TECHNICAL APPENDIX TO
OPTIMAL FISCAL POLICY IN A
STOCHASTIC GROWTH MODEL

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This appendix discusses the computation of the log–linear decision rules, the capital grid, the equilibrium debt rule and the ex post capital tax rates in the paper. The appendix also discusses our computational strategy for evaluating the implementability constraint.

A. The Decision Rules and the Capital Grid.

Log–linear decision rules are used to initiate the computations for finding the decision rules which solve the Ramsey allocation problem. This section shows how they are computed, and how they are used to construct our grid for the capital stock. The computational strategy outlined below actually differs somewhat from the one actually used in the project and explained in complete detail in Chari, Christiano and Kehoe (1991). The strategy described below, which follows Christiano (1990), yields numerically identical answers to the one actually used, but is simpler to present.

Log–Linearized Decision Rules

After substituting out for $c(s^t)$ from the resource constraint, (2.17)–(2.18) in the paper can be written as follows:

$$\text{(A.1)} \quad q(\tilde{k}_t, \tilde{k}_{t+1}, \tilde{\ell}_t, \tilde{g}_t, z_t) = 0,$$

(A.2) $E[h(\bar{k}_t, \bar{k}_{t+1}, \bar{k}_{t+2}, \bar{\ell}_t, \bar{\ell}_{t+1}, \bar{g}_t, \bar{g}_{t+1}, z_t, z_{t+1}) \mid \bar{k}_t, \bar{g}_t, z_t] = 0,$

for $t = 1,2,3,\dots$. Here, $\bar{k}_{t+1} \equiv \log k(s^t)$, $\bar{g}_t = \log g(s^t)$, $\bar{\ell}_t = \log \ell(s^t)$. Equations (A.1) and (A.2) are the intratemporal and intertemporal first order conditions, respectively. The functions q and h are replaced by Q and H, the linear Taylor–series expansions of q and h about the non–stochastic, steady–state values of their arguments. The non–stochastic version of the problem is obtained by fixing z_t and $\bar{g}_t$ at the unconditional mean of z_t and the log of the unconditional mean of $g(s^t)$, respectively. The functions Q and H are straightforward to construct given the parameters of our model. Our log–linear decision rules are found by solving the analogs of (A.1) and (A.2) with q and h replaced by Q and H.

Specifically, the condition $Q(\bar{k}_t, \bar{k}_{t+1}, \bar{\ell}_t, \bar{g}_t, z_t) = 0$ defines a linear function of $\bar{\ell}_t$, which can be used to substitute out for $\bar{\ell}_t$ and $\bar{\ell}_{t+1}$ in H. Call this new function $\bar{H}$. Substituting the linear expression, $\bar{k}_{t+1} = \alpha_0 + \alpha_1 \bar{k}_t + \alpha_2 \bar{g}_t + \alpha_3 z_t$, into $E[\bar{H} \mid \bar{k}_t, \bar{g}_t, z_t]$ gives:

$$E[H(\bar{k}_t, \bar{k}_{t+1}, \bar{k}_{t+2}, \bar{g}_t, \bar{g}_{t+1}, z_t, z_{t+1}) \mid \bar{k}_t, \bar{g}_t, z_t] = \Lambda_0 + \Lambda_1 \bar{k}_t + \Lambda_2 \bar{g}_t + \Lambda_3 z_t.$$

Here, we exploit the fact that $E[z_{t+1} \mid \bar{k}_t, \bar{g}_t, z_t]$ and $E[\bar{g}_{t+1} \mid \bar{k}_t, \bar{g}_t, z_t]$ are linear functions of z_t and $\bar{g}_t$. We then use the conditions $\Lambda_0 = \Lambda_1 = \Lambda_2 = \Lambda_3 = 0$ and $|\alpha_1\beta| < 1$ to determine values for α_0, α_1, α_2, α_3. The log–linearized decision rule is

(A.3) $k(s^t) = k(s^{t-1})^{\alpha_1} g_t^{\alpha_2} \exp\{\alpha_0 + \alpha_3 z_t\}.$

The ergodic set implied by this decision rule is defined by the maximum ($\equiv k_u$) and

minimum $(\equiv k_\ell)$ values of $\{g^{\alpha_2}\exp(\alpha_0+\alpha_3 z)\}^{1/(1-\alpha_1)}$ over the four possible combinations of (g, z).

