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The Optimal Quantity of Debt

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ABSTRACT

We describe a model for calculating the optimal quantity of debt and then apply it to the U.S. economy. The model consists of a large number of infinitely-lived households whose saving behavior is influenced by precautionary saving motives and borrowing constraints. This model incorporates a different role for government debt than the standard representative agent growth model and captures different trade-offs between the benefits and costs of varying its level. Government debt enhances the liquidity of households by providing additional assets for smoothing consumption (in addition to claims to capital) and effectively loosening borrowing constraints. By raising the interest rate, government debt makes assets less costly to hold and more effective in smoothing consumption. However, the implied taxes have wealth distribution, incentive, and insurance effects. Further, government debt crowds out capital (via higher interest rates) and lowers per capita consumption. Our quantitative analysis suggests that the crowding out effect is decisive for welfare. We also describe variations of the model which permit endogenous growth. It turns out that even with lump sum taxes and inelastic labor, government debt as well as government consumption have growth rate effects, thereby implying large welfare gains from reducing the level of debt.

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1. Introduction

The level, time path and nature of government debt are important issues in fiscal policy.[1] In this paper we describe a model for calculating the optimal level of public debt and then apply it to the U.S. economy.[2] The model consists of a large number of infinitely-lived households whose saving behavior is influenced by precautionary saving motives and borrowing constraints. This model incorporates a different role for government debt than the standard representative agent growth model and captures different trade-offs between the benefits and costs of varying its level. In this model government debt enhances the liquidity of households by providing additional assets for smoothing consumption (in addition to claims to capital) and effectively loosening borrowing constraints.[3] By raising the interest rate, government debt makes assets less costly to hold and more effective in smoothing consumption. However, the implied taxes have wealth distribution, incentive, and insurance effects. Further, government debt crowds out capital via higher interest rates and lowers per capita consumption.

The role of government debt and its welfare effects in our model may be contrasted with its role and welfare effects in the standard deterministic representative agent growth model. In the latter model, if lump sum taxes are permitted, then there is no role for government debt and, hence, no welfare effects. With distorting taxes there is a role for government debt as a means of smoothing tax distortions over time. Optimal debt policy in such a model (see Barro 1979 and Chamley 1985, 1986) generally implies that the steady state level of debt depends on the initial level of debt. If sufficient capital levies are available at the initial date then the optimal quantity of debt is the negative of the present value of government consumption evaluated using the undistorted sequence of interest rates.[4] Thus, in the standard representative agent growth model the optimal level

[1] By nature of government debt we mean the maturity structure of debt and the pattern of state contingent payments associated with it.

[2] Our analysis of the optimal level of debt is closely related to Bewley's (1980, 1983) analysis of the optimum quantity of money for which he used a pure exchange model with a finite number of consumers. Our analysis is richer and quantitatively oriented.

[3] A recent paper by Jappelli and Pagano (1994) analyzes the welfare effects of borrowing constraints in an overlapping generations model.

[4] This permits setting all distorting taxes to zero and thereby avoiding welfare losses from such taxes. Government consumption is financed by the interest earned on public assets.

of debt either depends on some unknown initial conditions or is indeterminate.[5]

The growth rate effects of government debt are also different in our model than in the standard model when the growth rate is endogenous. In our framework government debt has an effect on the growth rate even with lump sum taxes whereas there is no such effect in the standard model. Consequently, in our framework there can be large welfare effects from changing the level of debt.[6]

The class of models we consider are variants of the deterministic growth model modified to include a large number of individuals subject to uninsured idiosyncratic shocks to their labor productivities. Though there is no aggregate uncertainty in these models there is individual uncertainty due to the absence of insurance markets. This is the feature that generates precautionary saving. Further, there is ex-post heterogeneity among individuals; in the steady state there is a distribution of individuals according to asset holdings and earnings.

As Aiyagari (1994c) has argued, these models have many empirically plausible implications; furthermore, they are quite attractive for quantitative analysis.[7] When insurance markets are complete our model collapses to the representative agent growth model which facilitates comparison between the two frameworks. Most aspects of our model can be parameterized in exactly the same way as the representative agent growth model has been for quantitative analyses of growth and business cycles. The only additional aspect of our model that is not found in a representative agent growth model is the process governing the idiosyncratic labor productivity shocks. This process can be parameterized by using empirical microeconomic studies of earnings behavior. Thus, with the exception of the

[5] As we will see our model has some similarities with the overlapping generations model – in particular, the crowding out of capital by debt mentioned previously arises in both models. However, the role of debt in the two models is somewhat different. In the overlapping generations model government debt is equivalent to lump-sum redistributions across generations which are revenue neutral, and such redistributions affect equilibrium allocations and interest rates. In our model, due to the implicit altruism among generations, such lump-sum redistributions are completely neutral.

[6] In our model government consumption also affects the growth rate even with lump sum taxes and inelastic labor – effects which do not arise in the standard model. Further, our result obtains even though government consumption is not an argument in the aggregate production function as postulated by Barro (1990). In our model (with lump sum taxes) government consumption has a positive effect on the growth rate which is exactly as hypothesized by Barro. The growth rate effects of government debt and government consumption operate through their effects on precautionary saving. The recent paper by Devereux and Smith (1994) analyzes a similar mechanism.

[7] Bewley (undated) and Laitner (1979, 1993) were among the first to analyze such models. Aiyagari (1994a) uses such a model to evaluate the quantitative significance of precautionary saving.

process governing the idiosyncratic labor productivity shocks, the parameters of the two models could be exactly the same.

We quantify the welfare effects of government debt for several versions of our model using parameter values obtained from U.S. data. Our benchmark model is one with exogenous growth, a proportional income tax, and elastic labor supply.[8] We find that the optimal level of debt is quite negative, and the welfare gain of reducing the level of debt is economically significant. However, the welfare gain of moving from the current level to zero is modest. The optimal real interest rate is several percentage points below the current rate in the United States implying a significantly higher capital stock. The optimal income tax rate is a few percentage points below its current level. For the endogenous growth version the optimal level of debt is also very negative and, as expected, the welfare gains of reducing the level of debt are huge. The optimal growth rate is several percentage points higher than the current rate in the United States. For both exogenous and endogenous growth versions the optimal tax rate is somewhat lower than the current U.S. level.

The rest of this paper is organized as follows. In Section 2 we describe the exogenous and endogenous growth versions of our model with lump sum taxes and inelastic labor. We start with this specification even though our benchmark model has proportional taxes and elastic labor because this specification permits us to do several things. First, and most important, it is relatively easier to provide intuition for the workings of the model with lump sum taxes and inelastic labor than for the model with proportional taxes and elastic labor. Second, the simpler model illustrates the contrast between the growth rate effects (in the endogenous growth version) of government debt and government consumption in our model compared to the representative agent model. If taxes are proportional and labor supply is elastic, then growth rate effects arise in the representative agent endogenous growth model as well. Third, it permits us to separate the wealth distribution effect of taxes from the incentive and insurance effects which arise when taxes are proportional. After analyzing the model with lump sum taxes and inelastic labor we describe, in Section 3, the model with a proportional income tax and elastic labor and how the model is

[8] Since our focus in this paper is on the optimal quantity of debt and not on the structure of taxation we have chosen to work with a single tax rate on both labor and capital income. The steady state government budget constraint determines the income tax rate given the debt/GNP ratio. It should be noted, however, that in our model, as opposed to the representative agent model, it can be optimal to tax capital income even in the long run. See Aiyagari (1994b).

parameterized and how its steady state is computed. Results are described in Section 4. In Section 5 we discuss the robustness of some of the results for some alternative parameter values. Section 6 concludes.

