"Observations on Improper Methods of
Simulating and Teaching Friedman's Time
Series Consumption Model"

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Observations on Improper Methods of Simulating and
Teaching Friedman's Time Series Consumption Model

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Textbook accounts of the implications for time series of
Friedman's permanent income formulation of the consumption function
typically aim to demonstrate two things. The first is that Friedman's
theory supposedly implies that the "long-run" marginal propensity to
consume is larger than the "short-run" marginal propensity to consume.
The second is that Friedman's theory provides one way of rationalizing
the fact that regressions using Kuznets' overlapping ten-year averaged
data yield an estimated consumption function going through the origin
and possessing a high marginal propensity to consume, on the order of
.9. The slope of the regression on Kuznets' ten-year averages is often
taken as an estimate of the "long-run" marginal propensity to consume,
while the zero intercept of the regression is taken as evidence for the
hypothesis of proportionality of the long-run consumption function.

The experiments that are used to demonstrate the larger long-
run than short-run marginal propensity to consume implicit in Friedman's
time series model consist essentially of comparing the results of intro-
ducing alternative income paths, some that are "cyclical" with a period
roughly matching that of the business cycle, others that correspond to
once-and-for-all jumps or trends, and amount to long-term changes in
income. Rigorously, these experiments can be viewed as tracing out the
different consumption responses to inputs of income consisting of cosine
waves of different periodicities. It can then be correctly shown that
Friedman's model implies a larger consumption response to long period-
icity than to short periodicity fluctuations in income. But those
experiments all hold fixed the model linking consumption to income as
the income process is hypothetically varied. The hypothesis of rational
expectations, which is implemented extensively throughout Friedman's

book, implies that the consumption-income relationship will change in a determinate way with every change in the process generating income. This means that the standard experiments used to study the implications of Friedman's time series model are executed invoking assumptions that do violence to the underlying model. The key idea of Friedman's study, exhibited especially clearly in his cross-section work, is that objective and subjective probability distributions can be equated—which is the doctrine of rational expectations—and that because the consumption function reflects the probability distribution of subsequent income, it will vary as the statistical process generating income varies.[1/] This notion is ignored in the textbook experiments just described. In place of these faulty experiments, I will suggest a simple and correct way of describing the implications of Friedman's time series model.

On the second point, this paper calculates the regression on ten-period average data implicit in Friedman's statistical model of the consumption-income process. As n goes to infinity, the simple consumption-income regression on n-period average data does indeed recover the marginal propensity to consume out of permanent income. But for n equal to ten, it does not in general. Below I report a formula for the bias associated with interpreting the slope from the ten-year average regression as the marginal propensity to consume out of permanent income.

1. Rational Responses to Alternative Income Processes

Friedman's model can be written

$$(1) \qquad C_t = \beta Y_{pt} + U_t$$

where C_t is consumption, Y_{pt} is permanent income, and U_t is a random variable with mean zero and finite variance and obeying $E[U_t \cdot Y_{ps}] = 0$

for all t and s. This last condition is the statement that in (1) Y_{pt} is strictly econometrically exogenous.[2/] I will initially assume that permanent income is defined as

$$(2) \qquad Y_{pt} = \frac{1}{n} [E_t Y_{t+1} + E_t Y_{t+2} + \ldots + E_t Y_{t+n}],$$

where $E_t Y_{t+j}$ is the mathematical expectation of Y_{t+j} conditional on information dated t or earlier. Using mathematical expectations in definition (2) amounts to imposing rational expectations.[3/]

John F. Muth [4] showed that (1) and (2) would imply the consumption-income model that Friedman used for the time series, provided that income obeys the stochastic process

$$(3) \qquad (1-L)Y_{t+1} = a + (1-\lambda L)\varepsilon_{t+1}$$

where L is the lag operator defined by $L^n x_t = X_{t-n}$, and where a and λ are parameters and ε_t is a random term obeying $E_t \varepsilon_{t+1} = 0$. Equation (3) can be rearranged to read

$$(4) \qquad Y_{t+1} = \frac{1-\lambda}{1-\lambda L} Y_t + \frac{a}{1-\lambda} + \varepsilon_{t+1}.$$

Therefore we have

$$(5) \qquad E_t Y_{t+1} = \frac{1-\lambda}{1-\lambda L} Y_t + \frac{a}{1-\lambda}.$$

Since from (3) $Y_{t+2} = Y_{t+1} + a + \varepsilon_{t+2} - \lambda \varepsilon_{t+1}$, we have

$$E_t Y_{t+2} = a + E_t Y_{t+1}$$

and more generally,

$$E_t Y_{t+j} = (j-1)a + E_t Y_{t+1}.$$

Combining the preceding equation with (5) gives

$$(6) \qquad E_t Y_{t+j} = \frac{a}{1-\lambda} + (j-1)a + \frac{1-\lambda}{1-\lambda L} Y_t.$$

Substituting (6) into (2) gives

$$Y_{pt} = \frac{a}{1-\lambda} + \frac{1-\lambda}{1-\lambda L} Y_t + \frac{a}{n}(1+2+\ldots+n-1)$$

or

$$(7) \qquad Y_{pt} = (\frac{a}{1-\lambda}) + \frac{a(n-1)}{2} + (\frac{1-\lambda}{1-\lambda L}) Y_t.$$

Substituting (7) into (1) gives

$$(8) \qquad C_t = \beta[(\frac{a}{1-\lambda}) + \frac{a(n-1)}{2}] + \beta(\frac{1-\lambda}{1-\lambda L})Y_t + U_t,$$

which is the time series model used by Friedman. Equations (3) and (8)
together form a statistical model of the C-Y process. As Muth emphasized,
given a model like Friedman's (1) and (2) the consumption-income regres-
sion given by (8) depends on the nature of the stochastic process govern-
ing Y, as is testified to by the presence of the parameters a and λ of
the Y-process in the consumption-income regression (8). According to
the theory of rational expectations, the consumption-income regression
(8) will change whenever there is a change in the Y-process, as, for
example, will occur if λ or a changes in (3).

