

Federal Reserve Bank of Minneapolis
Research Department

Patterns of Exchange, Fiat Money, and the Welfare Costs of Inflation

Irasema Alonso*

Working Paper 510

March 1993

ABSTRACT

We seem to observe different patterns of exchange at different times and in different places. The first goal of this paper is to develop a model of money as a medium of exchange which allows multiple transaction patterns. A dynamic version of Shubik's trading post economy is used, and it is shown that this economy allows a role for fiat money, and that fiat money can coexist with barter in exchange. There are multiple decentralized equilibria, and one of these resembles the equilibrium of a cash-in-advance economy—indeed, the model can be viewed as a generalization of the cash-in-advance framework. The second goal of the paper is to show that the present model can help explain why inflation seems far more disruptive and costly than what is implied by empirical studies based on the cash-in-advance model. The argument for this is based on misestimations due to the unobservability of the patterns of exchange, which are variable in this model.

*Alonso, Federal Reserve Bank of Minneapolis and the University of Minnesota. I am grateful to my advisor Neil Wallace for his insightful comments and generous advice. I also want to thank Nobu Kiyotaki and Per Krusell for helpful suggestions. Financial support from Banco de España and the Fulbright Commission is gratefully acknowledged. The views expressed herein are those of the author and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1 Introduction

In this paper we develop a model of exchange based on Shubik's (1990) trading post framework. The model can be viewed as a generalization of the cash-in-advance (CIA) model. It relaxes the assumption "only money buys goods" in that model. It is desirable to relax the CIA assumption because we seem to see different patterns of exchange at different times and in different places. Relaxing the assumption also has consequences for measuring the welfare costs of inflation. In particular, the more general model can help explain why inflation seems more disruptive and costly than is implied by calibrated versions of the CIA model (see, for example, Cooley and Hansen (1989, 1991)).

The framework used is an infinite horizon, pure exchange, discrete time version of the trading post game studied by Shubik.¹ This game assumes that only two objects can be exchanged at a given trading post, and that there is one round of simultaneous trade at all trading posts in each period. Moreover, anonymity is assumed, thus implying that agents are not able to use credit in acquiring goods. The game played is noncooperative, with agents submitting nonnegative quantity offers of objects—consumption goods or fiat money—at each trading post. The CIA model shares these assumptions but adds that only certain trading posts are active, namely those where money is one of the

¹The closest relatives of the present framework, naturally, can be found in Shubik's own work. He and various collaborators have studied many aspects of the basic trading post game. However, in none of that work is there an infinite horizon with valued fiat money nor a study of the welfare costs of inflation. Hayashi and Matsui (1992) also explored a similar modeling approach.

objects. The more general setup here has multiple equilibria, among which we focus on two which have valued fiat money: a cash-in-advance equilibrium (CIA), in which only the trading posts where money is one of the objects are active, and the all markets equilibrium (AM) where monetary exchange coexists with barter and all trading posts are active.

We study the CIA and the AM equilibria under two money creation schemes. In the first, we assume that changes in the money stock are implemented using lump-sum taxes or transfers in terms of money. In this type of policy, the growth rate of the money stock is exogenous. In the other we assume that increases in the money stock finance a nonnegative real revenue.

For this experiment, we find that the number of equilibria and their welfare properties depend on the growth rate of money. When the growth rate exceeds a critical level, equilibria where money coexists with barter no longer exist, and a pure barter equilibrium dominates the CIA equilibrium. We also find that the Friedman rule is optimal in this economy.

When the real revenue collected by the government is the policy target, we find that any given positive real revenue is associated with a higher level of inflation and a lower welfare in the AM equilibrium than in the CIA equilibrium. That inflation is higher in the AM equilibrium is not surprising—there is more “avoidance” of the inflation tax in this equilibrium.

We take the view that our observations of actual economics leave open whether we are observing the CIA or the AM equilibrium. If it is presumed that the CIA equilibrium always occurs some errors may be made. In particular, I discuss two possible errors.

One error may arise in the interpretation of the data. If the available

observations reflect differences in exchange patterns, an econometrician basing his views on the CIA model will misestimate the preference parameter reflecting the elasticity of substitution between the "cash good" and the "credit good." This bias will lead to incorrect estimates of the welfare costs of inflation. We show that the effect of this bias can lead to either over- or underestimation of the welfare costs depending on parameters, on the data we have, and on which equilibrium generated those data.

The second problem is that even if we know the true parameter values, a literal interpretation of the multiplicity of equilibria says that we do not know what exchange pattern to expect for the contemplated policies. Assume, as an example, that the situation without the inflation tax is associated with the CIA equilibrium, and the situation with the inflation tax is associated with an equilibrium with money and barter. Now the experiment of using the inflation tax to replace other forms of revenue collection will lead to a much larger cost of inflation than if there were no change in exchange patterns: a higher inflation rate is needed to collect the given revenue, and this, as we will see, implies a higher distortion. Thus, it is possible that the use of the CIA framework understates the welfare costs of inflation; from the point of view of my model, its use amounts to selecting a particular equilibrium, namely, the equilibrium which has associated with it the lowest cost of generating revenues using the inflation tax.

Sections 2 and 3 set out the basic model in some generality. Section 4 then proceeds to describe equilibria for some example economies when the policy target is the money growth rate. Section 5 discusses the case of a real revenue target and the welfare costs of using the inflation tax. Section

6 concludes.

2 The environment

Time is discrete. The economy is one of pure exchange, and there is a large number of infinitely-lived agents. The agents are endowed with consumption goods which perish between periods; there is no intertemporal technology. In addition, each agent has an initial endowment of fiat money which can be costlessly stored over time. A given agent's preferences are defined over streams of consumptions of the different goods. We will assume that preferences can be described by a time-separable utility function with a constant discount factor. Throughout the paper, we will focus on a setup with I types of equally numerous agents, and I types of consumption goods. The type heterogeneity among agents takes the form of differences in preferences for goods and endowments of goods. The specific examples we will display have symmetry over agents and goods.