2. Growth Model With Uninsured Idiosyncratic Shocks

In this section we describe an augmented version of the model in Aiyagari (1994a) – augmented to permit growth and include government debt, lump sum taxes, and government consumption.

Our economy has a continuum of infinitely-lived agents of measure unity who receive idiosyncratic shocks to their labor productivities and supply one unit of labor inelastically. Let e_t denote an individual's labor productivity and suppose that it is i.i.d. across agents and follows some Markov process over time. We normalize per capita labor productivity to unity so that $E(e_t) = 1$. The assumption that all agents face the same stochastic process for productivity can be relaxed somewhat without affecting our main results.[9] We also assume that there are no aggregate shocks. We describe the balanced growth path of such an economy without insurance markets but with trading in risk-free assets – capital and government debt. Along the balanced growth path there will be fluctuations in an individual's consumption, income, and wealth but per capita variables will be growing at constant rates and cross-section distributions (relative to per capita values) will be constant over time. We will consider both exogenous and endogenous growth versions. We start with the exogenous growth case.

2.1. Exogenous Growth

There is a neoclassical aggregate production function, $Y_t = F(K_t, z_t N_t)$, where Y_t is per capita output, K_t is per capita capital, N_t is per capita labor input, and z_t is a measure of

[9] We could assume that there are several groups of agents where the groups are distinguished by the average level of productivity. This can be thought of as reflecting permanent skill level differences among groups of agents. Within each group agents are subject to idiosyncratic shocks to their productivity level. The distribution of shocks normalized by its mean can be assumed to be the same for all the groups. Taxes can also be assumed to be proportional to the average productivity level of a group. With homothetic preferences and constant returns technology (as we will assume) each group of agents is then just a scaled version of any other group. It will then turn out that the equilibrium interest rate and all per capita quantity variables will be the same as in the case that there is only one group of agents. Further, our results regarding the optimal quantity of debt and the welfare gains of changing the level of debt will also be unaffected.

labor-augmenting technical progress in period t. Note that per capita labor input equals unity in equilibrium. We assume that $z_t = z(1+g)^t$, where g is the rate of technical progress. Also, capital is assumed to depreciate at the constant geometric rate δ. Assuming competitive product and factor markets the wage rate w_t and the interest rate r are given by

$$w_t = z_t F_2(K_t, z_t), \tag{1}$$

$$r = F_1(K_t, z_t) - \delta. \tag{2}$$

Note that in a balanced growth equilibrium w_t, Y_t, and K_t will be growing at the rate g whereas r will be constant.

We now describe consumer behavior. Let c_t, a_t, and T_t denote an individual's consumption in period t, an individual's assets at the beginning of period t, and the lump sum tax in period t, respectively. The consumer derives utility in period t from consumption in period t; this utility is given by $c_t^{1-\nu}/(1-\nu)$ where $\nu > 0$ is the relative risk aversion coefficient. The typical consumer starts at date zero with some initial assets a_0 and some initial productivity shock e_0 and chooses stochastic processes for consumption and asset holdings in order to maximize the expected discounted sum of utilities of consumption subject to the sequence of budget constraints and nonnegativity constraints, i.e.,

$$\max_{\{c_t, a_{t+1}\}} E\left[\sum_{t=0}^{\infty} \beta^t c_t^{1-\nu}/(1-\nu) \,\middle|\, a_0, e_0\right] \tag{3}$$

$$\text{subject to} \quad c_t + a_{t+1} \leq w_t e_t + (1+r)a_t - T_t, \quad t \geq 0, \tag{4}$$

$$c_t \geq 0, \quad a_t \geq 0, \quad t \geq 0. \tag{5}$$

Note that the constraint $a_t \geq 0$ rules out borrowing.[10]

Let G_t denote per capita government consumption which we assume grows exogenously at the constant rate g. Let B_t denote per capita government debt. Then the government budget constraint is given by

$$G_t + rB_t = B_{t+1} - B_t + T_t. \tag{6}$$

[10] If $r < g$ some limit on borrowing must be imposed for the consumer's problem to be well defined. If $r < g$ then the present value of earnings is infinite (almost surely) and without a borrowing limit, nothing prevents the consumer from running a Ponzi scheme. When there is a limit on borrowing it is possible to have a steady state with $r < g$ which is very reminiscent of overlapping generations models.

Let A_t denote per capita assets held by consumers. In a balanced growth equilibrium of this economy we must have

$$A_t = K_t + B_t. \tag{7}$$

It will be convenient to transform the model into a stationary (nongrowing) form. Towards this end let $k = K_t/Y_t$, $\tilde{w} = w_t/Y_t$, $\tilde{c}_t = c_t/Y_t$, $\tilde{a}_t = a_t/Y_t$, $\tau = T_t/Y_t$, $\gamma = G_t/Y_t$, $b = B_t/Y_t$, and $\bar{a} = A_t/Y_t$. Note that in a balanced growth equilibrium T_t, B_t, and A_t will also be growing at the rate g.

We can now rewrite the model as follows. First divide through the consumer's budget constraint by Y_t and rewrite the consumer's preferences using the definition of $\tilde{c}_t$. Thus the consumer's problem becomes

$$\max_{\{\tilde{c}_t, \tilde{a}_{t+1}\}} E\left[Y_0^{1-\nu}\sum_{t=0}^{\infty}[\beta(1+g)^{1-\nu}]^t \tilde{c}_t^{1-\nu}/(1-\nu)\middle|\tilde{a}_0, e_0\right] \tag{8}$$

$$\text{subject to} \quad \tilde{c}_t + (1+g)\tilde{a}_{t+1} \le \tilde{w}e_t + (1+r)\tilde{a}_t - \tau, \quad t \ge 0, \tag{9}$$

$$\tilde{c}_t \ge 0, \quad \tilde{a}_t \ge 0, \quad t \ge 0. \tag{10}$$

Divide through the government budget constraint by Y_t so that it becomes

$$\gamma + (r-g)b = \tau. \tag{11}$$

Divide through the asset market equilibrium condition by Y_t to get

$$\bar{a} = k + b. \tag{12}$$

Using equation (2) we can express K_t/z_t as a function of r, and, hence, we can express k $(= K_t/Y_t)$ as a function of r, say $\kappa(r)$. Further, using (1) we can express $\tilde{w}$ $(= w_t/Y_t)$ as a function of r, say $\omega(r)$. Thus, we have

$$k = \kappa(r), \tag{13}$$

$$\tilde{w} = \omega(r). \tag{14}$$

The balanced growth equilibrium of the original economy corresponds to the steady state of the transformed economy which is characterized by an interest rate r^* that satisfies

$$\bar{a}(r; \gamma, b, g) = \kappa(r) + b, \tag{15}$$

where $\tilde{a}(r;\cdot)$ is the per capita assets desired by consumers (relative to per capita output) as a function of the interest rate, and $\kappa(r) + b$ is the per capita supply of assets (relative to per capita output) expressed as a function of the interest rate.[11]

In Figure 1 we show how the steady state interest rate is determined; the steady state is marked IM (for "incomplete markets"). The crucial feature of this picture is that $\tilde{a}(r;\gamma,b,g)$ tends to infinity as r approaches λ ($\equiv (1+g)^{\nu}/\beta - 1$) from below. The intuition is as follows. When r equals λ the consumer would like to maintain a smooth marginal utility of consumption profile. This can be seen by looking at the following version of the consumer's Euler equation

$$\tilde{c}_t^{-\nu} \geq [(1+r)/(1+\lambda)]E_t\tilde{c}_{t+1}^{-\nu}, \quad (= \text{ if } \tilde{a}_{t+1} > 0).$$

However, since there is some probability of receiving a long sequence of low labor productivity shocks, the only way for the consumer to maintain a smooth marginal utility of consumption profile would be to have an infinitely large amount of assets. Note also that under incomplete markets, due to the precautionary motive and borrowing constraints, the consumer will hold assets over and above the credit limit to buffer earnings shocks even when r is less than λ. This would not be the case under complete markets.

