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# SUSTAINABLE PLANS AND DEBT

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Abstract

In this paper we analyze the constraints imposed by dynamic consistency in a model of optimal taxation. We assume that only distorting taxes are available to finance government consumption. Optimal fiscal policy requires the use of debt to smooth distortions over time. Dynamic consistency requires that governments at each point in time not have an incentive to default on the inherited debt. We consider policy functions which map the history of the economy including the actions of past governments into current decisions. A sustainable plan is a sequence of history-contingent policies which are optimal at each date given that future policies will be selected according to the plan. We show that if agents discount the future sufficiently little and if government consumption fluctuates then optimal sustainable plans yield policies and allocations which are identical to those under full commitment. We contrast our notion of dynamic consistency with other definitions.

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Since the seminal paper on time consistency by Kydland and Prescott [1977] it has been widely recognized that dynamic consistency can impose severe constraints on the design of optimal policy. Strotz [1955] showed that individuals can have preferences which induce time inconsistencies in their decisions. The remarkable feature of Kydland and Prescott's examples is that even when the preferences of individual agents are time consistent, and there are no conflicting interests among members of a society, optimal social policy can be dynamically inconsistent. It hardly seems plausible that democratic societies can commit their future selves to policies which they would unanimously vote to change in the future. Consequently it seems appropriate to view consistency as a constraint on the set of feasible policies. The welfare of private agents is generally lower given this constraint than it would be if policymakers could somehow commit themselves in advance. In this paper we show that when agents discount the future sufficiently little, optimal policies which satisfy consistency constraints can lead to the same level of welfare as those under commitment.

Lucas and Stokey [1983] illustrated these issues in a model of optimal fiscal policy. They analyze the optimal restructuring of debt given that governments cannot default on debt. In this paper we consider the possibility of default. They consider an economy with a large number of identical, infinitely lived consumers. A single, nonstorable good is produced in each period using labor as the only input. This good can be either consumed privately or used to provide a public good referred to as "government consumption". The level of government consumption is exogenously specified to follow some stochastic process. The revenues required to provide government consumption can be raised only through a proportional, and therefore distorting, income tax.

The problem faced by society, then, is to structure the pattern of taxes over time and over the stochastic realizations of government consumption to minimize the excess burden of distorting taxation. From the perspective of society at date 0 this problem is formally identical to the optimal tax problem in a many-good static economy first studied by Ramsey [1927]. We follow Lucas and Stokey [1983] and use the Arrow-Debreu contingent claims formalism to represent the commodity space as the sequence of date and event contingent goods. In general, for this problem the revenues raised in a given contingency will not equal the value of government consumption at that contingency. To put it differently, debt can be used to smooth tax distortions over time and across states (see Barro 1979, Turnovsky and Brock 1980).

Consider then, the problem faced at time 1. The plan specifies contingent payments, possibly negative, by government to individuals. There is now an obvious incentive for all individuals in the society to agree to "default" on the debt if it is positive. Such a default acts as a lump sum tax and is, of course, preferred to distorting taxes. Private agents, correctly anticipating such a future default, will not accept promises to pay in the future. This logic suggests that it is impossible to smooth tax distortions. Paradoxically, it is precisely these considerations which suggest the possibility of better policies. Suppose, for the moment, that each government at each time believes that if it defaults on the inherited debt all future governments would also default, while if it continued along the originally devised plan no future governments would default. Then the incentives to default today are countered by the resulting loss in welfare in the future and it might even be possible to sustain the original plan.

An essential component of our argument is that the "strategies" of each government depend on the choices of preceding governments as well as on

the amount of debt that it inherits and the realizations of government consumption. In this paper, we refer to such policies as history-contingent policies. We will say that a plan is sustainable if the policies at each date induced by the plan are optimal given that future policies will be selected according to the plan. Such plans and the associated outcomes are similar to the subgame perfect equilibria of Lucas and Stokey [1983].

An alternative possibility is to restrict policies to depend only upon the state of the system at any time which is the inherited debt together with the realizations of government consumption. We use the term state-contingent policies to refer to such policies. Time consistent plans are then state-contingent policies which are optimal at each date given that future governments will choose optimally from the class of such policies. Lucas and Stokey define two equilibrium concepts. They define a subgame perfect equilibrium much as we define sustainable plans. They also define time consistent equilibria. In the latter they view the strategy spaces of the current government as consisting of not only the current policy but also all future policies of all future governments. They define an equilibrium to be time consistent if no government has an incentive to replan the entire future course of policy. For this environment the equilibrium allocations under their definition coincides with the allocations under our definition of consistency. Both involve no debt issue at any state.

In this paper we investigate the relationship among sustainable plans, full commitment plans and time consistent plans. We show that if agents discount the future sufficiently little and government consumption fluctuates then sustainable plans yield policies which are identical to those under full commitment. If government consumption settles down to a constant level forever starting at <u>any</u> date then the unique sustainable plan is the

time consistent plan. We also construct examples where sustainable plans do strictly better than time consistent plans but strictly worse than under full commitment.

These results are related in an obvious way to the folk theorems in the literature on repeated games. One distinguishing feature is that ours is not a repeated game. Technically, our environment induces a dynamic game, for which folk theorems have not been proven. In fact, as our examples illustrate, the equivalence between sustainable and full commitment plans depends critically on the nature of the stochastic process for government consumption. A second distinguishing feature is that within a given period there is no conflict among the agents in our economy. Recent applications of the folk theorems for repeated games to the study of aggregative policies (Barro and Gordon 1983a and Rogoff 1985) rely crucially upon the assumption that the objectives of policymakers and the public conflict.

In section 1, we describe our environment. In section 2 we examine the equilibrium outcomes if society can commit to a plan. In section 3, we define sustainable plans. In section 4, we prove our main result that sustainable plans can coincide with full commitment equilibria. Concluding comments are in section 5.

## 1. Environment

We consider an environment similar to that analyzed in Lucas and Stokey [1983]. Government consumption follows an exogenously defined stochastic process. We assume, in the tradition of the public finance literature, that revenues to finance government consumption can be raised only through proportional taxation of labor income. We will also assume that government debt is nonnegative, we relax this assumption in Section 6. The essential difference between our framework and that of Lucas-Stokey is that we assume

that the government can default on its inherited debt. We model such default as a tax on debt.

Consider a simple production economy populated by a large number of identical infinitely-lived consumers. There are two types of goods, leisure and a produced good. Each consumer is endowed with one unit of leisure each period and has access to a production technology for which one unit of leisure produces one unit of output. This output is split between private consumption and government consumption. Government consumption follows a given stochastic process for which the realizations up to and including time t are denoted $g^t = (g_0,\ldots,g_t)$. The probability of observing any particular history is known to all and denoted by $f(g^t)$. Let $G^t$ denote the finite set of all possible histories of length t. Each realization $g_t$ is assumed to be in the finite set $\{0,\gamma_1,\ldots,\gamma_K\}$ with $0 < \gamma_1 < \ldots < \gamma_K$.

There is no other uncertainty in the economy and so the commodity space L is the space of infinite sequences

$$L = \{c_t(g^t),x_t(g^t) \mid g^t \epsilon G^t, t=0,1,\ldots\}$$

where $c_t(g^t)$ and $x_t(g^t)$ are the private consumption of the produced good and leisure at time t given history $g^t$. Let L be endowed with the sup norm. Let c denote the collection of elements $c_t(g^t)$ one for each t and $g^t$ and define similar objects for other variables. An allocation (c,x) is feasible if it satisfies

$$(1.1) \qquad c_t(g^t) + x_t(g^t) + g_t \leq 1$$

for all dates t and histories $g^t$.

