Historical

Federal Reserve Bank of Minneapolis
Research Department Working Paper

EQUILIBRIUM EXISTENCE IN AN OVERLAPPING GENERATIONS MODEL WITH ALTRUISTIC PREFERENCES

S. Rao Aiyagari*

Working Paper 356

Revised July 1987

*Federal Reserve Bank of Minneapolis. I am grateful to the NSF for financial support. All errors are mine.

ABSTRACT

We prove the existence of a competitive equilibrium in an overlapping generations model in which each generation has a preference ordering over its own and its descendents' consumptions. The model is one of pure exchange with many goods in each period and two period lived generations. The bequest from one generation to the next is required to be non-negative and is endogenous. In equilibrium, some sequences of agents of successive generations may be continually "linked" by positive bequests and act as infinitely lived agents. Other sequences of agents may not be so linked and therefore behave as sequences of finite lived agents. We give three examples which illustrate the following points: (i) multiple equilibria may exist some of which resemble those of standard overlapping generations models, whereas in others some sequences of agents behave as if infinitely lived, (ii) multiple steady states of the above two types may exist in which the latter are unstable and the former are stable, and (iii) if agents have preferences given by discounted sums of utilities with different discount rates, then not all sequences of generations can be continually linked and hence behave as infinitely lived agents.

I. Introduction

We prove the existence of a competitive equilibrium for an overlapping generations (hereafter, OLG) model which may be viewed as an endogenously determined mix of infinitely lived agents and sequences of finite lived agents. The model is one of pure exchange with many goods in each period and sequences of overlapping two period lived agents.[1] We assume that each agent in a generation cares not only about its own lifetime consumptions but also about the welfare of its descendent in the next generation. This is characterized by specifying the preference ordering for an agent as defined over the infinite stream of consumptions consisting of its own lifetime consumptions, its descendant's lifetime consumptions and so on, ad infinitum. A consistency requirement is imposed on preferences in the following sense. Let $c^h(t)$ be the lifetime consumption vector of agent h in generation t. Then for any fixed $c^h(t)$, agent (h,t)'s subpreference ordering over $\{c^h(j), j>t\}$ is the same as agent (h,t+1)'s preference ordering over $\{c^h(j), j>t\}$.[2]

Each generation makes a bequest to the next and the bequest is required to be nonnegative; a constraint which may be binding. Agents of two successive generations may be said to be "linked" if the older agent is making a positive bequest to the younger. A sequence of agents of successive generations who are so linked act as a single infinitely lived agent, i.e., as one who not only has preferences over the entire infinite stream of consumptions of the linked agents but also faces a single infinite horizon budget constraint involving those same consumptions. A sequence of agents of successive generations for whom the constraint on bequests is binding at some dates act as a sequence of finite lived agents in terms of their preferences and budget constraints, both of which are effectively truncated. Bequests, and hence the effective horizon of various agents, is determined in equilibrium.

By means of examples, we discuss some possible types of behavior in such models. It is possible to have multiple equilibria with some of them corresponding to those of standard overlapping generations models whereas, in others some sequences of generations behave as infinitely lived agents. Similarly, there may be multiple steady states with steady states of the latter type being unstable whereas steady states of the former type are stable. In models of infinitely lived agents with time additive utilities and constant time preference rates, agents with the lowest rate of time preference (the most patient type) end up asymptotically owning all of the wealth and the consumptions of all the other agents converge to zero asymptotically (Lucas and Stokey [1984, p. 159]). This never happens in the present model for a similar specification of preferences. Instead, the constraint on bequests will become binding for some sequences of agents so that the infinitely lived agents structure cannot be maintained.

The paper borrows heavily from Wilson [1981]. It is also, obviously, related to the earlier work of Bewley [1972], Balasko and Shell [1980], and Balasko, Cass and Shell [1980]. Wilson's [1981] set up is general enough to accommodate both finite and infinitely lived agents. However, the distinction is made a priori and is not part of the determination of equilibrium. A similar comment applies to Muller and Woodford [1983]. Mention must be made of Barro [1974] which provides the motivation for considering models of this sort. Laitner [1986] is closely related to ours, though it has a different focus.

The rest of this paper is organized as follows. In section II, we lay out the model, establish existence of equilibrium and study optimality of equilibrium. Section III contains three examples which illustrate some of the possible types of equilibrium behavior mentioned previously. Section IV concludes. All of the proofs are relegated to the Appendix.

II. Model

At each date t (≥ 1), a constant number of agents H are born (indexed by h) who are alive at dates t and (t+1) and are referred to as members of generation t. At date 1, there are also alive H agents of generation 0, in their last period of life. Henceforth, agent h belonging to generation t is denoted by (h,t). The symbol $\mathcal{H}$ denotes the set $\{1,2,\ldots,H\}$. There are a fixed number $\ell(\geq 1)$ of completely perishable goods at each date, indexed by i.

Consumption Sets, Endowments and Preferences

For $t \geq 1$, let $(w_1^h(t), w_2^h(t)) \in R_+^\ell X R_+^\ell$, let $w^h(t) = \{(w_1^h(j), w_2^h(j+1)), j \geq t\} \in R_+^\infty$ and let $w^h(0) = (w_2^h(1), w^h(1)) \in R_+^\infty$. In a similar fashion, for $t \geq 1$, let $(x_1^h(t), x_2^h(t)) \in R_+^\ell X R_+^\ell$, let $x^h(t) = \{(x_1^h(j), x_2^h(j+1)), j \geq t\} \in R_+^\infty$ and let, $x^h(0) = (x_2^h(1), x^h(1)) \in R_+^\infty$.

For $t \geq 0$ and $h \in \hat{H}$, agent (h,t)'s consumption set is R_+^∞ with generic element $x^h(t)$ and his/her endowment is $w^h(t)$. We interpret $w_1^h(t)$ (respectively, $w_2^h(t)$) as the endowment when young (respectively, when old) of a type h agent at date $t \geq 1$. Similarly, we interpret $x_1^h(t)$ (respectively, $x_2^h(t)$) as the consumption when young (respectively, when old) of a type h agent at date t.

Each agent (h,t) has a strict preference relation $(>)_t^h$ and we let $P_t^h(x) = \{y | y\ (>)_t^h\ x\}$ and $(P_t^h)^{-1}(x) = \{y | x\ (>)_t^h\ y\}$. $P_t^h(x)$ denotes the set of commodity bundles that agent (h,t) strictly prefers to x and $(P_t^h)^{-1}(x)$ denotes the set of commodity bundles to which x is strictly preferred by (h,t).

Assumptions. The following assumptions are made regarding preferences and endowments. Except for assumption 2, they parallel those of Wilson [1981].

Assumption 1

(i) (Continuity) The sets $P_t^h(x)$ and $(P_t^h)^{-1}(x)$ are both open relative to R_+^∞ with respect to the product topology on R^∞.

(ii) (Convexity) If $y \in P_t^h(x)$ then $\lambda y + (1-\lambda)x \in P_t^h(x)$ for $0 < \lambda \leq 1$.

(iii) (Free Disposal) If $y \in P_t^h(x)$ and $z \geq y$ then $z \in P_t^h(x)$.

(iv) (Irreflexivity) $x \notin P_t^h(x)$.

Assumption 2 (consistency of preferences)

For any fixed $t \geq 0$, let $x^h(t)$ and $y^h(t)$ be such that $(y_1^h(t), y_2^h(t+1)) = (x_1^h(t), x_2^h(t+1))$. We then have: if $y^h(t+1) \in P_{t+1}^h(x^h(t+1))$ then $y^h(t) \in P_t^h(x^h(t))$.

With regard to endowments we have,

Assumption 3

The aggregate endowment of each good at each date is positive. That is,

$$\sum_{h=1}^{H} (w_1^h(t) + w_2^h(t)) \in R_{++}^{\ell} \text{ for all } t \geq 1$$

In addition to the above we need an assumption regarding irreducibility for the economy.

