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CO-EXISTENCE OF A REPRESENTATIVE AGENT TYPE EQUILIBRIUM
WITH A NONREPRESENTATIVE AGENT TYPE EQUILIBRIUM

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ABSTRACT

We provide an example of an overlapping generations model with bequest motives and a nonnegativity constraint on bequests which has at least two equilibria. In one, the bequest motive is operative at all dates and the equilibrium is formally equivalent to that of a representative infinitely lived agent model. In the other equilibrium, the bequest motive is never operative and the equilibrium is formally equivalent to that of an overlapping generations model without a bequest motive. The example has obvious implications for the Ricardian equivalence doctrine and the neutrality of (lump-sum) tax-transfers across generations.

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I. Introduction

We provide an example of an overlapping generations (henceforth, OLG) model with bequest motives and non-negativity constraints on bequests which has at least two equilibria. In one the bequest motive is operative at all dates and the equilibrium is formally equivalent to that of a representative infinitely lived agent model. There is another equilibrium in which the bequest motive is never operative, and hence, the equilibrium is formally equivalent to that of an OLG model *without* a bequest motive.

At least since Barro [2], it is usual to interpret a representative infinitely lived agent model as one consisting of a sequence of bequest linked overlapping generations. Barro used his construction to exhibit the Ricardian equivalence doctrine regarding the neutrality of debt versus (lump-sum) tax financing, and the neutrality of (lump-sum) tax-transfers (say, a pay-as-you-go social security program) across generations. The example we construct has the following implications for these types of interpretations and neutrality results. Clearly, the set of equilibrium allocations and prices is not invariant to even marginal changes in deficit finance or tax-transfer policies¹. The use of the representative agent model may implicitly imply the use of a particular selection criterion for choosing among the equilibria of the underlying sequence of overlapping generations with very different positive and possibly normative implications. The justification offered in Barro [2] for the Ricardian equivalence doctrine may be less compelling even from a purely theoretical standpoint².

The construction of the example is based on the transfer paradox of international trade theory³. With at least two agents and two goods, a transfer from one agent to another agent may make the receiver worse off and

the giver better off due to strong income effects and the induced relative price shifts.

The formal treatment of overlapping generations models with bequest motives and non-negativity constraints on bequests is given in Aiyagari [1]. The interested reader is referred to that paper for formal definitions of and results concerning equilibrium and optimality. That paper also contained an example (example 1) of multiple equilibria of the following kind. There were two types of preference linked dynasties with different intergenerational discount factors. There was one equilibrium (which was optimal) in which one of the types was linked by positive bequests and was behaviorally like a single infinitely lived agent, whereas the other type made no bequests. There were two other equilibria (one of which was non-optimal) in which neither type made positive bequests. However, none of the equilibria examined in that example were of the representative agent type. The example in this note fills this gap in my earlier paper. The example follows.

II. The Example

The time horizon is infinite and time is indexed by t which takes values $1, 2, 3, \dots$. At each time t , one agent is born who lives for two periods. The agent born at t is referred to as being young at t and old at $t+1$. In addition, at $t=1$, there is an initial old agent who only lives for one period. There are two goods at each date indexed by $i \in \{1, 2\}$. We let $c_t(t) \in \mathbb{R}_+^2$ and $c_t(t+1) \in \mathbb{R}_+^2$ be the consumption vectors of the agent born at t at times t and $t+1$, respectively. The consumption vector of the initial old agent at time 1 is denoted $c_0(1) \in \mathbb{R}_+^2$. Let $w_1 \in \mathbb{R}_+^2$ and $w_2 \in \mathbb{R}_+^2$ be the (non-storable) endowment vectors of the agent born at time t at times t and

$t+1$, respectively. The endowment vector of the initial old agent at time 1 is w_2 .

