Federal Reserve Bank of Minneapolis
Research Department

COMMUNICATION COSTS, THE BANKING
SYSTEM, AND AGGREGATE ACTIVITY

Stephen D. Williamson*

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Abstract

A model is constructed where banks provide access to a communication technology which facilitates trade. Bank liabilities may coexist with alternative means of payment in equilibrium, and there exist regions of the parameter space where banking dominates the payments system and where physical exchange media dominate. The model is consistent with some observations concerning the role of the banking system in economic development, and with characteristics of banking crises. In particular, in early stages of economic development: 1) rapid output growth is accompanied by an increasing share of banking in transactions activity and 2) there are recurrent banking "panics" where reductions in measured aggregate output coincide with increases in the use of alternative means of payment relative to bank liabilities. In later stages of development, growth slackens off, the share of banking in the payments system stabilizes and the economy is less likely to be subject to banking panics.

I. Introduction

One of the important distinguishing features of banks is that they supply transactions services. That is, they provide access to a communications technology which facilitates trade. In most economies, the transactions services supplied by banks compete with alternative transactions media, such as fiat currency or commodity monies. Authors who have studied the role of banking in development (e.g., Cameron 1967) or who have studied financial panics and banking crises (e.g., Friedman and Schwartz 1963 and Cagan 1965) focus considerable attention on substitution between bank liabilities and other means of payment.

The best-known traditional account of the demand for a medium of exchange is the inventory money demand model of Baumol (1952) and Tobin (1956) and its general equilibrium versions (e.g., Jovanovic 1982 and Romer 1987). These are essentially cash-in-advance models with a single means of payment and a technology for converting interest-bearing assets into "money." An inventory money demand model does not explain why interest-bearing assets cannot be used in exchange, and it does not model the demands for alternative means of payment.

In more recent approaches to modeling banking and financial intermediation, intermediaries serve to economize on monitoring or evaluation costs (Boyd and Prescott 1986, Diamond 1984, Williamson 1986), to insure against states that require liquid assets (Diamond and Dybvig 1983, Bhattacharya and Gabe 1987), or to provide a set of contracts rich enough to induce the revelation of private information (Williamson 1988). However, none of these environments includes an explicit transactions technology.

To model the provision of transactions services it seems necessary to explicitly specify an environment where there are barriers to communication

and trade. Two such approaches are Prescott (1987) in a representative agent framework, and Kiyotaki and Wright (1987), in a search environment. The model constructed here is much different from either Kiyotaki and Wright's or Prescott's, but it has some features which resemble Diamond and Dybvig's (1983) banking model. In particular, there are three periods, with agents being uncertain in the first period about their future preferences over consumption streams. However, in contrast to Diamond and Dybvig's model, the spatial separation of agents precludes the possibility of insurance against preference shocks. The provision of this insurance is the raison d'etre for banking in the Diamond-Dybvig model. Here, banks are essentially record-keepers who keep track of the ownership of assets. We abstract from other important features of banking such as diversification and asset transformation.

In the model, agents receive endowments in the first and second periods and they may produce assets in the first period and trade them in the second. Assets yield returns in the final period. Assets are of two types: the first can be transported costlessly between locations and traded without cost; the second cannot be transported and, if traded, costly communication with a bank is required. Asset 1 yields a lower two-period return than does asset 2. The first asset can be interpreted as cash, as a commodity money, or as private bearer notes, while the second has features associated with the deposit liabilities of banks.

With the preference uncertainity in the model, agents do not know in the first period whether they will be asset buyers or asset sellers in the second period. Given the nature of the communications technology, there are effectively two segmented asset markets in the second period, one of which requires a fixed entry cost (i.e., the cost of communication).

Though consumers in the model maximize identical objective functions, their equilibrium asset holdings may differ. That is, the model may generate endogenous heterogenity among agents because of the fixed costs of communication. Given equilibrium prices, agents may be indifferent among several different lifetime contingent asset-holding strategies. Agents choose among such strategies by submitting to lotteries. In equilibrium there may be 1, 2, or 3 types of agents, who differ according to their contingent asset-holding strategies.

A unique equilibrium exists in which either asset 1 is produced and traded, asset 2 is produced and traded, or both, depending on parameter values. Thus, the model predicts that there are circumstances where bank liabilities coexist with alternative means of payment, where banking dominates, and where alternatives dominate. In examples, the ranges of parameter values over which both assets are held represents a large subset of the parameter space. The fraction of transactions carried out through the banking system tends to increase as the costs of communication decrease, and as the returns on assets held by banks increase.

The response of equilibrium prices and quantities to changes in technological parameters differs quite markedly over the region of the parameter space where both assets are traded. Near the boundary with the region where only one asset is traded, small parameter changes lead to large changes in quantities and prices. However, over most of the region where both assets are traded, price and quantity responses are relatively small. The model thus has properties that are consistent with what is observed over stages of economic development, and with features of banking panics. That is, as the technology evolves stochastically over time, in such a way that there is trend growth in output, there will be an early stage of rapid growth where the

banking system also grows rapidly (see Cameron 1967), followed by a stage over which growth slackens off and the share of banking in transactions activity stabilizes. Over the early stage, there can be recurrent periods where large drops in output coincide with large decreases in the share of transactions carried out through the banking system, and with increases in the prices of transactions media (deflations). These "panic" periods subside as the banking system reaches a higher state of development.

The model thus provides an alternative explanation for the difference between pre-Depression and Depression-era U.S. macroeconomic behavior, on the one hand, and post-Depression observations, on the other. During and previous to the Great Depression, there were large recurrent drops in output accompanied by similarly large reductions in the demand for bank deposit liabilities relative to other means of payment (see Cagan 1965, and Friedman and Schwartz 1963). However, recurrent "banking panics" were absent in the post-Depression era. Usually, this observation is attributed to the government provision of deposit insurance and more appropriate behavior by the Federal Reserve System in the post-Depression era. However, in this model these observations are consistent with a natural evolution of the banking system as the technology changes over time.

In examples, parameter changes which lead to increases in measured output and in the size of the banking system in equilibrium can also lead to decreases in expected utility. This occurs because of the increased quantity of resources absorbed as the banking system expands. That is, in the early stages of development, the banking system can expand too quickly, so that welfare decreases.

The paper is organized as follows. In the second section the model is constructed. Section 3 contains a definition of competitive equilibrium,

and existence and uniqueness of equilibrium is established. In Section 4, some examples are used to illustrate properties of the model. The final section is a summary and conclusion.

II. The Model

There are three periods, 0, 1, and 2, and a continuum of consumers of measure 1. In period 0, consumers are distributed uniformly on the circumference of a unit circle. There is no communication or exchange of goods possible between locations in period 0. A consumer may be one of two types; a type i agent consumes in period i and has preferences given by $u(c_1)$ if type 1 and $v(c_2)$ if type 2. Here, c_j is consumption in period j. The measure of type 1 consumers is π, where $0 < \pi < 1$, and π is public knowledge in period 0. It is assumed that $u(c)$ and $v(c)$ are concave, increasing, finite for $0 < c < \infty$, that $\lim_{c \to 0} u(c) = \lim_{c \to 0} v(c) = -\infty$, $\lim_{c \to \infty} u(c) = \lim_{c \to \infty} v(c) = \infty$, and $cu''(c)/u'(c) \geq 1$. The last condition is sufficient for uniqueness of equilibrium.

Consumers do not know their types in period 0. In period 1, each consumer learns his type and then all consumers move to the center of the circle, where communication and trade among consumers is possible. Each consumer receives an indivisible endowment of x_0 units of an investment good in period 0, and type 2 consumers each receive x_1 units of a perishable consumption good in period 1.

Each consumer has a technology for producing two types of assets from the investment good in period 0. Since the period 0 endowment is indivisible, a particular consumer cannot diversify between the two assets.[1] Asset i yields a return of β_i units of consumption in period 2 for each unit invested in period 0, where $\beta_i > 0$, i = 1, 2. The first asset can be transported between locations following period 0, and its quality can be verified at zero cost. However, asset 2 cannot be moved from its investment location,

and it is necessary for agents to incur communication costs if asset 2 is bought or sold in period 1. Each consumer is paired with a record-keeper who stays at the initial location in period 0-2. Record-keepers operate accounting systems which keep track of which agents have claims to the type 2 asset at their location. Should an agent wish to transfer a claim to a type 2 asset to another agent in period 1, the asset seller must incur a cost a_1 (in terms of the period 1 consumption good) and the buyer a cost a_2. Here a_1 represents the cost to the seller of identifying herself and communicating with the record-keeper, and a_2 is the communication cost for the buyer.

At the period 1 trading location, agents meet and agree to asset trades, following which agents communicate with record-keepers. Communication costs are independent of the quantity of information transmitted or the number of locations to which communication is made. Therefore, each seller of asset 2 incurs a cost of a_1, and each buyer of asset 2 incurs a cost of a_2, no matter how many agents they engage in trade. Assume that $x_1 > a_2$, so that the endowment of an asset seller is always large enough to absorb the fixed cost.

In period 2, each consumer moves clockwise around the circle, starting at her period 0 position and consuming any returns to which she has a claim in transit. All consumers move at the same speed, so that no two ever meet in the final period. This precludes any arrangements which might allow for the sharing of communication costs in period 1, since with any such arrangement there is an incentive to cheat on the part of some agent.