Consumer preferences over (c,x) pairs are given by the expected utility functional U(c,x) where

$$(1.2) \qquad U(c,x) = \sum_{t=0}^{\infty} \beta^t \sum_{g^t \in G^t} f(g^t|g_0) U\left(c_t(g^t), x_t(g^t)\right)$$

where the period utility function U is smooth, monotone, strictly concave and bounded above and below by some finite constants $\bar{\mu}$ and $\underline{\mu}$. Also, for each fixed x, $U_c(c,x) \to \infty$ as $c \to 0$ and for each fixed c, $U_x(c,x) \to \infty$ as $x \to 0$.

Let $p_t(g^t)$ denote the price of a claim to one unit of the produced good at time t given history $g^t$ in an abstract unit of account. Letting $\tau_t(g^t)$ denote an ad valorem tax on labor income $\left(1-x_t(g^t)\right)$ we have the date 0 consumer budget constraint,

$$(1.3) \qquad \sum_{t=0}^{\infty} \sum_{g^t \in G^t} p_t(g^t)\left[c_t(g^t) - \left(1-\tau_t(g^t)\right)\left(1-x_t(g^t)\right)\right] = 0.$$

The consumer's problem is then to choose (c,x) to maximize (1.2) subject to (1.3). The first order conditions for this problem are

$$(1.4) \qquad \frac{U_x\left(c_t(g^t), x_t(g^t)\right)}{U_c\left(c_t(g^t), x_t(g^t)\right)} = 1 - \tau_t(g^t)$$

$$(1.5) \qquad \beta^t \frac{U_c\left(c_t(g^t), x_t(g^t)\right) f(g^t|g^0)\right)}{U_c\left(c_0(g_0), x_0(g_0)\right)} = \frac{p_t(g^t)}{p_0(g_0)}.$$

For later developments it is necessary to follow the government balance sheets through time. To accomplish this let us add to this economy one-period state contingent government debt that is subject to state contingent taxes. Let $b_{t+1}(g^{t+1}) = b_{t+1}(g^t, g_{t+1})$ denote the number of units of debt issued at time t given history $g^t$ that pays off in units of the produced good at t+1 given event $g_{t+1}$ occurs. Each such unit of debt represents a claim, before taxes, to one unit of the produced good in state $g^{t+1} = (g^t, g_{t+1})$. If

we denote the tax rate on such a claim by $\delta_{t+1}(g^{t+1})$, then, including taxes, each unit of debt is a claim to only $(1-\delta_{t+1}(g^{t+1}))$ units of the produced good in state $g^{t+1}$. For each date t and history $g^t$, the government period budget constraint, expressed in an abstract unit of account is

$$(1.6)\qquad \sum_{g_{t+1}} p_{t+1}(g^t,g_{t+1})(1-\delta_{t+1}(g^t,g_{t+1}))b_{t+1}(g^t,g_{t+1})$$

$$= p_t(g^t)(1-\delta_t(g^t))b_t(g^t) + p_t(g^t)g_t - p_t(g^t)\tau_t(g^t)(1-x_t(g^t)).$$

In addition we will assume debt issues are nonnegative

$$(1.7)\qquad b_t(g^t) \geq 0 \qquad \text{all } t,\ g^t \in G^t.$$

We can motivate this assumption in one of two ways. On the one hand the environment we consider allows only for proportional taxes on labor income and taxes on the debt. Lump sum taxation is thought to be infeasible in real world economies because of considerations of heterogeneity and bankruptcy. Precisely these considerations also seem to us to suggest that negative values for government debt should be ruled out. Of course, this argument can hardly be rigorous in the homogeneous, representative consumer environment we consider.

Alternatively, we believe the essential differences between sustainability and time consistency are easier to understand with this assumption. The additional complications that arise with negative debt are discussed in section 6. We then have:

A <u>competitive equilibrium</u> is an allocation (c,x) a price system p, and a government policy $(\tau,\delta,b)$ satisfying

- <u>Feasibility</u>: For each date and history, (c,x) satisfy (1.1).

- Consumer maximization: Given the price system and government policy the allocation $(c,x)$ maximizes (1.2) subject to (1.3).
- Government budget constraints: Given an allocation and price system, the government policy $(\tau,\delta,b)$ satisfies (1.6) and (1.7) for each date and history.

Note that feasibility and the consumer budget constraint imply that $\tau$ and $x$ satisfy the date-0 budget constraint,

$$(1.8) \qquad \sum_t \sum_{g^t} p_t(g^t)\left[g_t - \tau_t(g^t)\left(1-x_t(g^t)\right)\right] = 0.$$

## 2. Policy with Commitment

We will initially consider the case in which there is an institution or commitment technology through which a government can bind itself at date 0 to a policy for the entire future. In this situation the government chooses at time 0 a policy $a = (\tau,\delta,b)$, which is a description of actions to be taken for every possible history of realizations of government spending. Let A be the set of all such policies,

$$A = \{a=(a_t)_{t=0}^{\infty} \mid a_t : G^t \to B\}$$

where $a_t = (\tau_t,\delta_t,b_{t+1})$, where B is a compact subset of $R^3$.

For arbitrary policies a competitive equilibrium need not exist. Let F denote the subset of policies for which a competitive equilibrium exists. With suitable restrictions on the stochastic process for g, that we make precise in Section 4, F will be nonempty. Assume for each policy a in F there is unique equilibrium allocation. A sufficient condition for this to be true is that consumption and leisure are normal goods. Each a in F we denote the associated allocation by $(c(a),x(a)) = \{c_t(g^t;a),x_t(g^t;a) \mid \text{all } t,\ g^t \in G^t\}$. and we let $V(a)$ denote the expected utility of this allocation,

$$(2.1) \qquad V(a) = \sum_{t=0}^{\infty} \beta^t \sum_{g^t \in G^t} f(g^t|g_0)U(c_t(g^t;a),x_t(g^t;a)).$$

We then have:

Definition: A policy $\hat{a}$ is an optimal policy with full commitment if it solves

$$(2.2) \qquad \max_{a \in F} V(a).$$

This definition of optimal policy suggests an algorithm for solving for it. First, solve for allocations as a function of policy then maximize over such policies. However, a simpler way to characterize it is to write the conditions defining equilibrium as constraints on the maximization problem in which $(c,x,p,\tau,\delta,b)$ are all chosen. That is, choose $(c,x,p,\tau,\delta,b)$ to maximize (1.2) subject to (1.1) and (1.3)-(1.7). The following lemma will show that, except for some additional constraints that require debt to be nonnegative, this full commitment problem can be reduced to the full commitment problem of Lucas and Stokey in which governments are not allowed to tax debt. The reason for this is, basically, that letting governments tax debt does not expand the set of allocations attainable under a government policy. We have:

Lemma 1. For any allocation $(c,x)$, price system $p$, and tax policy $\tau$ that satisfy feasibility and consumer maximization, there is a policy of issuing and taxing debt $(b,\delta)$ that satisfies the government budget constraints (1.6) and in which the tax on debt is identically zero, $\delta_t(g^t) = 0$ all $t$ and $g^t$. The associated debt issue policy is

$$(2.3) \qquad b_t(g^t) = [\tau_t(g^t)(1-x_t(g^t))-g_t] + \frac{1}{p_t(g^t)} \sum_{s=t+1}^{\infty} \sum_{g^s|g^t} p_s(g^s)[\tau_s(g^s)(1-x_s(g^s))-g_s],$$

where the summation is taken over all future paths $g^s = (g^t, g_{t+1}, \ldots, g_s)$ that come after the given $g^t$ history.

The proof is immediate. If $(c,x,p,\tau)$ satisfy the feasibility condition (1.1) and the consumer budget constraint (1.3) then the specified $(\delta,b)$ satisfy (1.6).

