{w}'_t = w'_{\tau-1}+\beta\gamma' v'_\tau(w'_\tau) > \left[\frac{\gamma}{\gamma'}\right]^{\tau-1}[w_{\tau-1}+\beta\gamma v_\tau(w_\tau)]$$
$$= \left[\frac{\gamma}{\gamma'}\right]^{\tau-1}\bar{w}_t \geq \left[\frac{\gamma}{\gamma'}\right]^{t}\bar{w}_t$$

a contradiction to the above hypothesis.

Hence if $\bar{w}_t < \left[\frac{\gamma}{\gamma'}\right]^t \bar{w}_{t+1}$, then $w'_1 < \frac{\gamma}{\gamma'} w_1$. But then $\bar{w}'_t = \omega_o + \beta\gamma v'_1(w'_1) > \omega_o + \beta\gamma v_1(w_1) = \bar{w}_t$. This establishes that $\bar{w}'_t \geq \left[\frac{\gamma}{\gamma'}\right]^t \bar{w}_t$ □

<u>Lemma 2</u> $w'_1 \geq \frac{\gamma}{\gamma'} w_1$.

<u>Proof</u>. Suppose that $w'_1 < \frac{\gamma}{\gamma'} w_1$. Then

(*) $\bar{w}'_t = \omega_o + \beta\gamma' v_1(w'_1) > \omega_o + \beta\gamma' \frac{\gamma}{\gamma'} v_1(\frac{\gamma}{\gamma'} w_1) = \bar{w}_t$

We now show that the above implies that $w'_\tau < \left[\frac{\gamma}{\gamma'}\right]^\tau w_\tau$ for all $\tau < t$. Suppose that $w'_\tau < \left[\frac{\gamma}{\gamma'}\right]^\tau w_\tau$ for all $\tau < T$. Then if $w'_T \geq \left[\frac{\gamma}{\gamma'}\right]^\tau w_T$,

$$\bar{w}'_t = w'_{T-1}+\beta\gamma' v'_T(w'_T) < \left[\frac{\gamma}{\gamma'}\right]^{T-1}[w_{T-1}+\beta\gamma v_T(w_T)] = \left[\frac{\gamma}{\gamma'}\right]^{T-1} \bar{w}_t \leq w_t ,$$

contradicting (*). Hence $w'_1 < \frac{\gamma}{\gamma'} w_1$ implies that $w'_\tau < \left[\frac{\gamma}{\gamma'}\right]^\tau w_\tau$ for all $\tau < t$. But then

$$\bar{w}'_t = w'_{t-1}+\beta\gamma' v'_t(\bar{w}'_t) < w'_{t-1}+\beta\gamma' v'_t(\left[\frac{\gamma}{\gamma'}\right]^t \bar{w}_t)$$
$$< \left[\frac{\gamma}{\gamma'}\right]^{t-1}[w_{t-1}+\beta\gamma(\bar{w}_t)] \leq \bar{w}_t$$

contradicting (*). This establishes that $w'_1 \geq \frac{\gamma}{\gamma'} w_1$ □

<u>Proposition A.2</u> If $\gamma' \geq \gamma$ then $n(\tau,\mu') \leq n(\tau,\mu)$ where μ' and μ are the invariant distributions corresponding to γ' and γ, respectively.

Proof. We will show that for any truncated sequence

$$w(\tau,\mu') \geq \left[\frac{\gamma}{\gamma'}\right]^{\tau} w(\tau,\mu).$$

Since $n'_{\tau}[(\gamma/\gamma')^{\tau}w]=n_{\tau}(w)$, this suffices to prove the result.

By Lemma 1 $\bar{w}'_t \geq \left[\frac{\gamma}{\gamma'}\right]^{t}\bar{w}_t$. Suppose that $w_{\tau}\geq\left[\frac{\gamma}{\gamma'}\right]^{\tau}w_{\tau}$ for all $\tau\geq T$. We will show that

$$w'_{T-1}\geq\left[\frac{\gamma}{\gamma'}\right]^{T-1} w_{T-1}.$$

We will show that if this is not true then by induction (oh no..., not again!) $w'_1<\frac{\gamma}{\gamma'}w_1$ therefore contradicting Lemma 2. Hence suppose the contrary, that $w'_{T-1}<\left[\frac{\gamma}{\gamma'}\right]^{T-1} w_{T-1}$. Then

$$\text{(A.2)} \quad \bar{w}'_t < \left[\frac{\gamma}{\gamma'}\right]^{T-1}[w_{T-1}+\beta\gamma v_{T-1}(w_{T-1})] = \left[\frac{\gamma}{\gamma'}\right]^{T-1}\bar{w}_t .$$