The standard textbook experiments ignore this implication of
rational expectations and instead hold (8) invariant while in effect
examining the response of consumption to inputs consisting of cosine
functions of different frequencies. As a building block, I first con-
sider the response of C_t in (8) to a complex input $Y_t = e^{iwt}$. Suppress-
ing the constant and stochastic term in (8) we have that if $Y_t = e^{iwt}$,
then C_t is governed by

$$C_t = \beta\frac{1-\lambda}{1-\lambda L} e^{iwt}$$

$$= \beta(1-\lambda) \sum_{k=0}^{\infty} \lambda^k e^{iw(t-k)}$$

(9) $$C_t = \frac{\beta(1-\lambda) \cdot e^{iwt}}{1-\lambda e^{-iw}} \quad .$$

The response of C_t in (8) to the complex input e^{-iwt} is found as

$$C_t = \beta(1-\lambda) \sum_{k=0}^{\infty} \lambda^k e^{-iw(t-k)}$$

(10) $$C_t \frac{\beta(1-\lambda)e^{-iwt}}{1-\lambda e^{+iw}}$$

It is convenient to represent the complex quantity $\frac{\beta(1-\lambda)}{(1-\lambda e^{-iw})}$ in the polar form[4/]

(11) $$\frac{\beta(1-\lambda)}{1-\lambda e^{-iw}} = r(w)e^{i\theta(w)}$$

where

$$r(w) = \frac{\beta(1-\lambda)}{\sqrt{(1+\lambda^2-2\lambda\cos w)}}$$

and

$$\theta(w) = \arctan\left[\frac{-\lambda\sin w}{1-\lambda\cos w}\right] \quad .$$

Using (11), (9) and (10) can be written

(9') $$C_t = r(w)e^{i\theta(w)}e^{iwt}$$

(10') $$C_t = r(w)e^{-i\theta(w)}e^{-iwt}$$

Figure 2a

$$r(w) = \frac{(1-\lambda)}{\sqrt{1+\lambda^2-2\lambda\cos w}}$$

w \ λ	.3	.5	.8
0	1.0	1.0	1.0
π/8	.956	.876	.497
π/6	.927	.807	.396
π/4	.858	.679	.280
π/2	.670	.447	.156
π	.538	.333	.111

Figure 2b

$$\theta(w) = \arctan\left(-\lambda\sin w/(1-\lambda\cos w)\right)$$

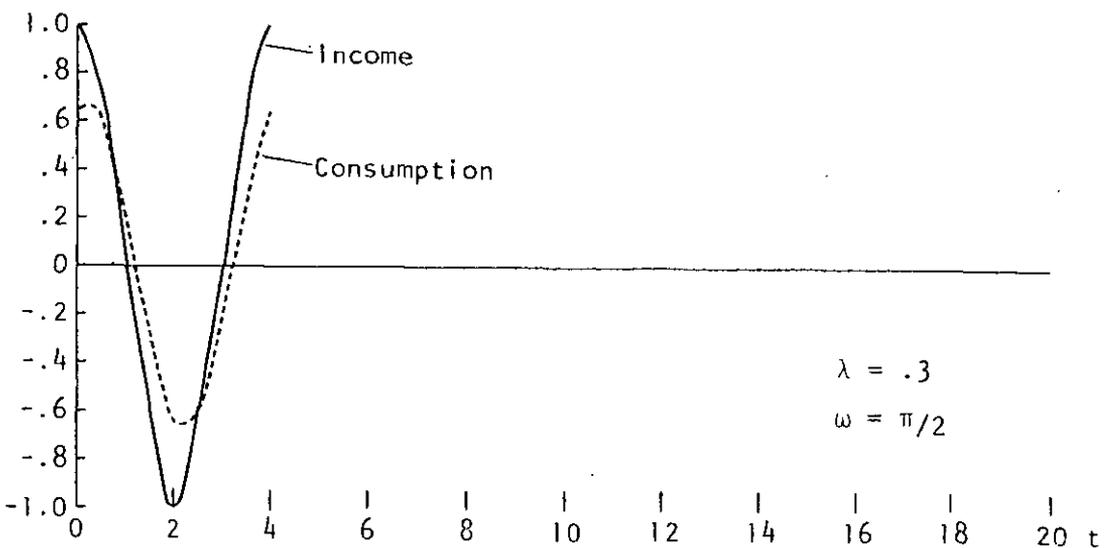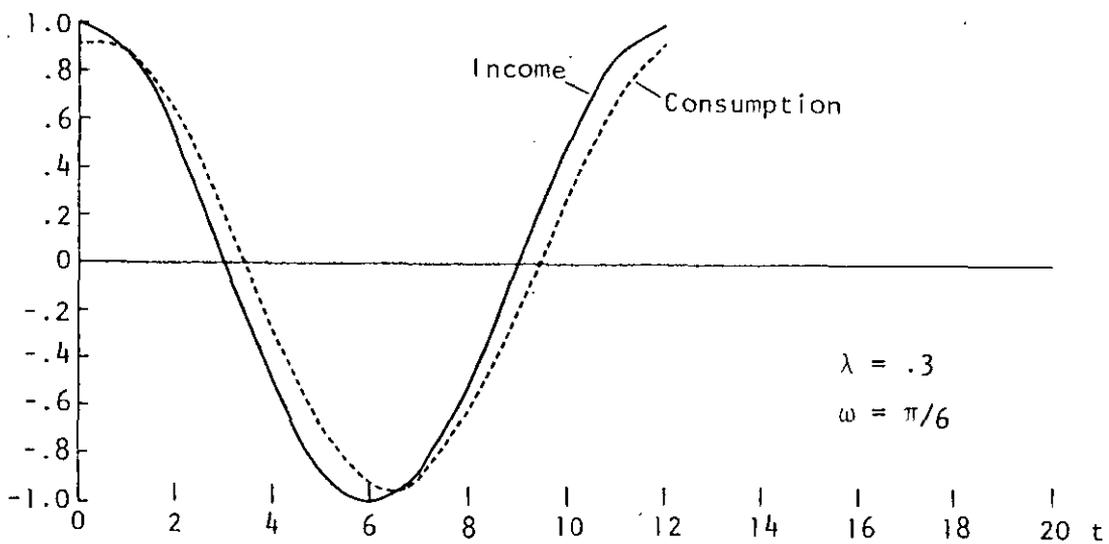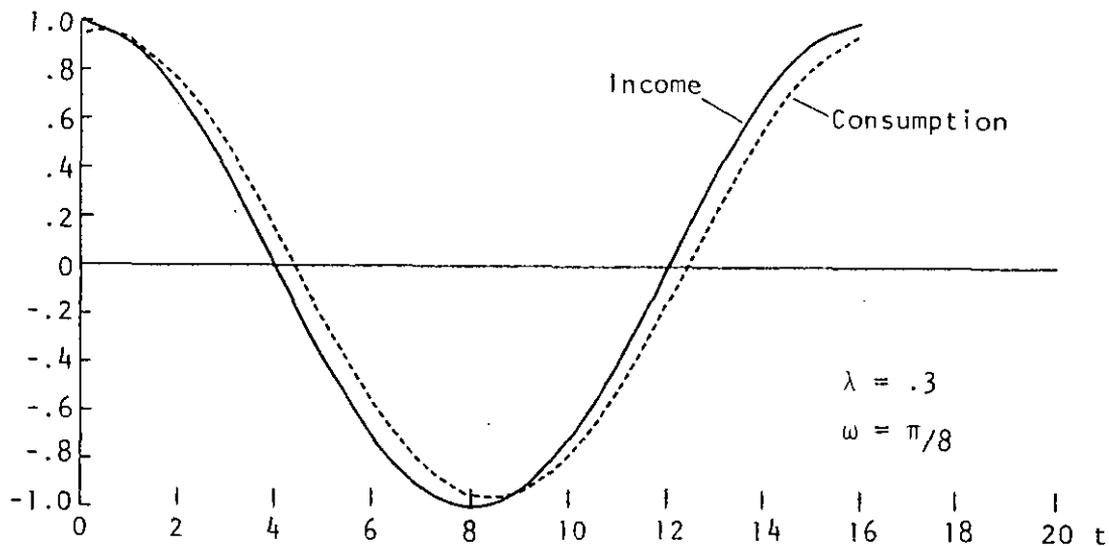| w \ λ	.3	.5	.8
0	.0	0	0
π/8	-.158	-.342	-.865
π/6	-.200	-.415	-.916
π/4	-.263	-.500	-.916
π/2	-.291	-.464	-.675
π	.0	.0	0