As in Shubik's static economy, trade takes place at trading posts. At a given trading post two objects are exchanged, and an implicit price emerges as a ratio between the total quantity offers of each of the two objects. No more than two objects can be exchanged at a given trading post. Each period there is one round of simultaneous trade at all trading posts: agents submit their offers and then receive objects in proportion to these offers where the proportion is an exchange ratio determined by the offers at each post. If no offers are made on one side of a market, the offers on the other side of the market are lost to the agents who made the offers. A situation can occur in

which no offers are made on either side of the market; in this case the term “inactive” market is used.

The restriction to one simultaneous round of pairwise exchange is important; it implies that offers, and hence total quantities purchased in each market, are constrained by endowments. In particular, the proceeds of a sale at one trading post cannot be offered for sale at a different trading post during the same period. The restriction to bilateral exchange is motivated by observed exchange patterns rather than derived as a function of economic fundamentals. The simultaneity of trade at the different trading posts has been relaxed by Dubey, Sahi and Shubik (1989). They found that an increase in the number of trading rounds each period will, under certain conditions, allow the competitive equilibrium allocation to be achieved.

The anonymity assumption implies that agents cannot issue personal debt contracts. The purpose for using this assumption is to rule out all (other than monetary) contracts which would be used by agents to partially or fully overcome the frictions implicit in the one-shot, simultaneous nature of the trading post game. Explicitly modelling credit is not straightforward. It would require assumptions about where and when redemption of private debt contracts (IOUs) take place, and these assumptions should respect the frictions implicit in the trading post framework. It should also be pointed out that if private credit were modelled as occurring entirely through trading post transactions, then the equilibria of the present model would also be equilibria in the model with private credit. This result is due to the multiplicity property of the Shubik framework, which will be discussed in more detail below.

The large number of agents assumed here differs from most of Shubik's work and is not a necessary element in the analysis of fiat money. The assumption is used for the purpose of tractability; it implies that the exchange ratios can be treated as parameters by the agents. Throughout the analysis, attention will be limited to equilibria in which agents of the same type behave the same way.

The government here has the ability to print fiat money and to implement lump-sum taxation/transfers in terms of fiat money. The government has no endowment of goods, and government consumption is made possible by submitting offers of fiat money in exchange for goods at the trading posts. We specify government behavior by taking as given a subset of its choice variables. Two different kinds of policies are considered. In one, the growth rate of the money stock is exogenous, and any changes in the money stock are implemented by lump-sum taxes/transfers of money. In this case, the government does not consume. In the other, the real revenue of the government is exogenous, and money printing finances the purchases of goods. Our assumptions are consistent with a lack of ability to transfer goods directly between consumers, i.e., to overcome the trading friction.

3 Nash Equilibrium

A simple Nash equilibrium concept is used. Private agents make offers to maximize their utility, taking other agents' offers and the government policy as given. Shubik shows that for the static economy and a Nash equilibrium concept, there are many equilibria that differ with respect to the set of mar-

kets that are active. It is possible to arbitrarily specify a set of trading posts to be inactive, and there will generally exist such an equilibrium. The reason, clearly, is that if no one makes an offer at a particular trading post, there is no incentive for a given agent to do so. This feature will be shared by the dynamic version of Shubik's model studied here. In fact, the multiplicity of Nash equilibria in this version is an order of magnitude larger. For each sequence of specifications of trading posts to be closed at each date, there will generally be a corresponding equilibrium.

We will now provide a formal definition of a Nash equilibrium in the dynamic trading economy with fiat money. This definition assumes that agents of the same type behave the same way, thus permitting us to treat agents and types synonymously.

Before displaying the formal definition, we need some notation. There is a finite number of types of agents, with a continuum of equal measure of each type. The types are indexed by i , with i belonging to the set of integers from 1 to I . We denote the set of agent types \mathcal{I} . There is also a set of objects indexed by j , and the set of objects, \mathcal{J} , which is the set of integers from 1 to $J + 1$. The first J objects are consumption goods, and object $J + 1$ is fiat money. Whenever possible, the subindex m will be used to refer to money. We measure money in fractions of the total stock of money in the current period. Finally, let the superindex G refer to the government.

We define

w_j^i = endowment of object j of a type i agent

$b_{j,j'}^i$ = offer of object j in the market of j and j' by a type i agent

- $p_{j,j'}$ = price, or exchange ratio, of object j in terms of object j'
in the market of j and j'
 m^i = holdings of fiat money of a type i agent
 g = growth rate of the total stock of money
 τ = lump-sum tax in terms of fiat money

for $j \neq j'$. Quantities are measured in units of goods except in the case of money.

The market of object j and object j' ($j \neq j'$) is said to be active if at least one type of agent on each side of the market offers a positive amount, i.e., if $b_{j,j'}^i > 0$ and $b_{j',j}^{i'} > 0$ for some i, i' with $i \neq i'$.

Let S be the set of pairs $s = (j, j')$ s.t. $j, j' \in \mathcal{J}, j \neq j'$. Note that the set of all possible trading posts is the set of members of S with the convention $j < j'$.

A consumer of type i solves the following problem:

The Problem of the Consumer:

$$\begin{aligned}
 & \max_{m^i, x, b} \sum_t \beta^t u^i(x_{1,t}, x_{2,t}, \dots, x_{J,t}) \text{ s.t.} & (1) \\
 & x_{j,t} = w_{j,t}^i + \sum_{s=1}^{J-1} (b_{j+s,j,t} p_{j+s,j,t} - b_{j,j+s,t}) + b_{m,j,t} p_{m,j,t} - b_{j,m,t} \\
 & g_{t+1} m_{t+1} = m_t + \sum_{j=1}^J b_{j,m,t} p_{j,m,t} - \sum_{j=1}^J b_{m,j,t} - \tau_t^i \\
 & \sum_{s=1}^{J-1} b_{j,j+s,t} + b_{j,m,t} \leq w_{j,t}^i
 \end{aligned}$$

$$\sum_{j=1}^J b_{m,j,t} \leq m_t$$

$$b_{s,t} \geq 0$$

$$m_0 \leq \bar{m}_0^i.$$

Here, variables with suppressed subscripts describe vectors. We also use the convention that $j + s$ is modulo J .