Under complete markets the consumer can fully insure against the idiosyncratic labor productivity shock and effectively eliminate any uncertainty in individual earnings. The asset demand function in this case is described by the dotted line in Figure 1, leading to the steady state marked CM (for "complete markets"). This is the usual result that the capital stock satisfies the modified golden rule, i.e., $F_1(K_t, z_t) - \delta = r = \lambda$.

It follows that the interest rate with incomplete markets is lower and, hence, the capital stock is higher than with complete markets. Further, increases in government debt are neutral with complete markets but not with incomplete markets. In general, we may expect an increase in b to raise the interest rate and, thereby, crowd out some capital. To see this last point more clearly and also to appreciate the role of government debt in

[11] Equation (15) is obtained in the following way. The solution to the consumer's problem yields a decision rule for asset accumulation: $\tilde{a}_{t+1} = \alpha(\tilde{a}_t, e_t; r, \gamma, b, g)$. This decision rule can be used together with the Markov process for the labor productivity shock (e_t) to calculate the stationary joint distribution of assets and the productivity shock, denoted by $H(\tilde{a}, e; r, \gamma, b, g)$. This stationary distribution then implies an expression for per capita assets, $\bar{a} = \int\int \tilde{a}\, dH = \tilde{a}(r;\gamma,b,g)$. This is the left side of (15). The right side of (15) is obtained from (12) by noting that $k = \kappa(r)$.

loosening borrowing constraints it is useful to rewrite the consumer's budget constraint by substituting for taxes τ from (11) into (9) and defining $a_t^* = \tilde{a}_t - b$. This leads to the following form of the consumer's budget constraint:

$$c_t + (1+g)a_{t+1}^* \leq \omega(r)e_t + (1+r)a_t^* - \gamma, \quad a_t^* \geq -b.$$

The steady state equilibrium condition in the asset market is now given by $\bar{a}^*(r;\gamma,b,g) = \kappa(r)$, where $\bar{a}^*$ is defined in a way analogous to $\bar{a}(\cdot)$, and $\bar{a}^*(\cdot) \equiv \bar{a}(\cdot) - b$. In this formulation government debt only enters the consumer's borrowing constraint. Higher levels of b, in effect, loosen the consumer's borrowing constraint and reduce the consumer's average asset holdings (net of government debt). In other words, when the consumer is allowed to borrow more he can smooth consumption by relying on loans and need not hold a large quantity of assets to buffer earnings shocks. Thus $\bar{a}^*(\cdot)$ is decreasing in b which explains why the interest rate rises and the capital stock falls with an increase in b.

Thus, in this framework the interest rate is not pinned down solely by the growth rate and preference parameters. It responds to policy variables like debt, taxes, and spending. Moreover, it is possible to have steady states in which the interest rate is less than the growth rate implying dynamic inefficiency – a result which is familiar from the overlapping generations model. If taxes were proportional rather than lump sum then not only the before-tax interest rate, but also the after-tax interest rate will vary with policy variables.

What about the effect of increases in the quantity of debt on welfare? The welfare criterion we will use is

$$\Omega = \int\int V(a,e)\,dH(a,e), \tag{16}$$

where $V(a,e)$ is the optimal value function and H is the steady state joint distribution of assets and productivity.[12] There are several effects present which makes it difficult to

[12] This criterion for optimality may be deemed reasonable for several reasons. First, it can be thought of as a utilitarian social welfare function. Second, it can be thought of as steady state ex-ante welfare, i.e., welfare of a typical consumer before he realizes his initial assets and the productivity shock which are assumed to be drawn from the steady state joint distribution H. The third reason is as follows. For the sake of exposition only, let's interpret an infinitely-lived household as a sequence of altruistically-linked, one-period-lived generations. Let U_t be the utility of a generation which depends on its own consumption as well as the utility of the next generation; specifically, $U_t = c_t^{1-\nu}/(1-\nu) + \beta E_t U_{t+1}$. It follows that under the optimal consumption and asset accumulation program $U_t = V(a_t, e_t)$. Let the welfare criterion be $\Omega' = p\lim_{t\to\infty}(1/T)\sum_{t=0}^{T-1} U_t$, where "plim" stands for probability limit. This welfare criterion weights the utilities of all generations equally. It can now be seen that

provide an analytical answer. First, an increase in debt (b) increases the return on assets and, thus, makes them less costly to hold for the consumer. The opportunity cost of assets to the consumer is λ and the closer is the interest rate to λ the less costly to hold are assets and the more effective they are in enabling the consumer to smooth consumption.[13]

The second effect on welfare of raising the level of government debt arises from the lump sum taxes levied to pay interest on government debt and can be understood in two ways. First, using the utilitarian interpretation of the welfare criterion we may note that lump sum taxes have distributional effects. They are relatively more onerous on individuals with low assets and low earnings as compared to individuals with high assets and high earnings.[14] Since the marginal value of assets is decreasing in the amount of assets held this tends to lower welfare. Second, using the ex-ante interpretation of the welfare criterion we may note that for a particular household, due to the uncertainty in earnings, lump sum taxes exacerbate the percentage variability in after tax earnings and, thereby, lead to greater consumption variability. Hence, welfare is lowered.

The third effect on welfare of raising the level of government debt is the crowding

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$\Omega' = \Omega$. Intuitively, this is because in any allocation that converges to a steady state infinitely many generations of households are within an arbitrarily small neighborhood of the steady state whereas only finitely many generations of households are outside such a neighborhood of the steady state. This argument can be easily extended to altruistically-linked generations of households who live for an arbitrary number of periods instead of just one period. A fourth justification for this welfare criterion is that it is computationally tractable for the class of models being used. Methods for computing equilibrium paths from arbitrary initial conditions are still being developed (see Krusell and Smith 1994) due to the difficulties involved in tracking the wealth distribution as an endogenous state variable. Fifth, this welfare criterion is commonly used in models of this type (e.g., Imrohoroglu 1991, 1989; Hansen and Imrohoroglu 1992; Alvarez et. al. 1992; and Mehrling 1993). It can also be shown that this welfare criterion is proportional to the unconditional expected value of period utility, i.e., to $\int\int[c(a,e)]^{1-\nu}\,dH$, where $c(a,e)$ is the optimal consumption plan.

13 However, there is a limit to how high an interest rate can be supported as an equilibrium by increases in b. This limit is strictly below λ. This is because when the debt level is high taxes are high and the consumer can no longer afford to pay taxes purely out of earnings. This happens when taxes exceed minimum earnings, i.e., $\tau > \bar{w}e_{\min}$. He is then forced to simply hold on to any additional debt the government issues in order to use the interest payments on debt to cover taxes. Thus, debt beyond a certain critical level provides no liquidity services to the consumer and has no effect on the equilibrium interest rate or welfare. See Aiyagari (1994a) for a more detailed explanation. The critical level of debt and the associated interest rate can be calculated as follows. Set $\tau > \tilde{w}_{\min}$, $b = (\tilde{w}e_{\min} - \gamma)/(r - g)$ and find the interest rate which clears the asset market. This is the highest interest rate that can be supported as an equilibrium and is strictly less than λ because desired assets go to infinity as $r \rightarrow \lambda$. The corresponding debt level can be found from the government budget constraint.