Let $\beta \in (0,1)$ be an intergenerational utility discount factor. Let $U: R_+^2 \rightarrow R$, and $V: R_+^2 \rightarrow R$ be twice continuously differentiable, bounded, strictly increasing and strictly concave, within period utility functions. We also assume that the closure of each indifference curve for the utility function U (as well as V) is contained in R_{++}^2 . The preferences of the initial old agent are described by the functional: $V(c_0(1)) + \sum_{t=1}^{\infty} \beta^t [U(c_t(t)) + V(c_t(t+1))]$. The preferences of the agent born at time t are described by the functional: $\sum_{j=0}^{\infty} \beta^j [U(c_{t+j}(t+j)) + V(c_{t+j}(t+j+1))]$. The notation V' and U' will be used to denote the gradient vectors of the functions V and U , respectively.

Since the economy is one of pure exchange, there are no assets, preferences are stationary and time separable, and the aggregate endowment as well as its distribution among agents is constant over time, we will focus only on stationary equilibria. These can be characterized by using recursive, dynamic programming methods.

Let c_1, c_2 be the consumption vectors of the young and the old, respectively. Let $p \in S^2$ be the price vector in the two dimensional simplex, $\gamma > 0$ be the market discount factor, and $b \geq 0$ be the bequest made by one generation to the next. A stationary competitive equilibrium may be defined as follows.

Definition: A stationary competitive equilibrium consists of c_1, c_2, p, γ, b and a bounded function $W: R_+ \rightarrow R$ which satisfy the following conditions,

$$(1) \quad W(z) = \text{Max}_{x, y, z'} [U(x) + V(y) + \beta W(z')] \\ \text{subject to: } p(w_1 - x) + \gamma p(w_2 - y) + z = \gamma z' \\ x, y \geq 0; z, z' \geq 0.$$

- (ii) $(y, z) = (c_2, b)$ attains $\text{Max}_{y, z} [V(y) + \beta W(z)]$
 subject to: $p(w_2 - y) = z$
- (iii) $(x, y, z') = (c_1, c_2, b)$ attains $W(b)$ in (i)
- (iv) $c_1 + c_2 = w_1 + w_2$ (market clearing).

Condition (i) defines the value function for an agent born at t as a function of the bequest received. Condition (ii) defines the problem solved by the initial old and its solution at equilibrium. Condition (iii) says that consumptions and bequest given be optimal for an agent born at t given the bequest received.

It is easy to use the conditions in the above definition to give simple characterizations of stationary equilibria with and without positive bequests. It will be convenient to define the artificial two agent, two good pure exchange economy consisting of an old agent (with utility function V and endowment vector w_2) and a young agent (with utility function U and endowment vector w_1). This artificial economy will be referred to as the 2×2 economy. We will also use c_{11} and c_{21} to denote the consumption of good 1 by the young and the old, respectively.

Proposition 1: (i) A stationary equilibrium allocation with positive bequests is a (c_1, c_2) such that $(x, y) = (c_1, c_2)$ attains $\text{Max}[V(y) + \beta U(x)]$ subject to $x + y = w_1 + w_2$ and $V'(c_2)(w_2 - c_2) > 0$. The supporting prices are given by $\gamma = \beta$ and $p \propto V'(c_2)$. The equilibrium level of bequest is $b = p(w_2 - c_2)$.

(ii) A stationary equilibrium allocation with zero bequests is a competitive equilibrium allocation for the 2×2 economy such that $V'(c_2) \geq \beta U'(c_1)^4$. The price vector p is the equilibrium price vector in the

artificial economy and $\gamma = (\partial V / \partial c_{21}) / (\partial U / \partial c_{11})$.

