Government Transaction Policy,

the Medium of Exchange,

and Welfare*

S. RAO AIYAGARI

Research Department, Federal Reserve Bank of Minneapolis,

P.O. Box 291, Minneapolis, Minnesota 55480-0291

AND

NEIL WALLACE

Research Department, Federal Reserve Bank of Minneapolis,

P.O. Box 291, Minneapolis, Minnesota 55480-0291

and

Department of Economics, University of Miami,

P.O. Box 248126, Coral Gables, Florida 33124-6550

[The proposed running head is as follows:]

Government Transaction Policy

[Please send the proofs of this article to:]

Neil Wallace

Department of Economics

Unversity of Miami

P.O. Box 248126

Coral Gables, Florida 33124-6550

[ABSTRACT]

Policy regarding what the government accepts in transactions is embedded in a version of the Kiyotaki-Wright model of media of exchange. In an example with two goods and one fiat money, the policies that are consistent with fiat money as the unique medium of exchange are identified. These policies have the government favoring fiat money in its transactions. A benefit and a cost accompany any such policy. The benefit is that a worse nonmonetary steady state is eliminated; the cost is that a better monetary steady state is eliminated. *Journal of Economic Literature* Classification Number: E40.

1. INTRODUCTION

A common view is that government policy has an influence on which objects are used as media of exchange—for example, on whether gold, silver, or both are used or on whether dollars, rubles, or a brand of cigarettes are used. Among the kinds of government policies that plausibly exert such an influence are policies about what is coined, what is legal tender, and what the government itself accepts in transactions. In this paper, we embed in a version of the Kiyotaki and Wright [4] model a particular interpretation of government policy regarding what the government accepts in transactions; we call this policy the *government transaction policy*. We then explore, by means of an example, the effect of this transaction policy on the number of steady states and on steady-state welfare. Our main result is that there can be a cost and a benefit in adopting a policy that supports a unique medium of exchange. To achieve such uniqueness, the government may have to favor particular objects in its transactions. Such favoring, however, implies that some *beneficial trades* do not occur; these are trades that occur in some steady state under a government transaction policy that does not favor particular objects. The existence of that *better steady state* is the sense in which there can be a cost in adopting a transaction policy that implies uniqueness of the medium of exchange. The benefit is that a *worse steady state,* which may exist when the government does not favor any object, is ruled out.

In Section 2, we describe how we amend the Kiyotaki and Wright [4] model to embed in it a government transaction policy. Then, in Section 3, we apply the resulting specification to an example with two symmetric goods and one fiat money. In that setting, we show that adopting a policy which ensures that there is a unique steady state in which people accept fiat money can give rise to the above sort of cost and benefit. Specifically, if the government does not favor fiat money in its transactions, then there can be a *nonmonetary steady state* in which private agents refuse fiat money as well as a *monetary steady state* in which private agents accept fiat money. If the

government does favor fiat money in its transactions, then there is a *unique steady state* and it is monetary. In the monetary steady state, when the government favors fiat money, welfare is higher than in the nonmonetary steady state when the government does not favor fiat money; but this welfare is lower than in the monetary steady state when the government does *not* favor fiat money. Thus a government transaction policy that favors fiat money is beneficial, since it eliminates a worse nonmonetary steady state, but it is also costly since it eliminates a better monetary steady state. In our concluding remarks, we comment on other possible applications of our conception of government transaction policy and on the robustness of our results.

2. A VERSION OF THE KIYOTAKI-WRIGHT MODEL
WITH GOVERNMENT TRANSACTIONS

We work within the framework of the Kiyotaki and Wright [4] model because it is the only attractive model with an endogenous transaction pattern. It is a discrete-time model with a [0,1] continuum of infinitely lived agents who meet in pairs at random at each date. The [0,1] continuum consists of a finite number of types, where the type is determined by preferences and technology. Objects are indivisible and can be stored. The initial conditions, preferences, and technology are such that no agent ever holds more than one unit of some object. The assumption that meetings are random means two things: (a) the probability that any particular agent meets an agent of a particular type with a particular object is equal to the fraction of all agents of that type with that object, and (b) the fraction of all meetings that involve meetings between agents of type i with object h and agents of type j with object k is equal to the product of the fraction of type i agents with object h and the fraction of type j agents with object k.

The original model, however, has no scope for policy—unless one views coordinating on one of several equilibria as a policy.[1] We, therefore, generalize the model to include an additional type,

called *government agents*. We let there be a [0,*G*] continuum of government agents and a [*G*,1] continuum of private agents of the same sort as the agents in the original model, with $G \in (0,1)$. Government agents differ from private agents only in that they do not consume or produce. Their role is to store objects and to trade, and each government agent can be thought of as a vending machine. With regard to meetings, government agents are exactly like private agents. For example, the probability that any agent will meet a government agent is *G*.

