Federal Reserve Bank of Minneapolis
Research Department

THE OUTPUT, EMPLOYMENT, AND INTEREST RATE EFFECTS OF GOVERNMENT CONSUMPTION

S. Rao Aiyagari, Lawrence J. Christiano
and Martin Eichenbaum*

Working Paper 456

April 1990

ABSTRACT

This paper investigates the impact on aggregate variables of changes in government consumption in the context of a stochastic, neoclassical growth model. We show, theoretically, that the impact on output and employment of a persistent change in government consumption exceeds that of temporary change. We also show that, in principle, there can be an analog to the Keynesian multiplier in the neoclassical growth model. Finally, in an empirically plausible version of the model, we show that the interest rate impact of a persistent government consumption shock exceeds that of a temporary one. Our results provide counterexamples to existing claims in the literature.

*Aiyagari, Federal Reserve Bank of Minneapolis; Christiano, Federal Reserve Bank of Minneapolis; Eichenbaum, Northwestern University, Federal Reserve Bank of Chicago, and the NBER. We are grateful to Lars Hansen for helpful comments. Eichenbaum acknowledges research support from the National Science Foundation.

1. Introduction

Early general equilibrium business cycle models, say as represented by the seminal works of Kydland and Prescott (1982) and Long and Plosser (1982), emphasized the importance of aggregate shocks to technology as impulses to post war US business cycles. Spurred on by both the empirical successes and shortcomings of these models, recent work has sought to identify and integrate other impulses to aggregate economic activity into equilibrium business cycle models.[1]

Curiously, modern equilibrium business cycle analysts have paid relatively little attention to the role of government expenditures as a possible contributor to business cycle activity. It is true that authors like Hall (1980) and Barro (1981,1987), approaching the problem from a neoclassical perspective, have argued that changes in aggregate government consumption can significantly alter aggregate economic activity. Still, very little work has been done on integrating the public sector into formal quantitative general equilibrium models of the business cycle.[2] This stands in sharp contrast to traditional Keynesian analyses which typically assign great importance to the role of government expenditures in determining aggregate output, employment and interest rates.

This paper investigates the impact of changes in government consumption on aggregate employment, output and interest rates. We proceed at both a theoretical and a quantitative level. Our theoretical analysis is conducted within the confines

[1]Hamilton (1983) for example has stressed the important role of oil shocks in business cycles while Barsky and Miron (1988) investigate the role of seasonal shocks arising from weather patterns and holidays. Other authors like Judd (1985) and Braun (1989) consider the role of movements in marginal tax rates on aggregate economic fluctuations. In contrast, authors like Kydland (1987) and Cooley and Hansen (1988) are incorporating monetary shocks into quantitative general equilbrium businesss cycle models.

[2]Some recent exceptions include Baxter and King (1988), Braun (1988), McGratten (1989), Christiano and Eichenbaum (1988), and Wynne (1988).

of a one sector stochastic neoclassical growth model. Our quantitative analysis is based upon a version of the model whose parameters were estimated in Christiano and Eichenbaum (1988). The reader may be puzzled that we find it necessary at all to consider the theoretical effects of variations in government consumption. Indeed, this question has been previously analyzed by Hall (1980) and Barro (1981,1987), ostensibly using the standard neoclassical growth model. However, our analysis indicates that the neoclassical growth model makes predictions about the impact of government consumption which are, in many ways, fundamentally different from those claimed by Hall and Barro.

It is convenient to begin by summarizing those claims. First, according to Hall (1980) and Barro (1981,1987), both transient and persistent increases in government consumption ought to increase aggregate output and hours worked. Second, Hall (1980) argues that the employment and output effects of persistent increases in government consumption are smaller than those of transient increases.[3] Third, according to Barro (1987), while positive, the ratio of the change in contemporaneous and steady state output to changes in government consumption must be less than one, i.e. there can be no analog to the Keynesian "multiplier" in the neoclassical growth model. Fourth, Barro (1981, 1987) argues that a transient increase in government consumption ought to impact positively on the interest rate, while a permanent increase ought to leave the interest rate unchanged.

While our theoretical analysis leads us to conclude, as do Hall (1980) and

[3]Barro (1981) seems to suggest this as well. In particular he writes that "Under plausible conditions, the temporary case involves an output response that is positive, less than one–to–one with the change in government purchases, and larger than that generated by an equal–sized, but permanent, shift in purchases." However he comments on the relative output effects of temporary and permanent changes in government purchases only in the abstract of the paper and in the section on optimal, non–lump–sum taxes—a case which we do not consider. Therefore, it is not clear whether the previous statement is meant to apply to both lump–sum and optimal non–lump–sum taxation situations, or only the latter.

Barro (1981,1987), that transient as well as persistent increases in government consumption do generate increases in output and employment, we find ourselves in disagreement along other important dimensions. First, we demonstrate analytically that—under standard assumptions spelled out in section 2 below—the employment and output effects of permanent increases in government consumption always *exceed* those of temporary increases.[4] From this perspective, the more persistent is government consumption, the more likely it is to be an important impulse to business cycle fluctuations. In our quantitative analysis, we find that the differential impact of persistent and temporary shocks to government consumption are quite substantial. Second, we use our parametric model to show that the steady state response of output to a permanent one unit change in government consumption can exceed unity. Further, the contemporaneous response of output to a one unit change in government consumption can also exceed unity if that change is sufficiently persistent. Thus, in principle there *can* be an analog to the Keynesian "multiplier" in conventional neoclassical models. Third, we show that, in our parametric model, an increase in government consumption causes the real interest rate to rise, regardless of whether the change in government consumption is temporary or persistent. Indeed, for that model, we find that a persistent increase in government consumption has a *larger* impact on the interest rate than does a transient increase.

---

[4]This verifies conjectures in Christiano and Eichenbaum (1988) and Baxter and King (1988) that such a result might hold for a reasonably general class of models. See also Barro and King (1982) who suggest in their footnote 13, that under lump sum taxes, a permanent change in government purchases would have a stronger effect than a temporary change in government purchases because "investment tends to decline more when the change in purchases is temporary."

In the remainder of this section we provide some intuition for the difference between our results and those in the existing literature.

The Relative Impact of Temporary Versus Persistent Government Consumption Shocks

The basic argument underlying Hall's conclusion about the relative impact of persistent and temporary changes in government consumption can be summarized as follows. A temporary increase of one unit in government consumption reduces consumers' permanent incomes by much less than one unit (approximately zero). Consequently, the demand for private consumption remains (roughly) unchanged, so that, for a given interest rate, net aggregate demand increases by one unit. Since there has been no change in labor suppliers' permanent incomes, other things equal, when viewed as a function of the interest rate, the aggregate supply curve of output does not shift. Consequently, real interest rates must rise to clear the commodity market. This transient increase in the interest rate induces a positive labor supply response for intertemporal substitution reasons.

In contrast, a permanent increase in government consumption by one unit reduces consumers' permanent incomes by one unit. Other things equal, this induces a fall in the demand for private consumption by one unit. With no net change in aggregate demand there is no need for any offsetting effects on interest rates. Consequently, equilibrium output and hours worked do not change.

The basic problem with Hall's argument is that it makes inconsistent assumptions regarding the income effect on leisure. An implicit assumption of his analysis of a permanent change in government consumption is that there is no income effect on leisure. But this cannot also be a maintained assumption of his analysis of a temporary change in government consumption. This is because, under that assumption, a temporary change in government consumption *also* has no impact on equilibrium output and employment.[5]

In fact, with a positive income effect on leisure, permanent changes in government consumption always have an effect on employment and output that is larger than the effect of transitory changes. We obtain this result by decomposing the effect on household hours worked of a change in government consumption as follows: (a) the direct effect of the change in government consumption on hours worked, holding constant private investment, (b) the effect on investment of the change in government consumption, and (c) the indirect effect of government consumption on hours worked that occurs via the change in investment.

As it turns out, in (a) and (c), increases in government consumption and investment can be viewed as exogenous reductions in income which raise hours worked. However, with respect to (b), a transient increase in government consumption *reduces* investment, reflecting agents' desire to smooth consumption over time. But, a *persistent* increase in government consumption either *increases* investment or does not reduce it by as much as in the transient case. Basically this is because a permanent increase in government consumption raises the steady state level of capital. It is the indirect effect of government that accounts for the fact that persistent changes in government consumption generate *larger* contemporaneous effects on output and employment than transient changes.

---

[5]This follows from two observations. First, if the income effect on leisure is zero, then the marginal rate of substitution between labor and consumption depends only on leisure. Therefore, households' labor supply depends only on the current real wage rate — interest rate considerations play no role in determining labor supply. Second, the marginal productivity schedule of labor, which determines the demand for labor, is fixed by the current stock of capital and the level of technology. Taken together, these two observations imply that equilibrium output and employment do not depend on the contemporaneous level of government consumption.