Proof: (i) Using standard dynamic programming arguments, it is easy to show that a unique value function W exists which is continuously differentiable, strictly increasing and strictly concave. The FONC and the envelope condition in part (i) of the definition together with part (iii) imply that $\gamma = \beta$ and $V'(c_2) = \beta U'(c_1)$. Part (ii) of the definition implies $V'(c_2)(w_2 - c_2) > 0$. The rest follows. \square

(ii) Since $b = 0$, $p(w_2 - c_2) = 0$ and hence $p(w_1 - c_1) = 0$. The FONC for bequest must hold with a weak inequality in part (i) of the definition. The remaining FONC and the envelope condition in part (i) together with part (iii) of the definition imply that $V'(c_2) = \gamma U'(c_1) \geq \beta U'(c_1)$. Therefore, (c_1, c_2) and p constitute a competitive equilibrium for the artificial 2×2 economy. \square

Part (i) of the above proposition can be used to show how a positive bequest equilibrium is equivalent to that of a representative infinitely lived agent model. Let $w = w_1 + w_2$ and $\Psi(c, \beta) = \text{Max} [V(y) + \beta U(x)]$ subject to $y + x = c$. Then, the equilibrium prices and allocations in an economy with a representative infinitely lived agent whose preferences are given by $\sum_{t=1}^{\infty} \beta^{t-1} \Psi(c_t, \beta)$ and who has the constant endowment w in each period will be exactly the same as in a positive bequest equilibrium. It might appear that the same holds true in the zero bequest equilibrium. The equilibrium prices and allocations are the same as those in a representative agent economy with preferences $\sum_{t=1}^{\infty} \gamma^{t-1} \Psi(c_t, \gamma)$. The crucial difference is that in the positive bequest case the results of policy experiments involving marginal changes in financing or intergenerational transfers are unaffected by adopting a representative agent representation of the underlying economy. This is clearly not true for the zero bequest equilibrium. This fact also

shows up in the representative agent representation of this equilibrium. The utility discount factor γ is not the true intergenerational utility discount factor and is clearly not invariant to changes in financing or transfer policies. The period utility function $\Psi(c_t, \gamma)$ is also, obviously, not invariant to policy changes.

The following proposition serves as the basis for constructing our example.

Proposition 2: Let $\gamma > 0$ and let $c_1(\gamma), c_2(\gamma)$ solve: $\text{Max } [V(y) + \gamma U(x)]$ subject to $x + y = w_1 + w_2$. Suppose that for some $\hat{\gamma} \leq 1$, $c_1(\hat{\gamma}), c_2(\hat{\gamma})$ is a competitive equilibrium allocation for the 2×2 economy and $[\partial[V'(c_2(\gamma))(w_2 - c_2(\gamma))]/\partial\gamma]_{\gamma=\hat{\gamma}} < 0$. Then there is a $\beta \in (0, \hat{\gamma})$ for which the economy with bequest motives has at least two equilibria; one of these has positive bequests and the other has zero bequests.

Proof: Obviously, $V'(c_2(\hat{\gamma}))(w_2 - c_2(\hat{\gamma})) = 0$. Hence, there is some $\beta \in (0, \hat{\gamma})$ such that $V'(c_2(\beta))(w_2 - c_2(\beta)) > 0$. Therefore, the resulting allocations constitute a positive bequest equilibrium. Moreover, for this β , $c_1(\hat{\gamma})$ and $c_2(\hat{\gamma})$ constitute a zero bequest equilibrium allocation, since $V'(c_2(\hat{\gamma})) = \hat{\gamma}U'(c_1(\hat{\gamma})) > \beta U'(c_1(\hat{\gamma}))$. \square

The proposition is illustrated in Figure 1 which shows how the construction of such equilibria is equivalent to those exhibiting the transfer paradox in the 2×2 economy. The consumption allocation in the post-transfer equilibrium is the one corresponding to $\gamma = \beta$ and the consumption allocation in the pre-transfer equilibrium is the one corresponding to $\gamma = \hat{\gamma} > \beta$. As shown in the figure, the post-transfer equilibrium is the equilibrium obtained after a transfer of purchasing power

from the old to the young (the value of the old's endowment exceeds the value of their consumption and conversely for the young). In this equilibrium the old are better off and the young are worse off relative to the pre-transfer equilibrium.