In Kiyotaki and Wright [4], storing an object from *t* to *t* + 1 imposes a utility cost at *t* (or, equivalently, at *t*+1), with different objects imposing different utility costs of storage. That specification is adopted to keep the state space small. For example, if, instead, objects depreciate over time in the usual fashion (that is, a fraction of the stock disappears per period), then the state space would be very large. However, for us the utility-cost-of-storage formulation is awkward. We do not want government agents to have utility functions, and we do not want private agents to be able to avoid storage costs by having government agents do the storage. After all, if objects did depreciate in the usual way, then that depreciation would not be avoided by having government agents do the storage. Instead, we assume that an object brought into period *t* has a probability of being misplaced during period *t*, so that during period *t* it is not available to be consumed or traded. In general, different objects have different probabilities of being misplaced for a period. These probabilities are uniform over time and across all agents, including government agents.[2]

At the initial date, each agent, including each government agent, is endowed with one unit of some object. A *government transaction policy* is a specification of the trading strategies of government agents. The agents can be programmed to always *trade* (to offer anything held for anything offered), to never trade, or to accept some trades and not others. Some government agents can be programmed in one way while others are programmed in a different way, and agents can be programmed to trade in a way that depends on the date.

Although the above specification is general, because we present results only for an example, we will not bother with a detailed general formulation—one with many goods and many fiat monies. Instead, we turn directly to the example.

3. An Example: Two Symmetric Goods and One Fiat Money

We describe, in turn, the environment, government transaction policy, the definition of a steady state, and the results. Almost all the proofs appear in the Appendix.

3.1. *The Environment*

There are two *symmetric goods,* indexed by 1 and 2, and there is a fiat money, indexed by 0. There are equal fractions, $(1-G)/2$, of each of 2 types of private agents, indexed 1 and 2. Private agents have the following preferences and technologies. Type 1 consumes good 1 and produces good 2, whereas type 2 consumes good 2 and produces good 1. Each agent maximizes discounted expected utility, with discount factor $\beta \in (0,1)$, and realizes period utility $u > 0$ from consuming one unit of the consumption good and realizes 0 from consuming nothing. An agent who consumes at t starts period $t + 1$ with one unit of his or her produced good. (That is, consumption at t is an input into production for $t+1$.)[3]

With probability $\delta \in (0,1)$, a good brought into period t is misplaced at t and becomes unavailable for consumption or trade during t. The fiat money, indexed by 0, is neither consumed nor produced and is never misplaced. A parameter of the environment is the amount of fiat money or the fraction of all agents, both government and private, who initially are endowed with one unit of fiat money. We call that fraction m and assume that $G > m$.

3.2. *Government Transaction Policy*

In order to be consistent with steady states, we assume that government policy is constant over time. Among constant policies, we study the following one-parameter class: each government agent (a) is always willing to accept fiat money, (b) is always willing to give up fiat money, and (c) is willing to exchange either good for the other good with probability θ. If $\theta = 0$, then fiat money must be on one side or the other of all government transactions; if $\theta = 1$, then government agents are always willing to trade. Thus we think of *low* θ as a policy of favoring fiat money in transactions.

3.3. *Equilibrium and Steady State*

The sequence of events and actions during a period is as follows. At the start of a period, each agent has one object. We let $p_{ij}(t)$ be the fraction of agents who are type i ($i = 0$ for government agents) and who hold one unit of object j at the start of period t, and we let $p(t)$ be the vector of these fractions. Next, the fraction δ of each type of agent experiences misplacement of the good held. Then agents meet in pairs at random according to the fractions $p_{ij}(t)$. Paired agents either trade or do not trade, except, of course, that an agent whose good has been misplaced cannot trade. The trade and any consumption that occurs determine the next period's beginning inventories according to the implied law of motion.[4] An agent who experiences misplacement of a good during period t ends up with that good at the beginning of period $t + 1$.

We can now give definitions of an equilibrium and a steady state. We assume, without loss of generality regarding steady states, that private agents always trade for their consumption good. Thus we have only to consider whether those who have their produced good trade for fiat money and whether those who have fiat money trade for their produced good. We let $s(t)$ denote the vector of these strategies, where a typical component of $s(t)$ is the probability that a type i agent with object

j offers to trade it for object k in period t. Given an initial inventory distribution, denoted by $p(1)$, and given a government policy, θ, an *equilibrium* is a sequence $\{p(t+1), s(t)\}$ (for $t \geq 1$) such that the law of motion is satisfied and such that each private agent's strategy as specified in the sequence is optimal for that agent under rational expectations. A *steady state* for government policy θ consists of constants (p,s) such that $\{p(t+1), s(t)\} = (p,s)$ is an equilibrium for $p(1) = p$. As noted above, we will be studying only steady states here.