Figure 1

Note: β=1 assumed

The response of C_t to a Y_t path consisting of a cosine wave can be easily derived from (9') and (10') by first representing the cosine wave in the polar form

$$\cos wt = \frac{e^{iwt} + e^{-iwt}}{2} .$$

Then we have

$$C_t = \frac{\beta(1-\lambda)}{1-\lambda L} \cos wt$$

$$= \frac{\beta(1-\lambda)}{1-\lambda L} \frac{e^{iwt} + e^{-iwt}}{2}$$

Using (9') and (10') we have

$$C_t = \frac{r(w)}{2} e^{i\theta(w)} e^{iwt} + \frac{r(w)}{2} e^{-i\theta(w)} e^{-iwt}$$

$$= r(w) \frac{(e^{i(wt+\theta(w))} + e^{-i(wt+\theta(w))})}{2}$$

$$(12) \qquad C_t = \frac{\beta(1-\lambda)}{\sqrt{(1+\lambda^2 - 2\lambda \cos w)}} \cdot \cos(wt+\theta(w))$$

Equation (12) shows that if (8) holds with the income path being a cosine wave of frequency w, then apart from the random term U_t, consumption will follow a cosine wave of the same frequency w and with amplitude $\beta(1-\lambda)/\sqrt{1+\lambda^2 - 2\lambda \cos w}$. Consumption will also be subjected to a phase shift of $\theta(w) = \arctan[-\lambda \sin w/(1-\lambda \cos w)]$. The amplitude $\beta(1-\lambda)/\sqrt{1+\lambda^2 - 2\lambda \cos w}$ equals β at w=0 and decreases monotonically to $\beta(1-\lambda)/(1+\lambda)$ at w=π. The cosine wave at w=π has a periodicity of two periods and corresponds to the highest frequency that can be considered with data at unit intervals. The cosine wave at w=0 is the constant $\cos 0 \cdot t = 1$, and corresponds to the "longest" periodicity. The period

Figure 3

$$\left| \frac{1}{n} \frac{\sin(\frac{w}{2}n)}{\sin\frac{w}{2}} \right|$$

w\n	5	10	20
0	1	1	1
π/8	.85	.47	.18
π/6	.75	.19	.17
π/4	.48	.18	.13
π/2	.2	.14	0
π	.2	0	0

from peak to peak of a cosine wave of frequency w is equal to $2\pi/w$. Thus, low w's correspond to very long swings.

Figure 1 shows graphs of C_t and Y_t implied by (8) where Y_t is a cosine wave of three successively different frequencies: one a long periodicity, a second a medium periodicity, and a third a short period-icity. Figure 2 graphs the amplitude of consumption $\beta(1-\lambda)/\sqrt{1+\lambda^2-2\lambda\cos w}$ for several values of λ.

I believe that Figures 1 and 2 and formula (12) summarize the ideas underlying the textbook manipulations of Friedman's model. As formula (12) shows, if income follows a high frequency cosine wave of unit amplitude, consumption follows a cosine wave with an amplitude that for sizable values of λ, will be much smaller than the "long-run" mar-ginal propensity to consume β. The amplitude of consumption swings induced by cosine waves in income of unit amplitude approaches β as the frequency w goes to zero and the periodicity $2\pi/w$ goes to infinity. This is the sense in which Friedman's model seems to imply bigger con-sumption responses to long swings in income than to short swings in income.