Using this lemma the full commitment problem reduces to choosing $(c,x,p,\tau)$ to maximize (1.2) subject to (1.1), (1.3)-(1.5) and (1.7). There are many policies for $(\delta,b)$ consistent with the resulting $(c,x,p,\tau)$, one of which is given in the lemma. Then we can simplify this problem further, as Lucas and Stokey do, by substituting (1.4) and (1.5) into (1.3) to eliminate $\tau$ and p and then using (1.1) to eliminate c. Likewise we can use (1.1), (1.4), (1.5) and (2.3) to simplify (1.7). The full commitment problem can thus be reduced to: choose x to maximize

$$(2.3) \qquad \sum_t \sum_{g^t} \beta^t f(g^t|g^0) U\left(1-x_t(g^t)-g_t, x_t(g^t)\right)$$

subject to

$$(2.4) \qquad \sum_t \sum_{g^t} \beta^t f(g^t|g^0)\left[\left(1-x_t(g^t)-g_t\right)U_c(g^t)-\left(1-x_t(g^t)\right)U_x(g^t)\right] = 0$$

and, for all t and $g^t$,

$$(2.5) \qquad \sum_{s=t}^{\infty} \sum_{g^s} \beta^{t-s} f(g^s|g^t)\left[\left(1-x_s(g^s)-g_s\right)U_c(g^s)-\left(1-x_s(g^s)\right)U_x(g^s)\right] \geq 0$$

where in (2.4) and (2.5), $U_c(g^s)$ and $U_x(g^s)$ are shorthand notation for the partial derivatives of U evaluated at $\left(1-x_s(g^s)-g_s, x_s(g^s)\right)$ for any $s \geq t$. In this problem constraint (2.4) is equivalent to the date 0 budget constraint of the government (1.8). The sequence of constraints (2.5) ensure that debt is always nonnegative. Then given the x that solves this problem the optimal

$(c,p,\tau)$ can be found using (1.1) and (1.3)-(1.5) and then $(\delta,b)$ can be filled in a variety of ways to be consistent with (1.6). In particular, they can be specified as in the lemma.

In short, if we let Y denote the set of allocations that are attainable under full commitment, that is,

$$Y = \{(c,x)\in L \mid \text{there exists an } a\in F \text{ s.t. } (c,x)=(c(a),x(a))\}$$

then we have shown

(2.6) $$Y = \{(c,x)\in L \mid (1.1),(2.4),(2.5) \text{ hold for all } t, g^t\in G^t\}.$$

Thus, under the full commitment the choice of an optimal policy $\hat{a}$ is equivalent to the choice of an optimal allocation $(\hat{c},\hat{x})$ that solves

(2.7) $$\max_{(c,x)\in Y} U(c,x)$$

where Y is given by (2.6).

For later it will be useful to contrast this full commitment solution with the "autarky solution" in which the government is constrained to keep its budget continually balanced, that is (1.6) is replaced by

(2.6) $$g_t = \tau_t(g^t)(1-x_t(g^t)).$$

Using (1.1) we can rewrite this as

(2.7) $$1 - x_t(g^t) - g_t = (1-\tau_t(g^t))(1-x_t(g^t)).$$

The autarky problem reduces to a sequence of static problems: at time t given history $g^t$, choose $x_t(g^t)$ to solve

$$\max U(1-x_t(g^t)-g_t, x_t(g^t))$$

subject to

$$(2.8) \qquad 1 - x_t(g^t) - g_t = \frac{U_x}{U_c}\left(1-x_t(g^t)\right)$$

where $U_x$ and $U_c$ are evaluated at $\left(1-x_t(g^t)-g_t, x_t(g^t)\right)$ and where we have combined (1.4) and (2.7) into (2.8) by eliminating taxes $\tau_t(g^t)$. The autarky allocation for leisure $x_t(g^t)$ is a time invariant function of the current $g_t$ say $x_t(g^t) = X^A(g_t)$. Denote the autarky consumption and tax rate for labor as $c_t(g^t) = C^A(g_t)$ and $\tau_t(g^t) = T^A(g_t)$ and let

$$(2.9) \qquad U^A(g_t) = U\left[C^A(g_t), X^A(g_t)\right].$$

In what follows we will use (2.9) repeatedly.

## 3. Sustainable Plans

The equilibrium described in section 2 has little counterpart in the world we observe. In terms of tax policy governments hardly seem restricted by commitments made by their predecessors. It might be argued that the possibility of default on government debt is not a serious one in an advanced democracy. However, the fact remains that governments do tax the interest payments on debt. Alternatively, one can think of the tax on real debt in our model as playing a role analogous to that the inflation tax plays in a model with nominal debt, as it does, for example, in section 5 of Lucas and Stokey [1983]. These considerations suggest that the possibility that future governments would want to deviate from the plans made by current governments should be an integral part of the analysis. The problem of dynamic consistency is particularly acute in our environment since each government would like to default on inherited debt but to constrain all future governments not to do so.

Consider, therefore, the case in which there is no technology through which a government can irrevocably bind itself to a particular policy path at date zero. In this case the government at time t, given the history up to time t, chooses a policy action at t. In this setting the history preceding time t is summarized by the history of realizations of government spending up to and including time t, namely $g^t = (g_0,\ldots,g_t)$, as well as the history of past policy actions taken $a^{t-1} = (a_0,\ldots,a_{t-1})$. We denote such a history by $h_t = (g^t,a^{t-1})$ where this history only records policies that were actually taken along path $g^t$, namely $\{a_0(g_0),a_1(g^1),\ldots,a_{t-1}(g^{t-1})\}$. Call $h_t$ a <u>feasible history</u> if it is consistent with the existence of an equilibrium, in the sense that there is some sequence of policies from t onwards denoted $\bar{a}^t = (a_t,a_{t+1},\ldots)$ such that $a = (a^{t-1},\bar{a}^t)$ is a feasible policy. Let $H_t$ denote the set of all feasible histories at t,

$$H_t = \{h_t=(g^t,a^{t-1}) \mid g^t \in G^t \text{ and } (a^{t-1},\bar{a}^t)\in F \text{ for some } \bar{a}^t\}.$$

For each feasible history $h_t = (g^t,a^{t-1})$ we define the set of feasible policies at t given $h_t$ as

$$F_t(h_t) = \{a_t \mid (a^{t-1},a_t,\bar{a}^{t+1})\in F \text{ for some } \bar{a}^{t+1}\}$$

For any feasible history $h_t = (g^t,a^{t-1})$ and feasible policy path $a = (a^{t-1},\bar{a}^t)$, let

$$(3.1) \qquad V_t(h_t,\bar{a}^t) = \sum_{s=t}^{\infty} \beta^{s-t} \sum_{g^s} f(g^s|g^t)U[c_s(g^s;a),x_s(g^s;a)].$$

Thus, $V_t(h_t,\bar{a}^t)$ is the expected discounted utility from time t onwards under the policy a, conditional on history $h_t$.

At each date t the government selects a policy plan at t which is a function $\sigma_t$ that assigns a feasible policy at t to every feasible history. Let $S_t$ denote the set of policy plans at t

$$S_t = \{\sigma_t | \sigma_t : H_t \to F_t(H_t)\}.$$

Call the collection $\sigma = (\sigma_t)_{t=0}^{\infty}$ with $\sigma_t \in S_t$ for all t a policy plan and let S denote the set of such policy plans.