Steady states may be of three types: (a) nonmonetary, (b) monetary, and (c) mixed. A *nonmonetary steady state* is one in which private agents who have their produced good reject fiat money and in which private agents with fiat money abandon it for their produced good. A *monetary steady state* is one in which private agents with their produced good accept fiat money and in which those with fiat money will not trade it for their produced good. A *mixed steady state* is one in which private agents of at least some type are indifferent between having fiat money or their produced good after trade and, hence, will trade one for the other with some probability between zero and unity.

Any steady-state p-vector has to satisfy the following conditions. First, because private agents do not hold their consumption goods, there are only seven nonzero components of the p-vector. Second, those seven components satisfy four independent adding-up constraints: holdings by government agents, the p_{0j}'s, sum to G; holdings by each type of private agent, the p_{1j}'s and the p_{2j}'s, sum to $(1-G)/2$; and holdings of fiat money, the p_{i0}'s, sum to m. In addition, steady-state p-vectors for monetary and nonmonetary steady states satisfy the conclusions of the following lemma, which are used later.

LEMMA 1. *There exists at most one nonmonetary steady state and one monetary steady state. Moreover, in each steady state the p_{ij}'s are symmetric (that is, they satisfy $p_{01} = p_{02}$ and $p_{12} = p_{21}$) and depend only on G and m (not on β, δ, or θ).*

3.4. *The Cost and Benefit of Favoring Fiat Money (Through Low θ)*

We now show that, for some parameters, there is both a cost and a benefit in having the government favor fiat money in its transactions. Proposition 1 describes the cost, while Proposition 2 describes the benefit.

PROPOSITION 1. *For each θ, there exists a monetary steady state, and welfare in that steady state is increasing in θ.*

The intuition behind this proposition is quite simple. Given that government agents always accept fiat money, as long as all other private agents are willing to accept fiat money, it is individually rational for a private agent to accept fiat money also. Hence, there always exists a monetary steady state. In such a monetary steady state, the effect of a high θ policy is that a larger fraction of the potentially beneficial trades between private agents and government agents occurs. Hence, welfare is increasing in θ.

Notice that there are no parameter restrictions other than those assumed at the outset in order to have a cost accompany a policy of favoring fiat money. However, additional restrictions are needed to have a benefit from such a policy.

PROPOSITION 2. *There exists a nonempty open set, A, in the parameter space such that for each $\alpha \in A$, there exists $\theta^*(\alpha) \in (0,1)$ such that (i) for $\theta < \theta^*(\alpha)$, there is a unique steady state that is a monetary steady state, and (ii) for $\theta > \theta^*(\alpha)$, there is a nonmonetary steady state with welfare lower than that for $\theta < \theta^*(\alpha)$.*

Figure 1 indicates the key aspects of Propositions 1 and 2. For some parameter values there is a critical value θ^* such that, if the government favors fiat money in its transactions more strongly (through a lower θ), then there is a unique steady state that is monetary. The cost of this policy is

that higher welfare levels are possible by not favoring fiat money (through a higher θ), provided that a monetary steady state obtains. The benefit of a low θ policy is that lower welfare levels are avoided, because a high θ policy is consistent with a nonmonetary steady state with even lower welfare levels.

As is indicated in Proposition 2, the existence of such a trade-off requires some parameter restrictions. Generally speaking, the size of the government parameter cannot be too large or too small. If G is too large, then even if the government always trades, private agents accept fiat money. After all, even though other private agents trade goods for goods, Proposition 1 shows that if other private agents accept fiat money, then it is individually optimal for a private agent to accept it. If G is large enough, then the government's acceptance of fiat money is sufficient to make any private agent accept it also. In this case there is never a nonmonetary or mixed steady state and, hence, there is no trade-off. Alternatively, if G is too small, then rejection of fiat money by other private agents is sufficient to imply that it is individually optimal to reject it. In this case, a nonmonetary steady state always exists, and once again there is no trade-off.

We now work up to the proof of Proposition 2 in steps. First we present two preliminary lemmas that provide the ingredients for the existence and uniqueness claims in (i) and (ii). Then we present a lemma that provides the ingredient for the welfare claim in (ii). Finally, we put them together and establish the existence of a set A, as is asserted in Proposition 2.

Lemmas 2 and 3 below give conditions that rule out the nonmonetary and mixed steady states and, hence, by Proposition 1, give conditions under which the monetary steady state is the unique steady state.

