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SUSTAINABLE PLANS AND DEBT

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Abstract

We analyze the incentive for a government to default on its debts in a variant of the Lucas and Stokey (1983) model of optimal taxation. Optimal fiscal policy requires the use of debt to smooth tax distortions over time. Dynamic consistency requires that governments not have an incentive to default on the inherited debt. We consider policy and allocation rules which map the history of the economy into current decisions. A sustainable equilibrium is a sequence of history-contingent functions which satisfy sequential rationality for the government and for private agents. We characterize sustainable equilibrium outcomes when the horizon is finite. We show that, under plausible assumptions, the loss in welfare due to the absence of a commitment technology to honor debts is small.

This paper analyzes a classic problem in the literature on time consistency, namely, the incentive for governments to default on their debts. We analyze this issue in a representative agent model of optimal taxation adapted from the seminal paper by Lucas and Stokey (1983).¹ Government consumption, which is exogenously given, fluctuates over time and the revenues to finance this consumption are raised through distortionary taxes on labor. The government chooses tax rates in each period to maximize the welfare of the representative agent. The public finance tradition, stemming from Ramsey (1927), is a useful starting point in thinking about this problem.

In this tradition, the government first chooses the entire sequence of tax rates. Prices and quantities are then determined in a competitive equilibrium, given the tax sequence. The government chooses a tax policy to maximize the representative agent's welfare over the resulting competitive allocations. An optimal policy, together with the resulting competitive equilibrium, is a Ramsey equilibrium. Implicit in this equilibrium is a sequence of debt issues and repayments. Consequently, the optimal tax policies imply an optimal debt policy.

This approach to the optimal taxation problem is appropriate in environments where societies have access to a commitment technology to bind the actions of future governments. In many situations, however, it is more appropriate to think of policies as being chosen at each date with no ability to commit to future policies. A solution to the optimal taxation problem must then require that policies be sequentially rational. That is, the policy rules must maximize the government's welfare function at each date, given that private agents behave optimally. Likewise, optimality by private agents requires that they forecast future policies as being sequentially rational for the government. Of course, competitive private agents must also take future

policies as unaffected by their actions. Various definitions of equilibrium have been offered in the literature (see, for example, Lucas and Stokey 1983) to satisfy these requirements. To distinguish our definition from the others, we call a sequence of rules for policies, allocations, and prices which satisfy sequential rationality a sustainable equilibrium.

Our formulation allows policies, allocations, and prices to depend on the entire history of past decisions by the government as well as past aggregate (or per capita) allocations. Thus, policies, allocations, and prices are defined as history-contingent functions. This break from the general equilibrium tradition of considering equilibria which are event-contingent functions is essential in imposing sequential rationality. Both governments and competitive agents must forecast how current decisions affect future outcomes. Allowing for history-contingent functions solves this forecasting problem. In a companion piece (Chari and Kehoe 1987b), we define and characterize sustainable equilibria in a model where there are no state variables linking actions between periods. In the model considered here, however, government debt is an essential link between periods. This makes the analysis both more complicated and more interesting.

In the model we allow the government to default on its debts. First, we consider a finite horizon version of the model. As might be expected, the Ramsey allocations are not, in general, outcomes of a sustainable equilibrium since the debt issues associated with a Ramsey allocation are positive at some dates. When the inherited debt is positive, the government has an incentive to default. Recognizing this, private agents will not buy such debt in previous periods. This does not imply, however, that a sustainable equilibrium must have a continuously balanced budget. The government can smooth tax distortions over time by issuing negative debt, that is, by

purchasing claims on private agents. We fully characterize sustainable equilibria with a finite horizon. We show that sustainable allocations solve a programming problem called the constrained Ramsey problem. The sustainable allocations maximize the welfare of the representative consumer at date zero, subject to the budget constraint and a sequence of constraints which require that the present value of the government's surplus be nonpositive at all future dates. All sustainable equilibria yield the same discounted value of utility at date zero.

With an infinite horizon, the set of sustainable equilibria becomes larger. The limit of the finite horizon equilibria is also an equilibrium with an infinite horizon. We show that an arbitrary sequence of policies, allocations, and prices is a sustainable outcome if it satisfies two simple conditions. We then show how our results generalize to the case of uncertainty.

A particularly interesting question is the magnitude of loss in welfare due to the absence of a commitment technology. We show that under plausible assumptions, the difference between utility in a sustainable equilibrium and the Ramsey utility can be made arbitrarily small by making the horizon sufficiently long and the discount factor sufficiently close to unity. Note that this result holds in a finite horizon and thus does not rely upon "trigger" strategies. Rather, the result holds because the ability to issue negative debt allows for almost as much tax smoothing as in a Ramsey equilibrium.² [Chari and Kehoe (1987a) characterize sustainable equilibria when the government cannot own claims on private agents. For such environments the results obtained are rather different.]

We conclude by characterizing sustainable equilibria in a series of examples.

1. Commitment

Consider a simple production economy populated by a large number of identical infinitely lived consumers. In each period there are two goods: labor and a consumption good. A constant-returns-to-scale technology is available to transform one unit of labor into one unit of output. The output can be used for private or for government consumption. The per capita level of government consumption in each period, denoted G_t , is exogenously specified. Let c_t and l_t denote the individual levels of consumption and labor, and let C_t and L_t denote the aggregate (or per capita) values of these variables. An aggregate allocation $(C, L) = \{C_t, L_t\}_{t=0}^{\infty}$ is feasible if it satisfies

$$(1.1) \quad C_t + G_t = L_t.$$

The preferences of each consumer are given by

$$(1.2) \quad \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)$$

where U is strictly increasing, strictly concave, and bounded, and where $0 < \beta < 1$. We also assume that consumption and leisure are normal goods.

Let p_t denote the price of the consumption good at time t in an abstract unit of account, and let $p = \{p_t\}_{t=0}^{\infty}$ denote the vector of such prices. Since the constant-returns-to-scale technology transforms a unit of labor into one unit of output, the wage rate equals the price of the consumption good. We assume that revenues can be raised only through a proportional tax on labor income. Let τ_t denote the tax rate on the labor income earned in period t , and let $\tau = \{\tau_t\}_{t=0}^{\infty}$ denote the sequence of such tax rates. The budget constraint of the representative consumer at time 0 is then

$$(1.3) \quad \sum_{t=0}^{\infty} p_t [c_t - (1 - \tau_t) l_t] = 0.$$

Notice that we have written the consumer's budget constraint in "date 0" or present-value form. Implicit in this constraint is a sequence of government debt held by consumers. To understand the government's incentives to tax (or to default) on the debt, it is useful to write out this sequence explicitly.

Following Lucas and Stokey (1983), we allow for government debt of all maturities. In each period t the government has outstanding net claims denoted ${}_{t-1}B = \{ {}_{t-1}B_s \}_{s=t}^{\infty}$, where ${}_{t-1}B_s$ is a claim to goods at time s . At time t the government raises revenues and issues new debt claims, which result in a net debt position of ${}_tB$. (One can think of ${}_tB$ as a single bond with time-varying coupon payments.) Let $\delta_t \in [0,1]$ denote the default rate on debt outstanding in period t . Here $\delta_t = 0$ corresponds to complete repayment, $\delta_t = 1$ to complete default and $0 < \delta_t < 1$ to partial default. (One can think of δ_t as a tax on debt.) Let ${}_tq_s$ be the price at time t of the debt claim maturing in period s . The value of the outstanding debt at time t is given by

$$\sum_{s=t}^{\infty} {}_tq_s {}_{t-1}B_s.$$

The government's budget constraint at time t is

$$(1.4) \quad p_t [\tau_t L_t - G_t] + \sum_{s=t+1}^{\infty} {}_tq_s {}_tB_s = (1-\delta_t) \sum_{s=t}^{\infty} {}_tq_s {}_{t-1}B_s$$

where ${}_{-1}B \equiv 0$.