The Multiplier

The basic argument underlying Barro's conclusion that contemporaneous and steady state output increase by less than a given permanent increase in government consumption can be summarized as follows. The negative wealth effect associated with a permanent increase in government consumption leads to a reduction in private consumption. At the same time, investment does not change, because according to Barro's analysis, interest rates do not change. Since aggregate output equals the sum of private consumption, investment and government consumption, the increase in output must be less than the increase in government consumption.

The key problem with this argument lies in the assertion that investment does not change in response to a permanent increase in government consumption. In general this claim is not true unless there is no income effect on leisure. To see why, it is convenient to consider the nonstochastic case. In standard versions of the neoclassical growth model, the steady state capital–hours worked ratio is determined by agents' subjective discount rate and is not affected by government consumption. But, given a positive income effect on leisure, steady state hours worked increase in response to a permanent increase in government consumption. It follows that the steady state stock of capital and investment must also increase. Given the increase in steady–state investment, there is no longer any reason for the steady state multiplier to be less than one. Similar considerations imply that the contemporaneous output effect of a permanent change in government consumption can also exceed one.[6]

[6]An alternative way to rationalize Barro's claim, which does not depend on the absence of an income effect on leisure, is to assume that the marginal productivity of capital does not depend on hours worked. We do not impose this restriction in our analysis.

The Interest Rate and Government Consumption

According to Barro (1981,1987) permanent increases in government consumption ought to have no impact on the interest rate. To see why this result does not hold in general, it is again useful to consider the nonstochastic case. With a positive income effect on leisure, permanent increases in government consumption drive up the steady-state stock of capital and hours worked. For a model economy like ours, the interest rate is above its steady state value whenever the capital stock is below its steady state. Moreover, convergence is monotonic, with the interest rate eventually returning to its unchanged steady-state value (see Aiyagari (1988), Proposition 3). Consequently, a permanent increase in government consumption induces a temporary increase in the interest rate. Indeed, in our parametric model, we find that this increase is actually larger than that which would result from a temporary change in government consumption. At the same time we wish to emphasize that this last result is by no means a general one. There exist alternative preference and technology assumptions for the one sector growth model in which temporary increases in government consumption have bigger impacts on the interest rate than permanent changes.[7]

The remainder of this paper is organized as follows. In section 2 we present our basic model and analytical results. All proofs are relegated to the appendix.

[7]To see this, suppose the economy is in nonstochastic steady state and there is no income effect on leisure. Then, a permanent increase in government consumption that is met by an equal decrease in private consumption and no change in hours worked is both feasible and optimal. But, this means that the interest rate is left unchanged by the permanent increase in government consumption. If the increase in government consumption is temporary, the proposed consumption and hours worked response would be feasible, but not optimal. In fact, investment would fall and the interest rate would rise.

Section 3 contains our numerical results. Finally section 4 contains some concluding remarks.

2. Dynamic Effects of Government Spending: Theoretical Results

In this section we analyze a variant of the neoclassical growth model modified to include elastic labor supply and government consumption. Our main result is that the contemporaneous effect on employment and output of a persistent increase in government consumption always exceeds the corresponding effect of a temporary increase in government consumption. A formal proof of this proposition is contained in the Appendix. The purpose of this section is to provide an intuitive description of the argument.

2.1 The Model.

We assume that the competitive equilibrium allocation can be represented as the solution to the following planning problem. Maximize,

$$(2.1) \qquad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^p, n_t) \qquad 0 < \beta < 1,$$

subject to,

$$(2.2) \qquad c_t^p + g_t + k_{t+1} \leq f(k_t, n_t),$$

and $0 \leq n_t \leq N$, $c_t^p \geq 0$, $k_{t+1} \geq 0$, $k_0 > 0$, given, by choice of contingency plans for

n_t, c_t^p, and k_{t+1}. Here, c_t^p, g_t and k_t denote private consumption, public consumption, and the beginning–of–period public plus private stock of capital, respectively. Also, n_t denotes hours worked and N is the total time endowment. The function, $f(k_t,n_t)$, relates date t gross output and the undepreciated part of capital to the factors of production.

Assumptions on Exogenous Shocks

We assume that government consumption, g_t, evolves according to

$$g_t = G(g_t^T + g_t^P). \tag{2.4}$$

Here, $G(\cdot) > 0$ is a strictly increasing function while g_t^T and g_t^P are the zero mean transitory and persistent components of government spending, respectively. Throughout, we assume that g_t^T is iid over time and that g_t^P and g_t^T are mutually independent. Both g_t^P and g_t^T are assumed to have bounded support, $[g_{min}^P, g_{max}^P]$, $[g_{min}^T, g_{max}^T]$, respectively. Further, g_t^P is a Markov process with conditional cumulative distribution function $\Psi(g^{P\prime}|g^P) = \mathrm{Prob}[g_{t+1}^P \le g^{P\prime}|g_t^P = g^P]$. We make the following assumption for Ψ:

E1 Ψ is decreasing in g^P.[8]

Assumption E1 formalizes our notion of persistence. Intuitively, g_t^P is persistent if

[8]Formally, Ψ is decreasing in g^P means that if $g_2^P > g_1^P$, then $\Psi(g^{P\prime}|g_1^P) \ge \Psi(g^{P\prime}|g_2^P)$ for all $g^{P\prime}$, with the inequality being strict for some $g^{P\prime}$.

higher values of g_t^P make higher values of g_{t+1}^P more likely.[9] To motivate the usefulness of this definition, consider the random variable

$$E[\omega(g_{t+1}^P)|g_t^P=g^P] = \int_{g_{min}^P}^{g_{max}^P} \omega(g^{P\prime})d\Psi(g^{P\prime}|g^P),$$

where ω is any continuous, increasing function. Integrating by parts we obtain:

$$(2.5) \qquad E[\omega(g_{t+1}^P)|g_t^P=g^P] = \omega(g_{max}^P) - \int_{g_{min}^P}^{g_{max}^P} \Psi(g^{P\prime}|g^P)d\omega(g^{P\prime}).$$

From this equation it is evident that our definition of persistence implies that the expectation in (2.5) is increasing in g_t^P. Below, we indicate where this implication of E1 is used in proving our result. Our final assumption on the exogenous shocks is:

E2 $\quad \text{Prob}\{g_{t+1}^P = g_{max}^P, g_{t+1}^T = g_{max}^T | g_t^P\} > 0, \quad \text{for all } g_t^P.$

Below we indicate how E2 helps guarantee an interior solution to the planning problem.

Assumptions on Technology

We begin by listing our assumptions on the function f, labeled T

[9]Our definition of persistence is equivalent to the concept of first order stochastic dominance. See, for example, Hadar and Russel (1969) and Hanoch and Levy (1969). We thank V.V. Chari for drawing our attention to this fact.

("technology"). These assumptions are typical in analyses of the standard neoclassical growth model.

T1 f is strictly increasing, concave and obeys constant returns to scale; it is twice continuously differentiable for $k > 0$, $n > 0$; and $f(0,N) - g_{max} < 0$.

T2 $f_k(n,k) \to \infty$ as $k \to 0$ for $0 < n \leq N$,
$f_n(n,k) \to \infty$ as $n \to 0$ for $0 < k$.

T3 $\lim_{k\to\infty} f_k(k,n) < 1$ for $0 < n \leq N$.

T4 there exists a k such that $f(k,N) - g_{max} > k$.

T5 $f_k > 0$, $f_n > 0$, $f_{kk} < 0$, $f_{nn} < 0$, $f_{kn} > 0$ for $k > 0$, $n > 0$.

Throughout, f_i is defined as the partial derivative of f with respect to its i^{th} argument, for $i = n, k$. Assumptions T1 and T4 assure that there exist values of k such that $f(k,N)-g_{max}-k = 0$, where $g_{max} \equiv G(g^P_{max}+g^T_{max})$. Denote the smallest such value of k by $\underline{k}$, so that

$$(2.6) \qquad f(\underline{k},N) - g_{max} = \underline{k}.$$

The existence of $\underline{k}$ is necessary for the economy to have the capability of reproducing its capital stock. An additional requirement needed to guarantee the survivability of this economy is:

T6 $k_0 \geq \underline{k}$.

It is straightforward to show that assumption T3 ensures the existence of a maximum possible value for k_t, which we denote by $\bar{k}$. Taken together, our assumptions imply that we can, without loss of generality, restrict ourselves to consider values of the capital stock that lie between $\underline{k}$ and $\bar{k}$. The upper bound is non–binding because it is technically infeasible to exceed it. Our assumptions on preferences (listed below) and E2 guarantee that the lower bound is also non–binding.