The first equation of the constraint set expresses the amount of final consumption of good j after the exchanges from all the trading posts that involve good j have taken place. The second equation links periods; it says that the fiat money carried to the next period is the sum of the money proceeds from the money trading posts, the money that was not spent in the current period, and any lump-sum taxes or transfers. Notice that g_{t+1} renormalizes this sum into m_{t+1} , which is measured in terms of the total stock of money next period. The two following equations capture the idea of the one shot game and the ruling out of short sales. The offer of each good (of money) in the current period cannot exceed the endowment (initial holding) of this good (of money). The next set of equations require that bids be non-negative, and the last equation specifies the initial holdings of money.

We now specify the elements of government policy.

Definition 1 *A monetary policy is a specification of sequences of government revenues $\{R_{j,t}\}$ for $j = 1, \dots, J$, lump-sum taxes $\{\tau_t^i\}$ for $i \in \mathcal{I}$, money growth rates $\{g_t\}$ and offers of money for goods $\{b_{m,j,t}^G\}$ for $j = 1, 2, \dots, J$.*

Definition 2 Given the sequences of prices of the money stock $\{p_{m,j,t}\}$ for $j = 1, 2, \dots, J$, a monetary policy is feasible if it satisfies

$$p_{m,j,t} b_{m,j,t}^G = R_{j,t} \text{ for } j = 1, 2, \dots, J$$

and

$$\sum_{j=1}^J b_{m,j,t}^G = (g_t - 1)/g_t + \sum_{i \in \mathcal{I}} \tau_{t-1}^i / g_t.$$

The equilibrium can therefore be defined as follows.

Definition 3 Given the policy sequences $\{R_{j,t}\}$, with $j = 1, 2, \dots, J$ and $\{\tau_t^i\}$ with $i \in \mathcal{I}$, a Nash equilibrium is a collection of sequences of prices $\{p_{j,j',t}\}$ with $j, j' \in \mathcal{J}, j \neq j'$, sequences of bids $\{b_{j,j',t}^i\}$ with $j, j' \in \mathcal{J}, j \neq j'$, consumption allocations $\{x_{j,t}^i\}$ with $i \in \mathcal{I}$ and $j = 1, \dots, J$, holdings of money $\{m_t^i\}$ with $i \in \mathcal{I}$, money growth rates $\{g_t\}$, and government bids $\{b_{m,j,t}^G\}$ $j = 1, 2, \dots, J$ with $j \neq j'$ such that

1. consumers solve (??)
2. prices satisfy

$$p_{j,j',t} = \begin{cases} \sum_{i \in \mathcal{I}} b_{j',j,t}^i / \sum_{i \in \mathcal{I}} b_{j,j',t}^i & \text{if } \sum_{i \in \mathcal{I}} b_{j',j,t}^i > 0 \\ 0 & \text{if } \sum_{i \in \mathcal{I}} b_{j',j,t}^i = 0 \end{cases} \quad (2)$$

3. the monetary policy given by $\{R_{j,t}\}$, $\{\tau_t^i\}$, $\{b_{m,j,t}^G\}$, and $\{g_t\}$ is feasible, given the sequences of prices of the money stock $\{p_{m,j,t}\}$.

Note that the equilibrium satisfies feasibility by construction.²

²Also note that the requirement that prices be zero whenever one side of a trading post lacks offers is not restrictive in the following sense. Any equilibrium with this convention is also an equilibrium with the alternative convention that offers on a given side of the trading post are returned if nothing is offered on the other side of the trading post.

An equilibrium where fiat money is used in exchange, i.e., such that $p_{m,j,t} > 0$ for some j and for some t , is called a monetary equilibrium, and we refer to an equilibrium where fiat money is never valued as a non-monetary equilibrium.

Notice that a particular set of government variables are specified as exogenous in the equilibrium definition, namely, the real revenues and the lump-sum taxes/transfers of fiat money. The remaining elements of government policy, e.g., the monetary growth rates, are therefore endogenously determined. In subsequent sections, we will also consider situations in which the growth rates of the money stock are exogenous, and in those contexts we are presuming that the equilibrium is redefined accordingly.

4 Equilibria for some example economies

This section uses special assumptions on preferences and endowments which imply the existence of symmetric, stationary equilibria. The purpose of the specialized setup is to allow for, in a very simple way, the possibility that there are two types of equilibria: one with goods vs. money trade only—the CIA equilibrium—and one with all trading posts active—the AM equilibrium. Both these equilibria hence have valued fiat money, but the trading patterns differ. The following section then uses this setup to show how the two kinds of possible exchange patterns may induce the econometrician who bases his estimations on the CIA model to mismeasure the welfare costs of inflation.

The preferences and endowments patterns studied here are among the simplest with sufficient heterogeneity to permit there to be absence of dou-

ble coincidence of wants. Thus, we will assume that preferences, endowments, and the government policy are stationary, and that there is a certain symmetry in preferences and endowments across the different types of agents. These features are not essential to the theory, but do, however, facilitate the description of some of the equilibria. The symmetry assumption makes it possible to make comparisons with the representative agent CIA model, and it also allows equilibria to be Pareto ranked, which simplifies the welfare analysis.

In this section we assume that the government does not consume, so that there is no government participation at the trading posts. The monetary policy is therefore simply to implement changes in the money supply via fiat money transfers to the consumers.

The endowment distribution has the form:

$$\begin{pmatrix} w & \epsilon_2 & \epsilon_1 \\ \epsilon_1 & w & \epsilon_2 \\ \epsilon_2 & \epsilon_1 & w \end{pmatrix}$$

where column i of the matrix represents agent i 's endowments of goods 1, 2, and 3, respectively. Consumers are also (symmetrically) endowed with an amount of fiat money. The rate of growth of the money supply, g where $g \geq \beta$, is constant over time, and in order to preserve symmetry the money transfers, τ , are distributed equally among agents.

In Alonso (1992), a number of special cases of this setup are studied. In the first one of these—the leading example—it is assumed that $c_1 = c_2 = 0$ and that consumers care only about goods i and $i + 1$. In the second, $c_1 > 0$ and $c_2 = 0$, and in the third, $c_1 = 0$ and $c_2 > 0$ with the same preferences

as in the first setup in both cases. Finally a version is considered with $\epsilon_1 = \epsilon_2 = 0$ and preference for all goods. Conditions under which stationary, symmetric monetary and non-monetary (barter or autarky) equilibria exist are provided, and welfare comparisons are made across equilibria in each case. For the leading example and small perturbations around it, monetary equilibria exist and dominate equilibria with other patterns of exchange. For the present purposes, we only consider one of the above examples.