Assumption 4 (Irreducibility)

For $t \geq 1$, let $H(t) = \{(h,s) | h \in \bar{H}, s \in \{t-1,t\}\}$ be the set of agents alive at date t. If $\alpha = (h,s) \in H(t)$ then $w^\alpha(j)$ denotes agent α's endowment at date $j \geq 1$. This is given by,

$$w^\alpha(j) = 0 \text{ for } j \neq s,\ j \neq s+1$$

$$w^\alpha(s) = w_1^h(s)$$

$$w^\alpha(s+1) = w_2^h(s+1).$$

Now, let $H_1(t)$ and $H_2(t)$ be any two nonempty, disjoint subsets of $H(t)$ whose union is $H(t)$. We then have:

There exists $\beta = (h,s) \in H_1(t)$ such that, if

$$y_1^h(s) = x_1^h(s) + \sum_{\alpha \in H_2(t)} w^\alpha(s)$$

$$y_2^h(s+1) = x_2^h(s+1) + \sum_{\alpha \in H_2(t)} w^\alpha(s+1)$$

$$y_1^h(j) = x_1^h(j),\ j \geq 1,\ j \neq s$$

$$y_2^h(j) = x_2^h(j),\ j \geq 1,\ j \neq s + 1$$

then,

$$y^h(0) \in P_0^h(x^h(0)).$$

Definition of Equilibrium

The notion of competitive equilibrium here is slightly different from the traditional Arrow-Debreu notion because of the nonnegativity restriction on bequests. This is reflected in the budget constraints which are developed as follows. Let $p_t \in R_+^\ell$ be the price vector of goods at date t and let $p = (p_1, p_2, \ldots) \in R_+^\infty$. Let $b^h(t) \in R_+$ be the value of bequests made by agent $(h,t-1)$ at date t to agent (h,t). Further, let $b^h = \{b^h(t), t\geq 1\} \in R_+^\infty$ and $b = (b^h, h\in \bar{H})$.

We then have,

$$\text{(2.1a)} \qquad b^h(1) = p_1(w_2^h(1) - x_2^h(1))$$

$$\text{(2.1b)} \qquad b^h(t+1) = p_t(w_1^h(t) - x_1^h(t)) + p_{t+1}(w_2^h(t+1) - x_2^h(t+1)) + b^h(t),\ t \geq 1.$$

This leads to the following formulation of the budget constraints for agents (h,t).

$$(2.2)\quad b^h(t) + \sum_{j=t}^{t+k-1} \{p_j(w_1^h(j)-x_1^h(j))+p_{j+1}(w_2^h(j+1)-x_2^h(j+1))\} \geq 0,\ k \geq 1$$

where we take $b^h(0) = 0$ and $p_0 = 0$.

A compact way to write these constraints is as follows. For $t \geq 1$, let $z_1(t)$, $z_2(t) \in R^\ell$ and let $z = (z_2(1),z_1(1),z_2(2),\ldots) \in R^\infty$. Then define

$$(2.3a)\qquad v(t+1,p,z) \equiv \sum_{j=0}^{t} p_j(z_1(j)+z_2(j)) + p_{t+1}z_2(t+1),\ t \geq 0$$

$$(2.3b)\qquad v(0,p,z) \equiv 0.$$

The constraints (2.2) may now be rewritten as follows.

$$(2.4)\qquad b^h(t) + v(t+k,p,w^h(0)-x^h(0)) \geq v(t,p,w^h(0)-x^h(0)),\ k \geq 1.$$

Now let $B_t^h(p,b^h(t))$, $D_t^h(p,b^h(t))$ be the budget set and the demand set, respectively of agent (h,t). We then have,

$$(2.5)\qquad B_t^h(p,b^h(t)) \equiv \{x^h(t)\in R_+^\infty | x^h(t) \text{ satisfies } (2.4)\}$$

$$(2.6)\qquad D_t^h(p,b^h(t)) \equiv \{x^h(t)\in B_t^h(p,b^h(t)) | B_t^h(p,b^h(t))\cap P_t^h(x^h(t))=\phi\}$$

where ϕ stands for the empty set.

Let $x = \{x^h(0),h\in\bar{H}\}$. An allocation x is said to be <u>attainable</u> if

$$\sum_{h=1}^{H} (w_1^h(t)+w_2^h(t)-x_1^h(t)-x_2^h(t)) = 0.$$

A competitive equilibrium for this economy is defined as follows.

Definition 1 (Competitive Equilibrium)

A competitive equilibrium for the above economy consists of a $\bar{p} \in R_{+}^{\infty}$, $\bar{b} \in R_{+}^{\infty}$ and $\bar{x}$ such that $\bar{x}$ is attainable and

(i) $\bar{x}^h(t) \in D_t^h(\bar{p}, \bar{b}^h(t))$ for all (h,t).

(ii) $\bar{b}^h(t) = v(t, \bar{p}, w^h(0) - \bar{x}^h(0))$ for all (h,t).

Because of the recursive structure of the model it is possible to simplify the definition of a competitive equilibrium. This is done by means of the following proposition.

Proposition 1

For any $h \in \bar{H}$, if $x^h(0) \in D_0^h(p,0)$ and $b^h(t) \equiv v(t,p,w^h(0)-x^h(0))$ for all $t \geq 1$, then $x^h(t) \in D_t^h(p,b^h(t))$ for all $t \geq 1$.

Proof: In the Appendix.

By virtue of the above proposition we may simplify the notation and rewrite the definition of a competitive equilibrium as follows. From now on we will write w^h and x^h in place of $w^h(0)$ and $x^h(0)$. The budget constraints (2.4) can be rewritten as

(2.7) $v(t+1,p,w^h-x^h) \geq 0, \; h \in \bar{H}, \; t \geq 0.$

The budget sets and the demand sets can be simplified to

(2.8) $B^h(p) = \{x^h \in R_{+}^{\infty} | x^h \text{ satisfies } (2.7)\}$

(2.9) $D^h(p) = \{x^h \in B^h(p) | B^h(p) \cap P_0^h(x^h) = \phi\}.$

Definition 2 (Competitive Equilibrium)

A competitive equilibrium is $(\bar{p},\bar{x})$ such that $\bar{x}$ is attainable and $\bar{x}^h \in D^h(\bar{p})$ for all $h \in \tilde{H}$.

The model consisting of equations (2.7)-(2.9) and definition 2 may be interpreted formally as a model of a fixed number H of infinitely lived agents but with budget sets which are not standard. This interpretation may be supported by distinguishing goods on the basis of date $(t\geq1)$, physical characteristics $(i=1,2,\ldots,\ell)$ and an additional type (k=1,2 for young and old, respectively). A production possibilities set may be defined by assuming that good (t,i,k) can be transformed into good (t,i,k') on a one to one basis where k, k' $\in \{1,2\}$. It follows that in equilibrium we must have $p_{tik} = p_{ti}$ for k = 1, 2.

An obvious implication of the budget constraints (2.7) is the following.

$$\text{(2.10)} \qquad \liminf_{t\to\infty} v(t+1,p,w^h-x^h) \geq 0.$$

In fact, it can be shown that at a consumer optimum, (2.10) must hold as an equality.

Proposition 2

Given assumptions 1 and 4, if $x^h \in D^h(p)$ then, $\liminf_{t\to\infty} v(t+1,p,w^h-x^h) = 0$.

Proof: In the Appendix.

The above proposition together with (2.10) suggests that if the nonnegativity restrictions on bequests are never binding then it ought to be possible to replace the budget constraints (2.7) by (2.10). This can be shown rigorously. Let us define an alternative budget set $\tilde{B}^h(p)$ and demand set $\tilde{D}^h(p)$ as follows

(2.11) $\tilde{B}^h(p) = \{x^h \in R_+^\infty | x^h \text{ satisfies } (2.10)\}.$

(2.12) $\tilde{D}^h(p) = \{x^h \in \tilde{B}^h(p) | \tilde{B}^h(p) \cap P_0^h(x^h) = \phi\}.$

We then have,

Proposition 3

(i) If $\liminf_{t\to\infty} v(t+1,p,w^h) > 0$, $x^h \in D^h(p)$ and the budget constraints (2.7) hold as strict inequalities, then $x^h \in \tilde{D}^h(p)$.

(ii) If $x^h \in \tilde{D}^h(p) \cap B^h(p)$, then $x^h \in D^h(p)$.

Proof: In the Appendix.

We, therefore, have the result that if the nonnegativity restrictions on bequests are not binding then the budget constraints (2.7) may be replaced by (2.10). This is exactly the form of the budget constraint imposed by Wilson (1981, section 3, p. 100). If, in addition, the consumption sets are restricted to ℓ_+^∞, the endowment sequences $w^h(0) \in \ell_+^\infty$ and the price sequence p $\in \ell_+^1$ then (2.10) becomes the standard Arrow-Debreu budget constraint for an infinitely lived agent. Thus, the principal distinction between this model and Bewley [1972] is in the nonnegativity constraints on bequests. On the contrary, if bequests are zero in every period then it can be seen that the budget constraints (2.7) break up into separate budget constraints for each generation and the equilibria correspond to those of the standard OLG model (Balasko and Shell [1980]).