The analogous sequential budget constraints for the aggregate allocations $\{C_t, L_t\}_{t=0}^{\infty}$ are

$$(1.5) \quad p_t [C_t - (1-\tau_t)L_t] + \sum_{s=t+1}^{\infty} {}_tq_s {}_tB_s = (1-\delta_t) \sum_{s=t}^{\infty} {}_tq_s {}_{t-1}B_s$$

where ${}_{-1}B \equiv 0$. Obviously, in a competitive equilibrium there is an arbitrage relation between the prices of the consumption goods and the prices of the debt claims, namely, ${}_tq_t = p_t$ and

$$(1.6) \quad {}_tq_s = p_s(1-\delta_{t+1})(1-\delta_{t+2})\cdots(1-\delta_s), \text{ for all } s \geq t + 1.$$

In this economy an individual agent's allocation is a vector of consumption and labor, denoted by $x = \{x_t\}_{t=0}^{\infty}$, where $x_t = (c_t, l_t)$. An aggregate allocation is defined analogously and denoted by $X = \{X_t\}_{t=0}^{\infty}$, where $X_t = (C_t, L_t)$. A policy for the government is a sequence of tax rates on labor, default rates on debt, and debt issues and is denoted by $\pi = \{\pi_t\}_{t=0}^{\infty}$, where $\pi_t = (\tau_t, \delta_t, B_t)$. We can now define a competitive equilibrium.

Definition: A competitive equilibrium is a set of individual allocations x , an aggregate allocation X , price systems p and q , and a policy π that satisfy

- Consumer maximization. Given π , p , q , and X , the individual allocation x maximizes utility (1.2) subject to (1.3).
- Sequential constraints for aggregate allocations. The aggregate allocation X satisfies (1.5) for each t .
- Sequential constraints for government policies. The policy π satisfies (1.4) for each t .
- No arbitrage. The price systems p and q satisfy (1.6) for all t .
- Representativeness. $x = X$.

Notice that the sequential constraints (1.4) and (1.5) imply the feasibility condition (1.1).

We comment briefly on the no-arbitrage condition and the sequence of constraints for aggregate allocations. We can derive these conditions from consumer maximization by including the sequence of period budget constraints for each of the consumers. These period budget constraints include the debt claims held on other consumers as well as on the government. Consumer maximization then implies the no-arbitrage condition. Market clearing in private debt and representativeness then imply the sequence of constraints for aggregate

gate allocations. For notational convenience we have simply imposed these conditions as part of the definition of equilibrium.

Since in any equilibrium the individual and aggregate allocations coincide, we refer to such a competitive equilibrium as (π, X, p, q) . Let E denote the set of policies for which an equilibrium exists. Assume that for each π in E there is a unique allocation $X(\pi)$. (A sufficient condition for this to be true is that consumption and leisure are normal goods.) The equilibrium value of utility under a policy π is given by

$$V(\pi, X(\pi)) = \sum_{t=0}^{\infty} \beta^t U(C_t(\pi), L_t(\pi)).$$

We say (π, X, p, q) is a Ramsey equilibrium if π solves

$$\max_{\pi \in E} V(\pi, X(\pi))$$

and $X = X(\pi)$, $p = p(\pi)$, and $q = q(\pi)$. The idea behind this equilibrium is that there is a commitment technology through which the government can commit to a policy π once and for all at the beginning of time.

In this model we have allowed government to tax labor and to default on debt. As we shall see, in the no-commitment equilibrium the incentive to default on debt drives the time inconsistency problem. However, interestingly enough, in the Ramsey equilibrium the ability to default is irrelevant; and, in terms of allocations, all that really matters is the tax on labor. Specifically, the Ramsey equilibrium for this economy coincides with the Ramsey equilibrium considered by Lucas and Stokey (1983) in which governments are assumed to honor their debts. The reason for this is that letting the government default does not expand the set of allocations attainable under a government policy. We then have

Proposition 1: (The Ramsey Equilibrium.)

The consumption and labor allocations C and L in the Ramsey equilibrium solve the problem

$$\max \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

subject to

$$(1.7) \quad C_t + G_t = L_t$$

$$(1.8) \quad \sum_{t=0}^{\infty} \beta^t R_t = 0$$

where $R_t \equiv U_c C_t + U_l L_t$ is the government surplus in period t in units of marginal utility.

Proof: First, the set of allocations attainable is the same as those in an economy where the government sets the default rate identically equal to zero. To see this, note that if (π, X, p, q) is an equilibrium with the default rate δ_t possibly positive for some t , then so is $(\hat{\pi}, X, p, \hat{q})$ with $\hat{\tau}_t = \tau_t$, $\hat{\delta}_t = 0$, $\hat{q}_s = p_s$, and $\hat{B}_s = B_s (1 - \delta_{t+1}) \cdots (1 - \delta_s)$ for all s , and t , with $s \geq t + 1$. Next, notice that in any competitive equilibrium, the consumer's first-order conditions imply

$$(1.9) \quad p_t = \beta^t U_c(C_t, L_t)$$

and

$$(1.10) \quad (1 - \tau_t) = -U_l(C_t, L_t) / U_c(C_t, L_t).$$

Substituting (1.9) and (1.10) into the consumer's budget constraint (1.3) gives (1.8). Clearly there are many debt sequences $\{B_t\}_{t=0}^{\infty}$ and default rates $\{\delta_t\}_{t=0}^{\infty}$ which satisfy the sequential budget constraints (1.4) and (1.5). \diamond

2. No Commitment

In an environment without commitment, we can no longer retain the fiction that all agents make decisions once and for all at the beginning of time and then simply execute those decisions at the appropriate time. Indeed, we need to ensure that these decisions are sequentially rational. In terms of the timing of decisions, we model the sequential decision making by assuming that governments in each period chooses a policy at the beginning of the period and then consumers choose their consumption and labor supply decisions.

In each period t the government chooses policies as a function of the aggregate history, which consists of past aggregate consumption and labor decisions and past policies. The aggregate history confronting the government at time t is

$$H_t = \{X_s, \pi_s | s=0, \dots, t-1\}.$$

Notice that this aggregate history does not include individual allocations. This omission is in keeping with the assumption that tax rates are the same across all consumers and cannot be altered by the decisions of any single consumer.

Consumers make their choices of consumption and labor at date t as functions of their individual histories. Such a history includes the policy choice π_t as well as the past individual decisions, past aggregate decisions, and past policy choices. The individual history h_{1t} is given by

$$h_{1t} = \{x_s, X_s, \pi_s | s=0, \dots, t-1\} \cup \{\pi_t\}$$

and the aggregate history H_t is

$$H_{1t} = \{X_s, \pi_s | s=0, \dots, t-1\} \cup \{\pi_t\}.$$

In keeping with the representative agent model used, only symmetric histories are considered.

For this environment, a sustainable equilibrium consists of an individual allocation rule f , an aggregate allocation rule F , a policy plan σ , and price systems p and q that satisfy certain sequential rationality conditions. An individual allocation rule is a sequence of functions $f = \{f_t\}_{t=0}^{\infty}$, where f_t maps each individual history h_{1t} into an agent's current choice of consumption and labor. Likewise, an aggregate allocation rule is a sequence of functions $F = \{F_t\}_{t=0}^{\infty}$, where F_t maps each aggregate history H_{1t} into an aggregate amount of consumption and labor. A policy plan σ is a sequence of functions $\sigma = \{\sigma_t\}_{t=0}^{\infty}$, where σ_t maps each history H_t into a current tax on labor, a default rate on debt, and new debt issues. Finally, price systems p and q are sequences of functions $p = \{p_t\}_{t=0}^{\infty}$ and $q = \{q_t\}_{t=0}^{\infty}$, where p_t maps each history H_{1t} into a price for the consumption good at t and where q_t maps each history H_{1t} into a vector of debt prices $\{q_s\}_{s=t}^{\infty}$.

In order to define a sustainable equilibrium, we need to explain how allocation and policy functions induce future histories. In what follows, we consider only symmetric histories. Let $f^t = (f_t, f_{t+1}, \dots)$ denote a sequence of individual allocation rules from time t onward. Let F^t and σ^t denote the corresponding objects for the aggregate allocation rules and policy plans. Given an individual history h_{1t} , the functions f^t , F^t , and σ^t induce future individual histories. For example, an agent's history at time $t + 1$ is

$$h_{1t+1} = \{h_{1t}, f_t(h_{1t}), F_t(H_{1t}), \sigma_{t+1}(H_{1t}, F_t(H_{1t}))\}.$$

In a similar fashion, given any aggregate history, say H_t , the allocation rules F^t and policy plans σ^t induce future aggregate histories $(H_{1t}, H_{t+1}, H_{t+1}, \dots)$ in the obvious way.

In a sustainable equilibrium, sequential rationality by consumers is modeled by assuming that the policy plans, allocation rules, and price functions form a competitive equilibrium for each aggregate history. In this equilibrium, each consumer is assumed to act competitively in that he assumes the evolution of policies and prices is not influenced by his actions. In particular, since future policies and prices are determined by aggregate histories, acting competitively implies that each consumer believes his actions have no effect on aggregate histories.