LEMMA 2. *A necessary and sufficient condition for the existence of a nonmonetary steady state is*

$$(1-G)(1-\delta) - (G-m)[1 - \theta(1-\delta)] \geq 0. \tag{1}$$

Note that condition (1) is likely to be satisfied when G is small or when θ is large. When G is small, even though government agents always accept fiat money, the rejection of fiat money by other private agents is enough to make it individually optimal to reject it and, hence, a nonmonetary steady state will exist. This argument applies even more strongly when θ is large, since a larger θ enhances the value of holding goods instead of money.

To complete the uniqueness part of claim (ii) in Proposition 2, we next show that (1) is necessary for the existence of a mixed steady state. In other words, if a mixed steady state exists, then a nonmonetary steady state also exists.

LEMMA 3. *Condition (1) is necessary for the existence of a mixed steady state.*

As noted above, Lemmas 2 and 3 provide the ingredients for the existence and uniqueness parts of claims (i) and (ii) in Proposition 2. In particular, whenever m, G, and δ are such that there exists a $\theta \in (0,1)$ satisfying (1) at equality, then such a θ can serve as the θ^* that satisfies the existence and uniqueness parts of claims (i) and (ii). Simple manipulation of (1) at equality shows that placing the following condition on G implies the existence of such a θ:

$$1 - \delta + m\delta > G > [m + (1-\delta)]/(2-\delta). \tag{2}$$

We now provide the ingredient for the welfare part of claim (ii).

LEMMA 4. *A sufficient condition for welfare in any nonmonetary steady state (for any θ) to be lower than welfare in any monetary steady state (for any θ) is that*

$$G \leq m\delta/\{(1-\delta) + [2(1-\beta)]/[\beta(1-m)]\}. \tag{3}$$

We are now in a position to prove Proposition 2.

Proof of Proposition 2. First, it may be verified that when $\delta = 0.6$, $\beta = 0.99$, $m = 0.55$, and $G = 0.7$, condition (2) is satisfied and condition (3) is satisfied with strict inequality. Second, since the functions of parameters that appear in conditions (2) and (3) are continuous, it follows that there is a nonempty open set of parameter values that satisfy those conditions. In view of Lemmas 2, 3, and 4, this completes the proof. (The shaded area in Figure 2 illustrates the shape of the set of (m, G) values that satisfies conditions (2) and (3) for $\delta = 0.6$ and $\beta = 0.99$.) $\square$

In summary, then, for some parameters there exists a *critical probability* that the government is willing to trade goods for goods that is positive and that is less than one. All policies with lower probabilities imply a unique steady state in which fiat money is accepted by everyone; all policies with equal or higher probabilities are consistent with multiple steady states. Among the multiple steady states is the monetary steady state, which has a higher utility than the steady state implied by a policy consistent with uniqueness, and the nonmonetary steady state, which has a lower utility than the steady state implied by a policy consistent with uniqueness. In this sense, there is a cost and a benefit to adopting a policy that implies uniqueness. Also, in general, $\theta = 0$ is not necessary for uniqueness and is not a desirable way to achieve uniqueness.

4. CONCLUDING REMARKS

Although we have dwelled on a particular application of our formulation of government transaction policy, we regard the formulation itself to be our main contribution. The formulation was devised to be consistent with the feature of the original model that trade must be quid pro quo (that is, it cannot involve credit of any sort). The formulation is not limited to studying the acceptability of a single fiat money. It can be applied to the study of commodity money in settings with more than two goods, and it can be applied to the study of more than one fiat money. In the latter context, one could study whether the acceptability of several fiat monies on the part of the

government ensures their acceptance. One could also study whether nonacceptance of one such money by the government ensures nonacceptance by the public (as when a government does not accept a foreign currency in its transactions).[5]

Although the example we have studied is special, most of the implications we have emphasized seem robust. First, our example tends to have multiple steady states if the government does not favor particular objects (if it is always willing to trade). There is one steady state in which fiat money is rejected by private agents (provided that the government is not too large), and there is one steady state in which fiat money is accepted. Such multiplicity will hold quite generally and is not limited, in more general settings, to a multiplicity concerning rejection or acceptance of fiat monies. Moreover, the multiplicity seems closely connected to the idea that government policies influence the objects used as media of exchange. Implicit in that idea is the notion that if the government does not favor particular objects, then the public, left on its own, could settle on one of several potential media of exchange—or perhaps on none. Second, in our example, the size of the government matters. That is certainly a robust implication for models in which trade is decentralized. Moreover, it suggests a way to explain historical instances in which a government failed to determine what the public used in its transactions; namely, the government was not large enough.[6] Third, our finding that there is a potential trade-off involved when the government favors particular objects also seems robust for models in which trade is decentralized, as does our finding that completely ruling out some trades is not necessary for uniqueness and is not a desirable way to achieve uniqueness. One seemingly nonrobust feature of our example is the uniqueness of the steady state. In general, even if a government transaction policy is successful in determining the medium of exchange, one would not expect that that policy alone would determine a unique steady state.