Assumptions on Preferences

Let u_i denote the partial derivative of u with respect to i for i = c^p, n. In addition, we denote the marginal rate of substitution between leisure and consumption by

$$(2.7) \qquad w(c^p,n) = \frac{-u_n(c^p,n)}{u_{c^p}(c^p,n)}, \text{ for } c^p > 0,\ 0 \leq n < N.$$

Our assumptions on preferences are as follows:

P1 u is strictly concave, strictly increasing in c^p and strictly decreasing in n for $c^p \geq 0$, $0 \leq n \leq N$; it is twice continuously differentiable for $c^p > 0$, $n < N$; $u_{c^p} > 0$, $u_n < 0$, $u_{c^pc^p} < 0$, $u_{nn} < 0$ and

$$\begin{vmatrix} u_{c^pc^p} & u_{c^pn} \\ u_{nc^p} & u_{nn} \end{vmatrix} > 0.$$

P2 $w(c^p,n) \to \infty$ as $n \to N$ for fixed $c^p > 0$,
$w(c^p,n) \to 0$ as $c^p \to 0$ for fixed $n < N$.

P3 If $c^p \to 0$ and $(-u_n/u_{c^p})$ is decreasing then $u_{c^p} \to \infty$.

P4 $w_{c^p} > 0$ for $c^p > 0$, $0 \leq n < N$

Assumption P3 is required to accommodate situations in which $c^p \to 0$ and $n \to N$ simultaneously. P4 is the assumption that leisure is a superior good. Assumptions P1 – P3 help guarantee that the optimal choices of k', c^p and n lie in the interior of the planner's constraint set.[10]

[10]With one exception, our assumptions on preferences are standard. The exception is that we do not permit g_t to affect the marginal utility of c_t^p and n_t. This rules out utility functions of the form considered in the literature on government consumption (see, e.g., Aschauer (1985), Barro (1987) and Kormendi (1983)), where utility functions of the form $u(c^p+\alpha g_t, n_t)$ for $\alpha \neq 0$ are studied. The numerical findings in Christiano and Eichenbaum (1988) suggest that the results in this paper are robust to this extension, as long as the appropriate analogue of $\alpha \neq 1$ obtains.

2.2 Government Consumption and Equilibrium Employment and Output

We discuss the qualitative features of the solution to our planning problem using standard dynamic programming methods. Bellman's equation for the planning problem is:

$$(2.8) \qquad v(k,g^P,g^T) = \max_{c^p,k',n \in A(k,g)} \{u(c^p,n) + E[\beta v(k',g^{P\prime},g^{T\prime}) | g^P]\}$$

where $k \in [\underline{k},\bar{k}]$ and

$$(2.9) \quad A(k,g) = \{c^p,k',n: c^p \geq 0,\ 0 \leq n \leq N,\ k' \geq \underline{k},\ c^p + k' + g \leq f(k,n)\}.$$

In (2.8)–(2.9), time subscripts have been deleted and the superscript $'$ denotes next period's value. Formulated in this way, the problem is to find a value function, v, which satisfies (2.8) and (2.9). Given our assumptions, it is straightforward to show that v exists, is unique and is strictly concave in k for fixed (g^T, g^P). In addition, for fixed g^P and g^T, the function v is continuously differentiable in k for all $k \in (\underline{k},\bar{k})$. The contingency plans for c^p, n, and k' that we seek are the unique plans which attain the maximum in (2.8).

For our purposes, it is convenient to rewrite (2.8)–(2.9) as

$$(2.10) \qquad v(k,g^P,g^T)$$
$$= \max_{\underline{k} \leq k' \leq f(N,k)-g} \left\{\left[\max_{n,c^p \in B(k,k'+g)} u(c^p,n)\right] + E[\beta v(k',g^{P\prime},g^{T\prime}) | g^P]\right\},$$

where,

$$(2.11) \qquad B(k,k'+g) = \{c^p,n:\ 0 \le n \le N,\ 0 \le c^p \le f(k,n)-(g+k')\},$$

for $k \in [\underline{k},\bar{k}\,]$, $k' \in [\underline{k},f(k,n)-g]$. Let the function defined in the first square brackets of (2.10) be denoted

$$(2.12) \qquad W(k,k'+g) = \max_{n\,,c^p \in B(k,k'+g)} u(c^p,n).$$

It is easy to show (see the Appendix) that there exist unique solutions

$$(2.13) \qquad n = h(k,k'+g),\ c^p = q(k,k'+g)$$

that attain the maximum in (2.12). It is also easy to establish (again, see the Appendix) that W is strictly concave in (k,k') for fixed g and is twice differentiable in the interior of its domain. With this notation, (2.8) can be written as

$$(2.14) \quad v(k,g^P,g^T) = \max_{\underline{k} \le k' \le f(k,N)-g} \{W(k,k'+g) + E[\beta v(k',g^{P\prime},g^{T\prime})\,|\,g^P]\}.$$

Thus, problem (2.8) is broken into two parts, a purely static part, (2.12), and a dynamic part, (2.14). This way of splitting the problem corresponds to our description in the introduction of the direct and indirect effects of movements in g_t. Problem (2.12) captures the direct effects of g and k' on (c^p,n) whereas problem (2.14) captures the indirect effect of g on (c^p,n) via k'.

Notice that the distinction between transient and persistent changes in g enters *solely* via the indirect effect in problem (2.14). One way to see this is to decompose the response of n to a change in g as:

$$\text{(2.15)} \qquad \frac{dn}{dg} = \frac{\partial h}{\partial g} + \frac{\partial h}{\partial k'}\frac{dk'}{dg} = \frac{\partial h}{\partial g}[1 + \frac{dk'}{dg}],$$

where h is defined in (2.13).[11] The expression after the second equality in (2.15) reflects the symmetry with which g and k′ enter the h function. Because g^P and g^T enter the static problem's constraint set, B, in a symmetric way, it follows that $\partial h/\partial g$ does not depend on whether the change in g reflects a change in g^P or g^T. However, because a change in g^P influences the distribution of g′ while a change in g^T does not, no such symmetry exists for dk'/dg. Hence, our problem reduces to understanding the relative magnitudes of dk'/dg^P and dk'/dg^T.

The Static Problem

In the first part of the Appendix we establish that h is strictly increasing in $k'+g$ (see equation (A5)). To understand this result, one need only consider the planner's static consumption–leisure choice problem. Here, an increase in $k'+g$ corresponds to a parallel downward shift in the planner's resource constraint set. When $w_{cp} > 0$, i.e., leisure is a superior good, the ensuing negative income effect generates an increase in hours worked. When the income effect on leisure is zero, i.e., $w_{cp} = 0$, then $\partial h/\partial g = 0$ (see (A.4a) in the Appendix). This last result, in combination with (2.15), shows why the absence of an income effect on leisure cannot be used to rationalize standard arguments regarding the relative impacts of

[11] Although our assumptions are sufficient to guarantee differentiability of h, we have not established differentiability of k′ with respect to g, nor does our proof in the Appendix rely on differentiability of k′. Nevertheless, we find this notation useful for pedagogical purposes.

transient and persistent changes in government consumption on output and employment.

The Dynamic Problem

In this subsection we discuss the impact of transient and persistent changes in government consumption on investment. In particular, we analyze the relative magnitudes of dk'/dg^T and dk'/dg^P. Denote the value of k' which attains the maximum in (2.14) by $F(k,g^P,g^T)$. Proposition 1 (ii) in the Appendix establishes that $F(k,g^P,g^T)$ lies in the interior of its constraint set, $[\underline{k},f(k,N)-g]$. This guarantees that the first order condition implied by (2.14) is satisfied as an equality, i.e.,

$$(2.16) \qquad -W_{k'}(k,k'+g) = E[\beta v_{k'}(k',g^{P\prime},g^{T\prime})\,|\,g^P]$$

for $k' = F(k,g^P,g^T)$. To understand how F varies with changes in g^T and g^P, we plot $-W_{k'}(k,k'+g)$ and $E[\beta v_{k'}(k',g^{P\prime},g^{T\prime})\,|\,g^P]$ as functions of k' in Figure 1.

Consider first the graph of $E[\beta v_{k'}(k',g^{P\prime},g^{T\prime})\,|\,g^P]$, the marginal benefit of an additional unit of k'. Two salient features of this curve are that it is downward sloping and it goes to infinity as $k' \to \underline{k}$. The first feature reflects the strict concavity of v in k'. The second feature follows from E2 and the fact that $\lim_{k' \to \underline{k}} v_{k'}(k',g^P_{max},g^T_{max}) = \infty$. To understand the latter, we use the result (established using Proposition 1 (iii) and (A6.a)) that $v_{k'} = u_{cp}f_k$, evaluated along the optimal plan, as well as the fact that when $(g^P,g^T) = (g^P_{max},g^T_{max})$ and $k' \to \underline{k}$, then $(c^{p\prime},n') \to (0,N)$. The behavior of u_{cp} corresponding to this $(c^{p\prime},n')$ sequence is governed by assumption P3, which asserts that $u_{c^P} \to \infty$.