For some given functions F^t , σ^t , p^t , q^t , and a history h_{1t} , the problem of the consumer at time t is to choose an allocation rule f^t to maximize

$$(2.1) \quad \sum_{s=t}^{\infty} \beta^s U(c_s(h_{1s}), l_s(h_{1s}))$$

subject to the budget constraint

$$(2.2) \quad \sum_{s=t}^{\infty} p_s(H_{1s}) [c_s(h_{1s}) - (1-\tau_s(H_s)) l_s(h_{1s})] \\ = (1-\delta_t(H_t)) \sum_{s=t}^{\infty} q_s(H_t)_{t-1} B_s.$$

In such a competitive equilibrium, the allocation rule F^t must satisfy the following sequence of constraints. For all $s \geq t$,

$$(2.3) \quad p_s(H_{1s}) [C_s(H_{1s}) - (1-\tau_s(H_s)) L_s(H_{1s})] + \sum_{r=s+1}^{\infty} q_r(H_{1s})_{s-1} B_r(H_s) \\ = (1-\delta_s(H_s)) \sum_{r=s}^{\infty} q_r(H_{1s})_{s-1} B_r(H_{s-1}).$$

We then have the following definition of sequential rationality.

Definition: Sequences of individual and aggregate allocation rules f^t and F^t , price functions p^t and q^t , and policy plans σ^t , are sequentially rational for consumers at time t , given a symmetric history h_{1t} , if they satisfy

- Consumer maximization. Taking F^t , p^t , q^t , and σ^t as given, f^t solves the consumer's problem of maximizing (2.1) subject to (2.2).
- Sequential constraints for aggregate allocations. F^t satisfies (2.3) for all $s \geq t$.
- No arbitrage. The price systems p^t and q^t satisfy

$${}_s q_r(H_{1s}) = p_r(H_{1r})(1-\delta_{s+1}(H_{s+1})) \cdots (1-\delta_r(H_r))$$
 and

$${}_s q_s(H_{1s}) = p_s(H_{1s})$$
 for all r and s , with $r \geq s \geq t + 1$.
- Representativeness. $f^t = F^t$.

It is important to note that, in this definition, the future histories h_{1s} , H_{1s} , and H_s are induced by σ^t , f^t , and F^t . Since representativeness is part of the definition of sequential rationality, we summarize these functions by $(\sigma^t, F^t, p^t, q^t)$.

Next consider the problem of the government. At time t the government, faced with an aggregate history H_t , takes as given that future aggregate allocations and prices evolve according to the functions F^t , p^t , and q^t . It is important to note that, in contrast to individual consumers, the government can influence the future allocations and prices by affecting the aggregate history. The objective function of the government at t is given by the utility of the representative agent from t onward under F^t and σ^t ; namely,

$$(2.4) \quad V_t(\sigma^t, F^t; H_t) = \sum_{s=t}^{\infty} \beta^s U(C_s(H_{1s}), L_s(H_{1s})).$$

The government's choice set at time t , given a history H_t , is the set of policy plans σ^t from t onward that satisfy the government budget constraints

$$(2.5) \quad p_s(H_{1s})[\tau_s(H_s)L_s(H_{1s})-G_s] + \sum_{r=s+1}^{\infty} q_r(H_{1s})_s B_r(H_s) \\ = (1-\delta_s(H_s)) \sum_{r=s}^{\infty} q_r(H_{1s})_{s-1} B_r(H_{s-1})$$

for all $s \geq t$, where the future histories are induced from H_t by σ^t and F^t . We denote this choice set by $\Sigma^t(F^t, p^t, q^t; H_t)$. We then define a sustainable equilibrium.

Definition: A sustainable equilibrium is a (σ, F, p, q) that satisfies

- Sequential rationality by consumers. For every history H_{1t} , the sequence of functions $(\sigma^t, F^t, p^t, q^t)$ are sequentially rational for consumers.
- Sequential rationality by the government. For every history H_t , the policy plan σ^t maximizes consumer welfare (2.4) over the set $\Sigma^t(F^t, p^t, q^t; H_t)$.

The sustainable equilibria have a simple feature which we will repeatedly use in our characterization results: the consumption and labor allocations in period t depend only on current policies and the prices for debt. More precisely, we have

Lemma 1: (Properties of the Sustainable Equilibrium Allocation Rules.)

In any sustainable equilibrium (σ, F, p, q) , the consumption and labor allocation rules satisfy $C_t(H_{1t}) = C(\tau_t, K_t)$ and $L_t(H_{1t}) = L(\tau_t, K_t)$, where the functions C and L are defined by $-U_c/U_l = (1-\tau_t)$ and $C + (1-\tau_t)L = K_t$, where K_t is defined by

$$(2.6) \quad U_c(C, L)K_t = (1-\delta_t(H_t)) \sum_{s=t}^{\infty} q_s(H_{1t})_{t-1} B_s - \sum_{s=t+1}^{\infty} q_s(H_{1t})_t B_s(H_t).$$

Proof: Since (σ, F, p, q) is a sustainable equilibrium, we know for any history

H_{1t} , the allocation rule f^t solves the consumer's problem. The necessary conditions include

$$(2.7) \quad - \frac{U_l(c_s(h_{1s}), l_s(h_{1s}))}{U_c(c_s(h_{1s}), l_s(h_{1s}))} = (1-\tau_s(H_s)), \text{ for all } s \geq t.$$

By representativeness, the same must hold for F^t . Since F^t must also satisfy the sequential constraints for aggregate allocations, the result follows. Notice that since consumption and labor are normal goods, the functions C and L are uniquely defined. \diamond

3. Finite Horizon

In this section we consider a finite horizon version of the model. We show that sustainable allocations and policies solve a certain programming problem we call the constrained Ramsey problem. As will be evident, our analysis builds on the work of Lucas and Stokey (1983).

For any vector of inherited debt ${}_{t-1}B$, the constrained Ramsey problem at t is to choose $\{C_s, L_s\}_{s=t}^T$ to solve

$$(3.1) \quad \max \sum_{s=t}^T \beta^s U(C_s, L_s)$$

subject to

$$(3.2) \quad C_s + G_s = L_s, \text{ for } s = t, \dots, T,$$

$$(3.3) \quad \sum_{s=t}^T \beta^s R_s \geq \min \left[\sum_{s=t}^T \beta^s U_c {}_{t-1}B_s, 0 \right],$$

and

$$(3.4) \quad \sum_{s=r}^T \beta^s R_s \leq 0, \text{ for } r = t + 1, \dots, T,$$

where $R_s = U_c C_s + U_l L_s$ is the government surplus at date s and the derivatives

of $U(\cdot, \cdot)$ are evaluated at C_s and L_s . Denote the solutions to this problem by $C_s(t_{-1}B)$ and $L_s(t_{-1}B)$ for $s = t, \dots, T$, and let $V^C(t_{-1}B)$ denote the maximized value of welfare.

A useful feature of the constrained Ramsey problem is that it is recursive. We have

Lemma 2: (Recursive Nature of Constrained Ramsey Problem.)

For any $t_{-1}B$ and any solution $\{C_s(t_{-1}B), L_s(t_{-1}B)\}_{s=t}^T$ to the constrained Ramsey problem, there is a unique sequence $\{B_s\}_{s=t}^{T-1}$ such that $C_r(sB) = C_r(t_{-1}B)$ and $L_r(sB) = L_r(t_{-1}B)$, for all $r \geq s$ and $s \geq t$.

Proof: For any given $t_{-1}B$, we first construct a candidate debt sequence and then show it is unique. Now consider the value of the debt under the constrained Ramsey problem: either $\sum_{s=t}^T \beta^s U_{c_t} B_s \leq 0$ or the right side of (3.3) can be set equal to zero without loss of generality. Obviously, if the present value of the debt is positive, the solution to the constrained Ramsey problem is identical to one where $t_{-1}B \equiv 0$. In what follows we consider only the case where the present value of the debt is nonpositive. By setting $t_{-1}B = 0$ where it appears, we obtain the solution in the case where the value of debt is positive.

Consider the constrained Ramsey problem at date t . Suppose constraint (3.4) binds at dates T_1, \dots, T_K , with $t \leq T_1 \leq T_2 \leq \dots \leq T_K \leq T$. It is clear that the constrained Ramsey problem is equivalent to $K + 1$ (unconstrained) "mini"-Ramsey problems. The k^{th} such problem is to maximize discounted utility between dates T_{k-1} and $T_k - 1$, subject to

$$(3.5) \quad \sum_{s=T_{k-1}}^{T_k-1} \beta^s R_s \geq 0$$

and (3.2), for $s = T_{k-1}, \dots, T_k - 1$. Notice that by hypothesis, constraint (3.4) is not binding and hence can be dropped.