The record-keepers in the model can be interpreted as banks. Here, we abstract from some important functions of banking, such as diversification and asset transformation, and focus solely on the role of the banking system in operating an accounting system to facilitate wealth transfers. The communication costs associated with the transfer of claims to asset 2 are similar

to the costs associated with the transfer of deposit claims by check. Asset 1 can be interpreted as currency, a commodity money, or as private bearer notes (to the extent that these are recognizable to all, and have a certain redemption value). This framework could have been embedded in an overlapping generations model, with asset 1 being an intrinsically worthless asset which could have value in equilibrium. However, it seemed better not to introduce the complication of determining an equilibrium with valued fiat money.

This study is similar to Diamond and Dybvig's (1983) model of deposit contracts and bank runs, in that consumers have uncertain preferences, and therefore uncertain demands for liquid assets. However, our framework differs in that we have precluded any insurance role for banking of the type that exists in the Diamond-Dybvig model. Also, all trading will take place here in a decentralized fashion in period 1, while the Diamond-Dybvig model involves a centralized risk-sharing arrangement which is set up in the initial period.

The Consumer's Choice Problem

A representative consumer's optimization problem is solved in two stages.

Stage 1

Let q_i denote the price of asset i in terms of the period 1 consumption good. In period 0, each consumer choose a_i, the quantity of asset i acquired in period 0, i = 1, 2, b_i, the quantity of asset i held at the end of period 1 if type 2, i = 1, 2, and z, the quantity of asset 2 sold in period 1 if type 1, to solve:

(1) $\max \left[\pi u(c_1) + (1-\pi)v(c_2) \right]$

subject to:

(2) $\qquad c_1 \leq q_1 a_1 + q_2 z - \delta_1 \alpha_1$

(3) $\qquad c_2 \leq \beta_1 b_1 + \beta_2 b_2$

(4) $\qquad q_1 a_1 + q_2 a_2 + \delta_2 \alpha_1 + \delta_3 \alpha_2 \leq q_1 a_1 + q_2 a_2 + x_1$

(5) $\qquad z \leq a_2$

(6) $\qquad a_1 + a_2 \leq x_0$

(7) $\qquad a_1 \in \{0, x_0\}$

(8) $\qquad a_2 \in \{0, x_0\}$

(9) $\qquad \delta_1 = 0 \text{ if } z = 0$

(10) $\qquad \delta_1 = 1 \text{ if } z > 0$

(11) $\qquad \delta_2 = 0 \text{ if } b_2 \geq a_2$

(12) $\qquad \delta_2 = 1 \text{ if } b_2 < a_2$

(13) $\qquad \delta_3 = 0 \text{ if } b_2 \leq a_2$

(14) $\qquad \delta_3 = 1 \text{ if } b_2 > a_2.$

Stage 2

Since the problem in Stage 1 is not concave, the solution need not be unique. However, there are at most 2^4 solutions, since for any $(\delta_1, \delta_2, \delta_3, a_1)$, the problem is concave and has a unique solution, and δ_1, δ_2, δ_3, and a_1 can each take on 2 values. Let $(a_1^j, a_2^j, z^j, b_1^j, b_2^j)$, $j = 1, \ldots, n$, denote the n solutions to stage 1. In stage 2, the consumer first

chooses probabilities p_j, $j = 1, \ldots, n$, where $\sum_j p_j = 1$. The consumer then submits to a lottery, the outcome of which determines which solution is chosen. That is, solution j is chosen with probability p_j.

Solution to Stage 1

Given q_1 and q_2, there are five candidate solutions to Stage 1. Let $U_j(q_1, q_2)$ denote expected utility if the consumer follows candidate solution j.

Candidate 1

$a_1 = x_0$, $a_2 = z = 0$, $b_1 = x_0 + x_1/q_1$, $b_2 = 0$, $U_1(q_1, q_2) = \pi u(q_1 x_0) + (1-\pi)v(\beta_1 x_1 + \beta_1 x_1/q_1)$.

Candidate 2

$a_1 = 0$, $a_2 = z = x_0$, $b_1 = x_1/q_1$, $b_2 = x_0$, $U_2(q_1, q_2) = \pi u(q_2 x_0 - \alpha_1) + (1-\pi)v(\beta_2 x_1 + \beta_1 x_1/q_1)$.

Candidate 3

$a_1 = 0$, $a_2 = z = x_0$, $b_1 = (x_0 - \alpha_1 + x_1)/q_1$, $b_2 = 0$, $U_3(q_1, q_2) = \pi u(q_2 x_0 - \alpha_1) + (1-\pi)v(\beta_1(q_2 x_0 + x_1 - \alpha_1)/q_1)$.

Candidate 4

$a_1 = x_0$, $a_2 = z = 0$, $b_1 = 0$, $b_2 = (q_1 x_0 + x_1 - \alpha_2)/q_2$, $U_4(q_1, q_2) = \pi u(q_1 x_0) + (1-\pi)v(\beta_2(q_1 x_0 + x_1 - \alpha_2)/q_2)$.

Candidate 5

$a_1 = 0$, $a_2 = z = x_0$, $b_1 = 0$, $b_2 = x_0 + (x_1 - \alpha_2)/q_2$, $U_5(q_1, q_2) = \pi u(q_2 x_0 - \alpha_1) + (1-\pi)v(\beta_2 x_0 + \beta_2(x_1 - \alpha_2)/q_2)$.

III. Competitive Equilibrium

Let $j = 1, \ldots, 5$ index the five candidate strategies in the previous section. Given the solution to the second stage of the consumer's problem, p_j is the fraction of consumers who choose solution j, as well as the probability of choosing j for an individual.

Definition. A competitive equilibrium is given by $\hat{q}_i$, $\hat{p}_j$, $i = 1, 2$, $j = 1, 2, \ldots, 5$, such that with $q_i = \hat{q}_i$, $p_j = \hat{p}_j$, $i = 1, 2$, $j = 1, 2, \ldots, 5$, (15)-(20) are satisfied.

(15) $\quad q_1 \, \pi(p_1 + p_4)x_0 = (1-\pi)(p_1 + p_2 + p_3)x_1$

(16) $\quad q_2 \, \pi(p_2 + p_3 + p_5)x_0 = (1-\pi)(p_4 + p_5)(x_1 - \alpha_2)$

(17) $\quad \sum_j p_j = 1$

(18) $\quad q_i \geq 0, \ p_j \geq 0, \ i = 1, 2, \ j = 1, 2, \ldots, 5.$

(19) $\quad$ If $p_j > 0$ and $p_k > 0$, then $U_j(q_1, q_2) = U_k(q_1, q_2)$

(20) $\quad$ If $p_j > 0$, then $U_j(q_1, q_2) \geq U_k(q_1, q_2)$ for all k.

Equations (15) and (16) are period 1 market-clearing conditions for assets 1 and 2, respectively. Condition (19) states that if two strategies are chosen with positive probability, then consumers are indifferent between the two. In condition (20), if a particular strategy is chosen with positive probability, then it must be weakly preferable to all other strategies.

Let $S = \{j : p_j > 0\}$. As a first step toward characterizing an equilibrium, we will determine what S cannot be in equilibrium.

Proposition 1. In equilibrium $q_i > 0$, $i = 1, 2$.

Proof: Suppose not. Then either $q_i = 0$, $i = 1, 2$, $q_1 > 0$ and $q_2 = 0$, or $q_1 = 0$ and $q_2 > 0$. In the first case, (15) and (16) imply that $p_j = 0$ for all j, which implies that (17) does not hold, a contradiction. In case two, (16) implies $p_4 = p_5 = 0$. Therefore, given (15) and (17), $p_1 > 0$. But if $q_1 > 0$ and $q_2 = 0$ then $U_1(q_1,q_2) < U_4(q_1,q_2)$. Therefore, (20) does not hold, a contradiction. In the third case, (15) implies that $p_1 = p_2 = p_3 = 0$. Then (16) and (17) imply that $p_5 > 0$. But if $q_2 > 0$ and $q_1 = 0$, then $U_2(q_1,q_2) > U_5(q_1,q_2)$ and (20) does not hold, a contradiction. ◊

Proposition 2. $3 \notin S$.

Proof: Suppose that $3 \in S$, i.e. $p_3 > 0$. Given (20), this implies that $U_2(q_1,q_2) \leq U_3(q_1,q_2)$. Therefore, $\beta_1/q_1 > \beta_2/q_2$. But then (20) implies that $p_4 = p_5 = 0$, which in turn implies that (16) does not hold, a contradiction. ◊

Proposition 3. $S \neq \{2\}$, $S \neq \{4\}$, $S \neq \{1,2\}$, $S \neq \{1,4\}$, $S \neq \{2,5\}$, $S \neq \{2,5\}$ $S \neq \{4,5\}$.

Proof: If $S = \{2\}$, $S = \{4\}$, $S = \{1,2\}$, $S = \{1,4\}$ $S = \{2,5\}$, or $S = \{4,5\}$, then proposition 1 implies that either (15) does not hold, or (16) does not hold, a contradiction. ◊

Proposition 4. $S \neq \{1,5\}$, $S \neq \{1,4,5\}$, $S \neq \{1,2,5\}$.