We can use this notation to formalize the idea of a sustainable plan. Loosely, a policy plan is sustainable if for any feasible history it is optimal for the government at t to choose a history-contingent policy consistent with the plan, given that this government takes it as given that future governments will choose history-contingent policies consistent with the plan. In our definition we will let $\bar{a}^{t+1}(h_t,a_t,\sigma) = \{a_s(h_t,a_t,\sigma)\}_{s=t+1}^{\infty}$ denote the history-contingent policy induced by history $h_t$, current action $a_t$, and policy plan $\sigma$, where

$$(3.2) \qquad a_{t+1}(h_t,a_t,\sigma)(h_{t+1}) = \sigma_{t+1}(h_{t+1})$$

where $h_{t+1} = (h_t;a_t,g_{t+1})$ and so on for $s > t+1$. We will also let $F_t(h_t,\sigma)$ denote the set of feasible policy paths at t given feasible history $h_t = (g^t,a^{t-1})$ under the plan $\sigma$,

$$F_t(h_t,\sigma) = \{a_t \mid (a^{t-1},a_t,\bar{a}^{t+1}(h_t,a_t,\sigma)) \in F\}.$$

We then have

Definition: A policy plan $\sigma \in S$ is sustainable if for every history $h_t = (g^t,a^{t-1}) \in H_t$,

$$(3.3) \qquad \sigma_t(h_t) \in \arg\max \{V_t(h_t,a_t,\bar{a}^{t+1}(h_t,a_t,\sigma)) | a_t \in F_t(h_t,\sigma)\}.$$

Let $\Sigma$ denote the set of sustainable plans.

When all governments follow a plan $\sigma$ a particular policy is realized. This policy is defined recursively by (3.2) with $h_{t+1} = (h_t;\sigma_t(h_t),g_t)$

for all t. If $\sigma$ is in $\Sigma$ we call such a policy the policy sustained by $\sigma$, we denote it by $a(\sigma)$ and we call $c(a(\sigma)), x(a(\sigma))$ the allocation sustained by $\sigma$. Likewise, we say an allocation $(c,x) \in Y$ is sustainable if there is some $a \in F$ and $\sigma \in \Sigma$ such that $\sigma$ sustains $(c,x)$.

Under a sustainable plan $\sigma$ the value of utility, $V(\sigma)$, is the value of utility for the allocation sustained by $\sigma$. That is, for any $\sigma \in \Sigma$,

$$V(\sigma) = \sum_{t=0}^{\infty} \beta^t \sum_{g^t \in G^t} f^t(g^t|g_0) U\, c_t(g^t;a(\sigma)), x_t(g^t;a(\sigma)).$$

We then have

Definition. A policy plan $\sigma$ is an optimal sustainable plan if it solves

$$\max_{\sigma \in \Sigma} V(\sigma). \tag{3.4}$$

Finally, we have

Definition. An equilibrium for society consists of a plan $\sigma$, an allocation $(c,x)$ and a price system $p$ such that $\sigma$ is an optimal sustainable plan and $(c,x,p,a(\sigma))$ is a competitive equilibrium.

There are at least two ways of requiring dynamic consistency in this environment. Our equilibrium allows policymakers in any period to deviate as they choose from the recommended plan. However, we require that they take the consequent policy functions that future governments use as given. These policy functions depend not only upon the realization of $g^t$, as they must, but also upon the actions chosen by past governments. Lucas and Stokey refer to this notion of dynamic consistency as subgame perfection, we call it "sustainability."

Lucas and Stokey consider the case in which the actions of past governments are restricted to influence the choices of current governments only

through the debt that current governments inherit. We refer to such policies as state-contingent policies. We argue, as they do, that the unique outcome in their formulation involves a balanced budget at each date and event. The argument is by contradiction. Suppose a government at t inherits some positive debt. It is always feasible to default on the part of the debt, reduce current tax rates by the implied amount, leave the debt passed on to future government unaffected and hence leave all future allocations unaffected. Clearly, this raises current welfare. Since it leaves the decision problems of future governments unaffected this deviation does not lead to lower welfare in the future. Thus, faced with any positive debt the current government will always default. The result then follows from induction.

Given our definitions we will show that a sequence of policies which involve continual budget balance form a sustainable plan. However, typically there are sustainable plans which yield higher utility than the plan of continual budget balance.

## 4. Characterization of an Equilibrium for Society

In this section we characterize an equilibrium for society as the solution to a particular programming problem. We start by showing that the "autarky" plan of continual budget balance is sustainable. This guarantees the set of sustainable plans is nonempty. We then use plans which we call the "revert to autarky plans" to characterize the set of sustainable allocations. (Such plans play a role in this dynamic model similar to the one that Abreu's [1982] optimal simple penal codes play in repeated games.) Finally, we prove an equilibrium for society exists.

If government consumption is too large an equilibrium for society will not exist. We require, loosely, that government consumption always be strictly below the peak of the Laffer curve. In particular, let $r(g)$ be the

maximal amount of revenue net of government consumption that can be raised if government consumption is g,

$$(4.1) \qquad r(g) = \max_{x} \left[1 - \frac{U_x(1-x-g,x)}{U_c(1-x-g,x)}\right](1-x)-g$$

and let $c(g)$, $x(g)$ denote the associated allocations. We require

Assumption 1. There exists an $r > 0$ such that

$$r(g) > r \text{ for all } g \in \{0,\gamma_1,\ldots,\gamma_k\}.$$

Note that this assumption implies $m < \infty$ where

$$m = \min_{(g_0,g_1)} \left\{ \frac{U_c(c(g_0),x(g_0))}{U_c(c(g_1),x(g_1))} \;\middle|\; (g_0,g_1)\in\{0,\gamma_1,\ldots,\gamma_k\} \right\}$$

Note also that this assumption implies that a competitive equilibrium will exist under a policy of continual budget balance.

In our existence proof we need to show the set of sustainable plans is nonempty. It is easy to show that the "autarky plan" is always sustainable where this plan is defined as follows: in any period t given any history $h_t$, default on all inherited debt, issue no new debt and set taxes on labor to balance the budget. Denote this plan by $\sigma^A$ and denote the autarky policy by $a^A$. We then have,

Lemma 2. The autarky plan is sustainable.

Proof. For every feasible history $h_t \in H_t$, $a_t^A \in F_t(h_t,\sigma^A)$. We need to show that for any history $h_t$ the policy $a_t^A$ is maximal in the sense of (3.3). Suppose to the contrary that for some t and $h_t$ some other policy $a_t' \neq a_t^A$ is maximal. The assumption that debt is nonnegative implies that under any such policy net government revenues $\tau_t(1-x_t) - g_t$ must be positive. However, under

$\sigma^A$ the price of any debt issued by the government at t is zero since the government at t + 1 will default on it. That is, under $\delta^A$, $S_{t+1}$ equals one regardless of the history. However, the period budget constraint of the government (1.6) implies that such a deviation is not feasible.◊

In theorem 1 we characterize the set of sustainable allocations. In the proof we use a modified version of the autarky plan. For any policy $a \in F$ define the "revert to autarky plan" as follows: at time t given history $h_t$ choose $a_t(g^t)$ if all preceding policymakers have chosen $a_s(g^s)$ for $s \leq t$. After any deviation revert to the autarky policy $a^A$. Denote this plan $\sigma^A(a)$. We have

Theorem 1. An allocation $(c,x) \in Y$ is sustainable iff for all t and $g^t$,

$$(4.2) \qquad \sum_{s=t}^{\infty} \sum_{g^s} \beta^{s-t} f(g^s|g^t) U\big(c_s(g^s), x_s(g^s)\big) \geq \sum_{s=t}^{\infty} \sum_{g^s} \beta^{s-t} f(g^s|g_t) U^A(g_s)/$$

(Sufficiency) since $(c,x) \in Y$ there is some $a \in F$ such that $(c,x) = \big(c(a),x(a)\big)$. We claim that if (c,x) satisfies (4.2) then $\sigma^A(a)$ sustains (c,x). We need to show for any history $h_t$ no policymaker will deviate from this plan. We need only check two types of histories. Suppose first that under $h_t$ no policymaker has deviated from policy a up until t. If the policymaker at t sticks with the plan and chooses $a_t(g^t)$ the discounted value of utility from t onward is given by the left side of (4.2). If the policymaker at t deviates by choosing policy $a_t^A(g^t) \neq a_t(g^t)$, the discounted value of utility from t onward is equal to the right side of (4.2). Since the autarky policy is the best one shot deviation from $\sigma^A(a)$, any other deviation at t will yield utility no greater than the right side of (4.2). thus, no deviations from this stage of $\sigma^A(a)$ are profitable.