Thus, we have a situation in which if the nonnegativity constraints in (2.7) are never binding then agent h behaves as an infinitely lived agent in terms of his/her preferences and budget constraints. If the constraints (2.$\bar{7}$) are binding in every period then the agent behaves as a sequence of independent overlapping generations in terms of preferences and budget constraints. It goes without saying that intermediate cases can also occur and that the situation may be different for different h's. However, a priori, we cannot assume that the equilibrium corresponds to one or the other case. Thus, we cannot appeal to Bewley [1972], Balasko and Shell [1980], Balasko, Cass and Shell [1980] or Wilson [1981] to claim the existence of a competitive equilibrium. However, as will be seen the methods involved are the same. We have chosen to follow Wilson [1981].[3]

Existence of Equilibrium

Here we will prove existence for the economy (denoted by E) defined by (2.7)-(2.9) and definition 2. We will construct an increasing sequence of truncated economies each with finite numbers of goods and agents and by a standard limiting argument establish existence for the infinite model. Let $T \geq 1$ be finite and let E_T be a truncated economy in which the set of agents is $\mathcal{H}$ and the commodity space of each agent is $R_+^{(2T+1)\ell}$ with generic element

(2.13) $$x_T^h = [x_{T,2}^h(1),\{(x_{T,1}^h(t),x_{T,2}^h(t+1)),t=1,2,\ldots,T\}].$$

The endowment vector of agent h is w_T^h where,

$$w_T^h = \left[w_2^h(1), \{(w_1^h(t), w_2^h(t+1)), t=1,2,\ldots,T\}\right]. \tag{2.14}$$

Let $w_2^h(i,T+1)$ be the endowment of good i for a type h old agent at date (T+1). Let M_T be a $\ell \times \ell$ matrix such that,

$$\begin{aligned} M_T(i,j) &= 0 \text{ if } i \neq j \\ &= 0 \text{ if } i = j \text{ and } \sum_h w_2^h(i,T+1) = 0 \\ &= 1 \text{ if } i = j \text{ and } \sum_h w_2^h(i,T+1) > 0. \end{aligned}$$

Let,

$$\tilde{x}_T^h = \left[\{(x_{T,2}^h(t), x_{T,1}^h(t)), t=1,2,\ldots,T\}, M_T x_{T,2}^h(T+1), \{(w_1^h(t), w_2^h(t+1)), t>T\}\right]. \tag{2.15}$$

For each h, a preference ordering $(>)_T^h$ is defined as follows: $x_T^h \ (>)_T^h \ y_T^h$ if and only if $\tilde{x}_T^h \ (>)_0^h \ \tilde{y}_T^h$.

Now, let

$$P_T^h(x) = \{y \mid y \ (>)_T^h \ x\}$$

$$p_t^T \in R_+^\ell, \ t = 1, 2, \ldots, T+1$$

$$p^T = (p_1^T, p_2^T, \ldots, p_{T+1}^T)$$

$$x_T = (x_T^h, h \in \tilde{H}).$$

The budget constraints are defined as follows:

$$v(t+1, p^T, w_T^h - x_T^h) \geq 0, \ t = 0, 1, \ldots, T. \tag{2.16}$$

The budget sets and the demand sets are then defined as

(2.17) $$B_T^h(p^T) = \{x_T^h \mid x_T^h \text{ satisfies equations (2.16)}\}.$$

(2.18) $$D_T^h(p^T) = \{x_T^h \in B_T^h(p^T) \mid B_T^h(p^T) \cap P_T^h(x_T^h) = \phi\}.$$

<u>A competitive equilibrium for the above truncated economy</u> is defined as $(\hat{p}^T, \hat{x}_T)$ such that $\hat{x}_T^h \in D_T^h(\hat{p}^T)$ for all $h \in \bar{H}$ and

(2.19a) $$\sum_{h=1}^{H} \left(w_1^h(t)+w_2^h(t)-\hat{x}_{T,1}^h(t)-\hat{x}_{T,2}^h(t)\right) = 0, \quad t = 1, 2, \ldots, T$$

(2.19b) $$\sum_{h=1}^{H} \left(w_2^h(T+1)-\hat{x}_{T,2}^h(T+1)\right) = 0.$$

It may be observed that the budget constraints together with the market clearing conditions (2.19) imply that in equilibrium,

(2.20) $$v(T+1, p^T, w_T^h - x_T^h) = \sum_{t=1}^{T} \hat{p}_t^T\left(w_1^h(t)+w_2^h(t)-\hat{x}_{T,1}^h(t)-\hat{x}_{T,2}^h(t)\right) + \hat{p}_{T+1}^T\left(w_2^h(T+1)-\hat{x}_{T,2}^h(T+1)\right)$$

$$= 0, \text{ for all } h \in \bar{H}.$$

We then have,

<u>Lemma 1</u>

For each $T \geq 1$, the economy E_T has a competitive equilibrium (p^T, x_T) with,

$$p_1^T \; w_2^h(1) > 0 \text{ for all } h$$

$$p_t^T \; w_1^h(t) + p_{t+1}^T \; w_2^h(t+1) > 0 \text{ for all } h \text{ and } t = 1, 2, \ldots, T.$$

<u>Proof</u>: In the Appendix.

Next, we have,

Lemma 2

There is a subsequence E_{T_k} of economies such that $\tilde{x}^h_{T_k} \to \bar{x}^h$ in the product topology of R^∞ where,

$$\sum_{h=1}^{H} \left(w_1^h(t)+w_2^h(t)-\bar{x}_1^h(t)-\bar{x}_2^h(t)\right) = 0, \ t \geq 1.$$

Proof: In the Appendix.

Henceforth, we restrict attention to the subsequence in the above lemma. In view of lemma 1, it is legitimate to restrict attention to equilibrium prices p^T which are normalized such that $p_1^T \ w_2^1(1) = 1$. From here on, we assume that equilibrium prices p^T are normalized in the above fashion.

For $t \geq 1$, let $H(t) = \{(h,s) | h \in \mathcal{H}, s=t-1,t\}$ be the set of agents alive at date t. If $\alpha = (h,s) \in H(t)$ then let $I_T^\alpha(t)$ be the income of agent α, so that

$$I_T^\alpha(t) = p_s^T \ w_1^h(s) + p_{s+1}^T \ w_2^h(s+1)$$

where, if necessary, we define $p_0^T \equiv 0$.

Lemma 3

For any fixed $t \geq 1$ and any two agents α, $\beta \in H(t)$, $I_T^\alpha(t)/I_T^\beta(t)$ is bounded and bounded away from zero as T goes to infinity.

Proof: In the Appendix.

We now let $\tilde{p}_t^T \in R_+^\ell$ be such that

$$\tilde{p}_t^T = p_t^T \text{ for } t = 1, 2, \ldots, T + 1$$

$$\tilde{p}_t^T = 0 \text{ for } t > T + 1.$$

Further, let $\tilde{p}^T = \{\tilde{p}_t^T, t \geq 1\} \in R_+^\infty$.

Lemma 4

There exists a subsequence of economies E_{T_k} such that $\tilde{p}^{T_k} \to \bar{p} > 0$ with $0 < \bar{p}_1 w_2^h(1) < \infty$ and $0 < \bar{p}_t w_1^h(t) + \bar{p}_{t+1} w_2^h(t+1) < \infty$ for all h and $t \geq 1$.

Proof: In the Appendix.

Theorem 1

$(\bar{p}, \bar{x})$ is a competitive equilibrium for the full economy E.

Proof: In the Appendix.

Optimality of Equilibrium

We will show that Theorem 4 of Wilson [1981, p. 105] extends to the present set up provided the convexity assumption 1 (ii) is replaced by a strong convexity assumption and assumption 2 is also suitably strengthened. We can then show that if the value of the aggregate endowment at competitive prices is finite then the competitive equilibrium allocation is pareto optimal. The argument for this is slightly more involved due to the difference in the budget constraints of agents. We now state the new assumptions.

Assumption 1'

Each agent (h,t) has a preference preordering $(\geq)_t^h$ that is complete, reflexive and transitive. The strict preference relation $(>)_t^h$ derived from $(\geq)_t^h$ satisfies assumption 1 except that 1 (ii) is replaced by: if $z\ (\geq)_t^h\ x$, $z \neq x$ then $\lambda z + (1-\lambda)\ x\ (>)_t^h\ x$ for $0 < \lambda < 1$.