Consider the constrained Ramsey problem at t . The first-order conditions for $s = t, \dots, T_1 - 1$ are given by

$$(3.6) \quad (1+\lambda_0)(U_c + U_\ell) + \lambda_0 \left[(C_s - {}_tB_s)(U_{cc} + U_{c\ell}) + L_s(U_{\ell\ell} + U_{c\ell}) \right] = 0,$$

where the derivatives of U are evaluated at (C_s, L_s) and λ_0 is the Lagrange multiplier on (3.3). If $T_1 = t + 1$, then set ${}_tB \equiv 0$. If $T_1 > t + 1$, then consider the constrained Ramsey problem at date $t + 1$ with inherited debt ${}_tB$. We wish to construct ${}_tB$ so that the solution to the date $t + 1$ constrained Ramsey problem coincides with the solution to the date t constrained Ramsey problem. Suppose, therefore, that these solutions coincide. By assumption, starting from date $t + 1$ with inherited debt ${}_tB$, the present value of the government surplus must also be zero at date T_1 . Furthermore, constraint (3.4) is not binding for $r = t + 2$ through $T_1 - 1$. The first-order conditions for the date $t + 1$ constrained Ramsey problem are

$$(3.7) \quad (1+\lambda_1)(U_c + U_\ell) + \lambda_1 \left[(C_s - {}_tB_s)(U_{cc} + U_{c\ell}) + L_s(U_{\ell\ell} + U_{c\ell}) \right] = 0$$

for $s = t + 1, \dots, T_1 - 1$. Subtracting (3.6) from (3.7), we obtain

$$(3.8) \quad \lambda_1 {}_tB_s = \lambda_0 {}_{t-1}B_s + (\lambda_1 - \lambda_0)a_s,$$

where

$$a_s = \frac{U_c + U_\ell + (U_{cc} + U_{c\ell})C_s + (U_{\ell\ell} + U_{c\ell})L_s}{U_{cc} + U_{c\ell}}$$

for $s = t + 1, \dots, T_1 - 1$. Equation (3.8) defines the debt sequence ${}_tB_s$ from $s = t, \dots, T_1 - 1$ in terms of the constrained Ramsey allocations and the

Lagrange multipliers λ_0 and λ_1 . For $s \geq T_1$, set ${}_t B_s = 0$. It follows by construction that for any such ${}_t B$, the solution to the date $t + 1$ constrained Ramsey problem coincides with the solution to the date t constrained Ramsey problem.

To verify uniqueness of the constructed debt sequence, it suffices to show that for some given solution $\{C_s({}_{t-1}B), L_s({}_{t-1}B)\}_{s=t}^T$, the Lagrange multiplier λ_1 is unique. Now by hypothesis the present value of the government surplus at date $t + 1$ is negative. From constraint (3.4) we have that this value equal the present value of the inherited debt. Hence we can multiply (3.8) by $\beta^s U_c$, add the resulting equations from $s = t + 1$ through $T_1 - 1$, and obtain

$$(3.9) \quad \lambda_1 \sum_{s=t+1}^{T_1-1} \beta^s [R_s - U_c a_s] = \lambda_0 \sum_{s=t+1}^{T_1-1} \beta^s [U_c {}_{t-1}B_s - U_c a_s].$$

This equation defines a unique value for λ_1 . Substituting this value into (3.8) gives a unique value for ${}_t B_s$.

Proceeding in the same way we construct the unique debt sequence ${}_s B$ for $s = t, \dots, T - 1$. \diamond

A useful corollary to this lemma can be obtained in the special case where ${}_{t-1}B \equiv 0$. We use (3.8) and (3.9) to get

$$(3.10) \quad {}_t B_s = \frac{\sum_{r=t+1}^{T_1-1} \beta^r R_r a_s}{\sum_{r=t+1}^{T_1-1} \beta^r U_c a_r}.$$

From (3.6), using normality, it follows that $U_c + U_\ell \geq 0$. We can rewrite (3.6) using the definition of a_s to get $(U_c + U_\ell) + \lambda_0 a_s (U_{cc} + U_{c\ell}) = 0$. Normality implies that $a_s \geq 0$. Recall that the present value of the government surplus at date $t + 1$ is negative. Hence, it follows that ${}_t B_s \leq 0$, for all s . We have established the following corollary.

Corollary: If the value of the initial inherited debt is nonnegative, then the debt sequence which supports the constrained Ramsey allocations is nonpositive at all dates. In particular, since $_{-1}B \equiv 0$, the constrained Ramsey debt sequence starting at date zero is never positive at any date.

So far we have assumed that at time t , given some inherited debt $_{t-1}B$, the government chooses a single default rate δ_t on all the coupons $\{_{t-1}B_s\}_{s=t}^T$. This corollary implies that for the debt sequence supporting the date-0 constrained Ramsey allocations, all of these coupons will be nonpositive. Now suppose that at time t we allowed the government to choose a different default rate for each such coupon. Clearly the government will set its default rate to zero for all nonpositive coupons. Thus, allowing the government to default separately on these coupons does not change the set of sustainable outcomes.

We now show that the set of sustainable outcomes coincides with the set of solutions to the constrained Ramsey problem. Denote the allocations that solve the constrained Ramsey problem at any time t for some inherited debt $_{t-1}B$ by $\{X_s(_{t-1}B)\}_{s=t}^T$. We will use the solutions to these various problems to construct a candidate equilibrium denoted $(\sigma^c, F^c, p^c, q^c)$. We will then use a type of backward induction argument to show that this candidate is indeed an equilibrium. When considering our candidate equilibrium, it is useful to remember that we must define policies for all histories H_t and prices and allocations for all histories H_{1t} .

We begin with the policy functions. For any history H_t there is some inherited debt $_{t-1}B$ and some associated set of constrained Ramsey allocations $X_s(_{t-1}B)$ from s through T . Let the labor tax $\tau_t^c(H_t) = 1 + U_\ell(X_t)/U_c(X_t)$, where $X_t = X_t(_{t-1}B)$. Let the debt tax $\delta_t^c(H_t) = 1$ if the value

of inherited debt ${}_{t-1}B$ evaluated at the constrained Ramsey allocations $X_s({}_{t-1}B)$ is positive, that is, if

$$\sum_{s=t}^T \beta^s U_c(X_s({}_{t-1}B)) {}_{t-1}B_s > 0$$

and let $\delta_t^c(H_t)$ equal zero otherwise. Finally, let ${}_tB^c(H_t)$ be the debt issued under the constrained Ramsey plan starting at period t with debt ${}_{t-1}B$.

Consider next the price function for the debt. Each history $H_{1t} = (H_t, \tau_t, \delta_t, {}_tB)$ specifies a debt ${}_tB$. For each such ${}_tB$, the allocations $X_s({}_tB)$ solve the associated constrained Ramsey problem at time $t + 1$. Let ${}_tq_s^c(H_{1t}) = 0$ if the value of the inherited debt ${}_tB$ evaluated at the constrained Ramsey allocations $X_s({}_tB)$ is positive, that is, if

$$\sum_{s=t+1}^T \beta^s U_c(X_s({}_tB)) {}_tB_s > 0$$

and let ${}_tq_s^c(H_{1t}) = \beta^s U_c(X_s({}_tB))$ otherwise. (Notice that these debt prices would be those that arise at time t if the government always followed the constrained Ramsey policies from time $t + 1$ on, regardless of what in the past history gave rise to the inherited debt ${}_tB$).

We use Lemma 1 and the definitions of q^c and σ^c to construct the allocation rules F^c and price functions p^c . For any H_{1t} , let $F_t^c(H_{1t}) = X(\tau_t, K_t)$, where $X(\tau_t, K_t)$ is defined as in Lemma 1, and where in (2.6) we use the constructed prices $q^c(H_{1t})$. Finally, we let $p_t^c(H_{1t})$ be the discounted marginal utility of consumption evaluated at the same allocations $X(\tau_t, K_t)$.

Proposition 2. (Sustainable Outcomes Solve the Constrained Ramsey Programming Problem.)