Assumption: $\epsilon_1 = 0$ and $\epsilon_2 = \epsilon > 0$

The endowments are therefore:

$$\begin{pmatrix} w & \epsilon & 0 \\ 0 & w & \epsilon \\ \epsilon & 0 & w \end{pmatrix}$$

i.e., agent i is endowed with w units of good i and ϵ units of good $i + 2$. We assume that agent i cares about goods i and $i + 1$, but not about good $i + 2$ (modulo 3). Agent i 's instantaneous utility is given by a function $u^i : R_+^2 \rightarrow R$ of x_i and x_{i+1} (modulo 3) with the following properties: u_i is bounded, continuously differentiable, strictly increasing, and strictly concave, and for all $w > 0$,

$$\begin{aligned} \lim_{x_i \rightarrow 0} V^i(x_i, w - x_i) &= \infty \\ \lim_{x_i \rightarrow w} V^i(x_i, w - x_i) &= 0 \end{aligned}$$

where V^i is the marginal rate of substitution function of agent i . Finally, it is assumed that x_i and x_{i+1} are normal goods for the type i agent.

If $\epsilon = 0$, there could not be any trade among individuals without an outside medium of exchange. Type i , the only holder of good i , would like to

trade good i only for good $i + 1$. Since this holds for all i , autarky is the only equilibrium candidate. The example illustrates, in the context of pairwise trading of goods in markets, the well-known problem of absence of double coincidence of wants. At each trading post one side of the market wants what the other one has but not reciprocally. When fiat money is introduced, the problem of absence of double coincidence of wants can be partially overcome. If fiat money is valued, agents can get access to the good they need but are not endowed with.

For positive ϵ 's, the absence of double coincidence of wants is not total. This implies that commodity versus commodity trading (barter) may arise, but only to an extent limited by the value of ϵ .

We will prove the existence of equilibria with valued fiat money. First, we study an equilibrium with all markets active, the AM equilibrium. In this equilibrium monetary exchange coexists with barter. Second, we look at an equilibrium where the only markets which are active are those where money is one of the objects; the CIA equilibrium.³

We first look for the existence of the AM equilibrium. The proof is by construction. The symmetric, stationary environment suggests the following conjectures on the prices: prices satisfy $p_{m,j} = p_m$ and $p_{j,j+1} = p$ for all $j = 1, 2, 3$ (modulo 3) and the following conjectured behavior: (a) $b_{i+2,i+1} = \epsilon$, (b) $b_{i,i+1}$, $b_{m,i+1}$, and $b_{i,m}$ are all > 0 , and (c) $b_{i+2,i} = b_{i+2,m} = b_{m,i} = 0$. We then have proposition 1.

³The parallel with the CIA model is clear. For agent i , good i is the "credit good," and good $i + 1$ the "cash good."

Proposition 1 *There exists a symmetric, stationary all-markets equilibrium for ϵ in an interval $(0, \bar{\epsilon})$.*

PROOF: Faced with constant prices, the agent i 's problem is recursive and can be stated in terms of the following functional equation:⁴

$$V(m) = \max_{m', x, b} \{u(x_i, x_{i+1}) + \beta V(m')\} \text{ s.t.}$$

$$x_i = w - b_{i,i+1} - b_{i,m}$$

$$x_{i+1} = \epsilon/p + b_{i,i+1}p + b_{m,i+1}p_m$$

$$gm' = m + b_{i,m}/p_m - b_{m,i+1} - \tau$$

$$b_{m,i+1} \leq m$$

$$b_{i,i+1} + b_{i,m} \leq w$$

$$b_s \geq 0.$$

The first order conditions are

$$u_1(x_i, x_{i+1}) - \beta V_m/(gp_m) = 0$$

$$u_2(x_i, x_{i+1})p_m - \beta V_m/g - \mu = 0$$

$$V_m = \beta V_m/g + \mu$$

where μ is the multiplier of the constraint $b_{m,i+1} \leq m$ and the last equation is an envelope condition.

Since the candidate equilibrium is symmetric, the following is satisfied:

$$x_i^i = x_{i'}^{i'} \text{ with } i, i' \in \mathcal{I} \text{ and } i \neq i'$$

⁴See Stokey and Lucas (1989).

$$x_{i+1}^i = x_{i'+1}^{i'} \text{ with } i, i' \in \mathcal{I} \text{ and } i \neq i'.$$

We can then verify that a solution with valued fiat money has to satisfy

$$V(x_i, x_{i+1}) = \beta/g$$

$$x_i + x_{i+1} = w + c$$

$$p = \beta/g$$

$$p_m = w - x_i - cg/\beta \geq 0$$

$$\mu = u_2(x_i, x_{i+1})(1 - \beta/g) \geq 0.$$

Existence of a solution to the equations above is guaranteed by the assumptions on the utility function. The normal goods assumption guarantees that the solution is unique. Note also that all the first-order conditions are met, including the non-negativity of the multiplier of the CIA constraint (since $\beta \leq g$). The given choices maximize the consumer's objective, since the Kuhn-Tucker conditions are sufficient conditions here. This is ensured by the concavity of the objective function and the convexity of the constraints.

We also need to verify the conjectures about the agent's chosen trade patterns. We have left some Kuhn-Tucker multipliers out of the formal maximization procedure for notational simplicity and instead look directly at the marginal considerations underlying these multipliers. First, good $i + 2$ is best offered in return for good $i + 1$. This exchange generates $u_2/p = u_2g/\beta$, whereas an exchange of $i + 2$ for good i only would give $u_1p = u_1\beta/g$, and an exchange for money would give βu_2 . Since $u_2 \geq u_1$ and g/β is greater than both β/g and β , we confirm the conjectured choice of how to trade good $i + 2$ as optimal. Second, it is clear that $b_{m,i}$ has to equal zero since $u_2 \geq u_1$.