14 Kehoe, Levine, and Woodford (1990) and Mehrling (1993) emphasize this point. With distorting taxes there are also incentive and insurance effects in addition to the distributional effect. The insurance effect is a positive role for distorting taxes in these types of environments as emphasized by Eaton and Rosen (1980) and Varian (1980).

out of capital and the consequent reduction in per capita consumption familiar from the overlapping generations framework. The intuition for why this effect arises in the present framework is as follows. First, note that the borrowing constraint in our framework effectively shortens a household's horizon and makes an infinitely-lived household's behavior similar to that of a sequence of finitely-lived households. Each time the borrowing constraint binds the household is in the same situation as a newly born household in an overlapping generations model.[15] If the dates at which the borrowing constraint binds were exogenous instead of endogenous then the household's optimization problem can be broken up into a sequence of optimization problems by truncating the preferences and the budget constraint at each such date. Second, note that in the overlapping generations model when the debt level is higher older households experience an increase in consumption whereas younger households experience a reduction in consumption. This leads younger households to cut their saving (net of government debt), i.e., their saving rises by less then the increase in government debt. Therefore, capital is crowded out and the interest rate increases.[16] In our framework, a household with a high probability of being constrained in its borrowing is like an older household in an overlapping generations model; a household with a low probability of being constrained in its borrowing is like a younger household in an overlapping generations model. This similarity between the present framework and the overlapping generations framework explains the crowding out effect. The crowding out of capital reduces consumption and welfare.

The net effect of these considerations on welfare is unclear.[17]

We now describe a variation of the above model in which growth is endogenous.

[15] Formally, the stochastic process for a household's assets is a renewal process with zero assets being the renewal point. The average number of periods between successive visits to zero assets may be regarded as a measure of the average life span.

[16] See Woodford (1990) for a demonstration of an exact analogy between a deterministic model of two infinitely-lived households with binding borrowing constraints and an overlapping generations model with two-period-lived agents.

[17] Note that in a pure exchange setting only the first two effects are present and there is no effect on per capita consumption arising from changes in the capital stock. In the pure exchange case, Bewley (1980) suggested that setting $r = \lambda$ would achieve a welfare maximum. Later (Bewley 1983), he realized that, in general, this policy is not even feasible. He showed that the supremum of interest rates that can be supported as equilibria is strictly less than λ. In general, because of the distribution effect of taxes, it need not be optimal to choose the level of government debt to achieve the supremum of feasible interest rates. See Mehrling (1993) and Kehoe, Levine, and Woodford (1990).

2.2 Endogenous Growth

We assume that there is a large finite number (say, I) of identical competitive firms (indexed by i) which produce output according to $y_{it} = F(k_{it}, K_t n_{it})$, where y_{it}, k_{it}, K_t, n_{it} are firm i's output, capital, the per capita capital, and firm i's labor input, respectively. Note the presence of per capita capital as a labor augmenting factor in the technology which is the feature that will generate endogenous growth. In a competitive equilibrium we will have $n_{it} = 1/I$, $k_{it} = K_t/I$, $y_{it} = Y_t/I = F(K_t, K_t)/I = K_t F(1,1)/I$, $w_t = K_t F_2(K_t, K_t) = K_t F_2(1,1)$, and $r = F_1(K_t, K_t) - \delta = F_1(1,1) - \delta$. The consumer's problem, the government budget constraint, and the asset market clearing condition are the same as in the exogenous growth version. Note that $k = K_t/Y_t = 1/F(1,1)$ and $\tilde{w} = w_t/Y_t = F_2(1,1)/F(1,1)$. Thus, in this model r, k, and $\tilde{w}$ are determined exogenously by the technology and the growth rate g will be determined endogenously from the asset market clearing condition.

The determination of the growth rate is shown in Figure 2. The number $([\beta(1+r)]^{1/\nu} - 1)$ is what the growth rate would be with complete markets. Note that as g approaches $([\beta(1+r)]^{1/\nu} - 1)$ from above per capita assets desired by households tend to infinity. The steady state value of g is that value at which desired household assets equal the exogenously determined supply of assets $(k + b)$.

Three points are worth noting about the steady state. First, it is immediately obvious that the growth rate under incomplete markets is always higher than what it would be with complete markets. Second, with complete markets changes in government debt have no effect on the growth rate whereas with incomplete markets such changes do affect the growth rate. This effect arises because changes in government debt shift the household's asset demand curve. An increase in government debt reduces the household's asset demand net of government debt by effectively loosening the borrowing constraint. Thereby, it lowers the growth rate. Note that this effect arises even though taxes are lump sum.

Third, an increase in government consumption also affects the growth rate. This effect operates through the reduction in the household's permanent income. Because the growth rate is higher than $([\beta(1+r)]^{1/\nu} - 1)$, the interest rate is lower than the effective discount rate λ $(= (1+g)^\nu/\beta - 1)$. Consequently, consumption drops by less than the reduction in permanent income. Thus, the household's saving and asset demand go up. Therefore, the equilibrium growth rate goes up. Thus, this model can explain the positive growth rate effect of government consumption emphasized by Barro (1990) even though government consumption is not an argument in the production function, taxes are lump sum, and labor supply is inelastic.

Regarding the welfare effects of government debt it seems quite likely that the optimal quantity of debt would be low due to the negative growth rate effects of increases in debt. Further, the welfare losses from deviating from the optimal level are likely to be quite large.

As the discussion in this section has, hopefully, made clear determining the welfare effects of government debt and finding its optimal level analytically are rather difficult even in the simpler case of lump sum taxes and inelastic labor supply. Therefore, we use computational methods to address this question. As our benchmark we use a model with exogenous growth, a proportional income tax, and elastic labor supply. This model is described in the next section. We also describe how this model is parameterized and how the steady state is computed.

3. The Benchmark Model

In the previous section we described the workings of a model with lump sum taxes and inelastic labor. While these features were useful for building intuition they are not adequately realistic characterizations of most actual economies for the purpose at hand. Actual tax systems distort labor supply and saving decisions and provide some insurance; these features are likely to be important for determining the optimal quantity of debt. In this section we describe our benchmark model which has a proportional income tax and elastic labor. This model is, hopefully, a more adequate characterization of actual economies.

We first describe the exogenous growth version. Let τ_y be a proportional income tax levied on the sum of labor income and interest income. The after tax wage and interest rate are denoted by $\bar{w}_t$ and $\bar{r}$, respectively, where

$$\bar{w}_t = (1-\tau_y)z_t F_2(K_t, z_t N) = (1-\tau_y)w_t, \tag{17}$$

$$\bar{r} = (1-\tau_y)(F_1(K_t, z_t N) - \delta) = (1-\tau_y)r \tag{18}$$

and N denotes per capita effective labor input.

Let the total time endowment for the household be normalized to unity and let ℓ_t denote leisure time. The household's period utility is now specified as $(c_t^\eta \ell_t^{1-\eta})^{1-\mu}/(1-\mu)$. The household's budget constraint and nonnegativity constraints now take the form

$$c_t + a_{t+1} \leq \bar{w}_t e_t(1-\ell_t) + TR_t + (1+\bar{r})a_t, \quad c_t \geq 0,\ 0 \leq \ell_t \leq 1,\ a_t \geq 0, \tag{19}$$

where TR_t are lump sum transfer payments.[18]

The government budget constraint takes the form

$$G_t + TR_t + rB_t = B_{t+1} - B_t + \tau_y(w_t N + rA_t). \tag{20}$$

The asset market equilibrium condition is as before, viz. (7). In addition we require that the labor market clear, i.e.,

$$N = Ee_t(1-\ell_t), \tag{21}$$

where the expectation is taken with respect to the steady state distribution.