[Footnotes]

[1]Aiyagari and Wallace [1, p. 912] show that the set of stationary incentive-feasible allocations and the set of steady states coincide when *incentive-feasible allocations* are defined as those consistent with (a) the obvious physical resource constraint in each meeting of a pair, (b) sequential individual rationality, and (c) privacy of individual trading histories.

[2]Our specification follows a suggestion made by Nobuhiro Kiyotaki of the University of Minnesota. He suggested that objects could have probabilities of disappearing when stored and that different objects could have different probabilities of disappearing. We adopted his suggestion, except that we make disappearance temporary in order to make it consistent with stationarity.

[3]In the general case with N types of private agents and N goods, the specification of preferences and technology would be that a type i agent consumes good i and produces good $i + 1$ (modulo N).

[4]See Aiyagari and Wallace [2, p. 449] for an explicit description of a law of motion that is easily adapted to the current setting, as well as for more formal definitions of an equilibrium and a steady state.

[5]This was suggested to us by Wayne Hickenbottom of the University of Texas at Austin.

[6]Friedman and Schwartz [3, p. 27] describe an instance of such failure. They note that, during the U.S. Civil War suspension period, the West Coast of the United States remained largely on a specie standard.

APPENDIX: PROOFS

LEMMA 1. *There exists at most one nonmonetary steady state and one monetary steady state. Moreover, in each such steady state the p_{ij}'s are symmetric (that is, they satisfy $p_{01} = p_{02}$ and $p_{12} = p_{21}$) and depend only on G and m (not on β, δ, or θ).*

Proof of Lemma 1. The proof deals separately with nonmonetary and monetary steady states. In the notation used below, the subscripts $i, j \in \{1,2\}$, and $i \neq j$.

(i) Nonmonetary Steady States

We begin by listing the equations that, together with the adding-up constraints, are necessary and sufficient for a p-vector to be part of a nonmonetary steady state:

$$(p_{10}+p_{20})(p_{01}+p_{02}) = 0, \tag{A.1}$$

$$p_{ij}\theta(1-\delta)p_{0i} = p_{0j}[p_{j0} + \theta(1-\delta)p_{ji}], \tag{A.2}$$

$$0 = p_{i0}p_{0i}, \tag{A.3}$$

$$p_{i0}(p_{0i}+p_{0j}) = 0. \tag{A.4}$$

There are seven such equations because Eqs. (A.2)–(A.4) must hold for $i = 1$ and $i = 2$. In each equation, the left side is the inflow and the right side is the outflow. For example, the left side of (A.1) is the inflow into p_{00}, which comes from meetings between private agents with money and government agents with goods, while the right side of (A.1) is the outflow from p_{00}, which is 0, since private agents do not accept the fiat object. Equations (A.2)–(A.4) are analogous equations for p_{0j}, p_{i0}, and p_{ij}, respectively.

We show that $p_{00} = m$, $p_{ij} = (1-G)/2$, $p_{0i} = (G-m)/2$, and $p_{i0} = 0$ for $i = 1$ and 2 is the unique solution to the adding-up constraints and Eqs. (A.1)–(A.4). That this is a solution is evident. That it is the only solution is established as follows. By (A.4), either private agents hold none of

the fiat money or the government holds no goods. However, since $G > m$, the latter is impossible. Therefore, private agents hold none of the fiat money, which implies that $p_{ij} = (1-G)/2$ for $i = 1$ and 2. However, this symmetry for private agents and (A.2) implies symmetry for government agents; namely, $p_{0i} = (G-m)/2$.

(ii) Monetary Steady States

Again, we begin by listing the equations that, together with the adding-up constraints, are necessary and sufficient for a p-vector to be part of a monetary steady state:

$$p_{10}p_{01} + p_{20}p_{02} = p_{00}(p_{12}+p_{21}), \qquad (A.5)$$

$$p_{ij}[p_{00} + \theta(1-\delta)p_{0i}] = p_{0j}[p_{j0} + \theta(1-\delta)p_{ji}], \qquad (A.6)$$

$$p_{ij}(p_{00}+p_{j0}) = p_{i0}(p_{0i}+p_{ji}), \qquad (A.7)$$

$$p_{i0}(p_{0i}+p_{ji}) = p_{ij}(p_{00}+p_{j0}). \qquad (A.8)$$

As above, there are seven equations because Eqs. (A.6)–(A.8) hold for $i = 1$ and $i = 2$. In each equation, the left side is the inflow and the right side is the outflow. Equations (A.5)–(A.8) apply to p_{00}, p_{0j}, p_{i0}, and p_{ij}, respectively.