The policy plans, allocation rules, and price functions $(\sigma^c, F^c, p^c, q^c)$ constitute a sustainable equilibrium. Furthermore, the set of sustainable outcomes coincides with the set of solutions to the constrained Ramsey problem at date 0.

Proof: Consider the last period T . In any equilibrium, it follows from Lemma 1 that for every history H_{1T} , the allocation rules and price functions depend only upon the current policy and the inherited debt ${}_{T-1}B$, and they must be identical to $F_T^C(H_{1T})$, $p_T^C(H_{1T})$, and ${}_{T}q_T^C(H_{1T})$ by construction of these functions. Hence, the problem of the government at time T , given history H_T , is to choose $\sigma_T(H_T)$ to solve

$$\max U(F_T^C(H_{1T}))$$

subject to

$$p_T^C(H_{1T})[\tau_T L_T^C(H_{1T}) - G_T] = (1 - \delta_T) {}_{T}q_T^C(H_{1T}) {}_{T-1}B_T,$$

where ${}_{T}q_T^C = p_T$ by definition.

It is clear that the optimal policy is to set $\delta_T = 1$ if ${}_{T-1}B_T > 0$ and to set $\delta_T = 0$ otherwise. Under F^C we know that $(1 - \tau_t) = -U_\ell / U_c$ and $p_t^C(H_{1T}) = U_c(C_T, L_T)$ and that the sequential constraints for aggregate allocations together with the government budget constraint imply $C_T + G_T = L_T$. Therefore, the problem facing the government at T is equivalent to choosing (C_T, L_T) to

$$\max U(C_T, L_T)$$

subject to

$$C_T + G_T = L_T.$$

and

$$R_T \geq \min[U_c {}_{T-1}B_T, 0]$$

This is, of course, the constrained Ramsey problem at date T . Hence, any equilibrium yields the same outcome at date T as the solution to the date T constrained Ramsey problem. This also determines the price of debt issued at

time $T - 1$. For any equilibrium price function q , ${}_{T-1}q_T(H_{1T-1}) = {}_{T-1}q_T^C(H_{1T-1})$.

Consider now the problem of the government at $T - 1$. Again, using Lemma 1, it follows that the allocation rules in period $T - 1$ can depend only upon the current policy, the inherited debt structure ${}_{T-2}B$, and the price function ${}_{T-1}q$. We have already argued that ${}_{T-1}q_T = {}_{T-1}q_T^C$. Since ${}_{T-1}q_{T-1} = p_{T-1}$, it follows that ${}_{T-1}q_{T-1}(H_{1T-1}) = p_{T-1}^C(H_{1T-1})$. Hence, in any equilibrium, $F_{T-1}(H_{1T-1}) = F_{T-1}^C(H_{1T-1})$. The problem facing the government at $T - 1$ is

$$\max U(F_{T-1}^C(H_{1T-1})) + \beta U(F_T^C(H_{1T}))$$

subject to

$$(3.11) \quad \beta^{T-1} U_c [\tau_{T-1} L_{T-1}^C(H_{1T-1}) - G_{T-1}] + {}_{T-1}q_T^C(H_{1T-1}) {}_{T-1}B_T \\ = (1 - \delta_{T-1}) \sum_{s=T-1}^T {}_{T-1}q_s^C(H_{1T-1}) {}_{T-2}B_s$$

and

$$(3.12) \quad \beta^T U_c [\tau_T L_T^C(H_{1T}) - G_T] = (1 - \delta_T) [{}_T q_T^C(H_{1T}) {}_{T-1} B_T].$$

It is clear that for any new debt ${}_{T-1}B_T$ issued in the solution to this problem, the period- T policies are chosen exactly as in the period T problem considered earlier. In particular, the value of δ_T which solves this problem is equal to $\delta_T^C(H_{1T})$; hence, ${}_{T-1}q_T^C = (1 - \delta_T) {}_T q_T^C$. Furthermore, it is clearly optimal to set $\delta_{T-1} = 1$ if the value of the inherited debt is positive. We add (3.11) and (3.12) and use the sequential constraints as before to derive

$$C_s + G_s = L_s, \text{ for } s = T - 1, T$$

and

$$\beta^{T-1}R_{T-1} + \beta^T R_T \geq \min \left[\sum_{s=T-1}^T \beta^s U_{c, T-2}^B, 0 \right]$$

and

$$R_T \leq 0.$$

Again, the problem of maximizing utility is identical to the constrained Ramsey problem. Furthermore, we have shown that there is a unique debt restructuring which supports the constrained Ramsey solution. Obviously, the same debt restructuring solves the problem faced by the government. Proceeding in the same fashion, it is easy to prove by backward induction that any equilibrium outcome solves the constrained Ramsey problem.

Sequential rationality by consumers is immediate since, in a finite horizon, the only implications of sequential rationality for consumers are the sequential constraints for aggregate allocations and the equality between the after-tax wage and the marginal rate of substitution. \diamond

4. Infinite Horizon

In this section we consider an infinite horizon version of the model. We first show that the constrained Ramsey allocations are the outcomes of a sustainable equilibrium. We then provide sufficient conditions for an arbitrary sequence of prices, policies, and allocations to be the outcome of a sustainable equilibrium. In particular, this sequence must be a date-0 competitive equilibrium and must satisfy a certain set of inequalities.

The constrained Ramsey problem for the infinite horizon economy is defined as in (3.1) with $T = \infty$. Using the solutions to these problems we construct the infinite horizon constrained Ramsey equilibrium $(\sigma^c, F^c, p^c, q^c)$ exactly as in the finite horizon case. We then have

Proposition 3: (Constrained Ramsey Solutions Are Sustainable Outcomes.)

The policy plans, allocation rules, and price functions $(\sigma^c, F^c, p^c, q^c)$ constitute a sustainable equilibrium when the horizon is infinite.

Proof: Consider some arbitrary history H_t . To verify sequential rationality by the government, we need to show that no policy $\hat{\sigma}^t$ that satisfies the government budget constraints can improve welfare. It suffices to show, however, that no one-shot deviations by the government can improve utility [see Whittle (1983), Chapter 24, Theorem 2.1 or Abreu (1984), Proposition 1]. That is, we need only show that for any policy $(\hat{\sigma}_t, \sigma^{ct+1})$ satisfying the government's budget constraints,

$$(4.1) \quad V_t(\sigma^{ct}, F^{ct}, p^{ct}, q^{ct} | H_t) \geq V_t(\hat{\sigma}_t, \sigma^{ct+1}, F^{ct}, p^{ct}, q^{ct} | H_t).$$

Clearly, to establish (4.1) it suffices to show that the allocations induced by such a deviation satisfy the constraints of the constrained Ramsey problem. Now given that the government will follow σ^{ct+1} from time $t + 1$ onward it follows from construction of q^c that

$$(4.2) \quad {}_s q_r^c(\hat{H}_{1s}) = p_r^c(\hat{H}_{1r}) (1 - \delta_{s+1}^c(\hat{H}_{s+1})) \cdots (1 - \delta_r^c(\hat{H}_r))$$

where for all $r \geq s \geq t$, the histories \hat{H}_r and \hat{H}_{1r} are induced by F^{ct} and $(\hat{\sigma}_t, \sigma^{ct+1})$. For each $s \geq t$, substitute the right side of (4.2) into government budget constraints of the form (2.5) and add the resulting equations from time t through infinity. Using the sequential constraints (2.3), the consumer budget constraint (2.2), and representativeness, the resulting sum can be written as

$$\begin{aligned}
 & p_t(\hat{H}_{1t}) [C_t^c(\hat{H}_{1t}) - (1 - \hat{\tau}_t(H_t)) L_t^c(\hat{H}_{1t})] \\
 & + \sum_{s=t+1}^{\infty} p_s^c(\hat{H}_{1s}) [C_s^c(\hat{H}_{1s}) - (1 - \tau_s^c(\hat{H}_s)) L_s^c(\hat{H}_{1s})] \\
 & = (1 - \hat{\delta}_t(H_t)) \sum_{s=t}^{\infty} q_s^c(H_t) t_{-1} B_s.
 \end{aligned}$$

Using the consumer's first-order conditions (1.9) and (1.10), we can write this as

$$(4.3) \quad \sum_{s=t}^{\infty} \beta^s R_s = (1 - \delta_t) \sum_{s=t}^{\infty} \beta^s U_c t_{-1} B_s.$$

Next, subtracting the sequential constraints for aggregate allocations from the government's budget constraints gives

$$(4.4) \quad C_s + G_s = L_s, \text{ for } s = t, t + 1, \dots$$

In addition, because the government follows σ^{ct+1} from $t + 1$ on, we have

$$(4.5) \quad \sum_{s=r}^{\infty} \beta^s R_s \leq 0, \text{ } r = t + 1, t + 2, \dots$$

Thus for any one-shot deviation, the resulting allocations satisfy (4.3)-(4.5). Since the constrained Ramsey allocations were constructed to maximize the government's utility subject to these constraints, (4.1) then follows. By induction it follows that no deviations of finite length can improve welfare. Since the utility function is bounded and $0 < \beta < 1$, the results of Whittle (1983) imply that no infinite deviations can improve welfare.