Next suppose that under $h_t$ some policymaker has deviated from a at some $s \leq t$. Under such a history $\sigma^A(a)$ reduces to the autarky plan which by Lemma 2 is sustainable. Thus $\sigma^A(a)$ is a sustainable plan such that $a = a(\sigma^A)$. (Necessity) We need to show any sustainable allocation $(c,x)$ satisfies (4.2) for all t and $g^t$. Since $(c,x)$ is sustainable there is some sustainable plan $\sigma$ such that $(c,x) = \big(c(a(\sigma),x(a(\sigma))\big)$. We first show that (4.2) holds for $t = 0$, that is

$$(4.3) \qquad V(\sigma) \geq \sum_{t=0}^{\infty} \sum_{g^t} \beta^t f(g^t|g_0) U^A(g_t).$$

Recall from the definition of a sustainable plan: for each t and $h_t \in H_t$,

$$(4.4) \qquad V_t\big(h_t,\sigma_t(h_t)\big),\bar{a}^{t+1}\big(h_t,\sigma_t(h_t|,\sigma)\big) \geq V_t\big(h_t,a_t',\bar{a}^{t+1}(h_t,a_t',\sigma)\big)$$

for all $a_t' \in F_t(h_t,\sigma)$. The idea of the proof is to repeatedly use (4.4) for a sequence of "autarky histories" to build up a sequence of inequalities that converge to (4.3). To that end let $a_t^A$ denote the autarky policy at t and $h_t^A$ denote of history of autarky policies, that is, $h_t^A = (g^t,a_0^A,\ldots,a_{t-1}^A)$. Clearly $h_t^A$ is a feasible history. Also for any history $h_t \in H_t$ and any plan $\sigma$, $a_t^A$ is contained in $F_t(h_t,\sigma)$. So, in particular, (4.4) implies

$$(4.5) \qquad V_t h_t^A,\sigma_t(h_t^A),\bar{a}^{t+1}\big(h_t^A,\sigma_t(h_t^A),\sigma\big) \geq U^A(g_t) + \beta E\big[V_{t+1}\ h_{t+1}^A,\bar{a}^{t+1}(h_{t+1}^A,\sigma)|g^t\big]$$

where the conditional expectation is under the probability law f. Since $\sigma$ is sustainable, a deviation to autarky at date 0 is not profitable. Thus,

$$(4.6) \qquad V(\sigma) \geq U^A(g_0) + \beta E\big[V_1\ h_1^A,\bar{a}^1(h_1^A,\sigma)|g_0\big].$$

Now evaluate (4.5) for t = 1, multiply both sides by β, sum over events $g^1$ and substitute the resulting expression into (4.6) to obtain

$$(4.7) \qquad V(\sigma) \geq \sum_{t=0}^{1} \sum_{g^t} \beta^t f(g^t|g_0) U^A(g_t) + \beta^2 E[V_2\ h_2^A, \bar{a}^2(h_2^A, \sigma)|g^1].$$

We can again use (4.5) to substitute for the second term on the right in (4.7). Continuing in this manner we obtain after T steps

$$(4.8) \qquad V(\sigma) \geq \sum_{t=0}^{T} \sum_{g^t} \beta^t f(g^t|g_0) U^A(g^t) + \beta^{T+1} E[V_{T+1}\ h_{T+1}^A, \bar{a}^{T+1}(h_{T+1}^A, \sigma)|g^T].$$

However, we know

$$(4.9) \qquad \beta^{T+1} E[V_{T+1}\ h_{T+1}^A, \bar{a}^{T+1}(h_{T+1}^A, \sigma)|g^T] \leq \beta^{T+1} \sum_{t=T+1}^{\infty} \beta^t \bar{u}$$

where $\bar{u}$ is the positive constant bounding utility. Taking limits as $T \to \infty$ in (4.8), and using (4.9) we obtain

$$(4.10) \qquad V(\sigma) \geq \sum_{t=0}^{\infty} \sum_{g^t} \beta^t f(g^t|g_0) U^A(g_t).$$

So far we have established the inequality for t = 0. We can establish it for arbitrary t and history $h_t = (g^t, a^{t-1})$ by considering histories of the form $h_{t+k} = (g^{t+k}, a^{t-1}, a_t^A, \ldots, a_{t+k-1}^A)$. For these histories the definition of sustainability yields a sequence of inequalities similar to those in (4.5). We can use these to obtain a sequence of inequalities similar to those in (4.8), taking limits then gives the desired result.◊

If we let X denote the set of sustainable allocations then theorem 1 together with (2.7) implies

$$(4.11) \qquad X = \{(c,x)|(1.1),(2.4),(2.5),(4.2) \text{ hold for all } t, g^t\}.$$

An implication of this theorem

Corollary. $(\hat{c},\hat{x})$ is an equilibrium allocation for society iff $(\hat{c},\hat{x})$ solve

$$\max_{(c,x)\in X} U(c,x)$$

where X is given in (4.11).

Finally, we have

Theorem 2. An equilibrium for society exists.

Proof. From Lemma 1 it is clear that X is nonempty. X is closed since the inequalities that characterize it are all continuous. X is compact since it is a closed subset of the compact set

$$L' = \{(c,x)\in L \mid c_t(g^t)+x_t(g^t)+g_t\leq 1 \text{ all } t,g^t\}.$$

Since the period utility function U is continuous given additive separability and discounting this implies the utility functional **U** is continuous in the sup norm on L'. Thus an equilibrium allocation $(\hat{c},\hat{x})$ exists. By construction this allocation is maximal in the set of sustainable allocations.

From $(\hat{c},\hat{x})$ we construct a prize system $\hat{p}$ from (1.5) and a policy $\hat{a} = (\hat{\tau},\hat{\delta},\hat{b})$ from (1.4) and (1.6). From the proof of theorem 1 and the corollary it is clear that $\sigma^A(\hat{a})$ is an optimal sustainable plan that sustains $\hat{a}$. ◊

## 5. Sustainable Plans and Full Commitment Equilibria

In this section we show that given certain assumptions on the stochastic process for government consumption and given a sufficiently large discount factor, the full commitment plan is sustainable. We also give examples to show that when these assumptions are not met full commitment is not sustainable for any discount factor less than or equal to one.

The key assumption is that no matter what the history, there is a probability, uniformly bounded away from zero, that government consumption will fluctuate in the future.

Assumption 2. There is a constant $\pi > 0$ such that for every $g^{t-1} \in G^{t-1}$, there is some $\gamma \in \{\gamma_1,\ldots,\gamma_k\}$ that satisfies

$$f(g^{t-1},\gamma,0 \mid g^{t-1}) > \pi > 0.$$

Let $\gamma(g^{t-1})$ denote the largest value of $\gamma$ that satisfies this assumption.

Assumption 2 is central to the proofs of the theorems in this section. We illustrate the need for such an assumption by giving examples of processes for government consumption which violate it in which the full commitment equilibrium is not sustainable for any nonzero discount factor.