The Capital Grid

The capital grid is constructed as follows. We first computed the 20 zeros of $T_{20}(x)$: $x_i \equiv \cos[\pi(i-.5)/20]$, $i = 1,...,20$. The capital grid is composed of $k_i = \exp\{.5[x_i(b-a) + (b+a)]\}$, $i = 1,...,20$. Here, $a = \log(.9k_\ell)$ and $b = \log(1.03k_u)$. The lower grid point for capital is only 90% of k_ℓ because the high period 1 ex ante tax rate on capital income leads to a very low period 1 capital stock. In all three of our model parameterizations, the ergodic set for capital lies strictly in the interior of the interval $[a,b]$.

B. The Equilibrium Debt Rule and Ex Post Capital Tax Rates.

This section describes our method for computing the equilibrium debt rule. We also describe the algorithm for computing the ex–post capital tax rates in the decentralization of the Ramsey allocations in which the return on government debt is not state–contingent. The ex ante capital tax rates can then be computed by substituting the ex post capital tax rates into the numerator of (3.10). Alternatively, the numerator in (3.10) may be computed directly from the solution to the Ramsey allocation problem using (3.3).

In the decentralization focused on here, $R_b(s^t,s_{t+1}) = \overline{R}_b(s^t)$ is determined by (3.2) and the Ramsey allocations. Consequently, from here on we treat it as a known quantity. We use (3.3) and (3.4) to compute the capital tax rates and the equilibrium debt rule. Before jumping into the details, we provide a brief sketch of

our procedure.

Fix some point in the state–space: k,s. Let the current date be denoted date t. Fix the piecewise linear debt rule which determines government debt at the end of period $t+1$: $b_{t+2} = \varphi(k_{t+1}, s_{t+1})$. (Here, $b(s^{t+1}) \equiv b_{t+2}$.) Then, the n period $t+1$ household budget constraints, together with the period t intertemporal Euler equation for capital, (3.3), are used to solve for the n period $t+1$ capital tax rates and the end–of–period t debt, b_{t+1}. Doing this for all m k's in the capital grid and all n values of s allows us to construct a new piecewise linear function, $b_{t+1} = \varphi'(k_t, s_t)$. The equilibrium debt rule we seek is a fixed point φ such that $\varphi = \varphi'$. The parameters of φ are an nm×1 vector b, which specify the value of φ at each of the nm possible combinations of k,s. Let b′ denote the nm × 1 vector of parameters of φ'. It turns out that the map from b to b′ is linear, being characterized by b′ = Z + Bb, where Z and B are nm × 1 and nm × nm, respectively. Thus, finding the fixed point that interests us is equivalent to finding a vector b such that b′ = b. This in turn may be found by solving a set of nm linear equations. Describing the construction of Z and B is the principle subject of the remainder of this section. The capital tax rule corresponding to the fixed point φ function is denoted by $\theta_{t+1} = \theta(k_t, s_t, s_{t+1})$.

We begin by developing a matrix representation of (3.2) and (3.4). Let $i \,\epsilon\, \{1,...,n{\times}m\}$ index the (k,s) combinations that we consider and fix some value of i. The solution to the Ramsey allocation problem, (2.1), (3.1), and the posited φ function deliver the n possible values of $c_{t+1} + k_{t+2} + b_{t+2} - [1 + F_{k,t+1} - \delta]k_{t+1} - (1 - \tau_{t+1})w_{t+1}\ell_{t+1}$. (Here, $k_{t+1} \equiv k(s^t)$, $c_t \equiv c(s^t)$, $F_{k,t} \equiv F_k(s^t)$, $\tau_t \equiv \tau(s^t)$, $w_t \equiv w(s^t)$.) Denote these by the n×1 vector $c_1^{(i)}$. Let $D^{(i)}$ denote the n×n diagonal matrix constructed from the n values of $F_{k,t+1} - \delta$ and let $\gamma^{(i)}$ denote the n×1 vector with each element equal to $R_{b,t+1}$. Also let the, as yet undetermined, n

values of θ_{t+1} be denoted by the n×1 vector $\theta^{(i)}$. Then, conditional on state i occurring at date t, the date t+1 household budget equation is written

$$\text{(B.1)} \quad c_1^{(i)} = D^{(i)}\theta^{(i)} + \gamma^{(i)}b^{(i)},$$

where $b^{(i)}$ is the as yet unknown value of the debt held at the end of period t.