 FIGURE 1 HERE

Our example is based on utility functions U and V which are separable in the two goods. This results in some simplification of the condition $\partial[V'(c_2(\gamma))(w_2 - c_2(\gamma))]/\partial\gamma < 0$ stated in Proposition 2. Therefore, let,

$$V(c_2) = \sum_1 V_1(c_{21}), \quad U(c_1) = \sum_1 U_1(c_{11})/\hat{\gamma}, \quad \hat{\gamma} \in (0, 1).$$

The allocations $c_1(\gamma)$, $c_2(\gamma)$ satisfy

$$V'_1(c_{21}) = \hat{\gamma}U'_1(c_{11})/\hat{\gamma}.$$

Differentiating with respect to γ and using the resource constraint we have,

$$\partial c_{21}/\partial\gamma = U'_1/[\hat{\gamma}V''_1 + \hat{\gamma}U''_1].$$

Let,

$$\alpha_{v1} = -c_{21}V''_1/V'_1, \quad \alpha_{u1} = -c_{11}U''_1/U'_1.$$

Hence,

$$\begin{aligned} (1) \quad \partial[V'(c_2(\gamma))(w_2 - c_2(\gamma))]/\partial\gamma &= \sum_1 [-V'_1 \partial c_{21}/\partial\gamma + (w_{21} - c_{21})V''_1 \partial c_{21}/\partial\gamma] \\ &= \sum_1 [U'_1/(\hat{\gamma}V''_1 + \hat{\gamma}U''_1)] [-V'_1 + (w_{21} - c_{21})V''_1]. \\ &= (1/\hat{\gamma}) \sum_1 U'_1 [1 + ((w_{21}/c_{21}) - 1)\alpha_{v1}] / [(\alpha_{v1}/c_{21}) + (\alpha_{u1}/c_{11})]. \end{aligned}$$

We now specify the endowments and preferences that satisfy the conditions of Proposition 2.

Let, $w_1 = (1.2, 0.8)$, $w_2 = (0.8, 1.2)$.

$$V_1(c_{21}) = c_{21}^{1-\alpha_{v1}} / (1-\alpha_{v1}), \quad U_1(c_{11}) = c_{11}^{1-\alpha_{u1}} / (1-\alpha_{u1})$$

$$\alpha_{v1} = 12, \quad \alpha_{u1} = 1.01, \quad \alpha_{v2} = 1.01, \quad \alpha_{u2} = 12.$$

It may be verified that $\gamma = \hat{\gamma}$ and the allocation $c_1 = c_2 = (1,1)$ satisfy the conditions of Proposition 2. For the given endowments this allocation together with the price vector $(0.5, 0.5)$ constitutes a competitive equilibrium in the 2×2 economy. This allocation also maximizes $[V(c_2) + \hat{\gamma}U(c_1)]$ subject to the resource constraint. Further, the expression on the right side of (1) is negative at this allocation, as required⁵.

Lastly, it may be noted that the example is robust to minor perturbations in the specifications of endowments and preferences.

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ENDNOTES

¹The neutrality results hold only if no one is bequest constrained before as well as after the policy change. It is possible that even if in the initial equilibrium bequests are positive, sufficiently large changes in such policies may lead to bequests being driven to zero and result in non-neutrality.

²Tobin [4, chapter III] has criticized Barro's revival of the Ricardian doctrine on grounds of the model's empirical limitations.

³See, for example, lecture 12 in Bhagwati and Srinivasan [3] and the references therein.

⁴If x and y are n -dimensional vectors then: (a) $x \geq y$ if $x_i \geq y_i$ for all i ; (b) $x > y$ if $x \geq y$ and $x \neq y$; (c) $x \gg y$ if $x_i > y_i$ for all i .