Figure 1 also displays the graph of $-W_{k'}$. In the Appendix we show that $-W_{k'}(k,k'+g)$ is an increasing function of k' and goes to infinity as k' goes to $f(k,N) - g$ (see equations (A7) and (A8) in the Appendix). Since $-W_{k'}(k,k'+g) = u_{c^P}(c^P,n)$ (see (A6.b) in the Appendix), we refer to this curve as the marginal cost curve of acquiring an extra unit of capital. The optimal value of k' is determined by the intersection of the marginal benefit and cost curves. In figure 1 this value is denoted by k^*.

In Figure 2, we analyze the impact of a change in g on the marginal cost and benefit curves. The benchmark marginal cost and benefit curves in that figure are labelled C_1 and B_1, respectively. The benchmark optimal value of k' is k_1'. Now suppose there is a purely transitory increase in g, i.e., $\Delta g^T = x > 0$. By construction, g^T is not an argument of $E[\beta v_{k'}(k',g^{P\prime},g^{T\prime})|g^P]$. Consequently, the marginal benefit curve does not shift. In contrast, g^T is an argument of $-W_{k'}(k,k'+g)$. Therefore, the marginal cost curve shifts to the left in response to an increase in g^T. We label the new marginal cost curve C_2, and the new intersection point, k_2'. It is evident that dk'/dg^T is *negative.*

Now suppose that the increase in g reflects $\Delta g^P = x$. In contrast to the previous experiment, the marginal benefit curve *is* affected by this change. This effect arises because the conditional distribution of $g^{P\prime}$ depends on the value of g^P. Under assumption E1 regarding the probability law of g^P, an increase in g^P shifts probability mass towards higher values of $g^{P\prime}$. To see the impact on $E\beta v_{k'}$, it is useful to exploit relation (2.5). Proposition 3 in the Appendix establishes that $v_{k'}$ is an increasing function of $g^{P\prime}$, so that we may set $\omega = \beta v_{k'}$. Consequently, assumption E1 guarantees that $E[\beta v_{k'}(k',g^{P\prime},g^{T\prime})|g^P]$ is an increasing function of g^P. In Figure 2, B_2 denotes the marginal benefit curve corresponding to the higher value of g^P. The new marginal cost curve coincides with C_1 since g^P and g^T enter $W_{k'}$ in a symmetric fashion. The optimal value of k' under these circumstances is denoted by k_3'. In general one cannot determine the sign of $k_3' - k_1'$. However, it *must* be the case that $k_3' - k_2'$ is positive so that $dk'/dg^P > dk'/dg^T$.

The previous result, in conjunction with (2.15), establishes the central result of our paper, namely, *the contemporaneous impact on n of a persistent increase in g exceeds the corresponding impact of a transient increase in g.* This result is formalized in Theorem 1 of the Appendix. It follows that the output effect of an increase in government spending is larger for increases that are persistent than for those that are transient.

3. Dynamic Effects of Government Spending: Quantitative Results.

In this section we accomplish four tasks. First, we provide a numerical example of the theoretical results discussed in section 2. Second, we show that both the contemporaneous and the nonstochastic steady state impacts of a persistent increase in government consumption on output can exceed unity. Third, we demonstrate that the contemporaneous impact on the interest rate of a persistent increase in government consumption is non-zero in general. Fourth, we show that the impact on interest rates of a persistent increase in government consumption can exceed the corresponding impact of a transient increase in government consumption.

We accomplish these tasks using a version of the parametric growth model considered by Christiano and Eichenbaum (1988).[12] That model is a special case of the one analyzed in section 2, except for the fact that we allow for shocks to the

[12]That model is the same as Kydland and Prescott's (1982) model and corresponds to the version of Gary Hansen's (1985) model in which labor is divisible.

aggregate production technology. This allows us to use values of the structural parameters estimated by Christiano and Eichenbaum (1988). In any event, it is trivial to extend the results of section 2 to allow for this additional source of uncertainty.

3.1 The Parametric Model.

Our parametric model assumes $u(c_t^p, n_t) = \log(c_t^p) + \gamma\log(N-n_t)$, $f(\lambda_t, k_t, n_t) = \exp(-\theta\lambda_t)n_t^{(1-\theta)}k_t^{\theta} + (1-\delta)\exp(-\lambda_t)k_t$ and $G(g_t^P + g_t^T) = \bar{g}\exp(g_t^P + g_t^T)$. Our specification of f reflects the presence of a technology shock, whose time t value we denote by λ_t. We assume that λ_t and g_t^T are mutually independent and independently distributed over time, with mean λ and zero, respectively.[13] Regarding Ψ, we suppose that g_t^P has the AR(1) representation,

$$(3.1) \quad g_t^P = \rho g_{t-1}^P + \mu_t, \quad |\rho| < 1,$$

where μ_t is fundamental for g_t^P and independent of λ_t.

In general it is not possible to obtain an analytic solution to this planning problem. Instead, we work with the log–linear approximation to the optimal decision rules for k_{t+1} and n_t described in Christiano and Eichenbaum (1988). These are of the form

---

[13] This specification actually corresponds to what Christiano and Eichenbaum (1988) refer to as the stationary version of their model. In the nonstationary version of their model the function f is given by $(z_t n_t)^{(1-\theta)}K_t^{\theta} + (1-\delta)K_t$ and the function u is given by $\log(C_t) + \gamma\log(N-n_t)$. Here, z_t is a technology shock with law of motion $z_t = z_{t-1}\exp(\lambda_t)$. The model in the text is obtained by rewriting the planning problem in terms of the stationary variables $c_t \equiv C_t/z_t$ and $k_{t+1} \equiv K_{t+1}/z_t$. The unorthodox specification of f in the text reflects this transformation.

$$(3.2)\quad k_{t+1} = k_s(k_t/k_s)^{r_k}\exp[d_k^T g_t^T + d_k^P g_t^P + e_k\lambda_t]$$

$$(3.3)\quad n_t = n_s(k_t/k_s)^{r_n}\exp[d_n^T g_t^T + d_n^P g_t^P + e_n\lambda_t],$$

where n_s and k_s are the values of n_t and k_t, respectively, in the nonstochastic steady state. The coefficients r_k, d_k^T, d_k^P, e_k, r_n, d_n^T, d_n^P, e_n are scalar functions of the model's structural parameters.

3.2 Non-Stochastic Steady State Results

Let $\eta_{n,g}$ and $\eta_{y,g}$ denote the elasticities of non-stochastic steady state employment and output with respect to $\bar{g}$. The fact that the steady state output-hours worked ratio is fixed by the discount rate implies

$$(3.4)\qquad \eta_{n,g} = \eta_{y,g} \equiv \frac{dy_s}{d\bar{g}}\frac{\bar{g}}{y_s}.$$

It can be shown that,

$$(3.5)\qquad \frac{dy_s}{d\bar{g}} = \left\{\frac{c_s^p + \bar{g}}{y_s} + \frac{c_s^p}{y_s}\frac{1}{[(N/n_s)-1]}\right\}^{-1}.$$

Here, c_s^p and y_s denote the nonstochastic steady-state values of private consumption and output.[14]

[14]We derive (3.5) as follows. In steady state the resource constraint and the planner's

To assess the magnitude of the multiplier and the elasticities, we used the same measures of n_t, c_t^p, g_t, and y_t analyzed in Christiano and Eichenbaum (1988). In particular c_t^p was measured using post–war quarterly U.S. time series data on purchases of nondurables and services plus an imputed service flow from the stock of household durable goods. A time series on g_t was constructed using data on government (federal plus state and local) purchases of goods and services minus a measure of government investment. Output, y_t, was measured as public plus private consumption, plus gross public and private investment. Finally, our hours worked data, n_t, where obtained from Gary Hansen (1984) (see Christiano and Eichenbaum (1988) for details). We estimated n_s, c_s^p/y_s and $\bar{g}/y_s$ by the sample averages of our measures of n_t, c_t^p/y_t and g_t/y_t over the period 1955.4 to 1983.4. These equal 320.2, .55 and .177, respectively. In addition, we set N = 2190, the number of hours in a quarter. Substituting these parameter values into (3.4) and (3.5), we obtain

$$\text{(3.6)} \qquad \frac{dy_s}{d\bar{g}} = 1.22,\ \eta_{y,g} = \eta_{n,g} = .22.$$

Since $dy_s/d\bar{g}$ exceeds unity, it follows that there can be a non–stochastic steady–state analog to the Keynesian multiplier in the neoclassical growth model.