Assumption 2'

Let $t \geq 0$ be given and let $x^h(t)$ and $y^h(t)$ be as in assumption 2. Then $y^h(t) \in P_t^h(x^h(t))$ if and only if $y^h(t+1) \in P_{t+1}^h(x^h(t+1))$.

In this subsection we will assume that assumptions 1 and 2 have been replaced by 1' and 2' respectively. We will say that an attainable allocation x is pareto optimal if there is no other attainable allocation z such that $z^h(t)\ (\geq)_t^h\ x^h(t)$ for all (h,t) and $z^h(t) \in P_t^h(x^h(t))$ for some (h,t) where $h \in \bar{H}$ and $t \geq 0$.

The need for a strong convexity assumption on preferences for optimality may be seen from the following example with just one agent in each generation and one good at each date. Let endowments be given by $(w_1(t),w_2(t)) = (w,w)$ where $w > 0$ and let preferences of the initial old be given by $x_2(1) + x_1(1) + \sum_{t=1}^{\infty} \beta^t[u(x_2(t+1))+u(x_1(t+1))]$ with $0 < \beta < 1$. The preferences of generation 1 are given by $x_1(1) + \sum_{t=1}^{\infty} \beta^t[u(x_2(t+1))+u(x_1(t+1))]$ and for $t \geq 2$, the preferences of generation t are given by $\sum_{j=0}^{\infty} \beta^j[u(x_1(t+j))+\beta u(x_2(t+j+1))]$. The function $u(\cdot)$ is assumed to be bounded, continuously differentiable, strictly increasing and strictly concave and further satisfies $u'(w) = 1$. It is obvious that $p_t = \beta^{t-1}$ and $(x_1(t),x_2(t)) = (w,w)$ is a competitive equilibrium with a finite value of the aggregate endowment. However, it is pareto dominated by the alternative attainable allocation, $\tilde{x}_2(1) = w - \varepsilon$, $\tilde{x}_1(1) = w + \varepsilon$, $(\tilde{x}_2(t), \tilde{x}_1(t)) = (w,w)$ for $t \geq 2$ where $0 < \varepsilon < w$.

We now state a preliminary lemma.

Lemma 5

Fix an h and suppose that $x^h \in D^h(p)$. If $y^h \in R^\infty_+$ is such that $y^h(t) \; (\geq)^h_t \; x^h(t)$ for all $t \geq 0$ and $y^h \neq x^h$ then $\liminf_{t \to \infty} v(t+1, p, w^h - y^h) < 0$.

Proof: In the Appendix.

We can now state the main result of this subsection as follows.

Theorem 2

Let $(\bar{p}, \bar{x})$ be a competitive equilibrium. If $\{\sum_h v(t+1, \bar{p}, w^h)\}$ is bounded then $\bar{x}$ is pareto optimal.

Proof: In the Appendix.

III. Examples

In our examples, preferences will be taken to be given by discounted sums of utilities as follows.

For agent h:

$$U_0^h(x_2^h(1)) + \sum_{t=1}^{\infty} (\beta_h)^t U^h(x_1^h(t), x_2^h(t+1)), \; 0 < \beta_h < 1 \tag{3.1}$$

(3.2) $\left(w_1^h(t), w_2^h(t)\right) = (w_1^h, w_2^h)$ for all h and $t \geq 1$.

(3.3a) $b^h(1) = p_1\left(w_2^h - x_2^h(1)\right)$

(3.3b) $b^h(t+1) = b^h(t) + p_t\left(w_1^h - x_1^h(t)\right) + p_{t+1}\left(w_2^h - x_2^h(t+1)\right)$, $t \geq 1$.

We assume that the functions $U_0^h(\cdot)$ and $U^h(\cdot,\cdot)$ are bounded, strictly increasing, concave and twice continuously differentiable. Maximizing (3.1) subject to (3.3) and assuming an interior solution for $\{x^h\}$ we have the first order conditions,

(3.4a) $U_{0,i}^h\left(x_2^h(1)\right) - \lambda^h(1)(p_1)_i = 0$

(3.4b) $(\beta_h)^t U_{2,i}^h\left(x_1^h(t), x_2^h(t+1)\right) - \lambda^h(t+1)(p_{t+1})_i = 0$, $t \geq 1$

(3.4c) $(\beta_h)^t U_{1,i}^h\left(x_1^h(t), x_2^h(t+1)\right) - \lambda^h(t+1)(p_t)_i = 0$, $t \geq 1$

(3.4d) $\lambda^h(t) \geq \lambda^h(t+1)$ with equality if $b^h(t) > 0$.

In equations (3.4), $\{\lambda^h(t)\}$ are the multipliers associated with the constraints (3.3).

Example 1 (Multiple Equilibria)

Suppose we have two types h = 1, 2 with $\beta_1 < \beta_2 = 1/4$ and only one good at each date. Let,

$$U^1(x,y) = \{(x^{1-\alpha}-1)+\gamma(y^{1-\alpha}-1)\}/(1-\alpha)$$

with $\alpha = 2$, $\gamma = 1/9$.

$$U^2(x,y) = x - x^2/2 + y/26.$$

Endowments are chosen as follows.

$$w_1^1 = 7,\ w_2^1 = 1,\ w_1^2 = 0,\ w_2^2 = 100.$$

Let $R_t = p_t/p_{t+1}$ be the gross interest rate at t. We will look for stationary equilibria with $R_t = R$ for all t. Overlooking the initial old for the moment (their utility functions $U_0^h(\cdot)$ will be chosen appropriately later) we can see that as long as $R < 4 = \beta_2^{-1} < \beta_1^{-1}$ bequests must be zero and hence we can treat the model as a standard OLG model. The savings functions of the two types are graphed in Figure 1 (at the end) and show that there exist <u>two</u> such equilibria denoted R_1 and R_2. We will focus on the R_1 equilibrium. We have to make sure that in this equilibrium there are no bequests being made by the initial old. This requires that,

$$(3.5) \qquad \left.\frac{\partial U_0^h}{\partial x}\right|_{R_1} > \beta_h U_1^h(x,y)|_{R_1},\ h = 1,\ 2.$$

where,

$$(3.6) \qquad x_2^h(1) = \begin{cases} 1\ , & h = 1 \\ 100, & h = 2 \end{cases}.$$

We now argue that there is also another stationary equilibrium with positive bequests for type 2 and zero bequests for type 1 in which $R = 4 = \beta_2^{-1} < \beta_1^{-1}$. Again, setting aside the initial old, we can see that since $R < \beta_1^{-1}$ there will be no bequests for type 1. Hence, they continue to behave as standard OLG agents and their saving can be calculated to be 11/14. As for type 2, let $b > 0$ denote bequests received when young. Then, their saving and consumption when young are given by, (3.7a) $s^2 = b - 1 + R/26 = b - 11/13$

(3.7b) $x_1^2 = 1 - R/26 = 11/13.$

It follows that there is a positive solution for bequests with b = 11/(13x14). For the initial old, we need

(3.8a) $$\left.\frac{\partial U_0^1}{\partial x}\right|_{R=4} > \beta_1 U_1^1(x,y)\Big|_{R=4}$$

(3.8b) $$\left.\frac{\partial U_0^2}{\partial x}\right|_{R=4} = \beta_2 U_1^2(x,y)\Big|_{R=4}$$

where

$$x_2^2(1) = 100 - 11/(13x14).$$

There is no problem in satisfying conditions (3.5) and (3.8) for type 1. We can simply pick $\partial U_0^1/\partial x$ to be arbitrarily large. However, for type 2 agents $\partial U_0^2/\partial x$ has to be picked appropriately. It can be seen that x_1^2 is larger at R_1 than at R = 4 and hence U_1^2 is smaller at R_1. That is,

$$U_1^2(x,y)\Big|_{R_1} < U_1^2(x,y)\Big|_{R=4}.$$

Therefore, we need to pick $U_0^2(\cdot)$ in such a way that,

$$\beta_2 U_1^2(x,y)\Big|_{R_1} < \left.\frac{\partial U_0^2}{\partial x}\right|_{R_1} \le \left.\frac{\partial U_0^2}{\partial x}\right|_{R=4} = \beta_2 U_1^2(x,y)\Big|_{R=4}.$$

It is obvious that such a choice can be made. We can pick,

$$U_0^2(x) \equiv x\beta_2 U_1^2(x,y)\Big|_{R=4}.$$

We thus have another stationary equilibrium with $R = 4$ and positive bequests being made every period for the type 2 agents. Note that with the above choice, R_2 is also an equilibrium of the same type as R_1. Moreover R_2 is optimal whereas R_1 is not.[4]