As before it is convenient to transform the model into a stationary (nongrowing) form. This will also help in explaining the parameterization that we choose later. Following the same procedure as in the previous section the consumer's problem can be rewritten as:

$$\max_{\{\tilde{c}_t, \tilde{a}_{t+1}\}} E\left[Y_0^{\eta(1-\mu)} \sum_{t=0}^{\infty} [\beta(1+g)^{\eta(1-\mu)}]^t (\tilde{c}_t^\eta \ell_t^{1-\eta})^{1-\mu}/(1-\mu) \middle| \tilde{a}_0, e_0\right] \tag{22}$$

$$\text{subject to} \quad \tilde{c}_t + (1+g)\tilde{a}_{t+1} \leq \bar{w}e_t(1-\ell_t) + \chi + (1+\bar{r})\tilde{a}_t, \quad t \geq 0, \tag{23}$$

$$\tilde{c}_t \geq 0, \quad 0 \leq \ell_t \leq 1, \quad \tilde{a}_t \geq 0, \quad t \geq 0, \tag{24}$$

where $\bar{w}$ is now defined as $(1-\tau_y)w_t/Y_t$ and χ equals TR_t/Y_t.

Substituting for per capita assets from the asset market clearing condition into the government budget constraint (20), using the definitions of w_t and r together with the first degree homogeneity of the production function we can rewrite (20) as

$$G_t + TR_t + \bar{r}B_t = B_{t+1} - B_t + \tau_y(F(K_t, z_t N) - \delta K_t). \tag{25}$$

[18] Whether government transfer payments to households should be treated as lump sum in nature is not so clear. A further discussion of this issue is given later when we describe the parameterization of the benchmark economy.

Now divide through (25) by Y_t so that it becomes

$$\gamma + \chi + (\bar{r} - g)b = \tau_y(1 - \delta k) \tag{26}$$

where $k = K_t/Y_t$.

The asset market equilibrium condition (7) can be rewritten in the form (12) by dividing through Y_t. Note that using equation (18) we can express $K_t/(z_t N)$ as a function of r, and, hence, we can express k $(= K_t/Y_t)$ as a function of r which is the function $\kappa(r)$ appearing in (13). Further, using (14) we can express $w_t N/Y_t$ as a function of r which is the function $\omega(r)$ appearing in (13). Thus, we have

$$\bar{w} = (1 - \tau_y)\omega(r)/N. \tag{27}$$

The steady state of this economy is characterized by an interest rate r^* and per capita effective labor input N^* which solve

$$\bar{a}(r, N; \gamma, b, g, \chi) = \kappa(r) + b \tag{28}$$

$$\bar{\varphi}(r, N; \gamma, b, g, \chi) = N \tag{29}$$

where $\bar{a}(r, N, .)$ is the per capita assets desired by consumers (relative to per capita output) as a function of the interest rate and per capita effective labor input, $\kappa(r) + b$ is the per capita supply of assets (capital plus government debt) relative to per capita output expressed as a function of the interest rate, and $\bar{\varphi}(r, N; \gamma, b, g, \chi)$ is the per capita effective labor supplied by households.

Note that for given values of $(r, N, \gamma, b, g, \chi)$ the government budget constraint can be used to find the income tax rate τ_y. Thus, $\bar{w}$ and $\bar{r}$ are known. The household's problem can now be solved to obtain decision rules for assets and leisure. The decision rule for assets can be used to determine the stationary distribution of assets and, thereby, per capita assets desired and effective labor supplied by households. The values of r and N are varied until equilibrium obtains in the asset and labor markets. In Appendix A, we describe the algorithm used to compute the decision rule for assets and the stationary distribution of assets.

The welfare standard we use is the one given by (16). This is converted into a welfare measure in units of consumption as follows. The welfare measure is that percent increase in benchmark consumption at every date and state (holding leisure at every date and state at its benchmark values) which leads to the same value of the welfare standard as its value at any alternative debt/GNP ratio.

We now describe how the above model is parameterized.

3.1. Parameterization of Benchmark Economy

The model period is specified to be one year. The production function is specified as Cobb-Douglas with θ denoting the capital share parameter. The stochastic process for e_t is specified as follows. The natural logarithm of e_t is assumed to be a first-order autoregressive process with a serial correlation coefficient ρ and a standard deviation σ. Note that with inelastic labor supply e_t is proportional to the household's labor earnings. We use studies by Abowd and Card (1989, 1987), Kydland (1984), and Heaton and Lucas (1993) to specify the values of ρ and σ (see Aiyagari 1994a for more details). After specifying ρ and σ we use the procedure of Tauchen (1986) to approximate the autoregression of $\log(e_t)$ by a first-order Markov chain with seven states. Thus, the parameters for the earnings shock are the values for the seven states and a probability transition matrix which we denote by π. Table 1 in Aiyagari (1994a) shows that this approximation is very good. In addition to the technology parameters, we must choose parameter values for preferences (β, η, μ), the depreciation rate (δ), the growth rate (g), transfers (χ), and government consumption (γ). We use the following values for the parameters:

$$\begin{array}{lllll} \theta = 0.356, & \sigma = 0.2, & \eta = 0.3394, & \delta = 0.0645, & \chi = 0.0817, \\ \rho = 0.6, & \beta = 0.9875, & \mu = 3.0, & g = 0.0185, & \gamma = 0.217. \end{array} \tag{30}$$

The methods and data sources used to calculate these values are described in Appendix B. We also try alternative values of μ, ρ, and σ to check on the robustness of our results. (See Section 5.) Values for r, N, and τ_y depend on our choice for the debt/GNP ratio. Our benchmark debt/GNP ratio is 0.667. The parameterization above implies that $r = 0.0698$, $N = 0.267$, and $\tau_y = 0.38$ when $b = 0.667$.

As we mentioned earlier whether to treat government transfer payments to households as lump sum in nature is not so clear. The largest component of transfers in recent decades is social security amounting to about 40 percent of total transfers. Since the payments depend to some extent on previous earnings an argument could be made that social security taxes and payments would be approximately neutral. The argument is that the social security system amounts to transferring income from younger and working members of the household to older and retired members of the household in a way that is approximately neutral with respect to labor supply decisions since future social security payments to younger working members depend on their current earnings. This argument suggests that social security payments could be treated as approximately proportional to income so that they offset social security taxes. A similar argument could be made for federal retirement benefits, state and local retirement benefits, unemployment insurance, and veterans benefits which together add up to about 35 percent of total benefits. This leaves about 25 percent of total benefits (amounting to less than 2 percent of GNP) which are accounted for by public assistance (about 15 percent of total benefits) and medical insurance (about 10 percent of total benefits).

Rather than consider a variety of possibilities for treating transfers we have chosen in our benchmark to focus on one of the two extremes. This is to treat all transfers as lump sum which implies the value 0.0817 for χ given above which is the post-war average of total government transfer payments as a proportion of GNP.[19] The other extreme would be to treat all transfers as proportional to income. This, effectively, amounts to setting χ to zero which would then imply a lower value of the income tax rate τ_y as determined by the government budget constraint (26). Our results are reported for the benchmark value of χ though we will note the more important differences that occur when all transfers are treated as proportional to income.

We now describe the endogenous growth version of the above model.