Of course, this does not necessarily imply that the most plausible empirical value for $dy_s/d\bar{g}$ exceeds one. Even if we condition on the strong functional form

---

intratemporal first order condition are: (i) $c_s + \bar{g} = Q_1 n_s$ and (ii) $Q_2(N-n_s) = c_s P$, respectively, where, $Q_1 = \{(y_s/n_s) - [1-(1-\delta)\exp(-\lambda)](k_s/n_s)\}$, $Q_2 = [(1-\theta)/\gamma](y_s/n_s)$. Using (i) and (ii), together with the fact that y_s/n_s and k_s/n_s (and, hence Q_1 and Q_2) are independent of $\bar{g}$ yields $dn_s/d\bar{g} = (Q_1+Q_2)^{-1}$. The result follows by exploiting the facts $dy_s/dn_s = y_s/n_s$, $Q_1 n_s = c_s P+\bar{g}$, and $Q_2 n_s/y_s = (1-\theta)/\gamma$.

assumptions which we have made in our example, several important caveats must be attached to (3.6). First, according to (3.4) and (3.5) $\eta_{n,g}$, $\eta_{y,g}$ and $dy_s/d\bar{g}$ are increasing functions of N/n_s. In fact, there is substantial uncertainty about the value of this ratio. Consider first the numerator, N. Various papers make different assumptions about the time endowment of the representative consumer. For example, Eichenbaum, Hansen and Singleton (1988) and Mankiw, Rotemberg and Summers (1985) set $N = 1460$, i.e., the representative consumer is endowed with 16 hours per day. Other things equal, this reduces the value of $dy_s/d\bar{g}$ to 1.13, while $\eta_{y,g}$ and $\eta_{n,g}$ fall to .20. Next, consider the denominator, n_s. The calculations underlying (3.6) utilize an estimate of n_s based upon average hours worked in the marketplace (i.e., private business and government). However, the model implies that our measure of n_t should also include the time required to produce services from the stock of household capital (i.e., durable goods and housing). Consequently, the estimates in (3.6) may well be biased upwards. Assessing the magnitude of this bias is difficult, absent a reliable measure of the missing component of hours worked.[15]

3.3 A Dynamic Analysis.

In order to evaluate the contemporaneous output and employment impacts of

[15] In principle, our model suggests a way to correct for this bias. In particular, we can use its implication that the capital–labor ratio in the market and household sectors are the same. Then, a measure of total hours, adjusted to include time spent operating household capital, is obtained by scaling market hours by the ratio of total capital to market capital. The average value of this scale factor in our data is 1.73, and our adjusted estimate of n_s is 554.59. This implies that N/n_s and the multiplier equal 3.95 and 1.09, respectively, when $N = 2190$. These calculations must be interpreted with caution, since the scale factor exhibits growth over the post–war period, reflecting relatively rapid growth in the stock of consumer durables. In the early 1960s the scale factor was around 1.64, while by the early 1980s it stood at roughly 1.76.

government consumption, we calculated the planner's decision rules using the values of the structural parameters estimated by Christiano and Eichenbaum (1988). We report results for three values of ρ: (i) $\rho = .99999$, so that g_t^P is highly persistent; (ii) $\rho = .97$, which corresponds (after rounding) to the value estimated by Christiano and Eichenbaum (1988); (iii) $\rho = .9$, the value used by Baxter and King (1988).

The Effects of Shocks to Government Consumption on Hours Worked and Output

Table 1 reports the log–linear decision rules for the three values of ρ which were considered. These can be used to deduce the contemporaneous multipliers. Consider first the contemporaneous elasticities of n_t and y_t with respect to a transitory shock to government consumption, g_t^T. These are given by d_n^T and $(1-\theta)d_n^T$, respectively. According to our calculations, these elasticities equal 0.018 and 0.012, respectively. We converted the output elasticity to a contemporaneous multiplier assuming that $\bar{g}/y_s = .18$. The corresponding multiplier, $(1-\theta)d_n^T/(\bar{g}/y_s)$, equals .07, i.e., a unit increase in g_t induced by an increase in g_t^T causes a .07 unit increase in y_t. At least according to our model, then, the impact of purely transitory movements in government consumption on output and employment is small.

Next we consider the impact of a persistent increase in g_t. When $\rho = .99999$, the elasticities of employment and output with respect to g_t^P equal 0.315 and 0.206, respectively. The corresponding output multiplier is 1.15. At least for this value of ρ the output and employment effects of a persistent increase in g_t are quite substantial. Indeed, the contemporaneous output multiplier actually exceeds one. More plausibly, when $\rho = .97$, the employment and output elasticities are 0.214 and

0.14, respectively, while the corresponding output multiplier in this case is .78. When $\rho = .9$ the employment and output elasticities are 0.12 and 0.078, respectively and the output multiplier is .44.

To summarize, we find that the contemporaneous impact on output and employment of a change in government consumption is an increasing function of ρ. This illustrates the basic result of section 2. Moreover, we find that given the values of the structural parameters estimated by Christiano and Eichenbaum (1988), the impact of a persistent change in government consumption is substantially larger than that of a temporary change. In addition, we provide a counterexample to the claim in Barro (1981,1987) that the contemporaneous output government multiplier must be less than one.

The Effect of Shocks to Government Consumption on the Risk Free Interest Rate

In equilibrium, the time t risk free rate of return on consumption loans, r_t, is equal to the expected value of the planner's intertemporal marginal rate of substitution in consumption. There is substantial evidence that indicates that the log–linear quadratic approximation method used to calculate the output and employment multipliers are quite accurate (see Christiano [1987,1990] and footnote 19). Less evidence exists regarding the quality of these approximations for asset returns. Consequently, we calculated r_t using decision rules for k_t and n_t obtained using the value function iteration procedure described in Christiano (1990). This involves discretizing the support of the exogenous processes of the model and of k_t. The very fine grid which we use for k_t gives us confidence in the accuracy of this approach.[16] In particular we assume that g_t^P is a realization from a 3 state, Markov chain with

$$(3.7) \qquad g_t^P = \{g_1^P, g_2^P, g_3^P\} = \{-.1444,\ 0,\ .1444\}$$

$$\text{Probability } [g_{t+1}^P = g_j^P \mid g_t^P = g_i^P\ ] = \pi_{ij},\ i,j = 1,2,3$$

$$\pi_{ij} = \begin{bmatrix} .97310 & .0250 & .00190 \\ .00625 & .9875 & .00625 \\ .00190 & .0250 & .97310 \end{bmatrix}.$$

With this specification, g_t^P has the first order autoregressive representation, $g_t^P = .97g_{t-1}^P + \mu_t$, where $\mu_t = g_t^P - \bar{E}[g_t^P | g_{t-\tau}^P,\ \tau > 0]$ and $\bar{E}$ denotes the linear least squares projection operator. While the variable μ_t is conditionally heteroscedastic, it has an unconditional standard deviation equal to .020. In stochastic steady state, g_t^P equals $-.1444$, 0, .1444 with probability 1/6, 4/6, and 1/6 respectively. The corresponding degree of kurtosis is 3, the value implied by the normal distribution. Also, the standard deviation of g_t^P is .083.

We assume that g_t^T is identically and independently distributed over time and can take on one of three values

---

[16]We used a grid with 15,000 points evenly distributed between $k_t = 7{,}000$ and 23,300. The unconditional means of n_t and k_t implied by the discrete decision rule are 318 and 11752, respectively. In addition, the associated ergodic set for capital has boundaries 7574 and 23121. For the sake of comparison, we computed the same objects using the log–LQ decision rules, (3.2) and (3.3). The mean of n_t and k_t implied by these decision rules are 318 and 11709, respectively. In addition, the ergodic set for k_t is defined by boundaries 7081 and 19364. The stochastic models for the exogenous shocks underlying these calculations are specified in equations (3.7) — (3.9) below.

$$(3.8) \qquad g_t^T = \{g_1^T, g_2^T, g_3^T\} = \{-.1444, 0, .1444\}$$

$$\text{Probability } \{g_t^T = g_i^T\} = \Pi_i, \ i = 1,2,3$$

$$\Pi_1 = \Pi_3 = p, \ \Pi_2 = 1\text{–}2p, \ p = .000311.$$

With this specification, g_t^T has an unconditional mean of zero and standard deviation of .0036.[17]

Finally, we assume that λ_t is identically and independently distributed over time and can take on one of three values. In particular,

$$(3.9) \quad \lambda_t = \{\lambda_1, \lambda_2, \lambda_3\} = \{-.02, .0047, .0295\}$$

$$\text{Prob}\{\lambda_t = \lambda_i\} = \mu_i, \ i = 1,2,3$$

$$\mu = \{.264, .471, .264\}.$$

It follows that λ_t has unconditional mean equal to .0047 and standard deviation of .018. These values correspond to the ones estimated in Christiano and Eichenbaum (1988).