The preceding exercises erroneously hold the forecasting rule (7) and hence equation (8) fixed in the face of hypothetical variations in the process generating income. The hypothesis of rational expecta-tions requires that for each hypothetical income process considered, one should compute anew the optimal forecasts that belong in the definition of permanent income (2). That will guarantee that people are being assumed to use the forecasting rules that are the optimal ones for the income process at hand. Now a cosine wave is perfectly predictable, so that if income is governed by

$$Y_t = \cos wt$$

it follows that[5/]

$$E_t Y_{t+j} = \cos w(t+j)$$

Substituting this into (2) gives[6/]

$$Y_{pt} = \frac{1}{n} \sum_{j=1}^{n} \cos w(t+j)$$

$$= \frac{1}{n} \cdot \frac{\sin \frac{w}{2} n}{\sin \frac{w}{2}} \cos[w(t+1) + \frac{w}{2}(n-1)]$$

Therefore, where the realization of income is a cosine wave and where (1) holds with permanent income being formed using optimal forecasts in (2), we have

$$(12') \qquad C_t \doteq \frac{\beta}{n} \frac{\sin \frac{w}{2} n}{\sin \frac{w}{2}} \cos[w(t+1) + \frac{w}{2}(n-1)] + U_t,$$

which does not agree with (12). Indeed, for $w > 0$, and in the limit as $n \to \infty$, (12') implies that

$$C_t = U_t,$$

permanent income not responding at all to current fluctuations in income. This can be seen directly from

$$\lim_{n \to \infty} Y_{pt} = \frac{1}{n} \sum_{j=1}^{n} \cos w(t+j) = 0$$

for $w > 0$, since $|\cos w(t+j)| \leq 1$ for $w > 0$. For several values of w and n, figure 3 reports the factor determining the amplitude of consumption fluctuations in (12'), namely

$$\left| \frac{1}{n} \cdot \frac{\sin \frac{w}{2} n}{\sin \frac{w}{2}} \right| \quad .$$

This figure should be compared with figure 2a.

The preceding assumes that there is no discounting in the definition of disposable income (2). It is of some interest to consider the consequences of replacing (2) with

$$(2') \qquad Y_{pt} = (1-\alpha) \sum_{j=0}^{\infty} \alpha^j E_t Y_{t+j}$$

where $0 < \alpha < 1$ is a discount factor equal to the reciprocal of one plus the discount rate. I have made the weights in (2') sum to unity to make Y_{pt} an income concept. It is easy to verify that where the income

process (3) is assumed, replacing (2) with (2') will lead to a version

of equation (8) with a modified constant term but exactly the same

distributed lag in Y_t. Now consider the implications of (1) and (2')

where income follows the process $Y_t = \cos wt$ so that $E_t Y_{t+j} = \cos w(t+j)$.

Under this circumstance Y_{pt} given by (2') becomes

$$Y_{pt} = (1-\alpha) \sum_{j=0}^{\infty} \alpha^j \cos w\ (t+j)$$

$$= \frac{(1-\alpha)}{2} \left(\sum_{j=0}^{\infty} \alpha^j e^{iw(t+j)} + \sum_{j=0}^{\infty} \alpha^j e^{-iw(t+j)} \right)$$

$$= \frac{(1-\alpha)e^{iwt}}{2(1-\alpha e^{iw})} + \frac{(1-\alpha)e^{-iwt}}{2(1-\alpha e^{-iw})} \quad .$$

As in the calculations leading to equation (11), we represent

$\frac{(1-\alpha)}{(1-\alpha e^{iw})}$ in the polar form

$$\frac{(1-\alpha)}{1-\alpha e^{iw}} = s(w)e^{-i\phi(w)}$$

where

$$s(w) = \frac{1-\alpha}{\sqrt{(1+\alpha^2 - 2\alpha\cos\ w)}}$$

$$\phi(w) = \arctan \left[\frac{-\alpha\sin\ w}{1-\alpha\sin\ w}\right] \quad .$$

It follows that

$$\frac{1-\alpha}{1-\alpha e^{-iw}} = s(w)e^{i\phi(w)} \quad .$$

Consequently, we can write Y_{pt} as

$$Y_{pt} = s(w)e^{-i\phi(w)} \cdot \frac{e^{iwt}}{2} + s(w)e^{i\phi(w)} \frac{e^{-iwt}}{2}$$

$$= s(w) \left(\frac{e^{i(wt-\phi(w))}+e^{-i(wt-\phi(w))}}{2}\right)$$

$$= s(w) \, \cos(wt-\phi(w)).$$

Therefore, using (1) we have that

$$C_t = \frac{\beta(1-\alpha)}{\sqrt{1+\alpha^2-2\alpha\cos w}} \cdot \cos(wt-\phi(w)).$$

It is interesting to compare this equation with (12). The equation just above shows the response of C to income where income is a cosine wave of frequency w and where expectations are formed rationally. Equation (12) shows the response of consumption to a cosine wave income path where permanent income is formed in the "irrational" fashion (irrational for this particular Y process under study) given by (7). It is of some interest that with $\alpha=\lambda$, equation (12) gives the correct amplitude of movements of C in response to Y_t, but gives a phase that is <u>minus</u> the phase given by the above equation. Some early discussions of the adaptive expectations scheme (7) conjectured that λ might turn out to equal the discount factor α. That conjecture is at best imperfectly borne out by the preceding calculations for income paths consisting of cosine waves. Setting $\lambda=\alpha$ and viewing cosine wave income paths as being inputs to a fixed (8) turns out to give the correct answer (under the rational expectations hypothesis) for the amplitude of consumption fluctuations, but a wrong answer for the phase shift.

So subjecting (8) to income paths consisting of cosine waves provides a faulty way of summarizing the implications of the model. The reason is that Friedman's model consists of both equations (8) <u>and</u> (3), and that (8) is predicted to hold only so long as (3) holds also.