<u>Example 2</u> (Steady States and Stability)

Assume that there is only one type (so that we will omit the h index) and only one good at each date. Let,

$$(3.9) \qquad U_0\left(x_2(1)\right) = U\left(\bar{x}_1(0), x_2(1)\right)$$

where $\bar{x}_1(0)$ is regarded as an initial condition. Let the endowments be given by (3.2) and let,

$$\bar{c}_1 + \bar{c}_2 = w_1 + w_2$$

$$\frac{U_1}{U_2}(w_1, w_2) = R < \frac{U_1}{U_2}(\bar{c}_1, \bar{c}_2) = \beta^{-1}$$

and $\bar{c}_1 > w_1$ and $\bar{c}_2 < w_2$. The above situation corresponds to assuming that in a neighborhood of $(\bar{c}_1, \bar{c}_2)$ where the marginal rate of substitution U_1/U_2 exceeds one, U_1/U_2 <u>decreases</u> as c_1 is reduced and c_2 increased by the same amount. This will happen provided,

$$(3.10) \qquad -\beta U_{11} + (1+\beta)U_{12} - U_{22} < 0 \text{ at } (\bar{c}_1, \bar{c}_2).$$

This example has two steady states. If $\bar{x}_1(0) = w_1$, then there is a steady state with zero bequests in every period so that, $\left(x_1(t), x_2(t)\right) = (w_1, w_2)$ and $R_t = R$ for all $t \geq 1$. On the other hand, if $\bar{x}_1(0) = \bar{c}_1$, then there is a steady state with positive bequests, $\bar{b} = w_2 - \bar{c}_2 = \bar{c}_1 - w_1 > 0$, and $R_t = \beta^{-1}$ for all t.

The second steady state can be shown to be unstable. Let $b(t) = w_2 - x_2(t) = x_1(t) - w_1$ be the bequest at date t and suppose to the contrary that the second steady state is stable. Then for b(1) in a small neighborhood of $\bar{b}$, the path $\{b(t)\}$ converges to $\bar{b}$ and is hence always positive. We therefore have,

$$U_2\left(x_1(t-1),x_2(t)\right) = \beta U_1\left(x_1(t),x_2(t+1)\right)$$

or, equivalently

$$U_2\left(\bar{c}_1+b(t-1)-\bar{b},\bar{c}_2-b(t)+\bar{b}\right) = \beta U_1\left(\bar{c}_1+b(t)-\bar{b},\bar{c}_2-b(t+1)+\bar{b}\right).$$

Linearizing the above equation around $(\bar{c}_1,\bar{c}_2)$ we have,

$$\beta U_{12}(b_{t+1}-\bar{b}) - (U_{22}+\beta U_{11})(b_t-\bar{b}) + U_{12}(b_{t-1}-\bar{b}) = 0.$$

In view of condition (3.10), it is easy to verify that both roots of the above equation are outside the unit circle which contradicts the assumed stability of the bequest equilibrium.

The first steady state with bequests being always zero is, however, stable. This is because,

$$U_2(w_1,w_2) > \beta U_1(w_1,w_2)$$

and hence for $\bar{x}_1(0)$ in a small neighborhood of w_1, we still have,

$$U_2\left(\bar{x}_1(0),w_2\right) > \beta U_1(w_1,w_2).$$

<u>Example 3</u> (different discount factors)

In models of infinitely lived agents with preferences analogous to (3.1), asymptotically the interest rate is determined by those agents with the lowest discount rate, i.e., the most patient type. Consumptions of all the other agents converge to zero (Lucas and Stokey [1984], p. 159). We will show that in the present context, this can never happen. Instead, at some point the nonnegativity constraint on bequests will become binding for some type of agents so that the infinite lived agent set up becomes untenable. In example 1, we have a situation with different discount factors but only one type of agent behaves effectively as infinite lived in the bequest equilibrium. We will show that this situation is general in the sense that when discount factors differ, not all types of agents can act as infinitely lived agents.

Suppose that preferences and endowments are as given in (3.1) and (3.2). We assume that the utility functions $U_0^h(\cdot)$ and $U^h(\cdot,\cdot)$ are strictly concave and that the marginal utilities $U_{0,i}^h(\cdot)$, $U_{1,i}^h(\cdot,\cdot)$ and $U_{2,i}^h(\cdot,\cdot)$ are bounded and bounded away from zero over compact subsets of R_{++}^{ℓ}, $R_{++}^{2\ell}$ and $R_{++}^{2\ell}$ respectively. We further assume that, $U_{0,i}^h(x) \to \infty$ for all i only if $x \to 0$, and $\left(U_{1,i}^h(x,y), U_{2,i}^h(x,y)\right) \to \infty$ for all i only if $(x,y) \to (0,0)$. For simplicity, let there be only two types of agents with $\beta_1 < \beta_2$ and suppose, if possible, that there is an equilibrium with $b^h(t) > 0$ for all t and h. It follows that $\lambda^h(t) = \lambda^h(t+1) = \lambda^h$ for all t.

Let us define,

(3.11a) $\quad q(t) = p(t)/(\beta_2)^t$

(3.11b) $\quad \tilde{b}^h(t) = b^h(t)/(\beta_2)^t.$

The equations (3.3) become,

$$\text{(3.12a)} \quad \tilde{b}^h(1) = q(1)\left(w_2^h - x_2^h(1)\right)$$

$$\text{(3.12b)} \quad \beta_2\tilde{b}^h(t+1) = \tilde{b}^h(t) + q(t)\left(w_1^h - x_1^h(t)\right) + \beta_2 q(t+1)\left(w_2^h - x_2^h(t+1)\right), \ t \geq 1.$$

It follows that,

$$\text{(3.13)} \quad \tilde{b}^h(t) = (\beta_2)^{-t}\sum_{j=1}^{t-1}(\beta_2)^j q(j)\left(w_1^h + w_2^h - x_1^h(j) - x_2^h(j)\right) + q(t)\left(w_2^h - x_2^h(t)\right), \ t \geq 1.$$

By market clearing we then have,

$$\text{(3.14)} \quad \sum_h \tilde{b}^h(t) = q(t) \sum_h \left(w_2^h - x_2^h(t)\right), \ t \geq 1.$$

Similarly, conditions (3.4) can be rewritten in terms of $q(t)$. The argument now proceeds along standard lines to show that $q(t)$ must be bounded and bounded away from zero. Therefore, $\left(x_1^1(t), x_2^1(t)\right) \to 0$. However, from (3.12b) this implies that $\tilde{b}^1(t) \to +\infty$ which contradicts (3.14).

The basic intuition behind this is quite straightforward. In the standard infinitely lived agents set up, the agent with the high discount rate (type 1 here) is facing a lower interest rate and hence would like to transfer consumption from the future to the present by borrowing and would asymptotically drive future consumption to zero. If one interprets such an agent as a succession of linked generations then this strategy can only be sustained if one is permitted to pass on debts to succeeding generations i.e., if bequests are allowed to be negative. If bequests have to be nonnegative, then this strategy is infeasible and the best that can be done is to set bequests to zero. Then the model becomes like that in example 1 where one type of agents is linked and behaves as infinitely lived whereas the other type is not linked and behaves as a succession of independent overlapping generations. Another way to see the above point is to note that when bequests have to be nonnegative any strategy that drives consumptions to zero is suboptimal for the type 1 agent. He/she would be better off to set bequests equal to zero for all large t and simply consume his/her endowment in each period of life at each date thereafter.

This feature of our example is similar to that in Becker [1980] in which the consumptions of the high discount rate individuals are not driven to zero asymptotically. He, however, obtains this result by imposing a restriction that agents cannot borrow against future labor income. The nonnegativity constraint on bequests serves a similar purpose. While agents may borrow freely against their <u>own</u> future labor income they cannot borrow against the labor income of their descendants.

<u>A Remark on Preferences</u>

It may be noted that if the utility functions $U^h(\cdot,\cdot)$ are time separable across the two periods of life and negative bequests are permitted, then example 3 can be mapped <u>exactly</u> into a model of a fixed number of infinitely lived agents who maximize discounted sums of utilities. Suppose that,

$$U^h(x_1,x_2) \equiv \beta_h^{-1} f^h(x_1) + g^h(x_2),$$

and

$$U_0^h(x_2) \equiv g^h(x_2).$$

Let,

$$w^h = w_1^h + w_2^h$$

$$x^h(t) = x_1^h(t) + x_2^h(t)$$

We may think of w^h and $x^h(t)$ as the total family endowment and consumption, respectively, of a type h infinitely lived agent at date t. We can now define a new period utility function $\nu^h(\cdot)$ as follows.

$$\nu^h(x) = \max\ [f^h(x_1)+g^h(x_2)]$$

subject to, $x_1 + x_2 \le x$.