We will now show that as a consequence of the above assumption, $w'_{T-2}<\left[\frac{\gamma}{\gamma'}\right]^{T-2} w_{T-2}$. Suppose not. Then

$$\bar{w}'_t \geq \left[\frac{\gamma}{\gamma'}\right]^{T-2}[w_{T-2}+\beta\gamma v_{T-1}(w_{T-1})] \geq \left[\frac{\gamma}{\gamma'}\right]^{T-1}\bar{w}_t ,$$

contradicting equation (A.2). Also, if $w'_{T-3} \geq \left[\frac{\gamma}{\gamma'}\right]^{T-3} w_{T-3}$ then $\bar{w}'_t \geq \left[\frac{\gamma}{\gamma'}\right]^{T-3}[w_{T-3}+\beta\gamma v_{T-2}(w_{T-2})] \geq \left[\frac{\gamma}{\gamma'}\right]^{T-1}\bar{w}_t$ contradicting equation (A.2). Ap-plying this same argument inductively we obtain that $w'_1<\frac{\gamma}{\gamma'}w_1$, a contradiction to Lemma 2.

Hence, as desired, $w'_{T-1}\geq\left[\frac{\gamma}{\gamma'}\right]^{T-1} w_{T-1}$. Since T was chosen arbitrarily, this establishes that $w'_{\tau} \geq \left[\frac{\gamma}{\gamma'}\right]^{\tau} w_{\tau}$, so the proof is complete □.

We establish here that the Euler equations for the maximization problem of the firm in section 5 are necessary and sufficient, together with a transversality condition that is satisfied.

Lemma 3. $n_T f_2(n_{T+1}, n_T) \to 0$ as $T \to \infty$.

Proof: $n_T f_2(n_{T+1}, n_T) = n_T v_T$. But $n_T \to 0$ and v_T is decreasing □.

Let $n = \{n_t^j\}$ where for notational convenience $n_{-1}^j = z_j^j$.

Lemma 4. If n satisfies

$$p_t \gamma^j f_1(n_t^j, n_{t-1}^j) + p_{t+1} \gamma^j f_2(n_{t+1}^j, n_t^j) - w_t \leq 0 \quad t=j, j+1, \ldots$$

$$p_0 \gamma^j f_2(n_0^j, n_{-1}^j) - w_j^o \leq 0$$

$$n_T^j f_2(n_{T+1}^j, n_T^j) \to 0 \quad \text{(transversality condition)}$$

then n is a profit maximizing allocation.

Proof: For notational convenience we will supress the index j. Consider an alternative input sequence $n' = \{n_t'\}$. For T fixed and letting $h_t = n_t' - n_t$ letting π_T' and π_T indicate the profits up to period T cor-responding to plans n' and n respectively,

$$\pi_T' - \pi_T \equiv \sum_{t=0}^{T} p_t \gamma^j f(n_t', n_{t-1}') - w_t n_t' - w_j^o n_{-1}' - \sum_{t=0}^{T} p_t \gamma^j f(n_t, n_{t-1}) - w_t n_t - w_j^o n_{-1}$$

$$= \sum_{t=0}^{T} p_t \gamma^j [f(n_t + h_t, n_{t-1} + h_{t-1}) - f(n_t, n_{t-1})] - w_t h_t - w_j^o h_{-1}$$

$$\leq \sum_{t=0}^{T} \alpha^{-1}\{p_t\gamma^j[f(n_t+\alpha h_t,n_{t-1}+\alpha h_{t-1})-f(n_t,n_{t-1})]-w_th_t-w_j^oh_{-1}$$

The last inequality follows from the concavity of f(.).

This implies that

$$\pi_T'-\pi_T \leq \sum_{t=0}^{T} p_t\gamma^j[h_tf_1(n_t,n_{t-1})+h_{t-1}f_2(n_t,n_{t-1})]-w_th_t-w_j^oh_{-1}$$

$$= \sum_{t=0}^{T-1} h_t[p_t\gamma^jf_1(n_t,n_{t-1})+p_{t+1}\gamma^jf_2(n_{t+1},n_t)-w_t]+h_T[\gamma^jp_Tf_1(n_T,n_{T-1})-w_T]$$

$$+h_{-1}[\gamma^jf_2(n_0,n_{-1})-w_j^o]$$

$$\leq h_T[p_T\gamma^jf_1(n_T,n_{T-1})-w_T] = -h_Tp_T\gamma^jf_2(n_{T+1},n_T)$$

$$= (n_T-n_T')p_T\gamma^jf_2(n_{T+1},n_T)$$

But then by lemma 3 as $T\to\infty$, $\pi_T'-\pi_T \to -p_Tn_T'\gamma^jf_2(n_{T+1},n_T) \leq 0$. This proves that there is no feasible plan with strictly positive profits, and hence the allocation $\{n_t\}$ is profit maximizing □.

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