Assuming that income is a cosine wave violates (3), so that (8) is no longer an implication of the fundamental equations of the model, (1) and (2).[7/]

A common alternative expositional device is to consider an income input of the form

$$Y_t = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad .$$

With the input, equation (8) implies a consumption path given by

$$C_t = \begin{cases} 0 & t < 0 \\ \beta(1-\lambda)(1+\lambda+\ldots+\lambda^t) & t \geq 0 \end{cases} \quad .$$

The t-period marginal propensity to consume is taken as $\beta(1-\lambda)(1+\lambda+\ldots+\lambda^t)$, which goes to β as t goes to infinity. Hence, β is interpreted as the long-run marginal propensity to consume. Once again, the problem with this expository device is that the income path used as the input to (8) does not resemble the kind of path generated by the income process (3) used to derive (8) under rationality. In a setup where income paths like the above one were the order of the day, (1) and (2) would not imply a relationship like (8). Thus, using the above income path as an input to (8) amounts to simulating the model by using an input path for income that is very unlikely to obtain if the bivariate model (8) and (3) is correct.

The preceding argument indicates that one correct way to illustrate Friedman's model is to examine the implications of income paths that are consistent with the income process (3). This can be done in an instructive way by examining the responses of income and consumption to an unexpected change in income. That ε_t is the unexpected part of current income comes from noticing that (3) implies

$$Y_t = Y_{t-1} + a + \epsilon_t - \lambda\epsilon_{t-1}$$

$$E_{t-1}Y_t = Y_{t-1} + a - \lambda\epsilon_{t-1}$$

$$Y_t - E_{t-1}Y_t = \epsilon_t.$$

Suppressing the constants in (3) and (8), we can write them as

$$Y_t = \frac{1-\lambda L}{1-L} \epsilon_t$$

$$C_t = \frac{\beta(1-\lambda)}{1-\lambda L} Y_t + U_t$$

$$= \frac{\beta(1-\lambda)}{(1-\lambda L)} \cdot \frac{(1-\lambda L)}{(1-L)} \epsilon_t + U_t$$

$$= \frac{\beta(1-\lambda)}{1-L} \epsilon_t + U_t \quad .$$

Writing out these expressions for C_t and Y_t we have

(13) $\qquad Y_t = \epsilon_t + (1-\lambda)\epsilon_{t-1} + (1-\lambda)\epsilon_{t-2} + (1-\lambda)\epsilon_{t-3} + \cdots$

(14) $\qquad C_t = \beta(1-\lambda) (\epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \cdots) + U_t$

Equation (13) shows that a random unexpected income of ϵ_t causes Y_t to jump by ϵ_t and can be expected to cause Y in <u>each</u> subsequent period to jump by $(1-\lambda)\epsilon_t$. Thus, an unexpected jump in income of ϵ_t causes a jump in permanent income of $(1-\lambda)\epsilon_t$. Equation (14) shows that the jump in ϵ_t causes C_t to jump by $\beta(1-\lambda)\epsilon_t$, which equals β times the change in permanent income. Equation (14) indicates that consumption in all subsequent periods can also be expected to increase by $\beta(1-\lambda)\epsilon_t$, so that the unexpected increase in income of ϵ_t can be expected to set off a <u>permanent</u> increase in consumption of $\beta(1-\lambda)\epsilon_t$.

Table 1--Kuznets' Data

Decade	National Income (billions of 1929 dollars)	Consumption Expenditures (billions of 1929 dollars)
1869-78	9.3	8.1
1874-83	13.6	11.6
1879-85	17.9	15.3
1884-93	21.0	17.7
1889-98	24.2	20.2
1894-1903	29.8	25.4
1899-1908	37.3	32.3
1904-13	45.0	39.1
1909-18	50.6	44.0
1914-23	57.3	50.7
1919-28	69.0	62.0
1924-33	73.3	68.9
1929-38	72.0	71.0

Source: S. Kuznets, National Product Since 1869, (National Bureau of Economic Research, 1946), p. 119.

Table 2
Regressions with Kuznets' Data

Regressions (1) and (2) <u>exclude</u> the observation for 1929-38.

$$(1) \quad \overline{C}_t = -1.72 + .928 \ \overline{Y}_t, \qquad \qquad \text{d.w.} = .77, \ \overline{R}^2 = .997$$
$$\phantom{(1) \quad \overline{C}_t =} (.67) \quad (.016)$$

$$(2) \quad \overline{C}_t = -1.28 + 1.137 \ \overline{Y}_t - 1.27t, \qquad \text{d.w.} = 1.65, \ \overline{R}^2 = .998$$
$$\phantom{(2) \quad \overline{C}_t =} (.57) \quad (.082) \quad (.49)$$

Regressions (3) and (4) <u>include</u> the observation for 1929-38.

$$(3) \quad \overline{C}_t = -2.48 + .958 \ \overline{Y}_t \qquad \qquad \text{d.w.} = .53, \ \overline{R}^2 = .993$$
$$\phantom{(3) \quad \overline{C}_t =} (1.09) \ (.024)$$

$$(4) \quad \overline{C}_t = -2.43 + .989 \ \overline{Y}_t - .18t \qquad \text{d.w.} = .57, \ \overline{R}^2 = .992$$
$$\phantom{(4) \quad \overline{C}_t =} (1.17) \ (.172) \quad (1.01)$$

The preceding exposition traces out the effects of drawing random terms from the distribution governing the ε's, taking care to insure that both (8) and (3) are held fixed, as is required by the rational expectations hypothesis. The exposition has the virtue that it gets along without reference to vaguely defined concepts like "short-run" and "long-run" marginal propensities to consume.

2. Do the Regressions on Kuznets' Data Recover β?

Table 1 records Kuznets' overlapping decade averages of consumption and income. Table 2 reports least squares regressions of average consumption on average income, both excluding and including a trend, and excluding and including the last observation corresponding to the decade of the Great Depression. It is the regressions without the trend terms that are widely regarded as recovering a good estimate of β as the coefficient on $\overline{Y}_t$. I report the regressions including the trend term to highlight how excluding it is required to deliver a coefficient on $\overline{Y}_t$ that seems plausible as an estimate of β. The calculations below provide reasons for expecting that the regression excluding the trend will provide a better estimate of β. However, those calculations also indicate that if Friedman's consumption-income model consisting of equations (3) and (8) is correct, then the regression including the trend term can be expected to underestimate β. The point estimates in Table 2 are not consistent with that prediction.