Then, each agent has preferences described by,

$$(3.15) \qquad \sum_{t=1}^{\infty}(\beta_h)^{t-1}\nu^h(x^h(t)).$$

As for budget constraints, we can rewrite (3.13) as,

$$b^h(t) = \sum_{j=1}^{t-1} p(j)[w^h-x^h(j)] + p(t)(w_2^h-x_2^h(t)).$$

It would be natural to impose the condition

$$\lim_{t\to\infty} b^h(t) \ge 0.$$

In equilibrium, it must be the case that $p(t) \to 0$ and hence $p(t)[w_2^h-x_2^h(t)] \to 0$. Therefore, we have the standard budget constraint,

$$(3.16) \qquad \sum_{t=1}^{\infty} p(t)[w^h - x^h(t)] \geq 0.$$

The market clearing conditions are obviously given by,

$$(3.17) \qquad \sum_{h} (w^h - x^h(t)) = 0.$$

The model defined by equations (3.15)-(3.17) is standard.

IV. Conclusion

We have formulated a pure exchange model of overlapping generations of two period lived agents with bequest motives and many goods in each period. Bequests are restricted to be nonnegative. The model resembles that of a fixed number of infinitely lived agents as in Bewley [1972] except that each agent faces a sequence of budget constraints rather than a single Arrow-Debreu budget constraint. We proved existence of a competitive equilibrium using the method of Wilson [1981].

In equilibrium, if a sequence of generations is linked by positive bequests, then that sequence behaves effectively as a single infinitely lived agent both in terms of preferences as well as the budget constraint. In this case, the sequence of budget constraints becomes equivalent to a single Arrow-Debreu budget constraint. However, if bequests are always zero for a sequence of generations, then that sequence of generations behaves as independent overlapping generations. Intermediate situations are also possible in which bequests may not be zero at every date but there is a subsequence of dates at which bequests are zero. The sequence of generations break up into a sequence of agents whose effective planning period runs from one date at which bequests are zero to the next. The equilibrium behavior of our model may, therefore, consist of some infinitely lived agents and some sequences of finite lived overlapping generations. This is in contrast to Wilson [1981] or Muller and Woodford [1983] in which the set of infinitely lived agents and the set of finitely lived agents are taken as data.

Because of the above feature, we cannot appeal directly to Bewley [1972], Balasko and Shell [1980] or Wilson [1981] to claim existence of equilibrium. However, slight modifications in Wilson's [1981] method have been used to establish existence.

We have given three examples of interesting behaviors that can emerge from models of this type. The first exhibits multiple equilibria. Some equilibria look like standard OLG equilibria whereas in others some sequences of agents behave as infinitely lived. In the second example, there are (at least) two steady states; one which looks like a standard OLG equilibrium while the other looks like that of an infinitely lived agent. In addition, the second steady state is unstable, whereas the first is stable. The third example is designed to illustrate what happens if agents have preferences given by discounted sums of utilities with different discount rates. The standard answer is that asymptotically, the interest rate converges to the lowest discount rate and agents with that discount rate end up owning all of the wealth in the economy whereas the wealth and consumptions of all the other agents are driven to zero. This cannot happen in the present model due to the nonnegativity restriction on bequests. Instead, bequests must become zero for some sequences of agents so that they no longer behave as if infinitely lived.

We believe that the above results are interesting and enhance our understanding of models of infinitely lived agents.

Footnotes

[1]We believe that the restriction to two period lived generations is inessential even though it is helpful for expositional clarity. In footnote 3, page 11, we outline a possible way of handling any demographic structure.

[2]This way of specifying preferences together with the consistency condition is along the lines of Gale [1983, p. 61].

[3]cf. footnote 1. Assume that there is a finite set of agents $\bar{H} = \{1,2,\ldots,H\}$. The commodity space is R^∞ and each agent's consumption set is R^∞_+. Each agent has a strict preference relation $(>)^h$ and we define $P^h(x)$ and $(P^h)^{-1}(x)$ as usual and assume that $(>)^h$ satisfies assumption 1. For each $h \in \bar{H}$, there is a sequence $\{w^h_k\}^\infty_{k=1}$ where each $w^h_k \in R^\infty_+$ and $\sum_{k=1}^\infty w^h_k = w^h \in R^\infty_+$. For $p \in R^\infty_+$, the budget sets and the demand sets are defined as follows.

$$B^h(p) \equiv \{x^h \in R^\infty_+ | x^h = \sum_{k=1}^\infty x^h_k, x^h_k \in R^\infty_+, \sum_{j=1}^k p(w^h_k - x^h_k) \geq 0, k \geq 1\}$$

$$D^h(p) \equiv \{x^h \in B^h(p) | B^h(p) \cap P^h(x^h) = \phi\}.$$

An allocation $x = \{x^h, h \in \bar{H}\}$ is <u>attainable</u> if $\sum_h (w^h - x^h) = 0$. A competitive equilibrium is a $(\bar{p},\bar{x})$ such that $\bar{x}$ is attainable and $\bar{x}^h \in D^h(\bar{p})$ for all $h \in \bar{H}$.

The preference relations $(>)^h$ may be viewed as arising as follows. For each h, there is a countable infinity of agents indexed by $k \geq 1$ each with consumption sets R^∞_+ and preference relations $(>)^h_k$ which satisfy assumption 1. Let $\{x^h_k\}$ and $\{y^h_k\}$ be such that $x^h_k, y^h_k \in R^\infty_+$ for all $k \geq 1$, $\sum_{k=1}^\infty x^h_k = x^h \in R^\infty_+$ and $\sum_{k=1}^\infty y^h_k = y^h \in R^\infty_+$. Analogous to assumption 2, we assume that for any fixed $k \geq 1$, if $\sum_{j=k+1}^\infty x^h_j \, (>)^h_{k+1} \sum_{j=k+1}^\infty y^h_j$ then $\sum_{j=k}^\infty x^h_j \, (>)^h_k \, x^h_k + \sum_{j=k+1}^\infty y^h_j$. The preference relation $(>)^h$ is then defined as being identical to $(>)_1^h$. Assumption 3 can be rewritten as $\sum_h w^h \in R_{++}^{\infty}$. Assumption 4 (irreducibility) would have to be suitably modified.

[4]That R_2 is optimal follows at once from Theorem 2. The nonoptimality of R_1 can be verified easily by constructing a pareto superior allocation in the usual fashion by transferring goods from the young to the old.

Appendix

Proof of Proposition 1

By construction, $x^h(t) \in B_t^h(p,b^h(t))$. Suppose, if possible, that for some (h,t) there is a $y^h(t) \in B_t^h(p,b^h(t)) \cap P_t^h(x^h(t)) \neq \phi$. Consider the alternative consumption plan $z^h(0) = (x_2^h(1),x_1^h(1),\ldots,x_2^h(t),y_1^h(t),y_2^h(t+1),\ldots)$. Clearly, $z^h(0) \in B_0^h(p,0)$ and by repeated application of assumption 2, $z^h(0) \in P_0^h(x^h(0))$ which contradicts the fact that $x^h(0) \in D_0^h(p,0)$. Therefore, $B_t^h(p,b^h(t)) \cap P_t^h(x^h(t)) = \phi$ and hence $x^h(t) \in D_t^h(p,b^h(t))$ for all (h,t).