Example 1. Eventually constant government consumption.

Let there be some T such that government consumption is zero after T, that is $f(g^t,0) = 1$ for all $t \geq T$. As long as government consumption is positive for some $t < T$, it is easy to show that in the full commitment equilibrium the inherited debt at T is positive and that tax rates are constant and positive for all $t \geq T$. In particular, for $g_t = 0$ the autarky allocation $c^A(0)$, $x^A(0)$ solves

$$\max_{c_t,x_t} U(c_t,x_t) \quad \text{s.t. } c_t + x_t \leq 1.$$

Thus, for all $t \geq T$ any allocation $(c,x) \in F$ satisfies

$$\sum_{s=t}^{\infty} \beta^{s-t} U(c_s,x_s) \leq \sum_{s=t}^{\infty} \beta^{s-t} U^A(0) \tag{5.1}$$

Combining (4.2) for $t \geq T$ with (5.1) implies $b_t(g^t)$ and $\tau_t(g^t)$ are zero for all $t \geq T$. Recursively using (1.6) and (1.7) implies $b_t(g^t) = 0$ for all $t < T$.

We can explain this result intuitively as follows. Consider the problem faced by the government at some $t \geq T$. If it inherits any positive debt then regardless of the policies of future governments it is optimal to default on this debt, that is set $\delta_t$ equal to 1. Consider next the problem faced by the government at T - 1. The equilibrium price of any debt it issues is zero and hence no revenues can be raised by selling debt. Consequently if this government inherits any debt it is optimal to default on it by setting $\delta_{T-1}$ equal to one. Repeating this argument for all $t < T$ it follows that all governments default and no debt is ever issued. Thus the full commitment policy is not sustainable for such a process of government consumption. Similar arguments apply if government consumption eventually settles down to any constant amount. Notice that in this example date T acts like a finite endpoint and then backward induction unwinds any strategy other than "autarky forever."

Example 2. Markov chain with absorbing states.

Let government consumption be a Markov chain with an absorbing state of 0. It is easy to show that under the full commitment plan the debt inherited in any zero state is strictly positive. Using a backward induction argument similar that of example 1, it is clear that the full commitment plan is not sustainable. However, one can show that the debt contingent on states other than zero is typically nonzero. This implies the sustainable plan does strictly better than the autarky plan which requires continuous budget balance but it does strictly worse than the full commitment policy.

This example points out how important uncertainty is in this environment. In particular, this is an example for which every history ending with a 0, namely any $(g^{t-1},0)$, acts as a finite endpoint. However, contrary to the certainty case backward induction from any such endpoint does not

unwind any nonautarky plan. Consequently, optimal sustainable plans can dominate autarky but still be worse than full commitment.

Example 3. Eventually absorbing states.

Let $K = 1$ and let $g_t \in \{0, \gamma_1\}$ be a time dependent Markov chain with transition matrix.

$$\Pi_t = \begin{matrix} 1/t & 1 - 1/t \\ 1 - \pi_{11} & \pi_{11} \end{matrix}$$

For large t arguments similar to those used in example 2 imply that full commitment is not sustainable.

In the rest of this section we establish that given our assumptions on government consumption and given a sufficiently high discount factor, the full commitment policy is sustainable. In order to prove this result we will need to show that the single period deviation from full commitment is outweighed by the discounted value of losses from reversion to autarky. We prove this in several steps. We start with a preliminary lemma, the proof of which is contained in the appendix.

Consider a two period (t=0,1) deterministic analogue of our environment with an arbitrary discount factor $\alpha \leq 1$. Let $g_0 = g > 0$ and $g_1 = 0$. Preferences are given by

$$U(c_0, x_0) + \alpha U(c_1, x_1) \text{ with } \alpha \in (0,1]. \tag{5.2}$$

Budget balance and feasibility require

$$\left[c_0 U_{c_0} - (1-x_0) U_{x_0}\right] + \alpha\left[c_1 U_{c_1} - (1-x_1) U_{x_1}\right] = 0 \tag{5.3}$$

and

$$(5.4) \qquad c_t + x_t + g_t \leq 1 \qquad t = 0, 1.$$

where $U_{c_t}$ and $U_{x_t}$ are similar to our earlier shorthand notation. If we also require that the budget be balanced in each period then

$$(5.5) \qquad c_0 U_{c_0} - (1-x_0)U_{x_0} = 0.$$

Let $\bar{w}(g,\alpha)$ be the maximized value of (5.2) subject to (5.3) and (5.4). Let $w^A(g,\alpha)$ be the maximized value of the autarky plan, maximize (5.2) subject to (5.3)-(5.5). We then have,

Lemma 3. Given $g \in \{\gamma_1,\ldots,\gamma_k\}$ and $\underline{\alpha} > 0$, there exists a constant $D > 0$ such that for all $\alpha \in [\underline{\alpha},1]$

$$(5.6) \qquad w(g,\alpha) \equiv \bar{w}(g,\alpha) - w^A(g,\alpha) > D.$$

We prove this lemma in the appendix.

Finally, we will assume that the level debt under the full commitment policy is uniformly bounded in $\beta$. That is,

Assumption 3. There exists a constant $B < \infty$ and a $\beta_0 \in (0,1)$ such that for any $\beta \in (\beta_0,1]$ the full commitment debt say, $\hat{b}_t(g^t,\beta)$ satisfies

$$\hat{b}_t(g^t,\beta) < B \text{ all } t \text{ and } g^t \in G^t.$$

We show in Theorem 4 that this assumption is satisfied if government consumption follows a Markov process with a strictly positive transition matrix. For now consider,

Theorem 3 Given assumptions 1-3, there is a $\underline{\beta} \in (0,1)$ such that for all $\beta \in (\underline{\beta},1)$ the full commitment policy is sustainable.

Proof. Let $\hat{V}_t(g^t,\beta)$ and $V_t^A(g^t,\beta)$ denote the discounted utility of consumption from time t onwards given some $g^t$ and discount factor $\beta$ under the full commitment and autarky policies respectively. In view of Theorem 1 it suffices to show there is some $\underline{\beta} \in (0,1)$

$$\hat{V}_t(g^t,\beta) \geq V_t^A(g^t,\beta) \text{ for all } \beta \in (\underline{\beta},1)$$

We will construct a plan which is feasible and which sufficiently dominates autarky for large $\beta$. For any history $g^t$ and associated debt level $b^t(g^t)$ the plan is as follows. For $k(g^t,\beta)$ periods set taxes high enough to just retire the outstanding debt. For any history $g^{s-1}$ with $s > t + k(g^t,\beta)$ do the following: If $g^s = (g^{s-1},\gamma(g^{s-1}))$ let the allocations $c_s(g^{s-1},\gamma(g^{s-1}))$, $x_s(g^{s-1},\gamma(g^{s-1}))$ and $c_{s+1}(g^{s-1},\gamma(g^{s-1}),0)$, $x_{s+1}(g^{s-1},\gamma(g^{s-1}),0)$ solve the problem maximize (5.2) subject to (5.3) and (5.4) where $g_0 = \gamma(g^{s-1})$, $g_1 = 0$ and

$$\alpha = \beta f(g^{s-1},\gamma(g^{s-1}),0|g^{s-1},\gamma(g^{s-1})).$$

For all other $g^s$ let the allocations be the autarky allocations. Let $\Delta(g^t,\beta)$ denote the difference discounted utility between this plan and the autarky plan.