The calculations in this section are intended to illustrate a rigorous method for evaluating descriptive interpretations such as the following one offered by Daniel Suits:

In the discussion of long-run and short-run
effects two things are sometimes confused: the
nature of the problem under investigation, and the
nature of the data employed. It is possible to use
quarterly data and still analyze a very long-run
consumption function. The data set a lower limit
to the "length of run" that can be investigated--the
Kuznets' estimates for decades cannot be used to
investigate quarterly variations in consumption, but
they do not, of themselves, set an upper limit. A
regression fitted to annual, quarterly, or even
monthly data for the period 1865 to the present
would yield results essentially no different from
that obtained from decade averages. When we use a
time span covering nearly a hundred years, the
regression analysis is going to be most sensitive
to the big overall changes, to the general drift
of the data and not to the relatively minor differ-
ences between one year and the next. [7, pp. 34-35].

In this section, I calculate the simple regression coefficient

that would obtain an n-period average data if Friedman's model consisting

of equations (3) and (8) were correct. To simplify the calculations, I

will work in first differences and write equations (3) and (8) as

(15) $y_t = (1-\lambda L)\varepsilon_t + a$

$$c_t = \frac{\beta(1-\lambda)}{1-\lambda L} y_t + u_t \, ,$$

or

(16) $c_t = \beta(1-\lambda)\varepsilon_t + \beta a + u_t$

where $c_t = C_t - C_{t-1}$, $y_t = Y_t - Y_{t-1}$, $u_t = U_t - U_{t-1}$.

Now consider forming n-period moving averages of y_t and c_t:

$$\bar{y}_t = \frac{1}{n} (1+L+\ldots+L^{n-1})y_t$$

$$\bar{c}_t = \frac{1}{n} (1+L+\ldots+L^{n-1})c_t \, .$$

Taking n-period moving averages on both sides of (15) and (16) gives

$$\bar{y}_t = a + \frac{1}{n} (\varepsilon_t + \ldots + \varepsilon_{t-n-1} - \lambda\varepsilon_{t-1} - \ldots - \lambda\varepsilon_{t-n})$$

(15') $$\bar{y}_t = a + \frac{1}{n} (\varepsilon_t + (1-\lambda)\varepsilon_{t-1} + \ldots + (1-\lambda)\varepsilon_{t-(n-1)} - \lambda\varepsilon_{t-n})$$

(16') $$\bar{c}_t = \beta a + \frac{\beta(1-\lambda)}{n} [\varepsilon_t + \varepsilon_{t-1} + \ldots + \varepsilon_{t-(n-1)}] + \bar{u}_t .$$

Since successive ε's are orthogonal, we have from (15') that the variance of $\bar{y}_t$ is

$$\sigma_{\bar{y}_t}^2 = \frac{\sigma_\varepsilon^2}{n^2} [1+\lambda^2+(n-1)(1-\lambda)^2]$$

$$= \frac{\sigma_\varepsilon^2}{n^2} [1+\lambda^2+(n-1)(1-2\lambda+\lambda^2)]$$

$$\sigma_{\bar{y}_t}^2 = \frac{\sigma_\varepsilon^2}{n^2} [n(1+\lambda^2)-2\lambda(n-1)].$$

The covariance between $\bar{y}_t$ and $\bar{c}_t$ is calculated by using (15') and (16') to calculate $\bar{y}_t - E\bar{y}_t$ and $\bar{c}_t - E\bar{c}_t$, multiplying, and taking expected values:

$$\sigma_{\bar{c}\,\bar{y}} = \sigma_\varepsilon^2 \frac{\beta(1-\lambda)}{n^2} [1+(n-1)(1-\lambda)]$$

$$\sigma_{\bar{c}\,\bar{y}} = \sigma_\varepsilon^2 \frac{\beta(1-\lambda)}{n^2} [n-(n-1)\lambda].$$

The studies using Kuznets' ten-year averaged data in effect calculated the simple regression through the origin of $\bar{c}_t$ on $\bar{y}_t$ for n chosen to be ten (with annual data). That is they presented the regression

(17) $\bar{c}_t = \gamma \bar{y}_t + \text{residual}$

Since we are working here with first differences, computing a regression of the level of the averaged C on the average Y and a constant term with no trend term corresponds to running the regression on first differences through the origin. Before considering regression (17) it is interesting to analyze the regression with an intercept term,

(18) $\bar{c}_t = \delta \bar{y}_t + \text{constant} + \text{residual}.$

The counterpart to (18) is a regression of the level of averaged C on the level of averaged Y, a constant, and a trend. The population value of δ is given by

$$\delta = \frac{\sigma_{\overline{cy}}}{\sigma_{\bar{y}}^2} \quad .$$

Using our formulas for $\sigma_{\overline{cy}}$ and $\sigma_{\bar{y}}^2$ we have

(19) $\delta = \dfrac{\beta(1-\lambda)(n-\lambda(n-1))}{[(1+\lambda^2)n-2\lambda(n-1)]} \quad .$

How closely does δ approximate β? For n sufficiently large, δ approximates β closely, since

$$\lim_{n\to\infty} \frac{(n-\lambda(n-1))}{(1+\lambda^2)n-2\lambda(n-1)} = \lim_{n\to\infty} \frac{(1-\lambda(\frac{n-1}{n}))}{(1+\lambda^2)-2\lambda(\frac{n-1}{n})}$$

$$= \frac{1-\lambda}{1+\lambda^2-2\lambda} = \frac{1}{1-\lambda} \quad .$$

Table 3

λ	$(1-\lambda)\ (\dfrac{10-9\lambda}{(1+\lambda^2)\cdot 10-2\lambda\cdot 9})$
0	1.0
.1	.987
.2	.965
.3	.929
.4	.873
.5	.786
.6	.657
.7	.483
.8	.280
.9	.100
1.0	.0

Thus, we have

$$\lim_{n \to \infty} \delta = \beta.$$

For $n=10$ and various values of λ, Table 3 reports values of the bias factor

$$(1-\lambda) \; \left(\frac{n-\lambda(n-1)}{(1+\lambda^2)n-2\lambda(n-1)} \right)$$

which is associated with taking δ as an estimate of β. For $\lambda=.3$, the value Friedman found for the annual time series, the bias factor is .929, which is substantial.