We finally have to show that $p_m \geq 0$ or, equivalently, that $\epsilon < (\beta/g)(w - x_i)$. We know that x_i is a continuous, increasing function of ϵ . For $\epsilon = 0$, $(w - x_i) > 0$ and as ϵ increases, $(\beta/g)(w - x_i)$ decreases. It follows that there exists a unique $\bar{\epsilon}$ for which $\bar{\epsilon} = (\beta/g)(w - x_i)$, with $\bar{\epsilon} < w$. Consequently, for $\epsilon < \bar{\epsilon}$, $\epsilon < (\beta/g)(w - x_i)$. \square

Note also that in this equilibrium the utility attained is decreasing in g for $g \geq \beta$ (if $g < \beta$, no stationary equilibrium exists): the higher the rate of deflation, the higher the return on money becomes.⁵ Hence, the higher g is, the less attractive it is to hold money and to consume the good the agents need but are not endowed with. Observe also that for $g = \beta$ (the Friedman rule), we find that $V(x_1, x_2) = 1$, and we achieve the full optimum. This result is also obtained in the CIA model.

For a range of values of ϵ larger than $\bar{\epsilon}$ but less than another cut-off point $\bar{\bar{\epsilon}}$, it can be shown that there is no equilibrium with money and barter. When $\epsilon \geq \bar{\bar{\epsilon}}$, there is a set of monetary equilibria giving rise to the same, Pareto optimal allocation. Some of these equilibria have a coexistence of monetary and barter exchange, and another, as we will see below, has monetary exchange only. The value $\bar{\bar{\epsilon}}$ is given by $V(w, \bar{\bar{\epsilon}}) = 1$.

In a CIA equilibrium the conjectured behavior on prices remains as above

⁵Notice that all stationary symmetric equilibrium allocations lie on a line with slope -1 which cuts the axes at $w + \epsilon$. Strict concavity of the utility function together with the normal goods assumption guarantee that the closer the marginal rate of substitution is to one, the higher the welfare obtained by the allocation under consideration. Note that the marginal rate of substitution for any stationary monetary equilibrium here is β/g , so the higher g is, the lower the attained utility.

whereas the conjectures on the bids reflect that all transactions are carried by using fiat money: $b_{i+2,m} = \epsilon$, $b_{i,i+1} = 0$, $b_{i+2,i+1} = 0$, $b_{i,m} > 0$, and $b_{m,i+1} > 0$.

The agent's problem is

$$V(m) = \max_{m',x,b} \{u(x_i, x_{i+1}) + \beta V(m')\} \text{ s.t.}$$

$$x_i = w - b_{i,m}$$

$$x_{i+1} = b_{m,i+1} p_m$$

$$gm' = m + b_{i,m}/p_m + \epsilon/p_m - b_{m,i+1} - \tau$$

$$0 \leq b_{m,i+1} \leq m$$

$$b_{i,m} \leq w$$

$$b_s \geq 0.$$

Notice that the constraint $x_{i+1} \leq m p_m$ has a CIA constraint interpretation. Agent i cannot purchase more of good $i+1$ than the real amount of money he is currently holding. This CIA constraint arises endogenously as a result of the structure of this economy and of the primitives. In this model, different agents face this constraint for different goods.

It is possible to demonstrate that this equilibrium exists and that it gives rise to the same consumption allocation as the AM equilibrium.⁶ It thus satisfies

$$V(x_1, x_2) = \beta/g$$

$$x_1 + x_2 = w + \epsilon$$

⁶See Alonso (1992).

$$p_m = x_2.$$

We note that the price of money is higher in the CIA than in the AM equilibrium, and that the same amount of trade is carried out only with money—the value of money equals the value of the goods it is exchanged for. This CIA equilibrium exists until the value of ϵ reaches a high enough level, $\tilde{\epsilon}$ ($> \bar{\epsilon}$), that it is no longer optimal for the agent to use good i to purchase money. At this point, the agent only offers good $i+2$ for money, and then, as before, uses all the money to buy good $i+1$. This behavior is optimal for the agent as long as $\epsilon \in (\tilde{\epsilon}, \bar{\epsilon})$. In this allocation, the marginal rate of substitution between goods i and $i+1$ is between β/g and 1, and the value $\tilde{\epsilon}$ is given by $V(w, \tilde{\epsilon}) = \beta/g$.⁷ Finally, when $\epsilon > \bar{\epsilon}$ the character of the CIA equilibrium changes once again. Now the acquired money holdings are used also to purchase good i . In this equilibrium, we have $V(x_i, x_{i+1}) = 1$, and the allocation is optimal. The optimality here is independent of the value of g ; there is no distortion in the accumulation of real money balances, since there is enough endowment of the good the agent does not derive utility from good $i+2$ to equalize the marginal utilities of goods i and $i+1$. When $\epsilon \leq \bar{\epsilon}$, the accumulation of real money balances is too low. Although the accumulation using good $i+2$ is costless and takes place to the full extent, the endowment of good $i+2$ is not high enough so as to achieve the full optimum. We conclude that for values of ϵ larger than or equal to $\bar{\epsilon}$, there are monetary equilibria which are Pareto optimal and yet do not satisfy the Friedman rule.

It will prove useful in the study of the welfare costs of inflation in the

⁷It is easy to see that $\tilde{\epsilon}$ satisfies $\bar{\epsilon} < \tilde{\epsilon} < \bar{\bar{\epsilon}}$.

following section that the CIA and the AM equilibria give rise to the same consumption allocation for $\epsilon < \bar{\epsilon}$. That the result is true is intuitive. Clearly the return on money is the same in both equilibria, β/g , and in this economy this is the only relevant margin, since agents do not care about good $i + 2$. With this margin determined, the allocation is pinned down. The fact that goods are traded in different ways in the two equilibria does not affect the margin between goods i and $i + 1$ for the same money creation rate.

It can also be shown that there exists a barter equilibrium for all positive values of ϵ . The CIA equilibrium dominates the barter equilibrium if and only if ϵ is less than $\bar{\epsilon}$. For large enough values of ϵ , the barter equilibrium attains the full optimum.