We now check sequential rationality for consumers. Sequential constraints for consumers and maximization of utility follow by construction. Finally, from representativeness we have that the budget constraint in the constrained Ramsey problem is identical to the consumer's budget constraint. ◊

We can use the constrained Ramsey equilibrium to help characterize other possible sustainable outcomes. To characterize such outcomes we use a modified version of the above equilibria, which we call the revert-to-constrained-Ramsey equilibria. (These equilibria are the natural competitive analogues of the trigger-strategy equilibria of repeated games.)

For an arbitrary sequence (π, X, p, q) , define the revert-to-constrained-Ramsey plans $(\sigma^r, F^r, p^r, q^r)$ as follows. Consider first the policy plan σ^r . For any history H_t , σ^r specifies the policy π_t given by π if the tax rates $(\pi_0, \dots, \pi_{t-1})$ have been chosen according to π and if the allocations (X_0, \dots, X_{t-1}) have been chosen according to X . If they have not, then revert to the constrained Ramsey policies; that is, let $\sigma^r(H_t) = \sigma^c(H_t)$. For any history H_{1t} , the allocation rules F^r and pricing functions p^r and q^r are defined analogously. We then have

Proposition 4: (A Set of Sustainable Outcomes.)

An arbitrary sequence (π, X, p, q) is the outcome of a sustainable equilibrium if

- (π, X, p, q) is a date-0 competitive equilibrium
- for every t the following inequality holds:

$$(4.6) \quad \sum_{s=t}^{\infty} \beta^s U(C_s, L_s) \geq V_t^c(t_{-1}B).$$

Proof: Suppose some arbitrary sequence (π, X, p, q) satisfies the two conditions just given. We show that the associated revert-to-constrained-Ramsey plans constitute a sustainable equilibrium. Consider histories under which there have been no deviations from (π, X) before time t . Since (π, X, p, q) is a competitive equilibrium at date 0, it is clear that the continuation of X is

sequentially rational for consumers. Consider the situation of the government. Confronted with allocation rules F^r and given that σ^c specifies the constrained Ramsey policies from $t + 1$ on, the best one-shot deviation is simply the constrained Ramsey policies at t . Thus (4.6) guarantees that σ^r is sequentially rational for such histories.

Consider now histories for which there has been a deviation before time t . The plans $(\sigma^r, F^r, p^r, q^r)$ specify the constrained Ramsey plans from then onward. By Proposition 3 these plans are sequentially rational. \diamond

5. Uncertainty

In this section we extend the analysis of the previous sections to allow for stochastic government consumption. We are interested in comparing the difference in utility between the Ramsey and the constrained Ramsey problems. This difference gives us a measure of the value of a technology through which a government can commit its policies. We find it convenient to normalize this difference by dividing utilities by $\sum_{t=0}^T \beta^t$. In our comparison we make two assumptions: First, government consumption follows a persistent Markov process. Second, under the Ramsey plan tax revenues are smoother than government consumption. Given these assumptions, we prove our main result: With long enough horizons and sufficiently little discounting, the difference in the normalized value of utility between these allocations tends to zero. (It is important to note that this result is quite distinct from the folk theorems of repeated games.)

Government consumption follows a given stochastic process for which the realizations up to and including time t are denoted $G^t = (G_0, \dots, G_t)$. The probability of observing any particular event G^t is $\mu(G^t)$. The initial realization G_0 is given. Each realization G_t is assumed to be in a finite set $\{\gamma_1, \dots, \gamma_k\}$, with $\gamma_1 < \dots < \gamma_k$. There is no other uncertainty in the econ-

omy, so the commodity space is the space of infinite sequences $(c, \ell) = \{c_t(G^t), \ell_t(G^t) \mid \text{for all } t \text{ and } G^t\}$, where $c_t(G^t)$ and $\ell_t(G^t)$ are contingent on the events G^t . Let (C, L) denote the aggregate allocation. We define similar objects for the policy π and price systems p and q .

An aggregate allocation is feasible if

$$(5.1) \quad C_t(G^t) + G_t = L_t(G^t), \text{ for all } t \text{ and } G^t.$$

The preferences of each agent are given by the expected utility function

$$(5.2) \quad \sum_t \sum_{G^t} \beta^t \mu(G^t) U(c_t(G^t), \ell_t(G^t)).$$

The budget constraint of the representative consumer at time 0 is

$$(5.3) \quad \sum_t \sum_{G^t} p_t(G^t) [c_t(G^t) - (1 - \tau_t(G^t)) \ell_t(G^t)] = 0.$$

The sequential constraints for aggregate allocations and government policies and the no-arbitrage conditions are the obvious stochastic analogues of (1.3), (1.4), and (1.6). Note that the implied debt issues are now contingent on the events G^t . It is immediate that the Ramsey equilibrium allocations maximize the aggregate value of utility, subject to (5.1) and

$$(5.4) \quad \sum_t \sum_{G^t} \beta^t \mu(G^t) R_t(G^t) = 0,$$

where $R_t(G^t) = U_c C_t(G^t) + U_\ell L_t(G^t)$ and the derivatives of U are evaluated at $C_t(G^t)$ and $L_t(G^t)$.

The first-order conditions for this problem are (5.1), (5.4), and for all t and G^t ,

$$(5.5) \quad (1 + \lambda_0)(U_c + U_\ell) + \lambda_0 [C_t(G^t)(U_{cc} + U_{c\ell}) + L_t(G^t)(U_{c\ell} + U_{\ell\ell})] = 0,$$

where λ_0 is the Lagrange multiplier on (5.4). Clearly the allocations that solve this problem depend on only the current value of government consumption G_t and the multiplier λ_0 . Suppressing the multiplier, we let $R(G_t)$ denote the value of the government surplus (in marginal utility units) under the Ramsey allocations. We will use the fact that $R(\cdot)$ is time invariant in the next proposition.

Next we incorporate uncertainty in the environment without commitment by letting the histories H_t and H_{1t} also include the past realizations of government spending G^t . For such an environment it is immediate to show that the sustainable outcomes solve the constrained Ramsey problem. For any vector of inherited debt ${}_{t-1}B(G^{t-1})$, this problem is to choose $\{C_s(G^s), L_s(G^s)\}$ for all $s = t, \dots, T$, to solve

$$(5.6) \quad \max \sum_{s=t}^T \sum_{G^s} \beta^s \mu(G^s | G^t) U(C_s(G^s), L_s(G^s))$$

subject to (5.1) as well as

$$(5.7) \quad \sum_{s=t}^T \sum_{G^s} \beta^s \mu(G^s | G^t) R_s(G^s) \geq \min \left[\sum_{s=t}^T \sum_{G^s} \beta^s \mu(G^s | G^t) U_c {}_{t-1}B_s(G^s), 0 \right]$$

and

$$(5.8) \quad \sum_{s=r}^T \sum_{G^s} \beta^s \mu(G^s | G^t) R_s(G^s) \leq 0, \quad \text{for } r = t + 1, \dots, T$$

where $\mu(G^s | G^t)$ denotes the conditional probability of G^s given G^t .

In our comparison of the Ramsey and the constrained Ramsey allocations, we will use two assumptions.

Assumption 1: Government consumption follows a stationary Markov process with strictly positive elements. Furthermore, it is persistent in that $\text{Prob}\{G_{t+1} \leq \gamma | G_t\}$ is a decreasing function of G_t for all γ .

Let $\mu_{ij} = \text{Prob}(G_{t+1} = \gamma_j | G_t = \gamma_i)$. Note that the persistence condition requires that higher values of government consumption at t give stochastically larger values of government consumption at $t + 1$.

Assumption 2: The value of the government surplus under the Ramsey plan, $R(G_t)$, is decreasing in G_t .

This assumption requires that the value of tax revenues be smoother than the value of government spending. In the next section, we give several examples for which the assumption is satisfied.