3.2. Endogenous Growth

The description here is very similar to the endogenous growth model in Section 2, so we will briefly note the few differences. Obviously, in equilibrium $n_{it} = N/I$. Therefore,

[19] Lucas (1990) adopts the same treatment. His figure for the share of transfers in NNP is 15 percent which works out to a share of transfers in GNP of 12.4 percent. Our figure is somewhat lower; we suspect this is because we use the post-war average whereas Lucas apparently uses the 1985 figure. Our value of the income tax rate is about 38 percent which is roughly consistent with Lucas' figures for labor and capital income tax rates. He takes the labor income tax rate to be 40 percent and the capital income tax rate to be 36 percent. The high figure for the income tax rate in our model is because we lump all consumption taxes into the labor income tax since both types of taxes affect the same consumption/leisure margin. This is also the case in Lucas (1990).

$Y_t = F(K_t, K_t N) = K_t F(1, N)$, $w_t = K_t F_2(1, N)$, and $r = F_1(1, N) - \delta$. For given values of (g, N, γ, b, χ) we can calculate r and k and use these values in the government budget constraint to calculate τ_y and, hence, $\bar{r}$. We can also calculate $\bar{w}$ $(= (1 - \tau_y)$ $F_2(1, N)/F(1, N))$. Now we can solve the household's problem and obtain the household's desired per capita assets and leisure. The values of g and N are varied until the asset market and the labor market clear.

In the next section we describe our findings for the parameter values we have reported. In the section following the next, we discuss the robustness of some of our findings to changes in some parameter values. We find that our results are quite robust to empirically plausible changes in some of the parameters.

4. Results

Our basic finding is that for the benchmark model with exogenous growth, elastic labor, and proportional income tax the optimal quantity of debt is significantly negative – about negative 250 percent of GNP – and the welfare gain (measured as a percent of per capita consumption) from moving to the optimum is also economically significant – about 4 percent of per capita consumption. The optimal interest rate is a little under 5 percent whereas the interest rate at the current value of the debt/GNP ratio is about 7 percent. The optimal income tax rate is about 33 percent which is lower than its current value of about 38 percent. Figure 3 shows the graphs of the welfare gain, the before and after-tax interest rates, the income tax rate, and total hours versus the debt/GNP ratio for our benchmark economy.[20]

The magnitude of the change in the debt/GNP ratio called for by optimal policy in our model economy may seem very large. However, it is not out of line with changes in the debt/GNP ratio implied by the tax experiments analyzed by others in the standard representative agent economy. For example, King and Rebelo (1990) and Stokey and Rebelo (1993) analyze the effects of 10 percent changes in various tax rates. It is easy to

[20] If all transfers are treated as proportional to income so that χ is set to zero then the results change as follows. The optimal debt/GNP ratio is approximately the same – about negative 250 percent of GNP – and the welfare gain from moving to the optimum is lower – only about 2.8 percent of consumption. The optimal interest rate is somewhat lower – about 4.8 percent – and the optimal income tax rate is about 21 percent which is quite a bit lower than its value of 29 percent at the current debt/GNP ratio.

see that a 10 percentage point reduction in the income tax rate will produce a debt/GNP ratio close to our optimum. Consider the budget constraint in Eq. (26) where the per capita capital/output ratio is given by $k = \alpha/[\bar{r}/(1-\tau)+\delta]$. Suppose that we set γ, χ, g, δ, and α equal to the benchmark values given in (30). In the standard representative agent economy, the after-tax interest rate $\bar{r}$ is independent of fiscal variables; set it equal to 0.0434 which is its equilibrium value in our benchmark economy. Now consider a 10 percentage point reduction in the income tax rate from 38 percent to 28 percent. If spending and transfers are not adjusted, then the debt/GNP ratio must be lowered to -2.8 – a level close to our optimum.

Notice that the tax rate in Figure 3 does not rise monotonically with debt; this is because changes in the level of debt induce changes in the before- and after-tax interest rates. An increase in debt raises the after-tax interest rate which lowers the capital/output ratio and, hence, raises the ratio of net income to gross income. As a result, there is an increase in the tax base which has a negative effect on the tax rate. This is the reason why the tax rate changes non-monotonically with debt. Thus, in our framework, tax policies may be consistent with more than one level of debt or may be infeasible. In the standard representative agent growth model, the after-tax interest rate is independent of the level of debt and the tax rate increases monotonically with debt. This is why, in the representative agent model, one can interpret the exercise of changing the debt as equivalent to the exercise of changing the income tax rate.

Since the optimal interest rate is significantly lower than the current interest rate it follows that the optimal capital/output ratio, the capital stock, and the optimal saving rate are significantly higher than their current values. The optimal capital/output ratio is 3.12 compared to its current value of 2.65. This implies that the capital stock is 1.34 times higher at the optimum compared to its current value. Part of this increase – about 15 percent – is because total hours are higher at the optimum (see Figure 3). The rest of the increase in the capital stock reflects capital deepening (i.e., the percentage increase in the capital/labor ratio). The optimal saving rate (gross saving/GNP) is about 26 percent compared to its current value of 22 percent. Thus, the reduction in public debt to its optimal level allows for a significant increase in the capital/output ratio, the saving rate, total hours, and the capital stock and, thereby, in per capita consumption and welfare.

The above numbers strongly suggest that a large part of the welfare gain from reducing the level of debt to its optimum value is arising from the consequent increase in the capital stock and, thereby consumption. In order to gauge the strength of this effect we calculated a modified welfare measure which only considers the changes in the *mean* levels of consumption and leisure, i.e., ignores the welfare effects of smoothing consumption and leisure streams. It turns out that virtually all of the total welfare gain is accounted for by changes in the mean levels of consumption and leisure. Specifically, changes in means lead to a welfare gain of about 3.4 percent leaving the remaining 0.6 percent of welfare gain to be accounted for by the smoothing of consumption and leisure streams. We obtain similar results when labor supply is inelastic. Specifically, the overall welfare gain is about 2.4 percent of consumption and the welfare gain due to changes in means only is about 2.2 percent of consumption. Thus, if the capital stock and total hours were fixed and not variable the welfare gains of reducing the public debt to its optimum value would appear to be quite small.

The welfare consequences of the incentive and insurance effects of the income tax are substantial. Comparing the benchmark model to one with a lump sum tax we find that the welfare gain from moving to the optimum is quite small – approximately 0.2 percent. The optimal quantity of debt is, however, still substantially negative – about negative 100 percent of GNP. Some idea of the strength of the insurance effect may be obtained by comparing the model of Section 2 with inelastic labor and lump-sum taxes to a version of that same model with an income tax. When there is an income tax, the welfare gain from moving to the optimum is about 2.4 percent of consumption and the optimal quantity of debt is about negative 300 percent of GNP. When taxes are lump-sum, the welfare gain is slightly less than 1 percent of consumption and the optimal quantity of debt is about negative 150 percent of GNP.

Turning to the endogenous growth version of the benchmark model the results are as expected. Reducing the public debt raises the growth rate and leads to enormous welfare gains. Figure 4 shows the graphs of the welfare gain, the growth rate, the tax rate, and total hours versus the debt/GNP ratio. As the debt/GNP ratio decreases to about -1 the welfare gain and the increase in the growth rate are moderate. However, further reductions in the debt/GNP ratio accelerate the increase in welfare gains and the growth rate. Needless to say the welfare gain is dominated by the changes in the mean levels of consumption and leisure – smoothing considerations play a negligible role. Changes in the tax rate seem fairly small. As can be seen in Figure 2 when the debt/GNP ratio is reduced from its current level of about 0.67 to, say, -2.5, the tax rate only decreases from 38 percent to 36 percent. Yet, the growth rate increases from 1.9 percent to over 4 percent and the welfare gain is over 80 percent of per capita consumption.[21] However, for a wide range of values for the debt/GNP ratio – ranging from -1 to 2 – the welfare effects and the growth rate effects are quite moderate.