This setup offers an illustration of how changes in monetary policy may change the number and/or the Pareto ranking of these equilibria. Let $\epsilon = \epsilon^*$ be fixed, and let the money growth rate (and the resulting inflation rate) be a number $g^* \geq \beta$ such that the two monetary equilibria give higher welfare than barter, i.e., g^* satisfies $\epsilon^* < \bar{\epsilon}(g^*)$. If the inflation rate is increased to a g^{**} s.t. $\epsilon^* > \bar{\epsilon}(g^{**})$, the AM equilibrium vanishes and the CIA equilibrium is dominated by the barter equilibrium.⁸

In the kind of environments studied above, it can be shown that there exist many non-constant CIA equilibria of the type studied in Woodford (1988). Among these are “self-fulfilling inflations”: for all finite values of g there is a continuum of perfect foresight CIA equilibria in which the level of real money balances approaches zero asymptotically; and “self-fulfilling deflations”: if

⁸Note that $\bar{\epsilon}(g)$ is decreasing in g .

$g \in (\beta, 1)$ there exists a continuum of perfect foresight equilibria in which the real balances become unboundedly large asymptotically.

5 The welfare costs of inflation

To measure the costs of anticipated inflation remains an important task for monetary economists. Examples of recent work which use the cash-in-advance framework for this purpose are the papers by Cooley and Hansen (1989, 1991). These studies generally find that the inflation tax is not particularly costly; it is about as distortionary as the labor income and consumption taxes, but much less distortionary than the capital tax. The findings stand in contrast to the widespread notion that inflation is one of the most important problems in the economy. Here, we investigate one possible reconciliation between these widespread beliefs and the CIA-based estimates of the welfare costs. In particular, we show that the CIA-based measures of the welfare costs of inflation may incorporate a bias.

We measure the distortion generated by the inflation tax as the difference in welfare (in terms of a consumption equivalent) between raising a given amount of revenue using the inflation tax and raising the same amount using lump-sum taxes.

Two errors may arise in CIA-based studies that calculate these costs. To describe these possible errors let us first summarize the procedure used in these studies. The basic data set can be described by, say, pairs of money growth rates and real revenues from seignorage. The first step in using the

data is to assume that a CIA model generated these data, and the model can then be used to infer the preference parameter that represents the substitutability between the cash and the credit good. In the second step of the analysis, the estimated model is used to predict or calculate the welfare levels associated with different values of the monetary growth rate (or equivalently the inflation rate).

From the perspective of the present model we now describe the two possible errors that these studies might be making. The two errors refer to the two steps in the procedure outlined above. The first error is a misinterpretation of the data. Suppose that different pairs of data correspond to different exchange patterns. For example, suppose that the CIA equilibrium corresponds to the pair with the lower money growth rate and the AM equilibrium to the one with the higher money growth. In this case the econometrician who interprets the world—i.e., both data pairs—as generated with a CIA transaction pattern will infer a too high preference parameter. The econometrician will therefore think that the changes in the value of money due to changes in the money growth rate solely reflect substitutability between consumption goods, when a large part of the change in the value of money reflects changes in the patterns of exchange.

The second potential error may occur when the CIA model is used to predict the welfare effects of a change in the policy. This is due to the multiplicity in the model. Even if the preference parameter used is correct, the multiplicity makes any predictions hazardous. If the CIA exchange patterns are assumed to be realized, the welfare estimates will be incorrect if there is a change in exchange patterns.

5.1 Real revenue targets—constant policies and equilibria

As a basis for the welfare analysis outlined, we now study in more detail some properties of equilibria with positive real revenue collection. The revenues are collected by money printing and/or by levying lump-sum taxes in monetary terms, and subsequently submitting money offers at the trading posts. To preserve symmetry, we assume that taxes are levied in equal amounts on all types, and that the money offers at the trading posts are all the same. In symmetric equilibria, the amount of revenue raised in terms of each good is the same. The monetary policies are formulated in terms of a constant target, R and τ , specifying that the government will purchase and consume R units of good j every period using money issue and the proceeds from lump-sum taxes τ for each type i .

The consumer's problem is the same as in the previous section and the government budget constraint reads, for all j ,

$$p_m b_{m,j}^G = R$$

$$b_{m,j}^G = (g - 1)/g + \tau/g.$$

We study the same endowments and preferences as used previously and focus on the comparison between the CIA and the AM equilibrium. First, consider the CIA equilibrium. The equilibrium can be represented by the values (x_i, x_{i+1}, g) solving

$$V(x_i, x_{i+1}) = \beta/g = \frac{\beta}{(1 - \tau)(1 + R/x_{i+1})}$$

and

$$x_i + x_{i+1} + R = w + \epsilon.$$

The price of money in this equilibrium is

$$p_m = x_{i+1} + R.$$

The AM equilibrium is characterized by the similar system

$$V(x_i, x_{i+1}) = \beta/g$$

with g given by the second order polynomial equation

$$gR = (R + x_{i+1} - \epsilon(1 + g/\beta))(g - 1 + \tau)$$

and

$$x_i + x_{i+1} + R = w + \epsilon.$$

Here the price of money is

$$p_m = x_{i+1} + R - \epsilon(1 + g/\beta).$$

Unlike when the government did not consume, here the AM equilibrium does not give rise to the same allocation as the CIA equilibrium. For a given growth rate of the money supply and of the lump-sum tax, the government will collect different amounts of resources in the two equilibria, since the price of money will differ across the two.

We will now describe the CIA and the AM equilibria from a slightly different perspective. Assume for a moment that the target was a given growth rate of money and that taxes were set at zero, so that all the printed

money was used to purchase goods at the prices given at each trading post. What, then, would the revenues be in each equilibrium? This amounts to describing a point on the “Laffer” curve for each equilibrium. It is possible to prove the following.