The response of r_t to a shock in either g_t^T or g_t^P depends on the values of the date–t state variables, k_t, λ_t, g_t^P, g_t^T. Here, we report the expected value of the derivative of r_t with respect to g_t^P and g_t^T. These are denoted by Δ_r^P and Δ_r^T, respectively. The expectation is taken with respect to the steady state distribution

[17]Under our assumptions, the univariate Wold representation of log g_t is $\log g_t = .97\log g_{t-1} + c + \epsilon_t - .03\epsilon_{t-1}$, where c is a constant. In Christiano and Eichenbaum (1988) the moving average coefficient in the representation for log g_t equals 0.

of the capital stock, and is evaluated at the mean values of the exogenous shocks. To be precise, we calculated

$$(3.10)\ \Delta_r^P = E[r(k_t|\lambda_t{=}.0047, g_t^P{=}.1444, g_t^T{=}0) - r(k_t|\lambda_t{=}.0047, g_t^P{=}0, g_t^T{=}0)]/.1444$$

$$(3.11)\ \Delta_r^T = E[r(k_t|\lambda_t{=}.0047, g_t^P{=}0, g_t^T{=}.1444) - r(k_t|\lambda_t{=}.0047, g_t^P{=}0, g_t^T{=}0)]/.1444,$$

and found that

$$\Delta_r^P = .0048,\ \Delta_r^T = .0005.$$

Thus, a persistent change of one percent in government consumption induces a .5 basis point increase in the interest rate. In contrast, a transient change of one percent in government consumption induces a barely perceptible rise of .05 basis points.[18] The fact that $\Delta_r^P > 0$ shows that, the arguments in Barro and Hall notwithstanding, a persistent increase in government consumption does change the interest rate. The fact that $\Delta_r^P > \Delta_r^T$ provides a counterexample to their claim that the interest rate necessarily responds more to a temporary change in government consumption than to a persistent change.[19] Finally, we note that,

[18]The standard deviations associated with the random variables implicit in the definition of Δ_r^T and Δ_r^P are .002 in each case. Thus, the interest rate response to government consumption varies substantially with the capital stock. We believe that much of this variation reflects approximation error induced by discretizing the capital space. The *differential* response is much less sensitive to the value of the capital stock, however. For example, the standard deviation associated with $\Delta_r^P - \Delta_r^T$ is .001. Morever, we found that the probability that the interest rate response to a temporary shock exceeds that of a permanent one is only about 1 percent.

[19]We used the decision rules obtained by value iteration methods to check the employment responses of government spending shocks obtained using the log–linear quadratic decision

according to our model, the interest rate effects of movements in government consumption—temporary or persistent—are very small. This is the case even though the output and employment effects of government consumption are substantial. From this perspective, it is not surprising that Barro (1981,1987) fails to find markedly higher real interest rates during periods of temporarily high government consumption, such as wartime.

4. Conclusion

This paper has investigated the effects of changes in government consumption on aggregate economic activity, using a simple stochastic one sector neoclassical growth model. Our main results can be summarized as follows. First, persistent changes in government consumption always have contemporaneous employment and output effects which are larger than those due to transitory changes. Moreover, the distinction between temporary and persistent increases in government consumption on employment and output can be quantitatively very important. Second, there is no reason for either the contemporaneous or steady state output/government consumption multiplier to be less than unity. Finally, for the specific parametric model which we analyze, we found that both temporary and persistent increases in government consumption increase the interest rate. For our model, a persistent increase in government consumption actually has a larger effect

---

rules analysed in the text. In particular, we found $\Delta^{P}_{\log(n)} = .228$ (.0039) and $\Delta^{T}_{\log(n)} = .020$ (.0032). Numbers in parentheses are the associated standard deviation. Here, $\Delta^{P}_{\log(n)}$ and $\Delta^{T}_{\log(n)}$ are defined by replacing r by log(n) in equations (3.10) and (3.11). These are close to the corresponding quantities reported in Table 1, for $\rho = .97$. Assuming the decision rules obtained by value iterations are highly accurate, this suggests the multiplier implications of the log–linear quadratic decision rules are reliable.

on the interest rate than a transitory increase.

In order to obtain these results, we made a number of important simplifying assumptions. First, we assumed that agents' preferences are additively separable across public and private consumption. In fact a variety of authors, including Barro (1981,1987), Kormendi (1983), Aschauer (1985) and Christiano and Eichenbaum (1988) have considered parametric models in which this assumption is relaxed. Numerical results in Christiano and Eichenbaum (1988) indicate that the results of this paper could be extended to allow for certain forms of non—separabilities across public and private consumption goods. Second, we assumed that government purchases are financed solely via lump–sum taxes. To us, this seems like a natural benchmark for isolating the theoretical effects of changes in government consumption on aggregate economic activity. Nevertheless, this is clearly not a natural assumption to make from the perspective of empirical work. Allowing for distorting taxes is an important task which we leave to future research.

Table 1

A: Model Parameters (Standard Errors)[1]

$$N = 2190,\ \delta = \underset{(0.0003)}{0.0207},\ \beta = 1.03^{-.25},\ \theta = \underset{(0.003)}{0.347},$$

$$\gamma = \underset{(0.07)}{7.00},\ \lambda = \underset{(0.0004)}{0.0047},\ \bar{g} = \underset{(3.25)}{199.0}$$

B: Decision Rule Parameters

	$\rho = .9$	$\rho=.97$	$\rho=.99999$
k_s	11709	11709	11709
r_k	0.95	0.95	0.95
d_k^T	–0.015	–0.015	–0.015
d_k^P	–0.006	0.002	.011
e_k	–0.95	–0.95	–0.95
n_s	317.5	317.5	317.5
r_n	–0.45	–0.45	–0.45
d_n^T	0.018	0.018	0.018
d_n^P	0.120	0.214	0.315
e_n	0.45	0.45	0.45

[1]Standard errors are reported only for estimated parameters. N and β were set a priori.

Appendix

As noted in the text, the dynamic programming problem (2.8) can be decomposed into a static part which is (2.12) and a dynamic part which is (2.14).

The Static Problem

(A1.a) $$W(k,k'+g) \equiv \max_{\{c^P,n\}} u(c^P,n)$$

subject to

(A1.b) $$c^P \leq f(k,n) - k' - g$$

(A1.c) $$0 \leq n \leq N,\ 0 \leq c^P$$

given (k,k',g) such that

(A1.d) $$\underline{k} \leq k' \leq f(k,N) - g$$

(A1.e) $$\underline{k} \leq k \leq \bar{k}.$$

Since $u(\cdot,\cdot)$ is continuous and strictly concave and the constraint set defined by (A1.b) and (A1.c) is nonempty, compact and convex, there is a unique solution to (c^P,n) which we denote by

(A2.a) $$c^p = q(k,k'+g)$$

(A2.b) $$n = h(k,k'+g).$$

So long as $k' < f(k,N) - g$, the solutions (A2) will be in the interior (i.e., $c^p > 0$, $0 < n < N$ by virtue of the Inada conditions, P2) and will satisfy the following first order condition

(A3) $$-\frac{u_n(c^p,n)}{u_{c^p}(c^p,n)} \equiv w(c^p,n) = f_2(k,n).$$

Standard comparative statics exercise using (A3) and (A1.b) at equality shows that

(A4.a) $$\begin{bmatrix} \frac{\partial c^p}{\partial k} & \vdots & \frac{\partial n}{\partial k} \\ \frac{\partial c^p}{\partial(k'+g)} & \vdots & \frac{\partial n}{\partial(k'+g)} \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} f_n f_{kn} + f_n(w_n - f_{nn}) & \vdots & f_k w_{c^p} + f_{kn} \\ f_{nn} - w_n & \vdots & w_{c^p} \end{bmatrix}$$

where,

(A4.b) $$\Delta = f_n w_{c^p} + w_n - f_{nn}$$

$$= w w_{c^p} + w_n - f_{nn} > 0$$

by strict concavity of $u(\cdot,\cdot)$.

Therefore, by assumption P4 (leisure is superior) we conclude

(A5) $h(k,k'+g)$ is strictly increasing in $(k'+g)$ whenever $k' < f(k,N) - g$.