Proof: Suppose that $S = \{1,5\}$. Then, (20) implies that

(21) $(\beta_2/q_2 - \beta_1/q_1)x_1 \geq \beta_2\alpha_2/q_2$

and

(22) $(\beta_2/q_2 - \beta_1/q_1)(q_1 x_0 + x_1) \leq \beta_2 \alpha_2/q_2.$

But (21) implies that $\beta_2/q_2 - \beta_1 q_1 > 0$. Therefore, $(\beta_2/q_2 - \beta_1/q_1) \cdot (q_1 x_0 + x_1) > \beta_2 \alpha_2/q_2$, a contradiction. Now, suppose that $S = \{1,4,5\}$. Given (19) and (20), inequality (21) and the following equality must hold.

(23) $(\beta_2/q_2 - \beta_1/q_1)(q_1 x_0 + x_1) = \beta_2 \alpha_2/q_2.$

But (21) implies that $(\beta_2, q_2 - \beta_1/q_1)(q_1 x_0 + x_1) > \beta_2 \alpha_2/q_2$, a contradiction. Last, suppose that $S = \{1,2,5\}$. Then (22) must hold, in addition to:

(24) $(\beta_2/q_2 - \beta_1/q_1) x_1 = \beta_2 \alpha_2/q_2.$

But (24) implies that $(\beta_2/q_2 - \beta_1/q_1)(q_1 x_0 + x_1) > \beta_2 \alpha_2/q_2$, a contradiction. $\Diamond$

We are now left with the following possibilities in equilibrium

Case 1. $S = \{1\}$

Case 2. $S = \{5\}$

Case 3. $S = \{2,4\}$

Case 4. $S = \{1,2,4\}$

Case 5. $S = \{2,4,5\}$.

We will deal with each of these cases in turn.

Case 1 Equilibrium.

Here, $p_1 = 1$ and $p_j = 0$, $j = 2, \ldots, 5$. Therefore from (15), we get

(25) $q_1 = (1-\pi)x_1/\pi x_0.$

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From condition (20), $U_1(q_1,q_2) \geq U_2(q_1,q_2)$ or, substituting using (25):

(26) $\pi u\big((1-\pi)x_1/\pi\big) + (1-\pi)v\big(\beta_1 x_0/(1-\pi)\big) \geq \pi u(q_2 x_0 - \alpha_1)$

$$+(1-\pi)v\big(x_0\big[(1-\pi)\beta_2 + \pi\beta_1\big]/(1-\pi)\big).$$

Also, for $j = 1$, and $k = 4$ in condition (20), and substituting using (25);

(27) $\beta_1 x_0/(1-\pi) \geq (\beta_2/q_2)(x_1/\pi - \alpha_2)$.

Conditions (26) and (27) put upper and lower bounds, respectively, on q_2. If (26) and (27) hold then, using (25), we have

$$U_5(q_1,q_2) \leq \pi u\big((1-\pi)x_1/\pi\big) + (1-\pi)v\big(\beta_1 x_0/(1-\pi)\big)$$

$$- (1-\pi)V\big(x_0\big[(1-\pi)\beta_2 + \pi\beta_1\big]/(1-\pi)\big)$$

$$+ (1-\pi)v\big(\beta_2 x_1 + \pi(x_1-\alpha_2)\beta_1 x_0/(1-\pi)(x_2 - \pi\alpha_2)\big) < U_1(q_1,q_2),$$

and

$$U_3(q_1,q_2) \leq \pi u(q_2 x_0 - \alpha_1) + (1-\pi)V\big(\beta_1 x_0/(1-\pi)\big) < U_2(q_1,q_2).$$

Therefore, if (25) and (26) hold for some $q_2 > 0$, then (19) and (20) are satisfied. In addition, since $p_j = 0$, for $j = 2, 3, 4, 5$, therefore (16) holds. Thus, using (27) to substitute in (26), a case 1 equilibrium exists if and only if

(28) $\beta_2(x_1/\pi - \alpha_2)(1-\pi)/\beta_1 - \alpha_1 \leq 0$

or

(29) $\beta_2(x_1/\pi - \alpha_2)(1-\pi)/\beta_1 - \alpha_1 > 0$

and

(31) $\pi u\big((1-\pi)x_1/\pi\big) + (1-\pi)v\big(\beta_1 x_0/(1-\pi)\big) \geq \pi u\big(\beta_2(x_1/\pi-\alpha_2)(1-\pi)/\beta_1-\alpha_1\big)$

$$+ (1-\pi)v\big(\beta_2 x_0 + \beta_1 x_0 \pi/(1-\pi)\big).$$

If the case 1 equilibrium exists, it is essentially unique. That is, expected utility and the price of the traded asset are uniquely determined. However, the equilibrium price of asset 2, q_2, is in general not unique, though asset 2 is not traded. There remains the possibility that, if a case 1 equilibrium exists, it may not be unique if we allow for cases 2-5. However, we will prove uniqueness in what follows.

Case 2 Equilibrium.

In a Case 2 equilibrium, $p_5 = 1$ and $p_j = 0$ for $j = 1, 2, 3, 4$. From (16), the equilibrium price of asset 2 is then

(32) $q_2 = (1-\pi)(x_1-\alpha_2)/\pi x_0.$

From (20), with $j = 5$, and $k = 2$, and substituting using (32), the following must hold in equilibrium:

(33) $\beta_2 \pi x_0/(1-\pi) \geq \beta_1 x_1/q_1.$

Similarly, with $j = 5$ and $k = 4$ in (20), and substituting using (32),

(34) $\pi u\big((1-\pi)(x_1-\alpha_2)/\pi-\alpha_1\big) + (1-\pi)v\big(\beta_2 x_0/(1-\pi)\big) \geq \pi u(q_1 x_0)$

$$+ (1-\pi)v\big(\beta_2 \pi x_0(q_1 x_0 + x_1 - \alpha_1)/(1-\pi)(x_1-\alpha_2)\big).$$

Conditions (33) and (34) put lower and upper bounds, respectively, on q_1. If (33) holds then, given (32), $U_3(q_1,q_2) < U_5(q_1,q_2)$. In addition, (33) implies that

$$U_1(q_1,q_2) \leq \pi u(q_1 x_0) + (1-\pi)v\big(\beta_1 x_1 + \beta_2 \pi x_0/(1-\pi)\big)$$

and (33) and (34) imply that

$$U_5(q_1,q_2) \geq \pi u(q_1 x_0) + (1-\pi)V\big(\beta_1 x_1 x_0/(x_1-\alpha_2)+\beta_2 \pi x_0/(1-\pi)\big).$$

Therefore, (33) and (34) imply that $U_5(q_1,q_2) \geq U_1(q_1,q_2)$. Therefore, if (33) and (34) hold for some $q_1 > 0$, then a case 2 equilibrium exists. That is, substituting using (33) in (34), a case 2 equilibrium exists if and only if

(35) $\qquad (1-\pi)(x_1-\alpha_2)/\pi-\alpha_1 > 0$

and

(36) $\qquad \pi u\big((1-\pi)(x_1-\alpha_2)/\pi-\alpha_1\big) + (1-\pi)v\big(\beta_2 x_0/(1-\pi)\big) \geq \pi u\big(\beta_1 x_1(1-\pi)/\beta_2 \pi\big)$

$$+ (1-\pi)v\big(\beta_1 x_1 x_0/(x_1-\alpha_2)+\beta_2 \pi x_0/(1-\pi)\big).$$

If the Case 2 equilibrium exists, it is essentially unique, though q_1 is generally not uniquely determined.

Case 3 Equilibrium.

Here, $p_2 > 0$, $p_4 > 0$, and $p_1 = p_3 = p_5 = 0$. From (15) and (16), we get:

(37) $\qquad q_1 = (1-\pi)p_2 x_1/\pi p_4 x_0$

(38) $\qquad q_2 = (1-\pi)p_4(x_1-\alpha_2)/\pi p_2 x_0.$

Condition (19), with $j = 2$ and $k = 4$, gives

(39) $\qquad \pi u(q_2 x_0-\alpha_1) + (1-\pi)v(\beta_2 x_0+\beta_1 x_1/q_1) = \pi u(q_1 x_0)$

$$+ (1-\pi)v\big(\beta_2(q_1 x_0+x_1-\alpha_2)/q_2\big).$$

From (20), with $j = 2$ and $k = 5$,

(40) $\beta_1 x_1/q_1 \geq \beta_2(x_1-\alpha_2)/q_2,$

and with $j = 4$ and $k = 1$,

(41) $\beta_2(q_1 x_0 + x_1 - \alpha_2)/q_2 \geq \beta_1 x_0 + \beta_1 x_1/q_1.$

Condition (40) then implies that (20) holds for $j = 2$ and $k = 3$. From (37) and (38), we get

(42) $q_2 = \dfrac{(1-\pi)^2(x_1-\alpha_2)x_1}{\pi^2(x_0)^2 q_1}.$

Since (39) yields an upward-sloping schedule in the upper right-hand quadrant of the (q_1, q_2) plane, and (42) a downward-sloping schedule, and given the nature of these schedules, (39) and (42) yield a unique solution for q_1 and q_2. Given that $p_2 = 1 - p_4$, we can then use (37) or (38) to solve for p_2 and p_4. This solution is an equilibrium if and only if the values of q_1 and q_2 that solve (39) and (42) satisfy (40) and (41). Using (40) and (42) to substitute in (39), we get