For regressions on levels that do not include a trend, (17) is the corresponding model in terms of first differenced data. The population value of the slope coefficient γ is given by

$$\gamma = \frac{\sigma_{\overline{cy}} + E(\overline{c})E(\overline{y})}{\sigma_{\overline{y}}^2 + (E(\overline{y}))^2} \; .$$

From (15) and (16) we have that $Ey_t = a$, $Ec_t = \beta a$. Since taking moving averages does not change means, we have that

$$E\overline{y}_t = a$$

$$E\overline{c}_t = \beta a.$$

Consequently, we have that the population parameter γ obeys

$$\gamma = \frac{\sigma_{\overline{cy}} + \beta a^2}{\sigma_{\overline{y}}^2 + a^2} = \frac{\dfrac{\sigma_{\overline{cy}}}{a^2} + \beta}{\dfrac{\sigma_{\overline{y}}^2}{a^2} + 1}$$

For fixed values of $\sigma_{\overline{cy}}$ and $\sigma_{\overline{y}}^2$, we have that

$$\lim_{a^2 \to \infty} \gamma = \beta,$$

so that as the trend term a becomes relatively more and more important, γ approaches β. Using our expressions for $\sigma_{\overline{cy}}$ and $\sigma_{\overline{y}}^2$, we can express γ as

$$\gamma = \frac{\frac{\sigma_\epsilon^2}{n^2} \beta(1-\lambda)[n-\lambda(n-1)]+\beta a^2}{\frac{\sigma_\epsilon^2}{n^2}[(1+\lambda^2)n-2\lambda(n-1)]+a^2}$$

or

$$(20) \quad \gamma = \frac{\dfrac{\beta(1-\lambda)(1-\lambda(\frac{n-1}{n}))}{(1+\lambda^2-2\lambda(\frac{n-1}{n}))} + \dfrac{n\beta a^2}{\sigma_\epsilon^2(1+\lambda^2-2\lambda(\frac{n-1}{n}))}}{1 + \dfrac{na^2}{\sigma_\epsilon^2(1+\lambda^2-2\lambda(\frac{n-1}{n}))}} = \frac{\delta + \dfrac{n\beta a^2}{\sigma_\epsilon^2(1+\lambda^2-2\lambda(\frac{n-1}{n}))}}{1 + \dfrac{na^2}{\sigma_\epsilon^2(1+\lambda^2-2\lambda(\frac{n-1}{n}))}}$$

Holding β, λ, σ_ϵ^2, and a fixed, we have

$$\lim_{n \to \infty} \gamma = \frac{\dfrac{\beta(1-\lambda)(1-\lambda)}{(1-\lambda)^2} + \dfrac{n\beta a^2}{\sigma_\epsilon^2(1-\lambda)^2}}{1 + \dfrac{na^2}{\sigma_\epsilon^2(1-\lambda)^2}}$$

$$= \frac{\beta(1+ \dfrac{na^2}{\sigma_\epsilon^2(1-\lambda)^2})}{1+ \dfrac{na^2}{\sigma_\epsilon^2(1-\lambda)^2}}$$

$$= \beta,$$

so that for averages sufficiently long, the slope γ does approximate β well. Expression (20) shows that γ approximates β better the larger is n, the larger is a, and the smaller is σ_ϵ^2.

Comparing (19) and (20) for n=1 and n=10 permits evaluating the passage by Suits quoted earlier. With n=1, (19) gives

$$\delta = \frac{\beta(1-\lambda)}{(1+\lambda^2)} \quad ,$$

which is the population slope of a regression of the one-period level C_t against a constant and a trend. Notice that for $\lambda > 0$ this value of δ is less than $\beta(1-\lambda)$, which is often interpreted as the one-period marginal propensity to consume. For n=1, (20) gives,

$$\gamma = \frac{\beta(1-\lambda)\sigma_\epsilon^2 + \beta a^2}{\sigma_\epsilon^2(1+\lambda^2)+a^2} \quad ,$$

which is the population slope of a regression of the one-period level of C_t against a constant and the level of Y_t. Clearly, for sizable values of λ, δ for n=1 is very much smaller than δ for n=10. Whether γ for n=1 is close to γ for n=10 depends critically on the ratio of the income-innovation variance σ_ϵ^2 to the income trend parameter a. The smaller is this ratio, the closer will γ for n=1 be both to β and to γ for n=10. The remarks of Suits are thus approximately valid only under suitable restrictions on σ_ϵ^2 and a. Further, their validity is crucially dependent on excluding a trend term from the regressions in question.

The preceding calculations provide a rigorous framework for evaluating the claim that the regressions on Kuznets' data estimate the marginal propensity to consume out of permanent income. The calculations indicate that for sizable λ's, the presence of a strong trend in income

(a large a) <u>and</u> the ommission of a trend term in the regressions on

Kuznets' data are essential elements in recovering a good estimate of β.[8]

3.   Conclusion

The techniques illustrated above are obviously applicable to other versions of Friedman's permanent income consumption model, versions that differ from the Friedman-Muth model in the stochastic process posited for income. From the viewpoint of Keynesian models, it would be interesting to relax the assumption made here that income is strictly econometrically exogenous in the consumption function. It is natural to suspect that relaxing that assumption would be necessary if the permanent income, rational expectations model is fruitfully to be applied to time series data.

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Footnotes

[1]/ Lucas [3] makes the point that the notion of rational expectations is an important ingredient of Freidman's study.