As we showed in Section 2, even with inelastic labor and lump sum taxes changes in the debt/GNP ratio have growth rate effects in our model whereas there would not be any growth rate effects in a representative agent version of our model. Indeed, the growth rate effect of a large reduction in the debt/GNP ratio is stronger in this case as compared to the case of elastic labor and a proportional income tax. For example, when the debt/GNP ratio is reduced to -2.5 the growth rate of the economy rises to 8.5 percent. However, much of the growth rate effect occurs only at very low values of the debt/GNP ratio. For a wide range of values of the debt/GNP ratio ranging from -1 to 3 the growth rate effect is quite negligible.

As we showed in Section 2 increases in government consumption increase the growth rate in the endogenous growth model even with lump sum taxes and inelastic labor – an effect that does not arise in the representative agent version of the same model. However, our quantitative analysis reveals that the growth rate effect of government consumption is negligible. For example, when government consumption rises from 15 to 30 percent of GNP the growth rate rises approximately 0.05 percentage points. However, in the benchmark case with a proportional income tax and elastic labor the growth rate effect of government consumption is negative and quantitatively stronger. For example, when government consumption rises from 15 to 30 percent of GNP the growth rate falls by 0.75 percentage points. The negative growth rate effect in the case of a proportional income tax is obviously because the increase in government consumption raises the income tax rate which reduces the after tax return to capital.

[21] If all transfers are treated as proportional to income then we get similar results. When the debt/GNP ratio is reduced to -2.5 the growth rate rises to 5 percent and the welfare gain is close to 80 percent of consumption.

In the next section we discuss the robustness of some of our results to changes in some of the parameters.

5. Robustness of the Results

In this section we focus on the robustness of our main results to changes in the following key parameters: the risk aversion coefficient μ, the standard deviation of the earnings shock σ, and the serial correlation coefficient of the earnings shock ρ. These parameters govern households' precautionary saving in our model. By either making the household more sensitive to risk or exposing the household to greater earnings risk these parameters influence the average amount of assets that households desire to hold in order to buffer earnings shocks. Thereby, these parameters affect the optimal quantity of debt and the welfare gain from moving to the optimum.

We first describe the effects of changing the risk aversion coefficient μ from the benchmark value of 3 to 9.[22] In this case, the discount factor β is set equal to 1.025 to ensure that the interest rate is approximately 7 percent in equilibrium for the debt level observed in the U.S. ($b = 0.67$). These changes turn out to have no effect on the optimal quantity of debt which continues to be around -2.5 times GNP. The optimal interest rate and the optimal income tax rate are also about the same as in the benchmark. However, the welfare gain from moving to the optimum debt/GNP ratio is a little under 5 percent of consumption – 1 percentage point higher than our benchmark case.

If we change the serial correlation coefficient of the earnings shock from the benchmark value of 0.6 to a new value of 0.9, we have to set the discount factor equal to 0.9853 to generate a 7 percent interest rate for $b = 0.67$. Assuming that all other parameters stay at their benchmark values, we find that the optimal quantity of debt is still -2.5 times GNP and the welfare gain is still approximately 4 percent. Furthermore, the change in ρ has a negligible effect on the interest rate, the income tax rate, and total hours at the optimum debt level.

We find larger effects with a change in the standard deviation. If we increase the standard deviation of the earnings shock from the benchmark value of 0.2 to a new value

[22] Note that the risk aversion coefficient on consumption (denoted ν) is given by $\nu = 1 - \eta(1 - \mu)$. Therefore, a change in μ from 3 to 9 implies a change in ν from 1.68 to 3.72.

of 0.4 and set the discount factor equal to 0.983 (to keep the benchmark interest rate at 7 percent), we find that optimal level shifts slightly higher to -2 times GNP, and the welfare gain from moving to the optimum level is lower – approximately 3 percent of consumption.

When we change all three parameters, we find a higher optimal level of debt and higher gains in comparison to the benchmark case. In particular, assume that $\mu = 9$, $\rho = 0.9$, $\sigma = 0.4$. Then, we need to set $\beta = 1.001$ for the benchmark interest rate to be 7 percent. In this case, the optimal debt/GNP ratio is -2 and the welfare gain to moving there is approximately 5.5 percent of consumption. The optimal interest rate is 3.5 percent which is significantly smaller than the optimal level found in the benchmark case.

The above experiments suggest a range of welfare costs between 3 and 6 percent of consumption. Thus, all variations imply an economically significant welfare gain to reducing our current level of debt to the optimal level. Furthermore, in all cases, we find that the optimal level is very negative – about -2 or -2.5 times GNP.

6. Conclusions

In this paper, we have calculated the optimal quantity of debt for a model parameterized to mimic certain features of the U.S. economy. In our model, the optimal level of government debt will be high if it is effective in smoothing out consumption over the lifetime of an individual. The optimal level of debt will be low if it crowds out capital and, therefore, lowers consumption. Our quantitative analysis suggests that the crowding out effect is decisive for welfare. We find that the optimal quantity of government debt is approximately -2.5 times GNP, the optimal interest rate is a little under 5 percent, and the optimal tax rate is about 33 percent. The welfare gain to reducing our current level of debt to its optimal value is approximately 4 percent of consumption.

Appendix A

In this appendix we provide some details on computing the equilibrium for the benchmark economy.[23] Here, we focus on the two main tasks: computing the optimal asset holdings α and computing the distribution of assets H. Both of these tasks are accomplished by applying a finite element method.

In the case of the decision rules, we first derive the first order conditions for the following optimization problem:

$$\max_{\{\tilde{c}_t,\tilde{a}_{t+1}\}} E\left[\sum_{t=0}^{\infty}\tilde{\beta}^t\left\{\frac{(\tilde{c}_t^{\eta}\ell_t^{1-\eta})^{1-\mu}}{1-\mu}+\frac{1}{3}\zeta(\min(\tilde{a}_t,0)^3+\min(1-\ell_t,0)^3)\right\}\Big|\tilde{a}_0,e_0\right],$$

$$\text{subject to} \quad \tilde{c}_t+(1+g)\tilde{a}_{t+1}\leq(1+\bar{r})\tilde{a}_t+\bar{w}e_t(1-\ell_t)+\chi,$$

where $\tilde{\beta}=\beta(1+g)^{\eta(1-\mu)}$. Notice that we have changed the optimization problem described in section 3 by using penalty functions for two sets of inequality constraints ($\tilde{a}_t \geq 0$ and $\ell_t \leq 1$). The remaining inequality constraints ($\tilde{c}_t \geq 0$ and $\ell_t \geq 0$) do not bind for any of our parameter choices so we ignore them when computing the optimal decisions. The first order conditions imply that the Euler residual $R(x,i;\alpha)$ must be equal to 0 for all $x \in [0, x_{max}]$ and all i where

$$\begin{aligned}R(x,i;\alpha) &= \eta(1+g)c(\ell^*(x,i;\alpha))^{\eta(1-\mu)-1}\ell^*(x,i;\alpha)^{(1-\eta)(1-\mu)}\\ &\quad-\tilde{\beta}\sum_j\pi_{i,j}\{\eta(1+\bar{r})c(\ell^*(\alpha(x,i),j;\alpha))^{\eta(1-\mu)-1}\\ &\quad\cdot\ell^*(\alpha(x,i),j;\alpha)^{(1-\eta)(1-\mu)}-\zeta\min(\alpha(x,i),0)^2\},\end{aligned}$$

$\ell^*(x,i;\alpha)$ is the solution to $f(\ell)=0$, and x_{max} is such that no $x > x_{max}$ would be chosen by the consumer. The functions $c(\cdot)$ and $f(\cdot)$ are defined as follows:

$$\begin{aligned}c(\ell) &= (1+\bar{r})x+\bar{w}e(i)(1-\ell)+\chi-(1+g)\alpha(x,i),\\ f(\ell) &= (1-\eta)c(\ell)^{\eta(1-\mu)}\ell^{(1-\eta)(1-\mu)-1}-\zeta\min(1-\ell,0)^2\\ &\quad-\bar{w}e(i)\eta c(\ell)^{\eta(1-\mu)-1}\ell^{(1-\eta)(1-\mu)},\end{aligned}$$

[23] For more details on the computational methods used here, see McGrattan (1994) and our technical appendix which is available from the authors upon request.

where $c(i)$ is the earnings shock in state i.