⁵Like most examples in economics, this one also violates some of the assumptions made on the period utility functions U and V . In the example, the utility functions are unbounded (below), and either undefined at zero or discontinuous at zero. These are not serious defects and can be corrected by altering the form of the functions outside an ϵ -neighborhood of the initial allocation.

Figure 1

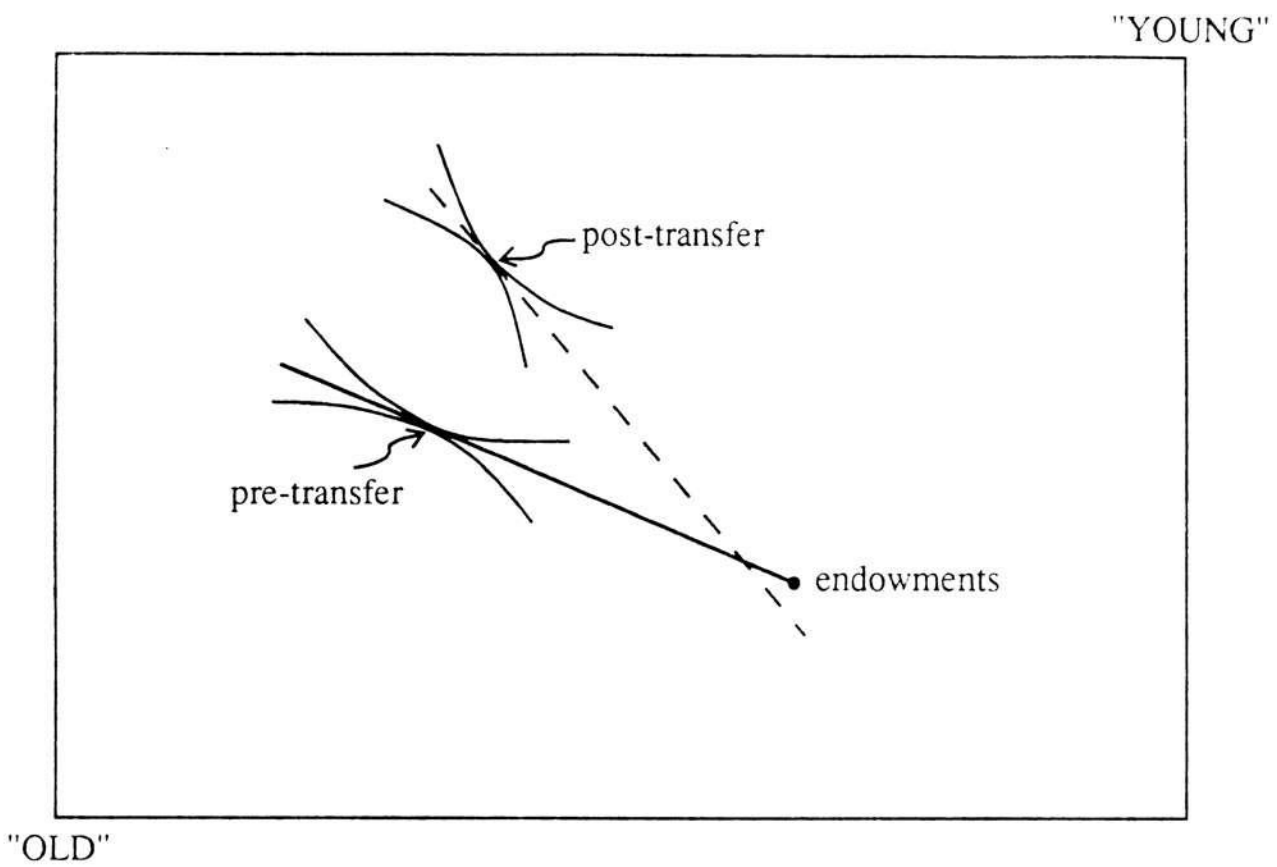


Figure 1