$W(k,k'+g)$ is continuous and strictly concave in (k,k') for each fixed g. Continuity follows from Berge's maximum theorem. Strict concavity follows from the fact that $u(\cdot,\cdot)$ is strictly concave and that if (c_1^p,n_1) and (c_2^p,n_2) are feasible for (k_1,k_1') and (k_2,k_2') respectively, and $\lambda \in (0,1)$, then $\lambda(c_1^p,n_1) + (1-\lambda)(c_2^p,n_2)$ is a feasible choice for $\lambda(k_1,k_1') + (1-\lambda)(k_2,k_2')$. Furthermore, the envelope theorem implies that $W(k,k'+g)$ is continuously differentiable whenever $k' < f(k,N) - g$ and its derivatives are given by

$$\text{(A6.a)} \qquad W_k = u_{c^p}(c^p,n)f_k(k,n)$$

$$\text{(A6.b)} \qquad W_{k'} = -u_{c^p}(c^p,n)$$

where (c^p,n) are given by (A2). Using (A4) it is easy to show that

$$\text{(A7)} \quad W_{k'k'} < 0, W_{kk'} > 0.$$

As $k' \to f(k,N) - g$, $c^p \to 0$, and $n \to N$ since in the limit $(c^p,n) = (0,N)$ is the only feasible point in the constraint set (A1.b and c). By (A3) this implies that $w(c^p,n)$ is decreasing. Assumption (P3) then implies that

$$\text{(A8)} \qquad W_{k'} \to -\infty \text{ as } k' \to f(k,N) - g.$$

A similar argument applies as $(k,g) \to (\underline{k},g_{max})$ because by (A1.d) this automatically implies that $k' \to f(\underline{k},N) - g_{max} = \underline{k}$. Therefore, from (A6.a) and using assumption (P3) we have

(A9) $$W_k \to \infty \text{ as } (k,g) \to (\underline{k},g_{max}).$$

The Dynamic Problem

Let C be the space of real valued, continuous functions on $S \equiv [\underline{k},\bar{k}] \times [g^P_{min},g^P_{max}] \times [g^T_{min},g^T_{max}]$ with the sup norm. The mapping T given by

(A10.a) $$(Ta)(k,g^P,g^T) \equiv \max_{k'}\{W(k,k'+g) + \beta E[a(k',g^{P\prime},g^{T\prime})|g^P]\}$$

subject to

(A10.b) $$\underline{k} \le k' \le f(k,N) - g$$

maps function $a \in C$ into function $(Ta) \in C$. Any fixed point of this mapping is a solution to (2.14). It is a standard result in dynamic programming that there exists a unique $v \in C$ (known as the optimal value function) that satisfies (2.14). Further, for any $a^o \in C$, $v = \lim_{n\to\infty} T^n(a^o)$.

Lemma 1. Let $a^o \in C$ and $a \equiv Ta^o$

(i) a is strictly increasing in k

(ii) If a^o is concave:

(a) there is a unique k' (denoted $F_{a^o}(k,g^P,g^T)$) that attains Ta^o. Further, if $(k,g) \neq (\underline{k},g_{max})$ then $F_{a^o} < f(k,N) - g$.

(b) a is strictly concave in k and is continuously differentiable in k for all $k \in (\underline{k},\bar{k})$ with

$a_k(k,g^P,g^T) \equiv W_k(k,F_{a^o}+g)$.

(c) If $(k,g) \neq (\underline{k},g_{max})$ then $F_a > \underline{k}$.

Proof.

(i) Follows from the facts that $W(\cdot,\cdot)$ is strictly increasing in k and the constraint set for k' in (A10.b) is also increasing in k.

(ii) (a) uniqueness follows since $W(\cdot,\cdot)$ is strictly concave. The second part follows from (A8).

(b) Strict concavity of a follows from the strict concavity of W. The Benveniste–Scheinkman (1979) theorem implies the differentiability result.

(c) $a\,(\underline{k},g^P_{max},g^T_{max}) = W(\underline{k},\underline{k}+g_{max}) + \beta E[a^o(\underline{k},g^{P\prime},g^{T\prime}) | g^P_{max}]$

since $k' = \underline{k}$ is the only feasible element. Further,

$a(k,g^P_{max},g^T_{max}) \geq W(k,\underline{k}+g_{max}) + \beta E[a^o(\underline{k},g^{P\prime},g^{T\prime}) | g^P_{max}]$

since $k' = \underline{k}$ is feasible.

It follows that,

$$\frac{a(k,g^P_{max},g^T_{max}) - a(\underline{k},g^P_{max},g^T_{max})}{(k-\underline{k})} \geq \frac{W(k,\underline{k}+g_{max}) - W(\underline{k},\underline{k}+g_{max})}{(k-\underline{k})}$$

$\to \infty$ as $k \to \underline{k}$ (see A9).

Further, for $(k,g) \neq (\underline{k},g_{max})$, $W_{k'}(k,\underline{k}+g)$ is finite by (A6.b). Therefore,

$$\max_{k'}[W(k,k'+g) + \beta E[a(k',g^{P\prime},g^{T\prime})|g^P]$$

must be attained (uniquely, by strict concavity) by some $k' > \underline{k}$. Note that assumption (E2) is used here. □

Proposition 1. Let v be the optimal value function.

(i) v is strictly increasing and strictly concave in k for each fixed (g^P,g^T).

(ii) There is a unique k' (denoted $F(k,g^P,g^T)$) that attains $v(\cdot)$. If $(k,g) \neq (\underline{k},g_{max})$ then $\underline{k} < F < f(k,N) - g$.

(iii) $v(\cdot)$ is continuously differentiable in k for all $k \in (\underline{k},\bar{k})$ with

$$v_k(k,g^P,g^T) \equiv W_k(k,F+g).$$

(iv) Let $a^o \in C$ be concave and let $a^n = T^n(a^o)$. Then $F_{a^n} \to F$ pointwise.

Proof.

(i) Start with $a^o \in C$ such that a^o is concave and let $a^n = T^n(a^o)$. By Lemma 1 (iib), a^n is strictly concave and $a^n \to v$. Therefore, v is concave. Using Lemma 1 (i and iib) with $a^o \equiv v$ implies that v is strictly increasing and strictly concave in k.

(ii) Follows from Lemma 1 (iia and c) with $a^o \equiv v$.

(iii) Follows from Lemma 1 (iib) with $a^o \equiv v$.

(iv) $a^n \rightarrow v$ in sup norm. Hence for each fixed (g^P, g^T), the sequence of unique maximizers $F_{a^n} \rightarrow F$. □

Lemma 2. If $f(g^{P\prime})$ is nondecreasing (strictly increasing) then $\bar{f}(g^P) \equiv E[f(g^{P\prime})|g^P]$ is nondecreasing (strictly increasing).

Proof. Follows from the fact that our assumption of persistence on $\Psi(g^P|g^P)$ is equivalent to the definition of first order stochastic dominance. That is, $\Psi(g^{P\prime}|g_1^P)$ first order stochastically dominates $\Psi(g^{P\prime}|g_2^P)$ whenever $g_1^P > g_2^P$. □

Proposition 2. Let $a^o \in C$ be concave in k and suppose that $(F_{a^o}+g)$ is nondecreasing in g^P. Let $a \equiv Ta^o$. Then (F_a+g) is strictly increasing in g^P whenever $(k,g) \neq (\underline{k}, g_{max})$.

Proof. From Lemma 1, a is strictly increasing, strictly concave in k, and continuously differentiable in k for all $k \in (\underline{k}, \bar{k})$ with $a_k = W_k(k, F_{a^o}+g)$. Since $W_{kk\prime} > 0$ (see A7), a_k is nondecreasing in g^P whenever $k \in (\underline{k}, \bar{k})$. Further, by Lemma 1 (iia) (with a in place of a^o) and Lemma 1 (iic) we know that

$$\underline{k} < F_a < f(N,k) - g, \quad (k,g) \neq (\underline{k}, g_{max}).$$

Hence, F_a satisfies the following first order condition

(A11) $$W_{k'}[k,F_a(k,g^P,g^T)+g] + E\{\beta a_k[F_a(k,g^P,g^T),g^{P\prime},g^{T\prime}] \mid g^P\} = 0,$$

for $(k,g) \neq (\underline{k},g_{max})$.

By Lemma 2, the second term in (A11) is nondecreasing in g^P. Therefore, when g^P goes up $F_a + g$ must go up. Otherwise, either F_a falls or both F_a and $F_a + g$ fall. In either case strict concavity of W in k′ and a in k will result in a contradiction. □

Proposition 3. $(F+g)$ as well as v_k are both strictly increasing in g^P, for $k \in (\underline{k},\bar{k})$.

Proof. From Proposition 1 (iii), v_k is strictly increasing in g^P if and only if $(F+g)$ is. To establish the result, we start with $a^o \equiv 0$ and let $a^n = T^n(a^o)$, $n \geq 1$. It is obvious that $F_{a^o} \equiv \underline{k}$ and hence $(F_{a^o}+g)$ is strictly increasing in g^P. Proposition 2 then implies that $(F_{a^n}+g)$ is nondecreasing in g^P for all n. Since $F_{a^n} \to F$ point wise (Proposition 1 (iv)), it follows that $(F+g)$ is nondecreasing in g^P. Applying Proposition 2 again with $a^o \equiv v$ shows that $F + g$ is strictly increasing in g^P. □

We now prove the main theorem which says that employment goes up with g^T well as g^P and that the rise in employment is larger for a given increase in g^P as compared to the same increase in g^T. Let

(A12) $H(k,g^P,g^T) \equiv h(k,F(k,g^P,g^T) + g)$ where the optimal policy function F satisfies,

(A13) $$W_{k'}(k,F+g) + E[\beta v_k(F,g^{P\prime},g^{T\prime}) \mid g^P] \equiv 0, \ (k,g) \neq (\underline{k},g_{max}).$$

Theorem 1. For any (k,g^P,g^T) such that $(k,g) \neq (\underline{k},g_{max})$ and $\Delta \neq 0$,

$$\frac{1}{\Delta}[H(k,g^P+\Delta,g^T) > \frac{1}{\Delta}H(k,g^P,g^T+\Delta)] > \frac{1}{\Delta}H(k,g^P,g^T).$$

Proof. The FONC (A13) implies that when g^T goes up F + g must go up. Therefore, the second inequality in the theorem is established since $h(\cdot)$ is strictly increasing in (F+g) (see A5). In proving the first inequality note that g remains fixed at $(g^P+g^T+\Delta)$. Since v_k is strictly increasing in g^P the second term in (A13) goes up with g^P but does not depend on g^T. It follows that $F(k,g^P+\Delta,g^T) > F(k,g^P,g^T+\Delta)$. Therefore, the first inequality is established since $h(\cdot)$ is strictly increasing in (F+g) (again, from A5). □

References

Aiyagari, S. Rao. 1988. What Are the Output and Interest Rate Effects of Government Purchases? Federal Reserve Bank of Minneapolis Working Paper 411.