We first show, by contradiction, that any solution is symmetric. Suppose, by way of contradiction, that

$$p_{12} > p_{21}. \qquad (A.9)$$

From the adding-up constraints for private-agent types, we then have that

$$p_{20} > p_{10}. \qquad (A.10)$$

It follows that

$$p_{12}p_{20} > p_{21}p_{10}. \qquad (A.11)$$

This and (A.7) for $i = 1$ imply that

$$p_{12}p_{00} < p_{10}p_{01}. \tag{A.12}$$

The sum of (A.7) for $i = 1$ and $i = 2$ is

$$p_{00}(p_{12}+p_{21}) = p_{10}p_{01} + p_{20}p_{02}. \tag{A.13}$$

It follows from (A.12) and (A.13) that

$$p_{21}p_{00} > p_{20}p_{02}. \tag{A.14}$$

If we divide (A.12) by (A.14), we get $p_{12}/p_{21} < (p_{10}p_{01})/(p_{20}p_{02})$, which can be rewritten as

$$p_{01}/p_{02} > (p_{20}/p_{10})(p_{12}/p_{21}). \tag{A.15}$$

Since (A.9) and (A.10) imply that the right side of (A.15) exceeds unity, it follows that $p_{01} > p_{02}$. This and (A.9) imply that $p_{12}p_{01} > p_{21}p_{02}$. This and (A.6) for $i = 1$ imply that

$$p_{12}p_{00} < p_{20}p_{02}. \tag{A.16}$$

But (A.14) and (A.16) contradict (A.9). Since the same argument can be used to rule out $p_{12} < p_{21}$, we conclude that any solution satisfies $p_{12} = p_{21}$. That symmetry conclusion and the adding-up constraints for private-agent types imply that $p_{10} = p_{20}$. But then (A.7) implies symmetry for the government type: namely, $p_{01} = p_{02}$.

The above argument shows that any monetary steady-state p-vector is symmetric. We now show that such a solution to (A.5)–(A.8) and the adding-up constraints exists and is unique. Under symmetry, Eqs. (A.6)–(A.8) are the same equation, namely,

$$p_{ij}p_{00} = p_{i0}p_{0i}. \tag{A.17}$$

Moreover, since the sum of (A.6) over i is (A.5), a necessary and sufficient condition for the existence of a symmetric solution to (A.5)–(A.8) is satisfaction of (A.17). In what follows, we let $p_{00} = x$. With regard to the adding-up constraints, the condition

$$p_{i0} = (m-x)/2 \geq 0 \qquad (A.18)$$

is necessary and sufficient for holdings of the fiat object to sum to m, while

$$p_{0i} = (G-x)/2 \geq 0 \qquad (A.19)$$

and

$$p_{ij} = (1-G)/2 - (m-x)/2 \geq 0 \qquad (A.20)$$

are necessary and sufficient for the satisfaction of the adding-up constraints for agent types. It follows that the set of *symmetric solutions* is the set of solutions to $x \in [\max(0, m+G-1), \, m]$ and the equation we get by substituting from Eqs. (A.18)–(A.20) into (A.17). That equation is

$$H(x) \equiv x^2 + x(2-G-m) - Gm = 0. \qquad (A.21)$$

Since $H(0) = -Gm < 0$, $H(m+G-1) = -(1-m)(1-G) < 0$, and $H(m) = 2m(1-G) > 0$, it follows that (A.21) has a unique solution in the interval $[\max(0, m+G-1), \, m]$. $\square$

PROPOSITION 1. *For each θ, there exists a monetary steady state, and welfare in that steady state is increasing in θ.*

Proof of Proposition 1. In what follows, we let x denote the steady-state magnitude of p_{00} in a monetary steady state. By Lemma 1, x does not depend on θ (or β and δ). In this and subsequent proofs, we let v_{12} and v_{10} denote the discounted expected utility for a type 1 private agent who holds his or her produced good (good 2) and money (object 0), respectively, and we let v_{21} and v_{20} denote those discounted expected utilities for a type 2 private agent. We show that $v_{10} > v_{12}$ and

that both are increasing in θ. Given the symmetry established in Lemma 1, analogous results will hold for v_{21} and v_{20} as well. Hence, a monetary steady state will exist and welfare in it will be increasing in θ.