(43) $\pi u\left((\beta_2/\beta_1)^{\frac{1}{2}}(1-\pi)(x_1-\alpha_2)/\pi-\alpha_1\right) + (1-\pi)v\left(\beta_2 x_0+(\beta_1\beta_2)^{\frac{1}{2}}\pi x_0/(1-\pi)\right)$

$\leq \pi u\left((\beta_1/\beta_2)^{\frac{1}{2}}(1-\pi)x_1/\pi\right) + (1-\pi)v\left(\beta_1 x_0 x_1/(x_1-\alpha_2)+(\beta_1\beta_2)^{\frac{1}{2}}\pi x_0/(1-\pi)\right).$

The solution to (39) and (42) satisfies (40) if and only if (43) holds. If (41) is satisfied with equality, then

(44) $q_2 = \beta_2(q_1 x_0 + x_1 - \alpha_2)q_1/\beta_1(q_1 x_0 + x_1).$

Using (44) to substitute in (42), we get

(45) $\beta_2(q_1 x_0 + x_1 - \alpha_2)q_1/\beta_1(q_1 x_0 + x_1) = (1-\pi)^2(x_1-\alpha_2)x_1/\pi^2(x_0)^2 q_1.$

The solution to (39) and (42) satisfies (41) if and only if the unique solution for q_1 and q_2 to (44) and (45) satisfies

$$(46) \qquad \pi u(q_2 x_0 - \alpha_1) + (1-\pi)v(\beta_2 x_0 + \beta_1 x_1/q_1) \le \pi u(q_1 x_0)$$

$$+ (1-\pi)v\big(\beta_2(q_1 x_0 + x_1 - \alpha_2)/q_2\big).$$

Therefore the Case 3 equilibrium exists if and only if (43) is satisfied and the solution to (44) and (45) satisfies (46). If the Case 3 equilibrium exists, it is unique.

Case 4 Equilibrium.

Here, $p_1 > 0$, $p_2 > 0$, $p_4 > 0$, and $p_3 = p_5 = 0$. From (15) and (16), we get

$$(47) \qquad q_1 = (1-\pi)(p_1+p_2)x_1/\pi(p_1+p_4)x_0$$

and

$$(48) \qquad q_2 = (1-\pi)p_4(x_1-\alpha_2)/\pi p_2 x_0.$$

Condition (20) implies that $U_1(q_1,q_2) = U_4(q_1,q_2)$ and $U_1(q_1,q_2) = U_2(q_1,q_2)$, that is

$$(49) \qquad \beta_2(q_1 x_0 + x_1 - \alpha_2)/q_2 = \beta_1 x_0 + \beta_1 x_1/q_1$$

and

$$(50) \qquad \pi u(q_1 x_0) + (1-\pi)v(\beta_1 x_0 + \beta_1 x_1/q_1) = \pi u(q_2 x_0 - \alpha_1)$$

$$+ (1-\pi)v(\beta_2 x_0 + \beta_1 x_1/q_1).$$

Using (49) to substitute for q_2 in (50), we get

(51) $\pi u(q_1 x_0) - \pi u(\beta_2(q_1 x_0 + x_1 - \alpha_2)q_1 x_0 / \beta_1 (q_1 x_0 + x_1) - \alpha_1)$

$$= -(1-\pi)v(\beta_1 x_0 + \beta_1 x_1 / q_1) + (1-\pi)v(\beta_2 x_0 + \beta_1 x_1 / q_1).$$

Equation (49) implies that $U_2(q_1, q_2) > U_3(q_1, q_2)$ and $U_2(q_1, q_2) > U_5(q_1, q_2)$. Therefore, if (49) and (50) hold, then (20) is satisfied. Given that $p_1 = 1 - p_2 - p_4$, equations (47) and (48) yield unique solutions, given q_1 and q_2, for p_2 and p_4, as follows:

(52) $$p_2 = \frac{(1-\pi)(x_1 - \alpha_2)\left[\pi x_0 / (1-\pi)x_1 - 1/q_1\right]}{\pi q_2 x_0 \left[(x_1 - \alpha_2)/q_2 x_1 - 1/q_1\right]}.$$

(53) $$p_4 = \frac{\pi x_0 / (1-\pi)x_1 - 1/q_1}{(x_1 - \alpha_2)/q_2 x_1 - 1/q_1}.$$

An equilibrium must satisfy $p_2 > 0$, $p_4 > 0$, and $p_2 + p_4 < 1$. That is, from (49), (52), and (53), we get

(54) $q_1 < (1-\pi)x_1 / \pi_0$

(55) $$(q_1)^2 > \frac{(1-\pi)^2(q_1 x_0 + x_1)x_1 \beta_1 (x_1 - \alpha_2)}{\pi^2 (q_1 x_0 + x_1 - \alpha_2)(x_0)^2 \beta_2}.$$

An equilibrium exists if and only if there is a q_1 satisfying (51), (54), and (55). Inequality (54) puts an upper bound on q_1, while (55) establishes a lower bound.

Differentiating the left-hand side of (51) with respect to q_1, we get

(56) $\quad \frac{\partial}{\partial q_1} \left[\pi u(q_1 x_0) - \pi u \left(\beta_2(q_1 x_0 + x_1 - \alpha_2) q_1 x_0 / \beta_1 (q_1 x_0 + x_1) - \alpha_1 \right) \right]$

$$= \pi x_0 \, u'(q_1 x_0) - \frac{\beta_2}{\beta_1} \left[1 - \frac{\alpha_2 x_1}{(q_1 x_0 + x_1)^2} \right]$$

$$\times u' \left(\beta_2(q_1 x_0 + x_1 - \alpha_2) q_1 x_0 / \beta_1 (q_1 x_0 + x_1) - \alpha_1 \right).$$

Similarly, differentiating with respect to q_1 on the right-hand side of (51) gives

(57) $\quad \frac{\partial}{\partial q_1} \left[v(\beta_1 x_0 + \beta_1 / q_1) - v(\beta_2 x_0 + \beta_1 x_1 / q_1) \right] = \frac{\beta_1 x_1}{(q_1)^2} \left[v'(\beta_1 x_0 + \beta_1 x_1 / q_1) \right.$

$$\left. - v'(\beta_2 x_0 + \beta_1 x_1 / q_1) \right] > 0.$$

If an equilibrium exists then, from (51),

(58) $\quad \beta_2(q_1 x_0 + x_1 - \alpha_2) q_1 x_0 / \beta_1 (q_1 x_0 + x_1) - \alpha_1 < q_1 x_0.$

Then if, on the one hand, $(\beta_2/\beta_1)[1 - \alpha_2 x_1 / (q_1 x_0 + x_1)^2] \geq 1$ in equilibrium, and given (58) and the concavity of $u(\cdot)$, the right-hand side of (56) is negative in equilibrium. On the other hand, if $\beta_2[1 - \alpha_2 x_1 / (q_1 x_0 + x_1)^2]/\beta_1 < 1$ in equilibrium, then

$$(\beta_2/\beta_1)[1 - \alpha_2 x_1 / (q_1 x_0 + x_1)^2] \, u'\left(\frac{(\beta_2/\beta_1)(q_1 x_0 + x_1 - \alpha_2) q_1 x_0}{(q_1 x_0 + x_1)} - \alpha_1 \right)$$

$$> \frac{\beta_2(q_1 x_0 + x_1 - \alpha_2)}{\beta_1(q_1 x_0 + x_1)} \, u'\left(\frac{\beta_2(q_1 x_0 + x_1 - \alpha_2) q_1 x_0}{\beta_1(q_1 x_0 + x_1)} - \alpha_1 \right)$$

$$> \frac{\beta_2(q_1 x_0 + x_1 - \alpha_2)}{\beta_1(q_1 x_0 + x_1)} \, u'\left(\frac{\beta_2(q_1 x_0 + x_1 - \alpha_2) q_1 x_0}{\beta_1(q_1 x_0 + x_1)} \right) > u'(q_1 x_0).$$

Here, the last inequality follows from the fact that $-cu''(c)/u'(c) \geq 1$. Therefore, the right-hand side of (56) is negative in equilibrium, which implies, given (57), that the Case 4 equilibria is unique if it exists.

Using (51), (54), and (55), a Case 4 equilibrium exists if and only if

$$(59) \quad \pi u\big((1-\pi)x_1/\pi\big) + (1-\pi)v\big(\beta_1 x_0/(1-\pi)\big) < \pi u\big(\beta_2(1-\pi)(x_1-\pi\alpha_2)/\beta_1\pi-\alpha_1\big)$$
$$+ (1-\pi)v\big(\beta_2 x_0+\beta_1 x_0\pi/(1-\pi)\big),$$

and the value of q_1 which solves

$$(60) \quad (q_1)^2 = \frac{(1-\pi)^2(q_1 x_0+x_1)x_1\beta_1(x_1-\alpha_2)}{\pi_2(q_1 x_0+x_1-\alpha_2)(x_0)^2\beta_2}$$

also satisfies

$$(61) \quad \pi u(q_1 x_0) + (1-\pi)v(\beta_1 x_0+\beta_1 x_1/q_1) > \pi u\,\frac{\beta_2(q_1 x_0+x_1-\alpha_2)q_1 x_0}{\beta_1(q_1 x_0+x_1)} - \alpha_1$$
$$+ (1-\pi)v(\beta_2 x_0+\beta_1 x_1/q_1).$$

Case 5 Equilibrium.