Proposition 2 *Assume that taxes are set to zero. Then for any given growth rate of the money supply, the CIA stationary equilibrium generates higher revenues than the money-and-barter equilibrium.*

PROOF: The CIA equilibrium is characterized by

$$V(x_i, (w + c - x_i)/g) = \frac{\beta}{g}$$

and the AM equilibrium by

$$V(x_i, (w + c - x_i)/g + c(1/g + 1/\beta)(g - 1)) = \frac{\beta}{g}.$$

Notice that the second argument of V in the AM allocation differs from the second argument of V in the CIA allocation by a term which is positive since $g > 1$. This means that x_i in the AM equilibrium has to exceed that of the CIA allocation since the normal goods assumption implies that V is decreasing in x_i . It follows from feasibility that $x_{i+1} + R$ is greater in the CIA equilibrium. This quantity equals the price of money in the CIA equilibrium and exceeds the price of money in the AM equilibrium. Hence it is clear that p_m is higher in the CIA equilibrium, and that the corresponding revenues raised must be higher. \square

The proposition is illustrated in Figure 1.⁹

Figure 1 shows that the CIA Laffer curve is above that of the AM equilibrium, and, from another viewpoint, that there may be more than one equilibrium with a fixed revenue target, R —one with a low level of money growth and one with a high level. Moreover, it shows that if we compare the equilibria with the lower level of money growth across exchange patterns, then it must be true that the CIA equilibrium generates higher utility. This is because the CIA equilibrium leaves the same amount of resources in the economy for the private agents as the AM equilibrium, but, as shown in the previous section, it has a less distortionary impact due to the lower money growth rate.

The reason for this welfare result is that the real price of money is lower in the AM equilibrium than in the CIA equilibrium for each rate of growth of money. Less resources would therefore be raised for each g in the AM economy. In terms of our welfare comparison of lump-sum taxes and money printing as alternative revenue sources, we now see clearly how the two equilibria with different exchange patterns give rise to different answers. In order to raise a given real revenue R , we need $g_{AM} > g_{CIA}$, and the CIA equilibrium is hence associated with a higher level of welfare than the AM equilibrium.

⁹Figure 1 assumes a period utility function given by $u^i(x_i, x_{i+1}) = (x_i^\alpha - 1)/\alpha + (x_{i+1}^\alpha - 1)/\alpha$, where $\alpha = 0.7$, $\beta = 0.9$, $w = 2$ and $\epsilon = 0.2$. Values of α that are greater than zero imply that the revenues raised in the CIA equilibrium will decrease for high values of g . The AM equilibrium gives zero revenues for high enough values of g , independently of the value of α .

5.2 Implications for the evaluation of the inflation tax

Here we study in more detail the two errors that the studies based on the CIA theory might make. To highlight the first error—the misinterpretation of the data—let us think of the following experiment. Assume that an econometrician observes constant equilibria of two separate economies without observing the actual exchange patterns. Also, assume that the economies are identical except in the government policy and in the realized pattern of exchange. Let (g^e, R^e) be the monetary growth rate and real revenue pair of economy e , and suppose that these variables are observable to the econometrician. The econometrician knows that the preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{x_{i,t}^{\alpha}}{\alpha} + \frac{x_{i+1,t}^{\alpha}}{\alpha} \right)$$

and we assume that he knows the value of β from microstudies, but that α needs to be inferred from these two economies. Furthermore, the econometrician does not know the magnitude of the aggregate resources $w + c$.

Suppose that the econometrician believes that the data corresponds to the CIA equilibria. In this case the theory, i.e., the equations corresponding to the CIA equilibrium in the previous section, can be used to infer the values of α and $w + c$. But suppose that the economy with the higher money growth rate has an AM equilibrium, and the economy with the lower growth rate has a CIA equilibrium. Then the econometrician would miscalculate α , inferring that it is higher than it really is. The observation of the AM equilibrium has associated with it a low R because of the exchange pattern difference, not because the agents have preferences with high substitutability between goods i and $i + 1$.

Having estimated α , the econometrician goes on to calculate the welfare gains from reducing g^e to 1 and replacing the revenue collection with lump-sum taxation. He thus takes an observed pair (g^e, R^e) and solves for the implied (unobserved) allocations (x_i^e, x_{i+1}^e) from

$$V(x_i^e, w + \epsilon - R^e - x_i^e) = \frac{\beta}{g^e}$$

$$x_{i+1}^e = w + \epsilon - R^e - x_i^e$$

and calculates the utility level accordingly.¹⁰ Finally, he calculates the lump-sum tax utility and finds the consumption equivalent, s , that satisfies

$$u(x_i^e, x_{i+1}^e) = u((1-s)x_i^{e,LS}, (1-s)x_{i+1}^{e,LS}),$$

where LS stands for the lump-sum allocation.

With the utility function used here the equation above simplifies to

$$s = 1 - \left(\frac{1 + \beta^{1/(\alpha-1)}}{1 + (\beta/g)^{1/(\alpha-1)}} \right) \left(\frac{1 + (\beta/g)^{\alpha/(\alpha-1)}}{1 + \beta^{\alpha/(\alpha-1)}} \right)^{1/\alpha}.$$

What, then, is the mistake in s , given the overestimation of α ? The mistaken α translates into mistaken consumption allocations; the true ones, $x^{e,*}$, are given by

$$V(x_i^{e,*}, w + \epsilon - R^e - x_i^{e,*}) = \frac{\beta}{g^e}$$

$$x_{i+1}^{e,*} = w + \epsilon - R^e - x_i^{e,*}$$

i.e., they are given by the same system as above, *but solved given the true α* . Accordingly, the true consumption equivalent, s^* , can be assessed.

¹⁰Observe that these equations apply independently of the patterns of exchange.

The mistake in s can in this case be seen using a graph plotting s against α , since s turns out to be a function $s(\alpha)$ of α .¹¹ Since α is overestimated we know that the true α is to the left of the estimated α in Figure 2.¹² The non-monotonicity of our graph hence implies that s may be overestimated, or underestimated, depending on the specific data of the problem. The non-monotonicity, moreover, does not depend on the specific data. This is because, first, $s(-\infty) = 0$. When the goods have no substitutability at all, the behavior is not distorted; and second, $s(1) = 0$. When goods are perfect substitutes, the agent is perfectly happy switching completely from good $i + 1$ to good i .

Note that the sign and size of the misestimation of the welfare loss depend on the particular data, because it determines the incorrect estimate of α .