We will first show there exists some $\beta_1 \in (0,1)$ and a $k_1 < \infty$ such that for all $\beta \in [\beta_1,1]$

$$k(g^t,\beta) \leq k_1 \text{ all } g^t \in G^t.$$

Now the discounted value of revenues raised by a policy of maximizing revenues in each state for some arbitrary number of periods, say k, is at least

$$(5.7) \qquad R(g^t,\beta,k) = r(g^t) + \sum_{s=t+1}^{t+k} \sum_{g^s} \beta^{s-t} f(g^s|g^t) \frac{U_c(g^s)}{U_c(g^t)} r(g^s)$$

where $U_c(g^s)$ denotes the marginal utility of consumption under this plan and where $r(g^t)$ is given in (5.1). From assumption 1 we have

$$(5.8) \qquad R(g^t,\beta,k) \geq r + \sum_{s=t+1}^{t+k} \sum_{g^s} \beta^{s-t} f(g^s|g^t) mr \geq h(\beta,k)$$

where $h(\beta,k) \equiv mr\ (1-\beta^{k-1})/(1-\beta)$. Clearly, $h$ is strictly monotone and continuous in $(\beta,k)$ for $k \geq 2$. Furthermore, for each fixed $k$,

$$\lim_{\beta \uparrow 1} h(\beta,k) = k - 1.$$

If we let $k_1 = 1 + 2B/mr$, then clearly there is a $\beta_1 \in (0,1)$ such that for all $\beta \in (\beta_1,1)$, $h(\beta,k_1) \geq B$. By construction it follows for all $\beta \in (\beta_1,1)$, $k(g^t,\beta) \leq k_1$ for all $g^t \in G^t$.

Now we will use this value of $k_1$ to prove the result. Given $g^t$ and $\beta$ let $\Delta_1(g^t,\beta)$ denote the difference in expected discounted utility under the above plan and the autarky plan for periods $t$ up to $t + k(g^t,\beta)$. Let $\Delta_2(g^t,\beta)$ denote the corresponding difference from $t + k(g^t,\beta)$ onwards. Since $k(g^t,\beta) \leq k_1$ we have

$$(5.9) \qquad \Delta_1(g^t,\beta) \geq -k_1(\bar{\mu}-\underline{\mu})$$

where $\bar{\mu}$ and $\underline{\mu}$ are the finite constants that bound the utility function $U$.

Next,

$$(5.10) \qquad \Delta_2(g^t,\beta) \geq \sum_{s=t+k_1}^{\infty} \sum_{g^{s-1}} \beta^{s-t} f(g^{s-1},\gamma(g^{s-1})|g^t)[w(\gamma(g^{s-1}),\alpha(g^{s-1}))]$$

where $\alpha(g^{s-1}) = \beta f(g^{s-1},\gamma(g^{s-1}),0|g^{s-1},\gamma(g^{s-1}))$ and $w$ is defined in (5.6). From assumption 2 we have $\alpha(g^{s-1}) > \beta\pi > 0$ for all $g^{s-1} \in G^{s-1}$. Choosing $\underline{\alpha} = \beta\pi$, it follows from Lemma 2 that for all $\alpha \in [\underline{\alpha},1]$

$$w(g,\alpha) > D > 0 \text{ for all } g \in \{\gamma_1,\ldots,\gamma_k\}.$$

Thus, for all $\beta \in (\beta_1,1)$

$$\Delta_2(g^t,\beta) > D \sum_{s=t+k_1}^{\infty} \sum_{g^{s-1}} f(g^{s-1},\gamma(g^{s-1})|g^t)$$

which, using assumption 2 again, implies for all such $\beta$

$$\Delta_2(g^t,\beta) > \beta^{k_1} D\pi/(1-\beta).$$

Notice that the right side of this expression is continuous and strictly increasing in $\beta$. Hence for any $N < \infty$ there is a $\beta_2 \in (0,1)$ such that for all $\beta \in [\beta_2,1]$

$$\Delta_2(g^t,\beta) > N \text{ all } g^t \in G^t.$$

Choose $\underline{\beta} = \max(\beta_0,\beta_1,\beta_2)$ and the proof is complete.◊.

Considering examples 1-3 it is clear that we need to rule out processes for government consumption which along some sequence of realizations settle down to a constant. However, it is clear from the proof of the theorem that we can weaken Assumption 2 to

Assumption 2': There is some constant $\pi > 0$ and integer $n < \infty$ such that for every $g^{t-1} \in G^{t-1}$ there is some $\gamma \in \{\gamma_1,\ldots,\gamma_K\}$ and integers i, j with $0 < i < j \leq n$ such that

$$f(g^{t-1},g_{t+1},\ldots,g_{t+i-1},\gamma,g_{t+i+1},\ldots,g_{t+j},0|g^{t-1}) \geq \pi > 0.$$

Notice that $\pi$ and n do not depend on the sequence $g^{t-1}$. Thus, for Theorem 3 to hold we need only assume that for all histories there is some positive probability that government consumption is positive and then zero within a finite amount of time.

We show in Theorem 4 that if government consumption is a Markov chain with a strictly positive transition matrix then Assumption 3 is satisfied. To prove this we use a simple result on strictly positive Markov chains. Let $\Pi$ be a transition matrix for a n-state Markov chain,

$$\Pi = (\pi_{ij} | i,j=1,\ldots,n)$$

with $\pi_{ij} \geq 0$ for all i,j and $\sum_j \pi_{ij} = 1$ for all i. For any vector $z \in R^n$ let $\|z\| = \sum_i |z_i|$. We then have

Lemma 4. If $\Pi$ is a transition matrix satisfying $\pi_{ij} \geq \varepsilon > 0$ for all i,j and $y,z \in R^n_+$ satisfy

$$\sum_i y_i = \sum_i z_i = 1$$

then

(5.11) $\quad \|x\Pi - y\Pi\| \leq (1-n\varepsilon)\|x-y\|$

Proof. $\quad \|y\Pi - z\Pi\| = \sum_i |\sum_i (y_i - z_i)\pi_{ij}| = \sum_j |\sum_i (y_i - z_i)(\pi_{ij} - \varepsilon)|$

since $\varepsilon \sum_i (y_i - z_i) = 0$. Rearranging the order of summation and using the Cauchy-Schwartz inequality we have

$$\|y\Pi - z\Pi\| \leq \sum_i \sum_j |(y_i - z_i)(\pi_{ij} - \varepsilon)| \leq \sum_i \left[ |y_i - z_i| \sum_j |\pi_{ij} - \varepsilon| \right]$$

$$< (1-n\varepsilon) \sum_i |y_i - z_i| = (1-n\varepsilon)\|y-z\|^*.$$

We will add Theorem 4 in the next draft.

## 5. Conclusion

In this paper we have analyzed the constraints imposed by dynamic consistency on the design of social policies. The classic problem in the time

consistency literature is the issue of levies on capital. In order to keep the framework as simple as possible we have abstracted away from the accumulation of capital. However, note that taxation or default on inherited debt functions exactly like taxation of capital in that does not distort current decisions. Anticipations of such taxes in the future, however, can have a sizable effect on current decisions. Introducing capital accumulation complicates our analysis but we believe the substantive conclusions will remain.

Our equilibrium concept is a derivative of subgame perfection in game theory. In our framework, policymakers in any period choose policies taking as given the policy functions that future policymakers use. We have shown that the class of feasible policies depends critically on the domain of the policy functions. If the state of the system is defined as the inherited debt and the realizations of government consumption, the only sustainable plan is never to issue any debt. If the domain is expanded to include past policies, then even full commitment plans are sustainable with sufficiently little discounting.