In the proposition that follows we use the normalized value of utility to compare allocations. For any allocation (C, L) , the normalized value of utility is

$$\sum_{t=0}^T \sum_{G^t} \beta^t \mu(G^t) U(C_t(G^t), L_t(G^t)) / \sum_{t=0}^T \beta^t.$$

Proposition 5: (The Value of Commitment.)

Given Assumptions 1 and 2 and any $\epsilon > 0$, there is some horizon length $T < \infty$ and some discount factor $\beta < 1$ such that the difference in the normalized value of utilities under the Ramsey and the constrained Ramsey allocations is, at most, ϵ .

Proof: Under our assumptions, a useful feature of the Ramsey allocations is that the present value of the government surplus at any date t is positive if and only if $G_t \leq G_0$. (Of course the present value of this surplus is equal to the value of the inherited debt.) To see this, consider the present value of the government surplus at date t with G_t equal to some γ_k . Using Assumption 1 and the fact that the government surplus function R is time invariant, we can write this present value as

$$(5.9) \quad R(\gamma_k) + \beta \sum_i \mu_{ki} [R(\gamma_i) + \beta \sum_j \mu_{ij} R(\gamma_j) + \dots].$$

Let $G_0 = \gamma_k$. From (5.4) it follows that the Ramsey plan satisfies

$$(5.10) \quad R(\gamma_k) + \beta \sum_i \mu_{ki} [R(\gamma_i) + \beta \sum_j \mu_{ij} R(\gamma_j) + \dots] = 0.$$

Subtracting the left side of (5.10) from (5.9) gives

$$(5.11) \quad R(\gamma_k) - R(\gamma_k) + \beta \sum_i \mu_{ki} R(\gamma_i) - \beta \sum_i \mu_{ki} R(\gamma_i).$$

Suppose $\gamma_k > \gamma_k$. Then, since R is decreasing and the conditional distribution μ_k stochastically dominates μ_k , the expression in (5.11) is nonpositive. It follows that the expression in (5.9) is nonpositive. Likewise, if $\gamma_k < \gamma_k$, then (5.9) is nonnegative.

Suppose first that $G_0 = \gamma_1$. By the above argument the Ramsey allocations satisfy (5.8). Hence, the Ramsey and the constrained Ramsey allocations coincide.

Suppose next that $G_0 > \gamma_1$. Consider the following plan: balance the budget continuously until the first realization of γ_1 , say at date t , and follow the constrained Ramsey plan from then on. Clearly this plan yields lower utility than the date-0 constrained Ramsey plan. Now, by the above argument, the value of the inherited debt at γ_1 is positive under the Ramsey plan. This implies that the present value of utility from date t on under the date- t constrained Ramsey plan is higher than the present value of utility under the original Ramsey plan. Thus the difference in utility between the Ramsey plan and the date-0 constrained Ramsey plan is bounded by the utility lost until the first realization of γ_1 .

We can compute an upper bound for this utility loss as follows. Let μ denote the smallest value of μ_{i1} for $i = 1, \dots, K$. Let \bar{u} denote the greatest possible difference in two utility levels in any period. Then the utility loss is bounded by

$$\sum_{t=0}^T \beta^t (1-\mu)^t \bar{u}$$

which in normalized utility terms can be written as

$$(5.12) \quad \frac{(1-\beta)}{(1-\beta^{T+1})} \frac{1-\beta^{T+1}(1-\mu)^{T+1}}{1-\beta(1-\mu)} \bar{u}.$$

Now, by hypothesis, $\mu > 0$ and $\bar{u} < \infty$. Thus for any $\epsilon > 0$, we can find a discount factor $\beta < 1$ and horizon length T such that the normalized utility loss is less than ϵ . \diamond

Thus, we have established that the value of a commitment technology becomes arbitrarily small as the horizon increases and the discount factor approaches one.

6. Examples

In this section we consider several examples. Examples 1-3 are parametric examples for which the surplus function $R(G_t)$ is decreasing in G_t . Examples 4-7 are simple deterministic examples which illustrate the logic behind Proposition 4. We begin with a simple quadratic example.

Example 1: Let U be quadratic. Manipulating the first-order conditions to the Ramsey problem, one can show that

$$R(G) = \alpha_0 - \alpha_1 G - \alpha_2 G^2$$

where the α_i are constants with $\alpha_1 > 0$ and

$$\alpha_2 = -(U_{cc} U_{ll} - U_{cl}^2) / (U_{cc} + 2U_{cl} + U_{ll}).$$

(See Appendix 1 of Lucas and Stokey (1983) for details.) Since we have assumed that U is concave and consumption and leisure are normal goods, we have that $\alpha_2 \geq 0$. Thus $R(\cdot)$ is a decreasing function.

We consider next a utility function with constant relative risk aversion in both goods. In addition to providing an example for which our assumption is satisfied, this next example is interesting in its own right. For such a utility function there is perfect tax smoothing in the sense that the Ramsey tax rates are constant.

Example 2: Let U be given by

$$U(C,L) = \frac{C^{1-\alpha_1}}{1-\alpha_1} - \alpha_3 \frac{L^{1-\alpha_2}}{1-\alpha_2}.$$

Note that $\alpha_3 \geq 0$ and, from concavity, we have $\alpha_1 \geq 0$ and $\alpha_2 \leq 0$. Given the additive separability of U , we can manipulate the first-order conditions (5.5) to get

$$(6.1) \quad (1+\lambda_0)\tau_t + \lambda_0 \frac{C_t U_{cc}}{U_c} - (1-\tau_t) \frac{L_t U_{ll}}{U_l} = 0$$

where we have suppressed dependence on G^t and have let $\tau_t = 1 + U_l/U_c$. Substituting for the derivatives of U in (6.1), we have that tax rates are constant over time and independent of the current realization of government consumption. Differentiating $R(G)$ gives

$$R'(G) = [C U_{cc} + U_c] C'(G) + [L U_{ll} + U_l] L'(G).$$

Again, substituting for the derivatives of U we get,

$$(6.2) \quad R'(G) = (1-\alpha_1)U_c C'(G) + (1-\alpha_2)U_l L'(G).$$

We also have that $C + G = L$ and

$$(6.3) \quad -\frac{U_l}{U_c} = (1-\tau)$$

where τ is a constant. Given the assumed utility function, (6.3) implies

$$(6.4) \quad \alpha_2 L'(G)/L = \alpha_1 C'(G)/C$$

and from feasibility we have $C'(G) + 1 = L'(G)$. Since $\alpha_1 \geq 0$, and $\alpha_2 \leq 0$, we have that $C'(G) \leq 0$ and $L'(G) \geq 0$. Furthermore, substituting (6.4) into (6.2) gives

$$R'(G) = [\alpha_1(1-\alpha_1) - \alpha_2(1-\alpha_2)(1-\tau)C/L] U_C C'(G)/\alpha_1.$$

Now since the tax rate is constant and the present value of government consumption is positive, we know $\tau > 0$. Thus, a sufficient condition for R to be decreasing is $\alpha_1(1-\alpha_1) \geq \alpha_2(1-\alpha_2)$. Since $\alpha_2 \leq 0$, R is decreasing whenever the coefficient of relative risk aversion $\alpha_1 \leq 1$. Even if $\alpha_1 > 1$, R will be decreasing if α_2 is negative enough.

Note that for the log case, $U(C,L) = \ln C - \alpha_3 \ln L$, $R(G)$ is always decreasing.

Example 3: Consider

$$U(C,L) = C^{\alpha_1} L^{\alpha_2}$$

with $\alpha_1 \geq 0$ and $\alpha_2 \leq 0$. The same type of arguments as in Example 2 imply that $R(G)$ is always decreasing.

We now consider several simple deterministic examples which help illustrate the intuition behind Proposition 4. In Example 4 there is no time consistency problem whatsoever: the Ramsey allocations are sustainable.

Example 4: Let $T = \infty$. Let $G_t = 0$, for t even, and $G_t = \gamma$, for t odd. Let U be as in Examples 1-3 so that $R(G_t)$ is decreasing. It is immediate that under the Ramsey plan, the budget is balanced over each two-period cycle; thus,

$$R(0) + \beta R(\gamma) = 0.$$

Since $R(G_t)$ is decreasing, $R(0)$ is positive and $R(\gamma)$ is negative. For t even, $\sum_{r=t}^{\infty} \beta^r R(G_r) = 0$ and for t odd, $\sum_{r=t}^{\infty} \beta^r R(G_r) = R(\gamma) < 0$. This implies the Ramsey allocations solve the constrained Ramsey problem. From Proposition 3, the Ramsey allocations are sustainable.