In a Case 5 equilibrium, $p_2 > 0$, $p_4 > 0$, $p_5 > 0$, and $p_1 = p_3 = 0$. From (15) and (16),

$$(62) \quad q_1 = (1-\pi)p_2 x_1/\pi p_4 x_0$$

$$(63) \quad q_2 = (1-\pi)(p_4+p_5)(x_1-\alpha_2)/\pi(p_2+p_5)x_0.$$

Condition (20) implies that $U_2(q_1,q_2) = U_5(q_1,q_2)$, and that $U_2(q_1,q_2) = U_4(q_1,q_2)$. That is,

$$(64) \quad q_2 = \beta_2(x_1-\alpha_2)q_1/\beta_1 x_1,$$

and

(65) $\pi u(q_2 x_0 - \alpha_1) + (1-\pi)v(\beta_2 x_0 + \beta_1 x_1/q_1) = \pi u(q_1 x_0)$

$$+ (1-\pi)v(\beta_2(q_1 x_0 + x_1 - \alpha_2)/q_2).$$

Equations (64) and (65) imply that $U_3(q_1,q_2) < U_2(q_1,q_2)$ and $U_1(q_1,q_2) < U_4(q_1,q_2)$. Therefore (20) holds. Substituting in (65) using (64), we get

(66) $\pi u(\beta_2(x_1 - \alpha_2)x_0 q_1/\beta_1 x_1 - \alpha_1) - \pi u(q_1 x_0)$

$$= (1-\pi)v(\beta_1 x_1 x_0/(x_1 - \alpha_2) + \beta_1 x_1/q_1) - (1-\pi)v(\beta_2 x_0 + \beta_1 x_1/q_1).$$

From (66), the following is a necessary condition for a Case 5 equilibrium to exist:

(67) $\beta_2 > \beta_1 x_1/(x_1 - \alpha_2).$

If (67) holds, we then have, differentiating the left-hand and right-hand sides of (66), respectively,

(68) $\dfrac{\partial}{\partial q_1} \left[\pi u(\beta_2(x_1 - \alpha_2)x_0 q_1/\beta_1 x_1 - \alpha_1) - \pi u(q_1 x_0) \right]$

$$= \pi x_0 \left\{ \frac{\beta_2(x_1 - \alpha_2)}{\beta_1 x_1} u'\left(\frac{\beta_2(x_1 - \alpha_2)x_0 q_1}{\beta_1 x_1} - \alpha_1 \right) - u'(q_1 x_0) \right\} > 0,$$

in equilibrium, and

(69) $\dfrac{\partial}{\partial q_1} \left[(1-\pi)v(\beta_1 x_1 x_0/(x_1 - \alpha_2) + \beta_1 x_1/q_1) - (1-\pi)v(\beta_2 x_0 + \beta_1 x_1/q_1) \right]$

$$= \frac{(1-\pi)\beta_1}{(q_1)^2} \left[v'(\beta_2 x_0 + \beta_1 x_1/q_1) - v'\frac{\beta_1 x_1 x_0}{x_1 - \alpha_2} + \beta_1 x_1/q_1 \right] > 0.$$

From (17), (62), (63), and (64), we get

(70) $p_2 = \dfrac{\pi x_0 \beta_2 q_1/(1-\pi)x_1 \beta_1 - 1}{\beta_2/\beta_1 - 1}$

and

$$(71) \qquad p_4 = \frac{\beta_2/\beta_1 - (1-\pi)x_1/\pi x_0 q_1}{\beta_2/\beta_1 - 1}.$$

For $p_2 > 0$, $p_4 > 0$, and $p_2 + p_4 < 1$, the solution to (66) must satisfy

$$(72) \qquad \beta_1(1-\pi)x_1/\beta_2\pi x_0 < q_1 < \beta_1^{\frac{1}{2}}(1-\pi)x_1/\beta_2^{\frac{1}{2}}\pi x_0.$$

Note that, given (64), (68), (69), (70), and (71), a Case 5 equilibrium is unique if it exists. From (66) and (72), given (68) and (69), necessary and sufficient conditions for the existence of a Case 4 equilibrium are that

$$(73) \qquad \pi u\big((1-\pi)(x_1-\alpha_2)/\pi-\alpha_1\big) + (1-\pi)v\big(\beta_2 x_0/(1-\pi)\big) < \pi u\big(\beta_1(1-\pi)x_1/\beta_2\pi\big)$$

$$+ (1-\pi)v\big(\beta_1 x_1 x_0/(x_1-\alpha_2)+\beta_2\pi x_0/(1-\pi)\big)$$

and

$$(74) \qquad \pi u\big(\beta_2^{\frac{1}{2}}(1-\pi)(x_1-\alpha_2)/\beta_1^{\frac{1}{2}}\pi-\alpha_1\big) + (1-\pi)v\big(\beta_2 x_0+\beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}}\pi x_0/(1-\pi)\big)$$

$$> \pi u\big(\beta_1^{\frac{1}{2}}(1-\pi)x_1/\pi\beta_2^{\frac{1}{2}}\big) + (1-\pi)v\big(\beta_1 x_1 x_0/(x_1-\alpha_2)+\beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}}\pi x_0/(1-\pi)\big).$$

Existence and Uniqueness of Equilibrium

So far, we have established necessary and sufficient conditions for particular equilibria to exist (i.e., Cases 1 through 5) and have shown that at most one equilibrium of a particular type can exist. It remains to be established that the equilibrium is unique, and that an equilibrium exists at each point in the parameter space.

Proposition 4. If an equilibrium exists, then it is unique.

Proof: We have shown that at most one equilibrium of a particular type can exist, and will proceed to prove that, working pairwise, no two types of

equilibria can co-exist. There are 5!/3!2! = 10 pairs to consider. Let
(i,j), where i, j = 1, ..., 5, denote the case i/case j pair. First, it is
clear from (31), (36), (43)-(46), (59)-(61), (73), and (74), that (1,4),
(2,5), (3,4), and (3,5) can be ruled out. To eliminate the other cases re-
quires some manipulation of the inequalities that are necessary and sufficient
conditions for each type of equilibrium. First,

(75) $\pi u\big((1-\pi)x_1/\pi\big) + (1-\pi)v\big(\beta_1 x_0/(1-\pi)\big)$

$- \pi u\big((\beta_2/\beta_1)[(1-\pi)x_1/\pi-(1-\pi)\alpha_2]-\alpha_1\big) - (1-\pi)v\big(\beta_2 x_0+\beta_1 x_0\pi/(1-\pi)\big)$

$< \pi u\big(\beta_1^{\frac{1}{2}}(1-\pi)x_1/\beta_2^{\frac{1}{2}}\pi\big) + (1-\pi)\,v\big(\beta_1 x_0+\beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}}\pi x_0/(1-\pi)\big)$

$- (1-\pi)v\big(\beta_2 x_0+\beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}}x_0\pi/(1-\pi)\big) < \pi u\big((\beta_1^{\frac{1}{2}}/\beta_2^{\frac{1}{2}})(1-\pi)x_1/\pi\big)$

$- \pi u\big((\beta_2^{\frac{1}{2}}/\beta_1^{\frac{1}{2}})(1-\pi)(x_1-\alpha_2)/\pi-\alpha_1\big) - (1-\pi)v\big(\beta_2 x_0+\beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}}\pi/(1-\pi)\big).$

In (75), the first inequality follows from $-cu''(c)/u'(c) \geq 1$ and $v'' < 0$.
Given $u' > 0$ and $v' > 0$, we get the last inequality. From (31) and (74), this
rules out (1,5). Next

(76) $\pi u\big(\beta_1^{\frac{1}{2}}(1-\pi)x_1/\pi\beta_2^{\frac{1}{2}}\big) + (1-\pi)v\big(\beta_1 x_0 x_1/(x_1-\alpha_2)+\beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}}\pi x_0/(1-\pi)\big)$

$- \pi u\big(\beta_2^{\frac{1}{2}}(1-\pi)(x_1-\alpha_2)/\pi\beta_1^{\frac{1}{2}}-\alpha_1\big) - (1-\pi)v\big(\beta_2 x_0+\beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}}\pi x_0/(1-\pi)\big)$

$< \pi u\big(\beta_1(1-\pi)x_1/\pi\beta_2\big) + (1-\pi)v\big(\beta_1 x_0 x_1/(x_1-\alpha_2)+\beta_2\pi x_0/(1-\pi)\big)$

$- \pi u\big((1-\pi)(x_1-\alpha_2)/\pi-\alpha_1\beta_1^{\frac{1}{2}}/\beta_2^{\frac{1}{2}}\big) - (1-\pi)v\big(\beta_2 x_0/(1-\pi)\big)$

$< \pi u\big(\beta_1(1-\pi)x_1/\pi\beta_2\big) + (1-\pi)v\big(\beta_1 x_0 x_1/(x_1-\alpha_2)+\beta_2\pi x_0/(1-\pi 0)$

$- \pi u\big((1-\pi)(x_1-\alpha_2)/\pi-\alpha_1\big) - (1-\pi)v\big(\beta_2 x_0/(1-\pi)\big).$

We use the fact that $-cu''(c)/u'(c) \geq 1$ and $v'' > 0$ to get the first inequality in (76), and the second inequality follows from $u' > 0$. Given (31), (36), (43), and (74), conditions (75) and (76) imply that we can rule out (1,2) and (2,3).