Straightforward manipulation of the stationary version of Bellman's equation implies that

$$2(1-\beta)v_{12} = (1-\delta)(m+x)\beta(v_{10}-v_{12}) + (1-\delta)^2[(\theta-1)(G-x) + (1-m)]\beta u \qquad (A.22)$$

and

$$2(1-\beta)v_{10} = -(1-\delta)(1-m)\beta(v_{10}-v_{12}) + (1-\delta)(1-m)\beta u. \qquad (A.23)$$

On the right side of (A.22), $(1-\delta)(m+x)$ is twice the probability that a type i private agent who holds his or her produced good is able to trade for money. That probability is $(1-\delta)(m-p_{10}) = (1-\delta)(m+x)/2$. The coefficient of βu in (A.22) is twice the probability that a type i private agent who holds his or her produced good is able to trade for his or her consumption good. In (A.23), $(1-\delta)(1-m)$ is twice the probability that a type i private agent who holds money is able to trade for his or her consumption good. By subtracting (A.22) from (A.23), we get

$$(v_{10}-v_{12}) = \beta u(1-\delta)[(1-m)\delta + (1-\delta)(1-\theta)(G-x)]/[2(1-\beta) + \beta(1-\delta)(1+x)]. \qquad (A.24)$$

It follows from (A.24) that $(v_{10}-v_{12}) > 0$. Therefore, there is a monetary steady state.

Since x does not depend on θ, it follows from (A.24) that $(v_{10}-v_{12})$ is decreasing in θ. Therefore, it is immediately seen from (A.23) that v_{10} is increasing in θ. From (A.22), it can be seen that

$$\text{sign}[\partial v_{12}/\partial\theta] = \text{sign}[(1-\delta)(m+x)\beta\partial(v_{10}-v_{12})/\partial\theta + (1-\delta)^2(G-x)\beta u],$$

which, by (A.24), is easily shown to be positive. $\square$

LEMMA 2. *A necessary and sufficient condition for the existence of a nonmonetary steady state is*

$$(1-G)(1-\delta) - (G-m)[1 - \theta(1-\delta)] \geq 0. \tag{1}$$

Proof of Lemma 2. In the proof of Lemma 1, we show that the p-vector in a nonmonetary steady state is

$$p_{01} = p_{02} = (G-m)/2 \text{ and } p_{12} = p_{21} = (1-G)/2. \tag{A.25}$$

We now verify that inequality (1) is necessary and sufficient for the nonmonetary strategies to be individually optimizing. Since the inequalities $v_{12} - v_{10} \geq 0$ and $v_{21} - v_{20} \geq 0$ are necessary and sufficient for such strategies to be individually optimizing, we proceed by deriving an expression for $v_{12} - v_{10}$.

The nonmonetary strategies and (A.25) imply that

$$v_{12} = [(1-G) + (G-m)\theta](1-\delta)^2\beta u/2 + \beta v_{12} \tag{A.26}$$

$$v_{10} = (G-m)(1-\delta)\beta u/2 + (G-m)(1-\delta)\beta v_{12} + [1 - (G-m)(1-\delta)]\beta v_{10}. \tag{A.27}$$

In (A.26), the coefficient of βu is the probability for type 1, who has a produced good, of getting good 1 in trade. Whether a trade occurs or not, type 1 leaves the period with a produced good. In (A.27), the coefficient of βu reflects the fact that type 1, who has fiat money, gets good 1 in trade only by meeting with a government agent who has good 1. The coefficient of βv_{12} in (A.27) reflects the fact that type 1 is willing to take any good and succeeds when he or she meets a government agent who has a good and is able to trade. Upon subtracting (A.27) from (A.26), we get

$$v_{12} - v_{10} = K(1-\delta)\beta u/2 + [1 - (G-m)(1-\delta)]\beta(v_{12}-v_{10}), \tag{A.28}$$

where K denotes the left side of inequality (1). Since $[1 - (G-m)(1-\delta)]\beta < 1$, the sign of $v_{12} - v_{10}$ is the same as the sign of K. By symmetry, this is also true of $v_{21} - v_{20}$. $\square$

LEMMA 3. *Condition* (1) *is necessary for the existence of a mixed steady state.*

Proof of Lemma 3. Suppose that there is a steady state in which one type of private agent is indifferent between holding his or her produced good or fiat money; i.e., the discounted expected utilities are equal. Without loss of generality, let $v_{10} = v_{12}$. Then $v_{20} \leq v_{21}$. If not, then type 2 always accepts fiat money, and, since the government accepts fiat money, it follows that fiat money strictly dominates the produced good for type 1, which is a contradiction. It follows that

$$v_{12} = (p_{21} + p_{01}\theta)(1-\delta)^2\beta(u + v_{12}) + [1 - (p_{21} + p_{01}\theta)(1-\delta)^2]\beta v_{12}.$$