Proof of Proposition 2

Suppose, if possible that $\liminf_{t\to\infty} v(t+1,p,w^h-x^h) > 0$ for some h, say for h = 1. Then, there is an $\varepsilon > 0$ and a $\hat{T}$ such that,

$$v(t+1,p,w^1-x^1) \geq \varepsilon \text{ for all } t \geq \hat{T}.$$

Consider the following alternative allocation

$$y_1^1(\hat{T}) = x_1^1(\hat{T}) + \sum_{h=2}^{H} w_1^h(\hat{T})$$

$$y_2^1(\hat{T}+1) = x_2^1(\hat{T}+1) + \sum_{h=2}^{H} w_2^h(\hat{T}+1) + \sum_{h=1}^{H} w_1^h(\hat{T}+1)$$

$$y_1^1(t) = x_1^1(t),\ t \geq 1,\ t \neq \hat{T}$$

$$y_2^1(t) = x_2^1(t),\ t \geq 1,\ t \neq \hat{T} + 1.$$

It is possible to choose $\lambda \in (0,1]$ such that,

$$\mu(\hat{T}) \equiv \lambda p_{\hat{T}}(y_1^1(\hat{T})-x_1^1(\hat{T})) + \lambda p_{\hat{T}+1}(y_2^1(\hat{T}+1)-x_2^1(\hat{T}+1)) \leq \varepsilon.$$

By assumption 4, we have that $y^1 \in P_0^1(x^1)$ and hence by assumption 1 (ii) we have that

$$z^1 = \lambda y^1 + (1-\lambda)x^1 \in P_0^1(x^1).$$

It follows that,

$$v(t+1,p,w^1-z^1) = v(t+1,p,w^1-x^1) \geq 0 \text{ for } t < \hat{T}$$

whereas for $t \geq \hat{T}$ we have,

$$\begin{aligned} v(t+1,p,w^1-z^1) &= v(t+1,p,w^1-x^1) - \mu(\hat{T}) \\ &\geq \varepsilon - \varepsilon \\ &= 0. \end{aligned}$$

Therefore, $z^1 \in B^1(p) \cap P_0^1(x^1)$ which is a contradiction.

Proof of Proposition 3

(i) Since $B^h(p) \subset \tilde{B}^h(p)$, it follows that $x^h \in \tilde{B}^h(p)$. Now, suppose, if possible that there is a $y^h \in \tilde{B}^h(p) \cap P_0^h(x^h) \neq \phi$. From proposition 2 and assumption 1 (i) we can find a $k < 1$ such that $z^h \equiv ky^h \in P_0^h(x^h)$ and $\liminf_{t\to\infty} v(t+1,p,w^h-z^h) > 0$. Therefore, there is a $\hat{T}$ such that $v(t+1,p,w^h-z^h) > 0$ for all $t > \hat{T}$. It follows that for λ positive and sufficiently small, $\lambda z^h + (1-\lambda)x^h \in B^h(p)$. From assumption 1 (ii), $\lambda z^h + (1-\lambda)x^h \in P_0^h(x^h)$ which is a contradiction. Therefore, $\tilde{B}^h(p) \cap P_0^h(x^h) = \phi$ and hence $x^h \in \tilde{D}^h(p)$.

(ii) Obvious, since $B^h(p) \subset \tilde{B}^h(p)$.

Proof of lemma 1

The proof is an application of McKenzie [1981, Theorem 3] with minor changes. In what follows, we point out the differences and indicate why the proof goes through.

First, the consumption sets of each agent may be restricted as follows: $x^h_{T,1}(t)$, $x^h_{T,2}(t) \leq \sum_h (w^h_1(t)+w^h_2(t))$, t = 1, 2, ..., T and $x^h_{T,2}(T+1) \leq \sum_h w^h_2(T+1)$. It should be noted that the aggregate endowment of every desired good is strictly positive. The only possible difficulty in this regard arises at date (T+1). However, as can be seen from the definition of the preference relation $(>)^h_T$, goods for which the aggregate endowment at (T+1) is zero do not matter in terms of preferences.

We may normalize prices by setting, $p^T e_T = 1$ where e_T is a $(T+1)\ell$ dimensional vector of ones. The budget correspondence $B^h_T(p)$ is upper hemicontinuous, compact valued and convex valued. In addition, it is continuous at p^T provided,

(A.1) $$p^T_1 \, w^h_2(1) > 0 \text{ and } p^T_t \, w^h_1(t) + p^T_{t+1} \, w^h_2(t+1) > 0$$

for all t = 1, 2, ..., T.

We may now use the mapping in McKenzie [1981, lemmas 2-6 and Theorem 3] from the product of the price simplex and the consumption sets into itself to find a fixed point. Such a fixed point will be an equilibrium provided (A.1) is satisfied for all h. Assumption 4 (irreducibility) guarantees this and hence we may claim the existence of a competitive equilibrium in which (A.1) holds for all h.

Proof of lemma 2

This follows from Tychonoff's theorem. See Wilson (1981, lemma 2, p. 102).

Proof of lemma 3

It is enough to prove that $I_T^{\alpha}(t)/I_T^{\beta}(t)$ is bounded away from zero. Suppose to the contrary that for some $t \geq 1$ and some pair of agents $\alpha, \beta \in H(t)$, $\liminf_{T \to \infty} I_T^{\alpha}(t)/I_T^{\beta}(t) = 0$.

Let,

$$H_2(t) = \{\gamma \in H(t) \mid \liminf_{T \to \infty} I_T^{\gamma}(t) \,|\, I_T^{\beta}(t) = 0\}$$

and $H_1(t) \equiv H(t) - H_2(t)$. We can select a subsequence of economies E_{T_k} such that $I_{T_k}^{\gamma}(t)/I_{T_k}^{\beta}(t) \to 0$ for all $\gamma \in H_2(t)$ and $I_{T_k}^{\gamma}(t)/I_{T_k}^{\beta}(t) \geq \varepsilon_{\gamma} > 0$ for all $\gamma \in H_1(t)$. By construction, $H_1(t)$ and $H_2(t)$ are nonempty, disjoint subsets of $H(t)$ whose union is $H(t)$. From assumption 4, we know that there is a $\delta = (h,s) \in H_1(t)$ such that, if we put,

$$\bar{y}_1^h(s) = \bar{x}_1^h(s) + \sum_{\gamma \in H_2(t)} w^{\gamma}(s)$$

$$\bar{y}_2^h(s+1) = \bar{x}_2^h(s+1) + \sum_{\gamma \in H_2(t)} w^{\gamma}(s+1)$$

$$\bar{y}_1^h(j) = \bar{x}_1^h(j), \; j \neq s$$

$$\bar{y}_2^h(j) = \bar{x}_2^h(j), \; j \neq s + 1$$

then,

$$\bar{y}^h \in P_0^h(\bar{x}^h).$$

From assumption 1 (i), we may choose $\lambda < 1$ such that for sufficiently large T and $\delta = (h,s) \in H_1(t)$, if

$$y^h_{T,1}(s) = x^h_{T,1}(s) + \sum_{\gamma\in H_2(t)} w^\gamma(s)$$

$$y^h_{T,2}(s+1) = x^h_{T,2}(s+1) + \sum_{\gamma\in H_2(t)} w^\gamma(s+1)$$

$$y^h_{T,1}(j) = x^h_{T,1}(j),\ j \neq s$$

$$y^h_{T,2}(j) = x^h_{T,2}(j),\ j \neq s+1$$

then, $\lambda\, y^h_T\ (>)^h_T\ x^h_T$. We will show that $\lambda\, y^h_T \in B^h_T(p^T)$. For this, it is enough to show that,

(A.2)
$$p^T_j\left(w^h_1(j)-\lambda y^h_{T,1}(j)\right) + p^T_{j+1}\left(w^h_2(j+1)-\lambda y^h_{T,2}(j+1)\right)$$
$$\geq p^T_j\left(w^h_1(j)-x^h_{T,1}(j)\right) + p^T_{j+1}\left(w^h_2(j+1)-x^h_{T,2}(j+1)\right)$$

for $\delta = (h,s) \in H_1(t)$ and all $j \geq 0$.

Clearly (A.2) holds for all $j \neq s$. For $j = s$, we have,

$$p^T_s\left(w^h_1(s)-\lambda y^h_{T,1}(s)\right) + p^T_{s+1}\left(w^h_2(s+1)-\lambda y^h_{T,2}(s+1)\right)$$

$$= I^\delta_T(t)\left[1-\frac{\lambda}{I^\delta_T(t)}\left\{p^T_s\left(x^h_{T,1}(s)+\sum_{\gamma\in H_2(t)} w^\gamma(s)\right)\right.\right.$$
$$\left.\left.+p^T_{s+1}\left(x^h_{T,2}(s+1)+\sum_{\gamma\in H_2(t)} w^\gamma(s+1)\right)\right\}\right]$$

$$= I^\delta_T(t)\left[1-\frac{\lambda}{I^\delta_T(t)}\left\{p^T_s x^h_{T,1}(s)+p^T_{s+1}x^h_{T,2}(s+1)+\sum_{\gamma\in H_2(t)} I^\gamma_T(t)\right\}\right]$$

$$\geq I_T^{\delta}(t)\left[1-\{p_s^T x_{T,1}^h(s)+p_{s+1}^T x_{T,2}^h(s+1)\}/I_T^{\delta}(t)\right]$$

$$= p_s^T\left(w_1^h(s)-x_{T,1}^h(s)\right) + p_{s+1}^T\left(w_2^h(s+1)-x_{T,2}^h(s+1)\right)$$

because $I_T^{\gamma}(t)/I_T^{\delta}(t) \to 0$ for all $\gamma \in H_2(t)$ and $\lambda < 1$. Therefore, $\lambda\, y_T^h \in B_T^h(p^T) \cap P_T^h(x_T^h)$ which contradicts the fact that (p^T, x_T) is a competitive equilibrium.