Let $\hat{q}_1$ be defined by

$$(77) \qquad (\hat{q}_1)^2 = \frac{(1-\pi)^2(\hat{q}_1 x_0 + x_1)x_1 \beta_1(x_1 - \alpha_2)}{\pi^2(\hat{q}_1 x_0 + x_1 - \alpha_2)(x_0)^2 \beta_2}.$$

Note that $(1-\pi)x_1/\pi x_0 > \hat{q}_1$. If

$$(78) \qquad \pi u\big((1-\pi)x_1/\pi\big) + (1-\pi)v\big(\beta_1 x_0/(1-\pi)\big)$$

$$- \pi u\big((\beta_2/\beta_1)\big[(1-\pi)x_1/\pi - (1-\pi)\alpha_2\big] - \alpha_1\big)$$

$$- (1-\pi)V\big(\beta_2 x_0 + \beta_1 x_0 \pi/(1-\pi)\big) \geq 0,$$

therefore $(\beta_2/\beta_1)\big[(1-\pi)x_1/\pi - (1-\pi)\alpha_2\big] - \alpha_1 < (1-\pi)x_1/\pi$. Condition (78) then implies that

$$(79) \qquad \pi u\big((1-\pi)x_1/\pi\big) + (1-\pi)v\big(\beta_1 x_0/(1-\pi)\big)$$

$$- \pi u\big((\beta_2/\beta_1)\big[(1-\pi)x_1/\pi - (1-\pi)\alpha_2\big] - \alpha_1\big) - (1-\pi)v\big(\beta_2 x_0 + \beta_1 x_0 \pi/(1-\pi)\big)$$

$$- \pi u\big(\beta_2(\hat{q}_1 x_0 + x_1 - \alpha_2)\hat{q}_1 x_0/\beta_1(\hat{q}_1 x_0 + x_1) - \alpha_1\big) - (1-\pi)v\big(\beta_2 x_0 + \beta_1 x_1/\hat{q}_1\big),$$

which follows from $\hat{q}_1 < (1-\pi)x_1/\pi x_0$, $-cu''(c)/u'(c) \geq 1$ and $v'' < 0$. We showed in the previous section that there exists a unique solution q_1^* to

$$F(q_1^*) = \pi u(q_1^* x_0) + (1-\pi)V(\beta_1 x_0 + \beta_1 x_1/q_1^*)$$

$$- \pi u\big(\beta_2(q_1^* + x_1 - \alpha_2)q_1^* x_0/\beta_1(q_1^* x_0 + x_1) - \alpha_1\big) - (1-\pi)v\big(\beta_2 x_0 + \beta_1 x_1/q_1^*\big) = 0.$$

If

$$(80) \qquad \pi u(\hat{q}_1 x_0) + (1-\pi)v(\beta_1 x_0 + \beta_1 x_0/\hat{q}_1)$$

$$- \pi u\big(\beta_2(\hat{q}_1 x_0 + x_1 - \alpha_2)\hat{q}_1 x_0/\beta_1(\hat{q}_1 x_0 + x_1) - \alpha_1\big)$$

$$- (1-\pi)v(\beta_2 x_0 + \beta_1 x_1/\hat{q}_1) < 0,$$

then $q_1^* < \hat{q}_1 < (1-\pi)x_1/\pi x_0$. Therefore,

$$(81) \qquad \pi u\big((1-\pi)x_1/\pi\big) + (1-\pi)v\big(\beta_1 x_0/(1-\pi)\big)$$

$$- \pi u\big((\beta_2/\beta_1)[(1-\pi)x_1/\pi - (1-\pi)\alpha_2] - \alpha_1\big)$$

$$- (1-\pi)v\big(\beta_2 x_0 + \beta_1 x_0 \pi/(1-\pi)\big) < 0.$$

Therefore, since (78) implies (79) and (80) implies (81), we can rule out (1,3). Next, let $\hat{q}_1 = (1-\pi)x_1 \beta_1^{\frac{1}{2}}(x_1 - \alpha_2)^\gamma/\pi x_0 \beta_2^{\frac{1}{2}} x_1^\gamma$. From (77), γ is the unique solution to

$$(82) \qquad \big[(1-\pi)\theta x_1 + \pi(x_1 - \alpha_2)\big]/\big[(1-\pi)\theta + \pi\big] = (x_1 - \alpha_2)^{1-2\gamma} x_1^{2\gamma},$$

where

$$\theta = (\beta_1^{\frac{1}{2}}/\beta_2^{\frac{1}{2}})\big[(x_1 - \alpha_2)/x_1\big]^\gamma.$$

Note that $0 < \gamma < 1/2$. Now,

$$(83) \qquad \pi u(\hat{q}_1 x_0) + (1-\pi)V(\beta_1 x_0 + \beta_1 x_1/\hat{q}_1)$$

$$- \pi u\big(\beta_2(\hat{q}_1 x_0 + x_1 - \alpha_2)\hat{q}_1 x_0/\beta_1(\hat{q}_1 x_0 + x_1) - \alpha_1\big) - (1-\pi)V(\beta_2 x_0 + \beta_1 x_1/\hat{q}_1)$$

$$= \pi u\big([(1-\pi)x_1/\pi](\beta_1^{\frac{1}{2}}/\beta_2^{\frac{1}{2}})(x_1 - \alpha_2)/x_1^{\hat{\gamma}}\big)$$

$$+ (1-\pi)v\big(\beta_1 x_0 + \beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}} x_0 \pi x_1^{\hat{\gamma}}/(1-\pi)(x_1 - \alpha_2)^\gamma\big)$$

$$- \pi u\left(\left[\beta_2^{\frac{1}{2}}\beta_1^{\frac{1}{2}}(1-\pi)/\pi\right]\left[(x_1-\alpha_2)/x_1\right]^\gamma\left[\hat\theta(1-\pi)x_1+\pi x_1-\pi\alpha_2\right]/\left[\hat\theta(1-\pi)+\pi\right]-\alpha_1\right)$$

$$- (1-\pi)v\left(\beta_2 x_0+\beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}}x_0\pi x_1^\gamma/(1-\pi)(x_1-\alpha_2)^\gamma\right) < \pi u\left((1-\pi)x_1\beta_1^{\frac{1}{2}}/\pi\beta_2^{\frac{1}{2}}\right)$$

$$+ (1-\pi)v\left(\beta_1 x_0+\beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}}x_0\pi x_1^\gamma/(1-\pi)(x_1-\alpha_2)^\gamma\right)$$

$$- \pi u\left(\left[\beta_2^{\frac{1}{2}}(1-\pi)(x_1-\alpha_2)^\gamma/\beta_1^{\frac{1}{2}}\pi x_1^\gamma\right]\left(\hat\theta(1-\pi)x_1+\pi x_1-\pi\alpha_2\right)/\left[\hat\theta(1-\pi)+\pi\right]-\alpha_1\right)$$

$$- (1-\pi)v\left(\beta_2 x_0+\beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}}x_0\pi/(1-\pi)\right) < \pi u\left((1-\pi)x_1\beta_1^{\frac{1}{2}}/\pi\beta_2^{\frac{1}{2}}\right)$$

$$+ (1-\pi)v\left(\beta_1 x_0 x_1/(x_1-\alpha_2)+\beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}}\pi x_0/(1-\pi)\right)$$

$$- \pi u\left(\beta_2^{\frac{1}{2}}(1-\pi)(x_1-\alpha_2)/\pi\beta_1^{\frac{1}{2}}-\alpha_1\right)- (1-\pi)v\left(\beta_2 x_0+\beta_1^{\frac{1}{2}}\beta_2^{\frac{1}{2}}x_0\pi/(1-\pi)\right).$$

The first inequality follows from $(x_1-\alpha_2)^\gamma/\hat x_1^\gamma < 1$, $u' > 0$, and $v' > 0$. Since (82) implies that

$$\frac{\theta(1-\pi)x_1+\pi x_1-\pi\alpha_2}{\hat\theta(1-\pi)+\pi} > (x_1-\alpha_2)^{1-\gamma}x_1^\gamma,$$

and

$$\frac{\theta(1\pi)x_1+\pi(x_1-\alpha_2)^{1+\gamma}/x_1^\gamma}{\hat\theta(1-\pi)+\pi} > x_1-\alpha_2,$$

we get the second inequality, given $u' > 0$ and $v' > 0$.

Conditions (76) and (82), along with (60), (61), (74), and (36), then imply that we can rule out (2,4) and (4,5). Having ruled out all possible pairs, this completes the proof. ◊

The proof of Proposition 4 should make it clear that an equilibrium exists for any $(\beta_1,\beta_2,\alpha_1,\alpha_2,x_0,x_1,\pi)$, where $\beta_1 > 0$, $\beta_2 > 0$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $x_0 > 0$, $x_1 > 0$, $0 < \pi < 1$. Also, note that the subset of the parameter space where a case i equilibrium exists is nonempty for i = 1, 2, ..., 5.