We now consider the second potential error—the misprediction of a change in the monetary policy. Here we assume that the true parameters of the economy are known. To calculate the welfare cost of inflation we have to calculate the consumption allocations and utility level corresponding to the lump-sum tax and the inflation tax schemes and calculate the consumption equivalent s as described above. Here, there are four different possibilities. The comparison between the equilibria with and without the inflation tax has four potential answers depending on whether the two equilibria are of the CIA and/or AM type. Here, however, the consumption allocation and corresponding utility of the lump-sum tax scheme are independent of the

¹¹In general, this function also depends on the perceived level of $w + \epsilon - R$. With the preferences used here, however, the resource level has no separate significance.

¹²The graph is based on the parameter values $\beta = 0.9$ and $g = 2$.

patterns of exchange. The equilibrium characterization of the previous section showed that the CIA and the AM equilibria for a given value of g are associated with identical consumption allocations and hence identical utility levels. Since the relevant margin, g , is the same in the two lump-sum tax equilibria and the government expenditures are the same, we know that the allocations have to be the same so that their corresponding utility levels are identical.

However, the consumption allocation and therefore also the utility level of the inflation tax scheme depend on the patterns of exchange. In the previous subsection it was shown that in order to get the same revenues in the CIA as in the AM equilibria, the money growth rate needed in the AM equilibrium has to be higher than the rate needed in the CIA equilibrium. This implies that there are two possible outcomes in terms of the difference in welfare between financing with money printing and financing with lump-sum taxes. It is clear that more distortion would be generated if the implementation of the inflation tax would have associated with it the AM equilibrium. We hence conclude that if the economist uses the CIA model when the relevant exchange pattern instead had money and barter, then the revenue collection ability of the inflation tax will be overestimated and the costs of inflation understated.¹³

¹³A similar kind of bias would be found in a richer model which studies the choice among distorting taxes, one of which is inflation. Using this comparison we could comment on the question asked in Aiyagari (1990). Would it be a good idea to lower the inflation rate by, say, $x\%$? For details, see Alonso (1992).

6 Conclusions

It has been shown how the Shubik model of exchange can be used to motivate valued fiat money in a decentralized economy, and how the cash-in-advance model can be viewed as an equilibrium outcome in this economy. The endogeneity of the determination of the patterns of exchange used by agents led to some comments on the appropriateness of using the CIA model for measuring the welfare costs of inflation. From the point of view of this model, the CIA model may lead the economist to an underestimation of the costs associated with the inflation tax. These findings should prove particularly relevant when studying the welfare consequences of very high inflation rates, such as the hyperinflation episodes experienced by some countries.

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Figure 2. Welfare loss

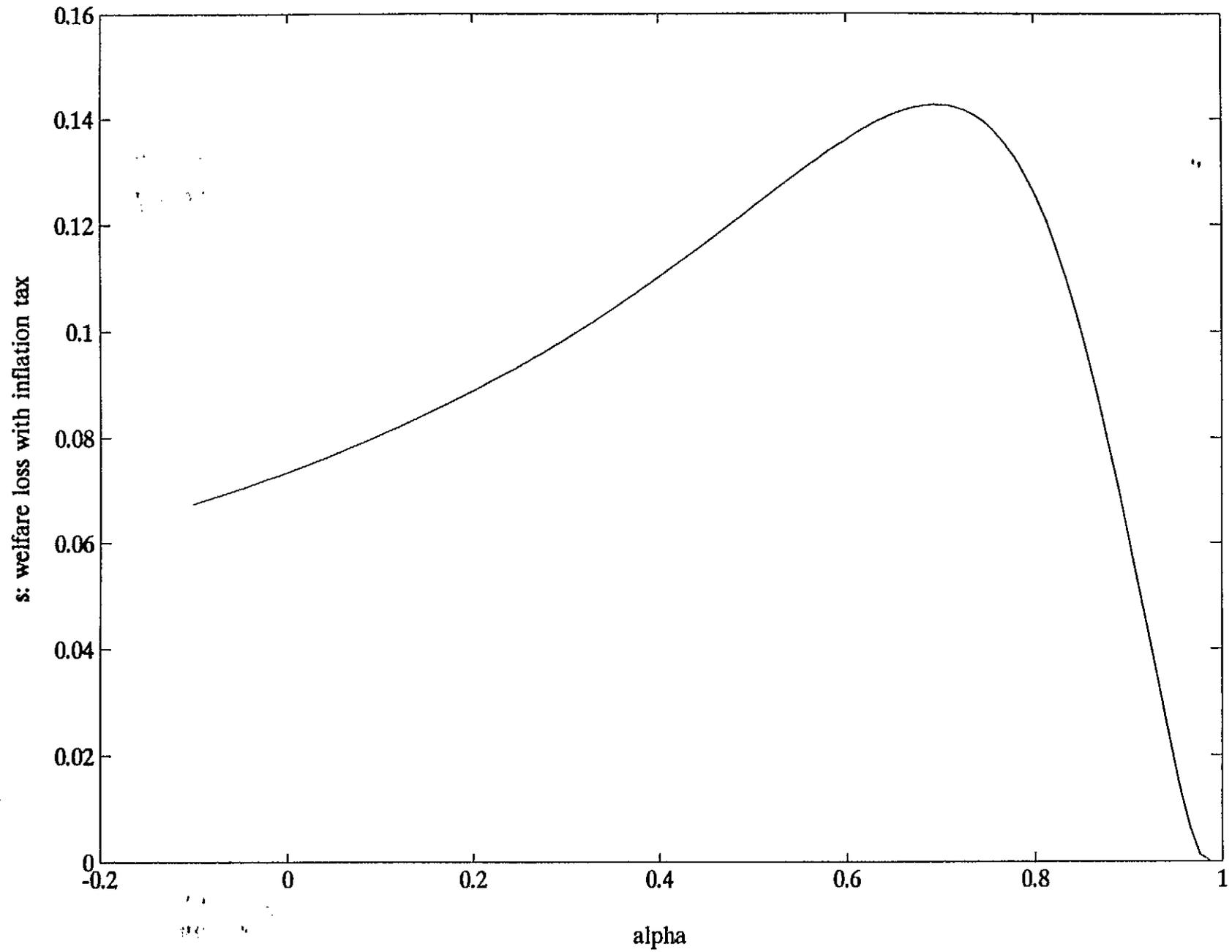


Figure 1. Seignorage revenue: - (cia); -- (am)