Here the first term on the right side is the product of the probability of having the opportunity to consume and the implied payoff, and the second term on the right side is the product of the probability of not having the opportunity to consume and the implied payoff. This equation can be rewritten as

$$(1-\beta)v_{12} = (p_{21} + p_{01}\theta)(1-\delta)^2\beta u. \tag{A.29}$$

The same inequalities imply the following lower bound on the discounted expected utility of having money after trade:

$$v_{10} \geq p_{01}(1-\delta)\beta(u + v_{10}) + [1 - p_{01}(1-\delta)]\beta v_{10}.$$

The first term on the right side is the product of the probability of meeting a government agent with good 1 who is able to trade and a lower bound on the implied payoff—a lower bound because $v_{10} \leq v_{12}$. The second term on the right side is the product of the remaining probability and a lower bound on the implied payoffs—a lower bound since holding money is always an option for an agent who starts with it. This inequality can be rewritten as

$$(1-\beta)v_{10} \geq p_{01}(1-\delta)\beta u. \tag{A.30}$$

But then $v_{10} \leq v_{12}$, (A.29), and (A.30) imply that

$$(p_{21} + p_{01}\theta)(1-\delta) \geq p_{01}. \tag{A.31}$$

Analogous reasoning for the type 2 private agent, using $v_{20} \leq v_{21}$, yields

$$(p_{12}+p_{02}\theta)(1-\delta) \geq p_{02}. \tag{A.32}$$

By rearranging (A.31) and (A.32) and summing we get

$$(p_{12}+p_{21})(1-\delta) \geq (p_{01}+p_{02})[1 - \theta(1-\delta)]. \tag{A.33}$$

Since $(1-G) \geq (p_{12}+p_{21})$ and $(p_{01}+p_{02}) \geq G - m$, (A.33) implies that

$$(1-G)(1-\delta) \geq (G-m)[1 - \theta(1-\delta)], \tag{A.34}$$

which is equivalent to inequality (1). □

LEMMA 4. *A sufficient condition for welfare in any nonmonetary steady state (for any θ) to be lower than welfare in any monetary steady state (for any θ) is that*

$$G \leq m\delta/\{(1-\delta) + [2(1-\beta)]/[\beta(1-m)]\}. \tag{3}$$

Proof of Lemma 4. By Eq. (A.26) and its analogue for a type 2 private agent, welfare of a private agent with his or her produced good is increasing in θ in a nonmonetary steady state. By Eq. (A.27) and its analogue for a type 2 private agent, welfare of a private agent with fiat money is also increasing in θ in a nonmonetary steady state. By Proposition 1, welfare is increasing in θ in the monetary steady state as well. Since a holder of fiat money is worse off than a holder of a good when money is rejected and is better off than a holder of a good when money is accepted, it is sufficient to show that a holder of goods is better off in the $\theta = 0$ monetary steady state than in the $\theta = 1$ nonmonetary steady state (see Fig. 1).

Let V_n denote the value of v_{12} in a nonmonetary steady state for $\theta = 1$. A straightforward calculation implies that

$$2(1-\beta)V_n = (1-m)(1-\delta)^2\beta u. \tag{A.35}$$

Let V_m denote the value of v_{12} in a monetary steady state for $\theta = 0$. It follows from (A.22) and (A.34) that

$$2(1-\beta)V_m = \phi(x)(1-\delta)^2\beta u, \tag{A.36}$$

where

$$\phi(x) = -(G-x) + (1-m) + \beta(m+x)[(1-m)\delta + (1-\delta)(G-x)]/[2(1-\beta) + \beta(1-\delta)(1+x)]$$

and where x denotes p_{00} in a monetary steady state. It follows from (A.35) and (A.36) that

$$\text{sign}(V_m - V_n) = \text{sign}[\phi(x) - (1-m)].$$

Since $\phi(x)$ is increasing in x, it is sufficient for $V_m > V_n$ that $\phi(0) - (1-m) \geq 0$. And, since

$$\text{sign}[\phi(0) - (1-m)] = \text{sign}\{\beta m(1-m)\delta - G[2(1-\beta) + \beta(1-\delta)(1-m)]\},$$

we get the asserted sufficient condition. $\square$

REFERENCES

1. S. R. AIYAGARI AND N. WALLACE, Existence of steady states with positive consumption in the Kiyotaki-Wright model, *Rev. Econ. Stud.* **58** (1991), 901–916.

2. S. R. AIYAGARI AND N. WALLACE, Fiat money in the Kiyotaki-Wright model, *Econ. Theory* **2** (1992), 447–464.

3. M. FRIEDMAN AND A. J. SCHWARTZ, "A Monetary History of the United States, 1867–1960," National Bureau of Economic Research, Studies in Business Cycles, Princeton University Press, Princeton, 1963.

4. N. KIYOTAKI AND R. WRIGHT, On money as a medium of exchange, *J. Polit. Econ.* **97** (1989), 927–954.