Aschauer, David A. 1985. Fiscal Policy and Aggregate Demand. *American Economic Review*, May, 75, 117–27.

Barro, R. J. 1981. Output effects of government purchases. *Journal of Political Economy*, vol. 89, no. 6.

Barro, R. J. 1987. *Macroeconomics*, 2nd edition, New York: John and Wiley and Sons.

Barro, R. J., and King, R. G. 1982. Separable preferences and intertemporal substitution models of business cycles. NBER Working Paper No. 888.

Barsky, R. B., and Miron, J. A. 1988. The seasonal cycle and the business cycle. NBER Working Paper No. 2688.

Baxter, M., and King, R. G. 1988. Multipliers in equilibrium business cycles models. Manuscript, University of Rochester.

Braun, A. 1989. Taxes and postwar US business cycle. Manuscript, Northwestern University.

Benveniste, L. M.; and Scheinkman, J. A. 1979. On the differentiability of the value function in dynamic models of economics. *Econometrica* 47, 3 (May).

Christiano, Lawrence J. 1987. Intertemporal substitution and the smoothness of consumption. Presented to the NBER's consumption study group, Philadelphia, October.

Christiano, Lawrence J. 1990. Linear–Quadratic Approximation and Value–Function Iteration: A Comparison, *Journal of Business and Economic Statistics*, Vol. 8, number 1, January, 99–113.

Christiano, L. J., and Eichenbaum, M. 1988. Is theory really ahead of measurement? Current real business of cycle theories and aggregate labor market fluctuations, NBER Working Paper No. 2700.

Cooley, Thomas F., and Hansen, Gary D. 1988. The inflation tax and the business cycle. Manuscript, University of Rochester.

Eichenbaum, M.; Hansen, L. P.; and Singleton, K. J. 1988. A time series analysis of representative agent models of consumption and leisure choice under uncertainty. *Quarterly Journal of Economics*, CH: 51–78.

Hall, R. E. 1980. Labor supply and aggregate fluctuations. *In On the State of Macroeconomics, Carnegie Rochester Conference on Public Policy*, XII, Amsterdam, North Holland.

Hamilton, James D. 1983. Oil and the macroeconomy since World War II. *Journal of Political Economy* 91, (April): 228–48.

Hadar, Josef and William R. Russell, 1969, Rules for Ordering Uncertain Prospects. *American Economic Review*, vol LIX, number 1, March.

Hanoch, G. and H. Levy, 1969, The efficiency analysis of choices involving risk. *The Review of Economic Studies* XXXVI (3) no. 107, July.

Hansen, Gary D. 1984. Fluctuations in total hours worked: A study using efficiency units. Manuscript, University of Minnesota.

Hansen, Gary D. 1985. Indivisible labor and the business cycle. *Journal of Monetary Economics* 16, 3 (Nov): 309–28.

Judd, K. L. 1985. Short–run analysis of fiscal policy in a simple perfect foresight model. *Journal of Political Economy* 93 (April): 298–319.

Kormendi, Roger C. 1983. Government Spending, Government Debt and Private Sector Behavior. *American Economic Review,* 73 994–1010.

Kydland, F. E. 1987. The role of money in a competitive model of business fluctuations. Manuscript, Carnegie Mellon University.

Kydland, F. E., and Prescott E. F. 1982. Time to build and aggregate fluctuations. *Econometrica* 50:6, 1345–70.

Long J., and Plosser C. 1983. Real business cycles. *Journal of Political Economy* 91, 1345–70.

Mankiw, N. G.; Rotemberg, J.; and Summers, L. H. 1985. Intertemporal substitution in macroeconomics. *Quarterly Journal of Economics* C: 225–52.

McGratten, Ellen, 1989, "The Macroeconomic Effects of Tax Policy in an Equilibrium Model," unpublished manuscript, Department of Economics, Duke University.

Wynne, M. 1988. The effects of government purchases in a perfect foresight model. Manuscript, University of Rochester.

FIGURE 1

DYNAMIC INVESTMENT PROBLEM

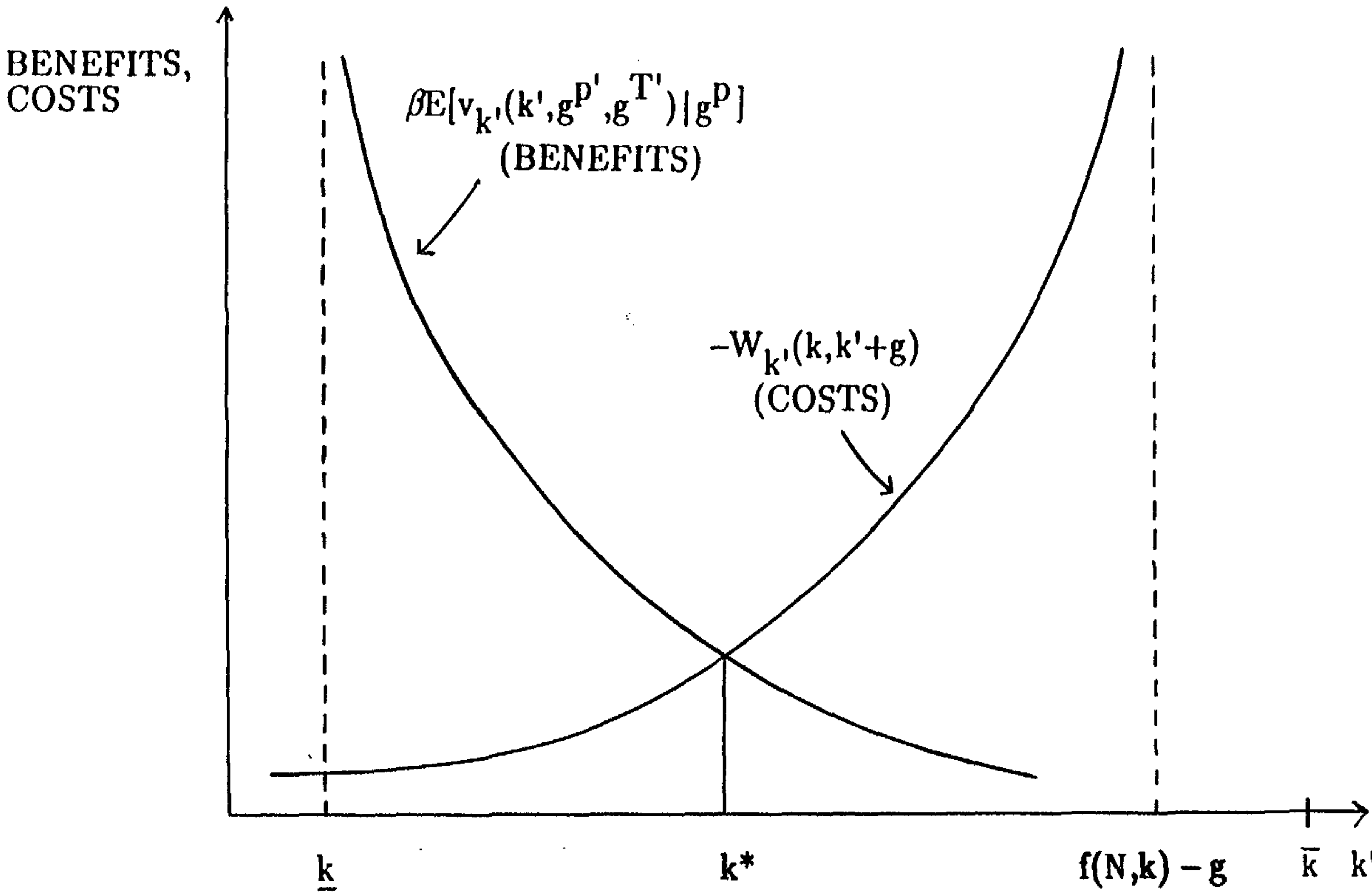

FIGURE 2

INVESTMENT EFFECTS OF TRANSITORY AND PERSISTENT GOVERNMENT CONSUMPTION SHOCKS