The value of c_t, k_{t+1}, ℓ_t and $U_{c,t}$ can be computed from the Ramsey allocation problem. Let the scalar, $c_2^{(i)}$, denote $U_{c,t} - \Sigma_{s_{t+1}} \beta\mu(s_{t+1}|s_t)[F_{k,t+1} + 1-\delta]$. Let $\psi^{(i)}$ denote the 1×n vector composed of the n possible values of $-\mu(s_{t+1}|s_t)(F_{k,t+1} - \delta)$. Then, the capital Euler equation is written

$$\text{(B.2)} \quad c_2^{(i)} = \psi^{(i)}\theta^{(i)}.$$

For later reference, it is useful to express k_{t+1} as a linearly interpolated function of the elements of the capital grid, $\underline{k} = (k_1,...,k_m)'$. In particular, there is some j(i) and $0 \leq \omega_i \leq 1$ such that $k_{t+1} = \omega_i k_{j(i)} + (1-\omega_i)k_{j(i)+1}$. Denote the 1×m vector with ω_i and $(1-\omega_i)$ in the j(i) –th and j(i)+1 –th locations, and zeros everywhere else by G_i. Then, $k_{t+1} = G_i\underline{k}$. The capital grid is constructed so that no $k_{t+1} > k_m$ or $k_{t+1} < k_1$ is ever encountered. Consequently, if j(i) = m, then $1-\omega_i = 0$. Also, $w_i > 0$ for all i.

Substitute out for $\theta^{(i)}$ in (B.2) from (B.1) and rearrange, to get

$$\text{(B.3)} \quad b^{(i)} = \frac{\psi^{(i)}[D^{(i)}]^{-1}c_1^{(i)} - c_2^{(i)}}{\psi^{(i)}[D^{(i)}]^{-1}\gamma^{(i)}}.$$

Now write

$$(B.4) \quad c_1^{(i)} = d_1^{(i)} + \tilde{b}^{(i)},$$

where $\tilde{b}^{(i)}$ denotes the n×1 vector composed of the n values of b_{t+2}.

Write the nm×1 vector of as yet undetermined b_{t+1}'s as

$$b' = \begin{array}{l} b^{(1)} \\ \vdots \\ b^{(nm)} \end{array} \quad .$$

Thus, stacking (B.3) and taking (B.4) into account, we get

$$(B.5) \quad b' = Z + \begin{bmatrix} \dfrac{\psi^{(1)} [D^{(1)}]^{-1} \tilde{b}^{(1)}}{\psi^{(1)} [D^{(1)}]^{-1} \gamma^{(1)}} \\ \vdots \\ \dfrac{\psi^{(nm)} [D^{(nm)}]^{-1} \tilde{b}^{(nm)}}{\psi^{(nm)} [D^{(nm)}]^{-1} \gamma^{(nm)}} \end{bmatrix},$$

where,

$$Z = \begin{bmatrix} \dfrac{\psi^{(1)} [D^{(1)}]^{-1} d_1^{(1)} - c_2^{(1)}}{\psi^{(1)} [D^{(1)}]^{-1} \gamma^{(1)}} \\ \vdots \\ \dfrac{\psi^{(nm)} [D^{(nm)}]^{-1} d_1^{(nm)} - c_2^{(nm)}}{\psi^{(nm)} [D^{(nm)}]^{-1} \gamma^{(nm)}} \end{bmatrix}$$

Next, we show how $\tilde{b}^{(i)}$ is linearly related to the parameters of the period t+1 debt rule, $\varphi(k_{t+1}, s_{t+1})$, which is parameterized by the nm×1 vector b. Write $b = (b_1', b_2', ..., b_m')'$, where b_j is n×1 for j = 1,...,m. The vector b_j denotes the n values of $b_{t+2} = \varphi(k_{t+1}, s_{t+1})$ corresponding to each of the n possible values of s_{t+1} for $k_{t+1} = k_j$, the j–th point on the capital grid. Now by construction, k_t is on the capital grid. However, typically k_{t+1} will not be. Instead, as noted previously, there will be some j and $0 \le \omega \le 1$ such that $k_{t+1} = \omega k_j + (1-\omega)k_{j+1}$. In this case, $\varphi(k_{t+1}, s_{t+1}) \equiv \omega\varphi(k_j, s_{t+1}) + (1-\omega)\varphi(k_{j+1}, s_{t+1})$. The n vector of all possible b_{t+2}'s is just $\omega b_j + (1-\omega)b_{j+1}$. This reasoning and an earlier discussion leads to the following result:

(B.6) $\quad \tilde{b}^{(i)} = [G_i \otimes I_m]b,$

where $\otimes$ denotes the Kronecker product, I_m is the m×m identity matrix, and G_i was defined earlier. Substituting this into (B.5), we get

(B.7) $\quad b' = Z + Bb,$

where

$$B = \begin{bmatrix} \dfrac{\psi^{(1)}[D^{(1)}]^{-1}}{\psi^{(1)}[D^{(1)}]^{-1}\gamma^{(1)}}\,(G_1 \otimes I_m) \\ \vdots \\ \dfrac{\psi^{(nm)}[D^{(nm)}]^{-1}\tilde{b}^{(nm)}}{\psi^{(nm)}[D^{(nm)}]^{-1}\gamma^{(nm)}}\,(G_{nm} \otimes I_m) \end{bmatrix}$$