We can use the infinite horizon version of (3.10) to calculate the debt issues. For t even,

$${}_t B_{t+1} = R(\gamma) / U_c(C(\gamma), L(\gamma)) < 0$$

and ${}_t B_s = 0$, for all $s \geq t + 2$. For t odd, ${}_t B_s = 0$, for all $s \geq t$. Notice that the debt issues are always nonpositive.

In the next example, a slight variant of Example 4, there is a time consistency problem, but the value of a commitment technology is not very large.

Example 5: Let $T = \infty$. Let $G_t = \gamma$, for t even, and $G_t = 0$, for t odd. Let U be as in Examples 1-3. Let $R(G_t)$ denote the surplus function for this pattern of government consumption. Under the Ramsey plan,

$$R(\gamma) + \beta R(0) = 0.$$

For t even, $\sum_{r=t}^{\infty} \beta^r R(G_r) = R(0) > 0$ and for t odd, $\sum_{r=t}^{\infty} \beta^r R(G_r) = 0$. Thus the Ramsey allocations do not solve the constrained Ramsey problems. Notice that the debt issues are as follows. For t even,

$${}_t B_{t+1} = R(0) / U_c(C(0), L(0)) > 0$$

and ${}_t B_s = 0$, for all $s \geq t + 2$. For t odd, ${}_t B_s = 0$, for all $s \geq t$.

Consider the following policy. Balance the budget in period 0 and follow the constrained Ramsey allocations from date 1 onward. From Example 4,

we know that from date 1 onward this policy gives the Ramsey allocations of that example. Thus, the utility difference between this plan and the original Ramsey plan is at most the utility lost from balancing the budget in the first period.

In Examples 4 and 5, government consumption followed a two-period cycle. We now consider examples where it follows a K-period cycle.

Example 6: Let $G_t = \gamma_k$ for $t = nk$, where $k = 1, \dots, K$, and the integer $n > 0$. Let $\gamma_1 < \gamma_2 < \dots < \gamma_K$. Notice that government consumption monotonically increases over each K-period cycle. Let U be such that R is decreasing. Under the Ramsey plan the budget is balanced over each cycle so that

$$\sum_{k=1}^K \beta^k R(\gamma_k) = 0.$$

Since $R(\cdot)$ is decreasing, we know that

$$\sum_{k=r}^K \beta^k R(\gamma_k) < 0, \text{ for } r = 2, \dots, K.$$

Thus the Ramsey plan coincides with the constrained Ramsey plan, and there is no time consistency problem.

Example 7: Consider the same pattern of government consumption as in example 6, except let government consumption start at some γ_J . That is, let $G_t = \gamma_{t+J}$ for $t = 1, \dots, K - J$, and $G_{t+J} = \gamma_k$ for $t = nk$, where $k = 1, \dots, K$, and the integer $n \geq 1$. Again, under the Ramsey plan the budget is balanced over each K-period cycle:

$$R(\gamma_J) + \beta R(\gamma_{J+1}) + \dots + \beta^K R(\gamma_{J-1}) = 0.$$

Notice that for appropriately chosen J , the debt will be positive under the Ramsey plan. Now consider a policy similar to the one used in Proposition 5: balance the budget from period 0 through period $K - J$ and follow the (date $K - J + 1$) constrained Ramsey plan from then on. Clearly this constrained Ramsey plan is simply the Ramsey plan of Example 6. Thus, the utility difference between the original Ramsey plan and the constrained Ramsey plan is bounded by the utility lost in the first $K - J$ periods.

7. Conclusion

In this paper we have analyzed the incentives of the government to renege on its debts. We completely characterized the set of sustainable outcomes when the horizon is finite. A somewhat surprising conclusion of this analysis is that under plausible assumptions, the value of a commitment technology become arbitrarily small as the horizon becomes long and the discount factor approaches unity. We have also given sufficient conditions for an arbitrary sequence of prices, policies, and allocations to be sustainable when the horizon is infinite.

Avenues for further research include using the techniques developed here and in Chari-Kehoe (1988) to analyze capital taxation and international borrowing and lending. We plan to address these issues in future work.

Footnotes

¹For some further work using the model of Lucas and Stokey, see Alesina and Tabellini (1987), Persson and Svensson (1984), Persson (1987), Persson and Svensson (1987), and Rogers (1987). For some other work on optimal taxation and debt, see Barro (1979). For some early work on time consistency, see Kydland and Prescott (1977), Prescott (1977), Calvo (1978), Fischer (1980). For more recent developments, see the surveys by Barro (1985) and Rogoff (1987).

²In an interesting paper, Bulow and Rogoff (1988) investigated the implications of allowing for negative debt in an open economy setting. See also Atkeson (1987) and Grossman and Van Huyck (1986).

References

- Abreu, Dilip. 1984. Infinitely repeated games with discounting: a general theory. Harvard University Discussion Paper 1083.
- Alesina, Alberto and Guido Tabellini. 1987. A positive theory of fiscal deficits and government debt. Carnegie-Mellon University. Photocopy.
- Atkeson, Andrew. 1987. Time consistency in a macroeconomic model. Stanford University. Photocopy.
- Barro, Robert J. 1979. On the determination of the public debt. Journal of Political Economy 87 (5, Pt. 1): 940-71.
- _____. 1985. Recent developments in the theory of rules versus discretion. University of Rochester Working Paper 12.
- Barro, Robert J., and David B. Gordon. 1983. Rules, discretion and reputation in a model of monetary policy. Journal of Monetary Economics 12 (1): 101-21.
- Bulow, Jeremy, and Kenneth Rogoff. 1988. Sovereign debt: Is to forgive to forget? Stanford University. Photocopy.
- Calvo, Guillermo A. 1978. On the time consistency of optimal policy in a monetary economy. Econometrica 46 (6): 1411-28.
- Chari, V. V., and Patrick J. Kehoe. 1987a. Sustainable plans and positive debt. Research Department Working Paper 354. Federal Reserve Bank of Minneapolis.
- _____. 1987b. Sustainable plans. Research Department Working Paper 377. Federal Reserve Bank of Minneapolis.
- Fischer, Stanley. 1980. Dynamic inconsistency, cooperation, and the benevolent dissembling government. Journal of Economic Dynamics and Control 2 (1): 93-107.

- Friedman, James W. 1971. A non-cooperative equilibrium for supergames. Review of Economic Studies 38 (113): 1-12.
- Fudenberg, Drew, and Eric Maskin. 1986. The folk theorem for repeated games with discounting or with incomplete information. Econometrica 54 (3): 533-54.
- Grossman, Herschel and John Van Huyck. 1986. Sovereign debt as a contingent claim: excusable default, repudiation, and reputation. Brown University Working Paper 86-27.
- Howard, R. 1960. Dynamic Programming and Markov Processes New York: John Wiley and Sons.
- Kydland, Finn E., and Edward C. Prescott. 1977. Rules rather than discretion: the inconsistency of optimal plans. Journal of Political Economy 85 (3): 473-91.
- Lucas, Robert E., Jr., and Nancy L. Stokey. 1983. Optimal fiscal and monetary policy in an economy without capital. Journal of Monetary Economics 12 (1): 55-93.
- Persson, Mats, Torsten Persson, and Lars E. O. Svensson. 1987. Time consistency of fiscal and monetary policy. Econometrica 55 (6): 1419-32.
- Persson, Torsten, and Lars E. O. Svensson. 1984. Time-consistent fiscal policy and government cash-flow. Journal of Monetary Economics 14 (3): 365-74.
- Prescott, Edward C. 1977. Should control theory be used for economic stabilization? In Optimal Policies, Control Theory and Technology Exports, Carnegie-Rochester Conference Series on Public Policy 7, ed. K. Brunner and A. H. Meltzer, pp. 13-38. Amsterdam: North-Holland.
- Ramsey, F. P. 1927. A contribution to the theory of taxation. Economic Journal 37: 47-61.

Rogers, Carol Ann. 1987. Debt restructuring with a public good. Georgetown University. Photocopy.

Rogoff, Kenneth. 1987. Reputational constraints on monetary policy. In Bubbles and Other Essays, Carnegie-Rochester Conference Series on Public Policy 26, ed. K. Brunner and A. H. Meltzer, pp. 141-82. Amsterdam: North-Holland.

Whittle, Peter. 1983. Optimization Over Time: Dynamic Programming and Stochastic Control, vol. II. New York: John Wiley and Sons.