Consider the following example. Let $u(c_1) = \ln c_1$, $v(c_2) = \ln c_2$, $x_0 = x_1 = 1$, $\pi = 0.5$, $\beta_1 = 1$, and $\alpha_1 = \alpha_2 = \alpha$. Regions in the (α, β_2) plane where particular types of equilibria exist were computed and plotted in Diagram 1. Note that as α decreases and β_2 increases, making asset 2 more attractive, that we move in the diagram from Case 1 to Case 4 to Case 3 to Case 5 to Case 2 equilibrium. The region over which both assets are traded is of primary importance for our purposes, since it is here where multiple transactions media are used in equilibrium.

IV. Examples

In the following example, we will let $u(c_1) = \ln c_1$, $v(c_2) = \ln c_2$, $x_0 = x_1 = 1$, $\pi = .5$, $\beta_1 = 1$, and $\alpha_1 = \alpha_2 = \alpha$. In the first example, $\beta_2 = 1.05$, and equilibria are computed for different values of α. Next, setting $\alpha = 0.05$, β_2 is allowed to vary, and the resulting equilibria are computed in Example II.

Example I.

Tables I and II show the computed equilibria when $\beta_2 = 1.05$ and α varies. Here c_i is aggregate consumption in period i and w is the expected utility (in period zero) of the representative consumer (an increasing function of expected utility is tabulated).

For $\alpha \geq 0$, the equilibrium takes on the following characteristics:

$0 \leq \alpha \leq 0.0285$: Case 2

$0.0285 < \alpha < 0.0288$: Case 5

$0.0288 \leq \alpha \leq 0.0470$: Case 3

$0.0470 < \alpha < 0.0488$: Case 4

$\alpha \geq 0.0488$: Case 1.

In Tables I and II, note that all equilibrium variables appear to be contin-uous functions of α. For small α, all consumers follow strategy 5, and asset 2 is the only asset produced and traded. As α increases, increasing fractions of consumers follow strategies 2 and 4 (i.e., asset 1 begins to be produced and traded) and the fraction following strategy 5 falls to zero. With in-creasing α, some consumers then begin taking strategy 1, and the fraction following strategies 2 and 4 fall to zero, at which point asset 2 is not produced or traded.

The region of interest is $0.0285 < \alpha < 0.0488$, i.e., Cases 5, 3 and 4, where both assets are produced and traded. In Tables I and II, note that prices and quantities are much more sensitive to changes in α for α in the intervals $(0.0285, 0.0288)$ (Case 5) and $(0.0470, 0.0488)$ (Case 4), than in the interval $(0.0288, 0.0470)$ (Case 3). For example, in the interval $(0.0470, 0.0488)$, a small increase in α causes a relatively large decrease in the size of the banking sector (i.e., p_2+p_5), a large decrease in period 2 consumption (because of the substitution of asset 1 for asset 2) and a large increase in period 1 consumption, as fewer resources are absorbed in carrying out transac-tions.

Of particular interest is the fact that expected utility is not monotonic in α. For example, if $\alpha \in (0.0470, 0.0488)$, an increase in α makes consumers better off (see Table I), though it leads to a reduction in measured output in period 2 and no change in period 1 output, and causes the banking sector to shrink. Note that measured output in period 1 consists of consump-tion of final goods plus the imputed value of transactions services, and is therefore always equal to the period 1 endowment. The non-monotonicity in expected utility is perhaps not surprising in light of what is known about the welfare effects of arbitrarily adding or dropping markets in an incomplete

market setting (see Hart, 1975, for example). Here, complete contingent claims markets are absent, due to spatial separation, and market structure is determined endogenously. However, due to the departures from a frictionless Arrow-Debreu setting, we should not necessarily expect agents to be better off as we push out the production possibilities frontier.

Example II.

In this example we retain the same functional forms and parameter values as in Example I, except that $\alpha = 0.05$ and equilibria are computed for different values of β_2. Results are displayed in Tables III and IV. Here, equilibria have the following characteristics:

$$\beta_2 \leq 1.0513: \quad \text{Case 1}$$

$$1.0513 < \beta_2 \ 1.0533: \quad \text{Case 4}$$

$$1.0533 \leq \beta_2 \leq 1.0914: \quad \text{Case 3}$$

$$1.0844 < \beta_2 < 1.0914: \quad \text{Case 5}$$

$$\beta_2 \leq 1.0914: \quad \text{Case 2.}$$

Tables III and IV yield some of the same results as Tables I and II. In particular, prices and quantities are much more sensitive to changes in β_2 in a Case 4 or a Case 5 equilibrium than in a Case 3 equilibrium. A small decrease in β_2 in a Case 4 equilibrium causes a large decrease in measured output and the fraction of transactions carried out using the banking system. As in the previous example, expected utility is not monotonic in β_2. For example, from Table III, expected utility increases with β_2 in a Case 3 equilibrium but decreases as β_2 increases in cases 4 and 5, in spite of the fact that measured output increases with β_2 over both of these intervals. Case 4 and Case 5 equilibria are characterized by large changes in the size of

the banking system in response to changes in underlying parameters. Thus, a parameter change that leads to an increase in second period consumption because of the higher returns available from investments made by banks, also causes a fall in first period consumption due to the higher costs of transacting in bank liabilities. Over the Case 4 and Case 5 regions, this increase in transactions costs has a more than offsetting effect on expected utility.

Remarks

Suppose an economy where, every three periods, the environment analyzed in the previous sections is replicated. That is, in each period $t = 1, 4, 7, \ldots$, a new population of agents is born, who have lifetimes as specified in Section II and interact in the same manner. Thus, there is no intergenerational trade. Also, suppose that in each period $t = 1, 4, 7, \ldots$, a new set of parameters is drawn from a probability distribution. For simplicity, fix x_1, x_2, β_1, and r, let $\alpha_1 = \alpha_2 = \alpha$ (as in the examples), and let (β_2, α) follow a stochastic process such that there is positive trend growth in β_2 and negative trend growth in α.

From the examples in the previous section, we would observe this economy going through an early stage where there is no growth and where asset 1 is the only means of payment. This stage is followed by one where there is rapid growth in measured output and in the size of the banking sector, followed by a period where output growth slows and the relative size of the banking sector stabilizes. This appears to be typical of the manner in which development occurred in many industrialized economies (see Cameron 1967). However, in spite of large increases in output and banking activity in the early stages of development, it may not be the case that there is an immediate welfare improvement (see the examples).

In the early stages of development of the banking system, the economy will be relatively sensitive to shocks in underlying parameters. Small negative shocks can cause large reductions in output which coincide with increases in the quantity of asset 1 produced and traded relative to asset 2, and with increases in the prices of transactions media (i.e., deflations, see Tables I and II). In more advanced stages of development (Case 3 equilibria), fluctuations in fundamentals cause relatively small fluctuations in output and have little effect on the relative size of the banking sector.

These characteristics of the model are reminiscent of the recurrent banking crises which occurred prior to and during the Great Depression in the United States. These periods were characterized by large reductions in banking activity, decreases in output, and increases in the ratio of currency to bank deposit liabilities (see Friedman and Schwartz 1963 and Cagan 1965). However, financial crises have been virtually absent in the post-Great Depression U.S. economy. This development is conventionally viewed as being the result of the introduction of government-provided deposit insurance which acted to prevent bank runs (see Friedman and Schwartz 1963) and/or it is attributed to more appropriate behavior by the monetary authority. The bank runs model of Diamond and Dybvig (1983) is consistent with these conventional views.

Alternatively, in our model, the difference between pre-1930's and post-1930's U.S. macroeconomic behavior can be seen as consistent with the natural evolution of the banking system. This does not contradict the views of Friedman and Schwartz (1963) and Hamilton (1987), for example, who argue that monetary policy accentuated the downturn in the Great Depression.

Summary and Conclusions

In this paper, a model was constructed where banks provide access to a communications system which permits the trading of goods for assets. Trade can also be carried out using alternative means of payment. In equilibrium, there is in general endogenous heterogeneity of agents, in that different agents may follow different contingent asset-holding strategies. Depending on parameter values, there may be no banking activity, all transactions may be carried out through the banking system, or bank liabilities and alternative means of payment may coexist.

The model's predictions are consistent with studies of the role of banking in the stages of economic development (e.g., Cameron 1967). In particular, in early stages rapid growth is accompanied by increases in banking's share in transactions activity, while in later stages of development, growth in output levels off, as does the relative quantity of banking activity. In addition, the model predicts that, when deposit banking is in its infancy, small technological fluctuations will cause recurring periods with large reductions in output accompanied by increases in the use of alternative means of payment relative to deposit liabilities. At higher stages of development, technology shocks on the same order of magnitude cause only small changes in prices and quantities.

This model can thus reconcile the differences between macroeconomic behavior in the U.S. prior to the 1930s, on the one hand, and following the 1930s, on the other. This reconciliation is brought about without relying on conventional wisdom concerning the role of government deposit insurance and central bank behavior in stemming financial panics.

Footnote

[1]In this model, diversification can potentially complicate the analysis substantially. However, there are a wide range of circumstances under which agents would not diversify, even if they could.