The fixed point of (B.7) is given by

$$\text{(B.8)} \quad b = (I - B)^{-1}Z.$$

To find $\theta^{(i)}$, simply solve (B.1) using (B.4):

$$\text{(B.9)} \quad \theta^{(i)} = [D^{(i)}]^{-1}[d_i^{(i)} + \tilde{b}^{(i)} - \gamma^{(i)}b^{(i)}]$$

$$= [D^{(i)}]^{-1}\{d_i^{(i)} + [(G_i \otimes I_m) - \gamma^{(i)}\tau_i]b\},$$

where τ_i is a 1×nm vector with a 1 in the i–th location and zeros elsewhere. Equation (B.9) for i = 1,...,nm yields a rule for the ex post tax rate on date t+1 capital earnings, $\theta(k_t, s_t, s_{t+1})$. To evaluate this function at points not on the capital grid, we use linear interpolation.

To compute b and $\theta^{(1)},...,\theta^{(nm)}$ for the baseline model requires nearly 3 minutes of CPU time using MATLAB on a DOLCH computer with a 386–chip, 25 Mhz and a math co–processor. When these objects are computed using the log–linearized decision rule for capital and hours worked, the CPU time is around 50 seconds. (The largest eigenvalue (in an absolute value sense) of B is .97.) One of the reasons for this substantial reduction in computer time is that the time–intensive computations involve solving the non–linear equation, (2.17), to compute equilibrium hours worked.

C. Evaluating the Implementability Constraint.

Our strategy for solving the Ramsey allocation problem solves (2.15) conditional on a value for λ. Given this solution, the objects on the left and the right sides of the equality in (2.5) can be evaluated. The equality is not be satisfied for an arbitrary value of λ. We repeatedly solve (2.15) for different values of λ until one is found which causes the equality to be satisfied within a specified level of accuracy. A direct way to evaluate the implementability constraint, (2.5), is by Monte Carlo simulation. However, this procedure seems inefficient, as it requires a very large number of simulations to compute the left and right sides of (2.5) to an acceptable level of accuracy. We therefore adopted the following procedure instead.

Note that $b(s^0)$ can be computed from the solution to the Ramsey allocation problem using the date 0 household budget constraint, (1.3). Call this value of the debt b_1^f. At the same time, given that the debt carried out of period 1 satisfies the steady–state debt rule computed in section B, it is possible to compute a value for the debt carried out of period 0 working backward using the strategy outlined there. Call this debt level b_1^b. This is found by solving the appropriate analogs of (B.1) and (B.2), with $b^{(i)} = b_1^f$, and where $\theta^{(i)}$ is the vector of n period 1 capital tax rates in the decentralization in which government debt returns are not state–contingent and capital tax rates are. The index i corresponds to the k, s pair which obtains in period 0. In particular,

$$(\text{C.1}) \quad b_1^b = \frac{\boldsymbol{\psi}^{(i)}[\mathbf{D}^{(i)}]^{-1}\{\mathbf{d}_1^{(i)} + [G_i \otimes I_m]\mathbf{b}\} - \mathbf{c}_2^{(i)}}{\boldsymbol{\psi}^{(i)}[\mathbf{D}^{(i)}]^{-1}\boldsymbol{\gamma}^{(i)}},$$

where b is given by (B.7) and G_i is a 1×m vector discussed in section B. In (C.1), $\boldsymbol{\psi}$, $\mathbf{d}_1$, $\mathbf{D}$, $\boldsymbol{\gamma}$, and $\mathbf{c}_2$ have been bolded, in order to differentiated them from their un–bolded counterparts in equations (B.1) and (B.2). The reason they need to be

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differentiated is that the latter are computed using the stationary–state decision rules for capital and labor, which are active from period 1 on. The bolded objects in in equation (C.1) are computed using the period 0 capital and hours worked decisions. When $b_1^f = b_1^b$, then the implementability constraint is satisfied. It is clear that $b_1^f - b_1^b = f(\lambda)$, where f is quite complicated. We solve the Ramsey allocation problem by finding a λ such that $f(\lambda) = 0$. To evaluate f once when the log–linearized decision rules are used requires around 70 seconds of CPU time, while these computations requires around 7 hours using the method underlying the calculations in the paper. All calculations were done in 386MATLAB using a DOLCH computer with 386 chip, 25 Mhz and a math co–processor.

References

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