Proof of lemma 4

By the price normalization, we have that $p_1^T\, w_2^1(1) = 1$ for all T. It, then, follows from lemma 3 that for each (h,t),

(A.3) $\qquad p_t^T\, w_1^h(t) + p_{t+1}^T\, w_2^h(t+1)$

is bounded and bounded away from zero. Therefore, for each $t \geq 1$, $\sum_{h=1}^{H} p_t^T\left(w_1^h(t)+w_2^h(t)\right)$ is bounded and hence by Tychonoff's theorem, there is a subsequence $\{p^{T_k}\} \to \bar{p}$. (A.3) then implies that $\bar{p} > 0$ and that, $0 < \bar{p}_1\, w_2^h(1) < \infty$ and $0 < \bar{p}_t\, w_1^h(t) + \bar{p}_{t+1}\, w_2^h(t+1) < \infty$ for all h and $t \geq 1$.

Proof of Theorem 1

Since $(p^T, x_T) \to (\bar{p}, \bar{x})$ we have from (2.16) that for each fixed (h,t), $v(t+1, \bar{p}, w^h-\bar{x}^h) \geq 0$. Therefore, $\bar{x}^h \in B^h(\bar{p})$ where $B^h(p)$ is given by (2.8). By lemma 2, $\bar{x}$ is attainable. Hence, it only remains to show that $B^h(\bar{p}) \cap P_0^h(\bar{x}^h) = \phi$ for all h. Suppose to the contrary that for some h, $\bar{y}^h \in B^h(\bar{p}) \cap P_0^h(\bar{x}^h) \neq \phi$. From assumption 1 (i) and lemma 4 we can choose some $\lambda < 1$ such that

(A.4) $\qquad \lambda\, \bar{y}^h \in P_0^h(\bar{x}^h)$

(A.5) $\qquad v(t+1, \bar{p}, w^h-\lambda\bar{y}^h) > 0$ for all $t \geq 1$.

From (A.4) and assumption 1 (i) we can find a T' sufficiently large such that,

$$\tilde{y}^h \equiv [\lambda y_2^h(1), \lambda y_1^h(1), \lambda y_2^h(2), \ldots, \lambda y_2^h(T'+1), w_1^h(T'+1), w_2^h(T'+2), \ldots]$$

$\in P_0^h(\bar{x}^h)$.

Applying assumption 1 (i) again, we can find a T sufficiently large such that $\tilde{y}^h \in P_0^h(\tilde{x}_T^h)$ where $\tilde{x}_T^h$ is given by (2.15). Now let,

$$y_T^h \equiv [\lambda y_2^h(1), \lambda y_1^h(1), \lambda y_2^h(2), \ldots, \lambda y_2^h(T'+1),$$

$$w_1^h(T'+1), w_2^h(T'+2), \ldots, w_2^h(T+1)].$$

We then have that $y_T^h \ (>)_T^h \ x_T^h$. Since (p^T, x_T) is a competitive equilibrium we must have that $y_T^h \notin B_T^h(p^T)$. Therefore,

$$\text{(A.6)} \qquad v(t+1, p^T, w_T^h - y_T^h) < 0$$

for some t = 0, 1, 2, ..., T'. Now, letting $T \to \infty$ and noting that $p^T \to \bar{p}$ we see that (A.6) contradicts (A.5). Therefore, $B^h(\bar{p}) \cap P_0^h(\bar{x}^h) = \phi$ for all h and the theorem is proved.

<u>Proof of Lemma 5</u>

If $y^h \in B^h(p)$ then for $0 < \lambda < 1$, $z^h = \lambda y^h + (1-\lambda)x^h \in B^h(p) \cap P_0^h(x^h)$ which is a contradiction. Therefore, $y^h \notin B^h(p)$ and there is a t_1 such that $v(t_1+1, p, w^h - y^h) < 0$. Consider the allocation $z_{t_1}^h = (x_2^h(1), \ldots, x_2^h(t_1+1), y_1^h(t_1+1), \ldots)$. If $z_{t_1}^h = x^h$ then $y^h = (y_2^h(1), \ldots, y_2^h(t_1+1), x_1^h(t_1+1), \ldots)$. For $t > t_1$, $v(t+1, p, w^h - y^h) = v(t_1+1, p, w^h - y^h) + v(t+1, p, w^h - x^h) - v(t_1+1, p, w^h - x^h)$. By proposition 2, $\lim\inf v(t+1, p, w^h - x^h) = 0$ which implies that lim inf $v(t+1,p,w^h-y^h) < 0$. If $z^h_{t_1} \neq x^h$ then $z^h_{t_1} \notin B^h(p)$ and therefore, there is a $t_2 > t_1$ such that $v(t_2+1,p,w^h-y^h) < v(t_1+1,p,w^h-y^h) < 0$. We may now define $z^h_{t_2} = \{x^h_2(1),\ldots,x^h_2(t_2+1),y^h_1(t_2+1),\ldots\}$ and proceed as before to conclude that ultimately lim inf $v(t+1,p,w^h-y^h) < 0$.

Proof of Theorem 2

Suppose to the contrary that there is an attainable allocation y such that $y^h(t)\ (\geq)^h_t\ \bar{x}^h(t)$ for all (h,t) and $y^h(t) \in P^h_t(\bar{x}^h(t))$ for some (h,t). From proposition 2 and lemma 5 we have that $\liminf_{t\to\infty} v(t+1,\bar{p},w^h-y^h) \leq 0$ for all h with strict inequality holding for some h. By assumption, $v(t+1,\bar{p},w^h)$, and $v(t+1,\bar{p},y^h)$ are monotone increasing, bounded sequences and are hence convergent. Therefore,

$$\lim_{t\to\infty} v(t+1,\bar{p},w^h) = \lim_{t\to\infty} \sum_{j=1}^{t} \bar{p}_j\left(w^h_1(j)+w^h_2(j)\right) \equiv \bar{p}.(w^h_1+w^h_2)$$

and similarly for y^h. Therefore, $\bar{p}.(w^h_1+w^h_2) \leq \bar{p}.(y^h_1+y^h_2)$ for all h with strict inequality for some h. Summing over h and noting that the left hand side is finite by assumption, we obtain a contradiction. This shows that there cannot be any attainable allocation that pareto dominates $\bar{x}$ and hence that $\bar{x}$ is pareto optimal.

References

Balasko, Y. and Shell, K. (1980). "The Overlapping Generations Model, I. The Case of Pure Exchange Without Money," Journal of Economic Theory, 23, pp. 281-306.

Balasko, Y., Cass, D. and Shell, K. (1980). "Existence of Competitive Equilibrium in a General Overlapping Generations Model," Journal of Economic Theory, 23, pp. 307-22.

Barro, R. J. (1974). "Are Government Bonds Net Wealth?" Journal of Political Economy, 82, pp. 1095-1117.

Becker, R. A. (1980). "On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium With Heterogeneous Households," Quarterly Journal of Economics, September, pp. 375-82.

Bewley, T. (1972). "Existence of Equilibrium in Markets With Many Commodities," Journal of Economic Theory, 4, pp. 514-40.

Gale, Douglas. (1983). Money: In Disequilibrium, Cambridge University Press, Cambridge and New York.

Laitner, J. (1986). "Bequests, Gifts and Social Security," Discussion Paper No. C-75, Department of Economics, University of Michigan.

Lucas, Robert E. Jr. and Stokey, Nancy L. (1984), "Optimal Growth With Many Consumers," Journal of Economic Theory, 32, pp. 139-71.

McKenzie, L. W. (1981). "The Classical Theorem on Existence of Competitive Equilibrium," Econometrica, 49 (4), pp. 819-41.

Muller, W. J. III. and Woodford, M. (1983). "Stationary Overlapping Generations Economies With Production and Infinite-Lived Consumers: I. Existence of Equilibrium," Working Paper

No. 325, Department of Economics, Massachusetts Institute of Technology.

Wilson, C. (1981). "Equilibrium in Dynamic Models With an Infinity of Agents," Journal of Economic Theory, 24, pp. 95-111.