[2]/ In standard Keynesian models in which the interest elasticity of the demand for money is not zero, the strict exogeneity of Y_{pt} in (1) will not obtain. In such models, there is a simultaneity problem on top of the problems I am discussing in this paper. I am ignoring that problem here to avoid complicating the argument. It should be noted, though, that there do exist macroeconomic models in which assuming exogeneity of Y_{pt} in (1) is legitimate. An example is Tobin's "Dynamic Aggregative Model."

[3]/ If the reader wishes, he can interpret $E_t Y_{t+j}$ as the linear least squares projection of Y_{t+j} against information dated t or earlier. Under this interpretation, all of the subsequent developments go through under the condition that ε_t is serially uncorrelated, which is weaker than the condition imposed in the test that $E_t \varepsilon_{t+1} = 0$, making ε_t serially independent.

[4]/ To represent $\dfrac{1}{1-\lambda e^{-iw}}$ in polar form we first write it in the form

$$\frac{1}{1-\lambda e^{-iw}} = a + bi.$$

To solve for a and b notice that the above equality implies

$$1 = (a+bi)[(1-\lambda\cos w)+i\lambda\sin w)]$$

$$1 + i\cdot 0 = [a(1-\lambda\cos w)-b\lambda\sin w] + i[a\lambda\sin w+b(1-\lambda\cos w)].$$

Equating the real and imaginary parts and solving for b and a gives

$$a = \frac{1-\lambda\cos w}{1+\lambda^2-2\lambda\cos w}$$

$$b = \frac{-a\lambda\sin w}{1-\lambda\cos w}$$

Next we can write $a + bi = re^{i\theta}$ where $r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}(b/a)$. We have $a^2 + b^2 = 1/(1+\lambda^2-2\lambda\cos w)$, $\theta = \tan^{-1}(-\lambda\sin w/(1-\lambda\cos w))$. Therefore, we have the formula used in the text.

The sequence $Y_t = \cos wt$ is a particular realization of the deterministic stochastic process

$$Y_t = R(\xi) \cos (wt - \theta(\xi))$$

where $$R(\xi)^2 = A(\xi)^2 + B(\xi)^2$$

$$\tan \theta(\xi) = B(\xi)/A(\xi)$$

and $A(\xi)$, $B(\xi)$ are random variables defined on a probability space (Ω, F, P), where $\xi \epsilon \Omega$, and which satisfy $EA(\xi) = EB(\xi) = 0$, $EA(\xi)B(\xi) = 0$, and $EB(\xi)^2 = EA(\xi)^2 = \sigma^2 < \infty$. The stochastic process is (linearly) deterministic, meaning it can be predicted arbitrarily well arbitrarly far into the future by suitable linear combinations of past values of itself.

[6/]We have

$$2 \sum_{j=1}^{n} \cos w(t+j) = \sum_{j=1}^{n} (e^{iw(t+j)} + e^{-iw(t+j)})$$

$$= \frac{e^{iw(t+1)} - e^{iw(t+n+1)}}{i - e^{iw}} + \frac{e^{-iw(t+1)} + e^{-iw(t+n+1)}}{i - e^{-iw}}$$

$$= \frac{e^{iw(t+1)} e^{i\frac{w}{2}n}(e^{-i\frac{w}{2}n} - e^{i\frac{w}{2}n})}{e^{i\frac{w}{2}}(e^{-i\frac{w}{2}} - e^{i\frac{w}{2}})} + \frac{e^{-iw(t+1)} e^{-i\frac{w}{2}n}(e^{i\frac{w}{2}n} - e^{-i\frac{w}{2}n})}{e^{-i\frac{w}{2}n}(e^{i\frac{w}{2}} - e^{-i\frac{w}{2}})}$$

$$= \frac{\sin\frac{w}{2}n}{\sin\frac{w}{2}} \cdot [e^{iw(t+1) + i\frac{w}{2}(n-1)} + e^{-iw(t+1) - i\frac{w}{2}(n-1)}]$$

$$= \frac{\sin\frac{w}{2}n}{\sin\frac{w}{2}} 2 \cos[w(t+1) + \frac{w}{2}(n-1)] \ .$$

Dividing by two gives the formula in the text.

The spectral density of $(1-L)Y_{t+1}$ governed by (3) is

$$f_{\Delta y}(w) = \sigma_\epsilon^2(1-\lambda e^{-iw})(1-\lambda e^{iw})$$

$$= \sigma_\epsilon^2(1+\lambda^2-2\lambda\cos w), \qquad w\epsilon[-\pi, \pi].$$

The quantity $\int_{w_1}^{w_2} f_{\Delta y}(w)dw$ $(w_2 > w_1)$ tells the proportion of the variance in Δy that can be accounted for by cosine waves with frequencies in the band $[w_1, w_2]$. Where $f_{\Delta y}(w) = \sigma_\epsilon^2(1+\lambda^2-2\lambda\cos w)$, which is a continuous function of w, we have that

$$\lim_{w_1 \to w_2} \int_{w1}^{w2} f_{\Delta y}(w)dw = 0, \qquad \pi > w_2 > w_1 > 0.$$

so that the proportion of the variance in Δy accounted for by cosine waves in a band near w_2 is approximately zero for small frequency bands. This indicates that subjecting (8) to cosine income paths is in effect to assume a process for income very different from the process (3) used to derive (8).

The calculations in this section amount to a way of evaluating the effects of omitting variables in (16'). According to (16'), $\bar{c}_t$ and $\bar{y}_t$ are related by

$$\bar{c}_t = \beta(1-\lambda)(\bar{y}_t+\lambda\bar{y}_{t-1}+\lambda^2\bar{y}_{t-2}+\ldots) + \bar{u}_t$$

Therefore, the simple regression of $\bar{c}_t$ on $\bar{y}_t$ obeys

$$E[\bar{c}_t|\bar{y}_t] = \beta(1-\lambda)\bar{y}_t + \beta(1-\lambda)\sum_{i=1}^{\infty} \lambda^i E[\bar{y}_{t-i}|\bar{y}_t]$$

The regressions $E[\bar{y}_{t-i}|\bar{y}_t]$ can be calculated by using (15'). These calculations seem more tedious than those done here, but lead to identical answers.