Diagram I

Equilibria when $u(c_1) = lnc_1$, $v(c_2) = lnc_2$, $\beta_1 = 1$, $x_1 = x_2 = 1$, $\alpha_1 = \alpha_2 = \alpha$, $\pi = .5$

CASE 3
(Strategies 2 and 4)

CASE 5
(Strategies 2, 4 and 5)

CASE 4
(Strategies 1, 2 and 4)

CASE 2
(Strategy 5)

CASE 1
(Strategy 1)

β_2

1.14

1.12

1.10

1.08

1.06

1.04

1.02

O   .02   .04   .06   .08   .10   .12   .14

α

Table I

Equilibria for $u(c_1) = \ln c_1$, $v(c_2) = \ln c_2$, $\beta_1 = 1$,

$\beta_2 = 1.05$, $x_1 = x_2 = 1$, $\alpha_1 = \alpha_2 = \alpha$, $\pi = \frac{1}{2}$

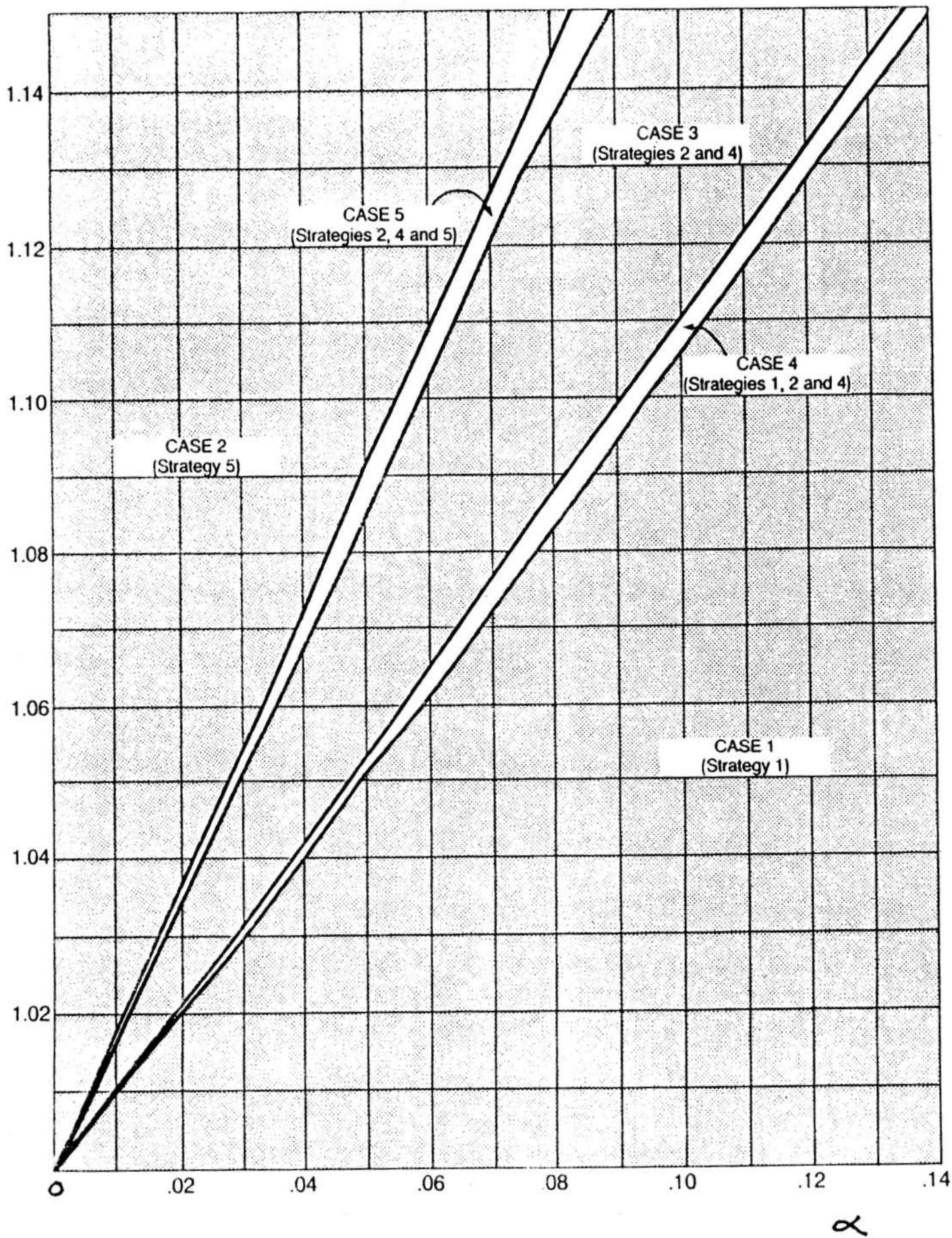| α	c_1	c_2	q_1	q_2	$e^{w/0.5}$ (w = expected utility)
0.0285	0.4715	1.0500	0.9527	0.9500	1.9803
0.02865	0.4784	1.0380	0.9634	0.9610	1.9918
0.0288	0.4851	1.0264	0.9743	0.9719	2.0032
0.0300	0.4850	1.0247	0.9751	0.9743	2.0020
0.0350	0.4825	1.0246	0.9720	0.9774	1.9912
0.0400	0.4800	1.0246	0.9688	0.9806	1.9800
0.0450	0.4775	1.0246	0.9656	0.9838	1.9689
0.0474	0.4811	1.0194	0.9714	0.9928	1.9714
0.0480	0.4893	1.0108	0.9838	1.0059	1.9838
0.0483	0.4934	1.0066	0.9900	1.0124	1.9900
0.0488	0.5000	1.0000	1.0000	1.0244	2.0000

Table II

Equilibria for $u(c_1) = \ln c_1$, $v(c_2) = \ln c_2$, $\beta_1 = 1$,

$\beta_2 = 1.05$, $x_1 = x_2 = 1$, $\alpha_1 = \alpha_2 = \alpha$, $\lambda = \frac{1}{2}$

α	p_1	p_2	p_4	p_5	$\dfrac{p_1 + p_4}{p_2 + p_5}$
0.0285	0	0	0	1.0000	0
0.02865	0	0.2522	0.2402	0.5076	0.3161
0.0288	0	0.4961	0.4724	0.0315	0.8954
0.0300	0	0.4937	0.5063	0	1.0255
0.0350	0	0.4929	0.5071	0	1.0288
0.0400	0	0.4921	0.5079	0	1.0321
0.0450	0	0.4912	0.5088	0	1.0358
0.0474	0.2053	0.3886	0.4061	0	1.5733
0.0480	0.5553	0.2160	0.2287	0	3.6296
0.04832	0.7271	0.1321	0.1408	0	6.5700
0.0488	1.0000	0	0	0	--

Table III

Equilibria for $u(c_1) = \ln c_1$, $v(c_2) = \ln c_2$, $\beta_1 = 1$,

$x_1 = x_2 = 1$, $\alpha_1 = \alpha_2 = 0.05$, $\pi = 0.5$

β_2	c_1	c_2	q_1	q_2	$e^{w/0.5}$
1.0513	0.5	1	1	1.0251	2
1.052	0.4910	1.009	0.9864	1.0116	1.9864
1.0525	0.4849	1.0154	0.9771	1.0023	1.9771
1.0530	0.4789	1.0218	0.9680	0.9934	1.9680
1.0540	0.4750	1.0265	0.9620	0.9875	1.9632
1.0600	0.4750	1.0294	0.9613	0.9882	1.9707
1.0700	0.4750	1.0343	0.9602	0.9894	1.9836
1.0800	0.4750	1.0392	0.9591	0.9905	1.9964
1.0850	0.4750	1.0416	0.9586	0.9910	2.0031
1.0897	0.4712	1.0506	0.9518	0.9853	2.0019
1.0900	0.4683	1.0367	0.9452	0.9788	1.9949
1.0905	0.4609	1.0701	0.9345	0.9681	1.9837
1.0910	0.4547	1.0747	0.9241	0.9578	1.9727
1.0914	0.45	1.0914	0.9163	0.95	1.9645

Table IV

Equilibria for $u(c_1) = \ln c_1$, $v(c_2) = \ln c_2$, $\beta_1 = 1$,

$x_1 = x_2 = 1$, $\alpha_1 = \alpha_2 = 0.05$, $\pi = 0.5$

β_2	p_1	p_2	p_4	p_5	$\dfrac{p_1 + p_4}{p_2 + p_5}$
1.10513	1	0	0	0	--
1.0520	0.6418	0.1735	0.1847	0	4.7637
1.0525	0.3966	0.2936	0.3098	0	2.4060
1.0530	0.1575	0.4118	0.4307	0	1.4284
1.0540	0	0.4903	0.5097	0	1.0396
1.0600	0	0.4901	0.5099	0	1.0404
1.0700	0	0.4898	0.5102	0	1.0416
1.0850	0	0.4896	0.5104	0	1.0425
1.0850	0	0.4894	0.5106	0	1.0433
1.0897	0	0.4145	0.4354	0.1501	0.7712
1.0900	0	0.3363	0.3558	0.2884	0.5696
1.0905	0	0.2107	0.2255	0.5638	0.2912
1.0910	0	0.0900	0.0974	0.8126	0.1079
1.0914	0	0	0	1	0

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