We have chosen to focus on the effect of the actions of current policymakers on the actions of future policymakers. It might be thought that in our framework similar results can be obtained by restricting the domain of policymakers' functions and allowing the beliefs of private agents about future policies to vary in response to current actions. For example, suppose that we assume that private agents believe that if current policymakers default, all future policymakers will also default. Suppose also policymakers are constrained to use functions which depend only on the inherited debt and government consumption. We will argue that no debt will be issued in equilibrium. The argument is by contradiction. Suppose along the equilibrium path, no future governments will default. Then, regardless of the decisions of

current governments, it will not be rational for private agents to believe that future governments will default. Consequently, it is best for current policymakers to default on the inherited debt and reduce current taxes appropriately. It follows that no debt is ever issued in equilibrium.

An alternative equilibrium concept we have not explored in this paper is one for which dynamic consistency requires policies to be optimal when "replanning" is permitted in the future. Within our framework, one could think of society at each instant selecting policy functions for the entire future given that future societies will also act similarly. It is easy to see that in our framework no debt will ever be issued in equilibrium. We do not find this replanning equilibrium concept appealing. If governments cannot commit themselves to actions in the future we hardly find plausible the idea that they can commit the functions future governments will use.

It is sometimes argued that the problems raised by time consistency suggest the superiority of rules over discretion. Since any sequence of policies can be interpreted as a rule, this distinction is not meaningful in economic theory. Rather the literature on time consistency emphasizes the value of rules as a form of commitment and the difficulties in binding future societies to policies current societies find desirable. In this paper we suggest that even though societies cannot make binding commitments about debt repayments this need not imply that governments can never sell any debt. Indeed, with sufficiently little discounting the policies and resulting allocations are identical to those in a world with commitment.

References

Barro, Robert J., 1979, "On the determination of the public debt," Journal of Political Economy, 87.

Barro, Robert J. and David B. Gordon, 1983a, "Rules, discretion and reputation in a model of monetary policy," Journal of Monetary Economics.

_____, (1983b) "A positive theory of monetary policy in a natural rate model," Journal of Political Economy.

Kydland, Finn E. and Edward C. Prescott, 1977, "Rules rather than discretion: The inconsistency of optimal plans," Journal of Political Economy, 85.

Lucas, Robert E., Jr. and Nancy L. Stokey, 1983, "Optimal fiscal and monetary policy in an economy without capital," Journal of Monetary Economics, 12.

Prescott, Edward C., 1977, "Should control theory be used for economic stabilization?" K. Brunner and A. H. Meltzer, eds., Optimal policies, control theory and technology exports (North-Holland, Amsterdam).

Ramsey, Frank P., 1927, "A contribution to the theory of taxation," Economic Journal, 37.

Rogoff, Kenneth, 1987, "Reputational constraints on monetary policy," in K. Brunner and A. H. Meltzer (eds.) Carnegie-Rochester Conference Series on Public Policy, 26 (North-Holland: Amsterdam).

Strotz, Robert H., 1955, "Myopia and inconsistency in dynamic utility maximization," Review of Economic Studies, 23.

Turnovsky, Stephen J. and William A. Brock, 1980, "Time consistency and optimal government policies in perfect foresight equilibrium," Journal of Public Economics, 13.

## Appendix

This appendix proves Lemma 3. In the proof we use arguments similar to those used in section 3 of Lucas and Stokey [1983].

Proof: We prove the lemma in two steps. First, we show that for any $\alpha > 0$ and $g > 0$, $w(\alpha,g) > 0$. Second, we show for any $\underline{\alpha} > 0$, $w(g,\cdot)$: $[\underline{\alpha},1] \to R$ is a continuous function. The result then follows immediately.

To establish the first step it suffices to prove that the solution to the problem: choose $(c_0,x_0,c_1,x_1)$ to maximize (4.2) subject to (4.2) and (4.3) involves a positive amount of debt, that is,

$$(A.1) \qquad b = \frac{U_{c0}}{\alpha U_{c1}} \left(g-\tau_0(1-x_0)\right) > 0.$$

Because then adding constraint (4.4) to this problem necessarily leads to a strictly lower level of utility.

The first order conditions for the $\bar{w}(g,\alpha)$ problem include:for t=0,1.

$$(A.2) \qquad (1+\lambda)U_c + \lambda\left[c_t U_{cc}+(x_t-1)U_{cx}\right]-\mu_t = 0$$

and

$$(A.3) \qquad (1+\lambda)U_x + \lambda\left[c_t U_{cx}+(x_t-1)U_{xx}\right]-\mu_t = 0$$

where $\lambda$ are the Lagrange multipliers for (4.3) and (4.4) respectively. We can manipulate these first order conditions to establish the first part of the proof. Multiply (A.2) by $c_t$ and (A.3) by $(x_t-1)$ and add them to get: for $t = 0, 1$,

$$(A.3) \qquad (1+\lambda)\left[c_t U_c+(x_t-1)U_x\right] - \lambda Q_t - \left[c_t+x_t-1\right]U_t = 0$$

where

$$Q_t = [c_t^2 U_{cc} + 2c_t(x_t-1)U_{cx} + (x_t-1)^2 U_{xx}].$$

Since U is strictly concave, $Q_t < 0$ for t = 0, 1. Evaluate (A.3) at t = 0 and add it to $\alpha$ times (A.3) evaluated at t = 1 to get

$$(A.4) \qquad (1+\lambda)[(c_0 U_c + (x_0-1)U_x) + \delta(c_1 U_c + (x_1-1)U_x)] + \lambda[Q_0 + \delta Q_1] - \sum_{t=0}^{1} \delta^t u_t(c_t + x_t - 1) = 0$$

Constraint (4.2) implies the first term is zero while constraints (4.3) evaluated at t = 0, 1 imply $c_0 + x_0 - 1 = -g$ and $c_1 + x_1 - 1 = 0$. Thus we can rewrite (A.4) as

$$(A.5) \qquad \lambda[Q_0 + \delta Q_1] + u_0 g = 0.$$

The multiplier $u_0 > 0$, since relaxing (4.3) by increasing the endowment of leisure strictly increases the value of utility given that U is strictly monotone. Since $u_0 > 0$ and $Q_t < 0$, we have $\lambda > 0$.

We use the fact that $\lambda > 0$ to show the debt is positive. Evaluating (A.3) for t = 1 gives

$$(A.6) \qquad (1+\lambda)[c_1 U_c + (x_t-1)U_x] + \lambda Q_1 = 0.$$

Since $\lambda > 0$ and $Q_1 < 0$, the bracketed term in (A.6) is positive. Since $U_c > 0$, this implies

$$0 < c_1 + (x_1-1)\frac{U_x}{U_c}$$

or

$$(A.7) \qquad 0 < c_1 - (x_1-1)(1-\tau_1) = \tau_1(1-x_1).$$

Rewriting (4.3) as

$$[\tau_0(1-x_0)-g_0] + \frac{\delta U_{c_1}}{U_{c_0}} [\tau_1'(1-x_1)] = 0,$$

it is clear that (A.7) implies the debt b given in (A.1) is strictly positive. Thus adding constraint (4.4) to this problem lowers the maximal value of utility for any $\alpha > 0$ and $g > 0$.

Next, for any $\underline{\alpha} > 0$ the maximum theorem of Berge implies the function $w(g,\cdot)$: $[\underline{\alpha},1] \to R$. is continuous. For any $g > 0$, let

$$w(g) = \min \; w(g,\alpha) \,|\, \alpha \in [\underline{\alpha},1]$$

Since $w(g,\cdot)$ is a continuous function and $[\underline{\alpha},1]$ is a compact subset of R, such a minimum exists. From the first part of the lemma, $w(g) > 0$ for any $g > 0$. Let

$$D = \frac{1}{2} \min \left[ w(g) \,|\, g_t \in \{\gamma_1, \ldots, \gamma_k\} \right].$$

This constant D satisfies (4.6) and the proof is complete.