The computational task is, therefore, to find an approximation for $\alpha(x,i)$, say $\alpha^h(x,i)$ that implies $R(x,i;\alpha^h)$ is approximately equal to zero for all x and i. We do this by applying a finite element method. In particular, we do the following. First, we choose some discretization of the domain of our functions. Since only x is continuous, we need to specify some partition of [0,x_{max}]. We refer to each subinterval of x as an element. On each element, we choose a set of basis functions for approximating α. Local interpolations are assembled to construct a globally-defined piecewise approximation. In our case, we chose linear basis functions for all elements, e.g.,

$$\alpha^h(x,i) = \psi_j^i N_j(x) + \psi_{j+1}^i N_{j+1}(x),$$

on the element $[x_j, x_{j+1}]$ where

$$N_j(x) = \frac{x_{j+1} - x}{x_{j+1} - x_j}, \quad N_{j+1}(x) = \frac{x - x_j}{x_{j+1} - x_j}.$$

This choice was motivated by related test problems described in the technical appendix. Notice that $\alpha^h(x_j,i) = \psi_j^i$ and $\alpha^h(x_{j+1},i) = \psi_{j+1}^i$. If we consider the approximation globally, we need to compute the asset holdings for all points x_j and all productivity levels i (i.e., ψ_j^i for all i and j). We choose these values for ψ_j^i by setting the weighted residual equal to zero, i.e.,

$$\int R(x,i;\alpha^h) N_j(x) dx = 0, \qquad \text{for all } i \text{ and } j \tag{31}$$

In effect, we solve a problem of the form: find $\vec{\psi}$ such that $h(\vec{\psi}) = 0$, where $\vec{\psi}$ is the vector of coefficients that we are searching over and $h(\cdot)$ is the system of equations in (31).

Asset holdings next period are a function of asset holdings this period and the earnings shock. Therefore, one can show that the invariant cumulative distribution function must satisfy

$$H(x,i) = \sum_j \pi_{j,i} H(\alpha^{-1}(x,j),j) I(x \geq \alpha(0,j)), \tag{32}$$

where I is an indicator function (e.g., $I(x > y)$ is equal to one if $x > y$ and zero otherwise). To compute H, we again apply the finite element method with linear basis functions. In this case, the residual is the difference between the right and left hand sides of (32). For this problem, we do not have to worry about inequality constraints directly. But we do have to deal with them indirectly. If inequality constraints bind in the consumer's problem, then the decision functions for low productivity levels will be set equal to zero for some interval $[0, x^*]$. This implies a mass point at $x = 0$. This also implies that mass points will exist throughout the distribution; these occur at the states traversed prior to reaching the zero asset position. The mass points in the distribution imply that the solution to (32) has discontinuities, possibly at a countably infinite number of points. In the technical appendix, we show that the finite element method generates a good approximation even when such discontinuities occur.

Appendix B

In this appendix we describe the methods and data sources used to parameterize our benchmark model.

- B_t/Y_t: Gross debt (beginning of period) as a percent of GDP, 1940-94 (sources: Debt figures are from the *Statistical Abstract of the United States,* U. S. Department of Commerce, various issues; GDP figures are from the *Economic Report of the President,* 1994, p. 362, Table B-77);
- G_t/Y_t: Government purchases as a percent of GDP, 1947:1–1994:2 (source: Citibase, variables GGEQ and GDPQ);
- TR_t/Y_t: Transfer payments as a percent of GDP, *Survey of Current Business,* various issues;
- N: Total hours for all industries divided by the 16+ population and the total number of discretionary hours per year, 1946:01–1993:12 (sources: Citibase, variables LHOURS and P16, and Hill (1985) who estimates discretionary time to be 1,134 hours per year);
- r: % Real Return on the S&P 500, 1889–1978 (source: Grossman and Shiller 1981);
- $w_t N/Y_t$: Compensation of employees divided by GDP less proprietors' income, 1946:1-1994:2 (source: Citibase, variables GCOMP, GDP, GPROBV, GPROFV, GCCPF, i.e., GCOMP/(GDP-GPROBV-GPROFV+GCCPF));

- K_t/Y_t: Total constant-cost net stock of fixed reproducible tangible wealth as a percent of GDP, 1947–92 (sources: Survey of Current Business, September 1993, Table 24 and Citibase, variable GDPQ);
- Y_{t+1}/Y_t: Growth of per capita GDP, 1947-93 (source: Citibase, variables GDPQ and POP).

We use sample means of the above series and derive estimates of our parameters as follows

$$
\begin{aligned}
r &= 0.0698 \\
N &= 0.267 \\
g &= (1/T)\sum_t Y_{t+1}/Y_t - 1 = 0.0185 \\
\theta &= 1-(1/T)\sum_t w_t N/Y_t = 0.356 \\
b &= (1/T)\sum_t B_t/Y_t = 0.667 \\
\gamma &= (1/T)\sum_t G_t/Y_t = 0.217 \\
\chi &= (1/T)\sum_t TR_t/Y_t = 0.0817 \\
\delta &= \frac{\theta}{(1/T)\sum_t K_t/Y_t} - r = 0.0645 \\
\tau_y &= \frac{\gamma+\chi+(r-g)b)}{1-\delta\theta/(r+\delta)+rb} = 0.380 \\
\eta &= \left[1+(1-\tau_y)\frac{(1-N)(1-\theta)}{N[1-\gamma-\theta(g+\delta)/(r+\delta)]}\right]^{-1} = 0.3394.
\end{aligned}
$$

The expression used to calculate τ_y is obtained from the government budget constraint (26). The expression used to calculate η is obtained from the consumer's first-order condition for labor/leisure choice.[24]

[24] Note that $N = Ee_t(1-\ell_t) = 1 - E(e_t\ell_t)$, since $E(e_t)$ is normalized to unity. Assuming an interior solution, the household's first-order condition for labor-leisure choice yields: $\ell_t = (1-\eta)\tilde{c}_t/(\eta\bar{\varpi}e_t)$. Therefore, $E(e_t\ell_t) = (1-\eta)E(\tilde{c}_t)/(\eta\bar{\varpi})$. Further, $\bar{\varpi}$ equals $(1-\tau_y)w_t/Y_t = (1-\tau_y)\omega(r)/N$. The resource constraint for the economy implies an expression for per capita consumption: $E(\tilde{c}_t) = 1-\gamma-(g+\delta)\kappa(r)$. These relationships can be used to solve for N in terms of r, γ, b, and g. Using the Cobb-Douglas form of the production function we can see that $\omega(r) = 1-\theta$ and $\kappa(r) = \theta(r+\delta)$. This leads to the following expression for N: $N = 1/(1+AB)$, where $A = (1-\eta)/[\eta(1-\tau_y)(1-\theta)]$ and $B = 1-\gamma-\theta(g+\delta)/(r+\delta)$. The solution for N implies the expression for η given in the text.

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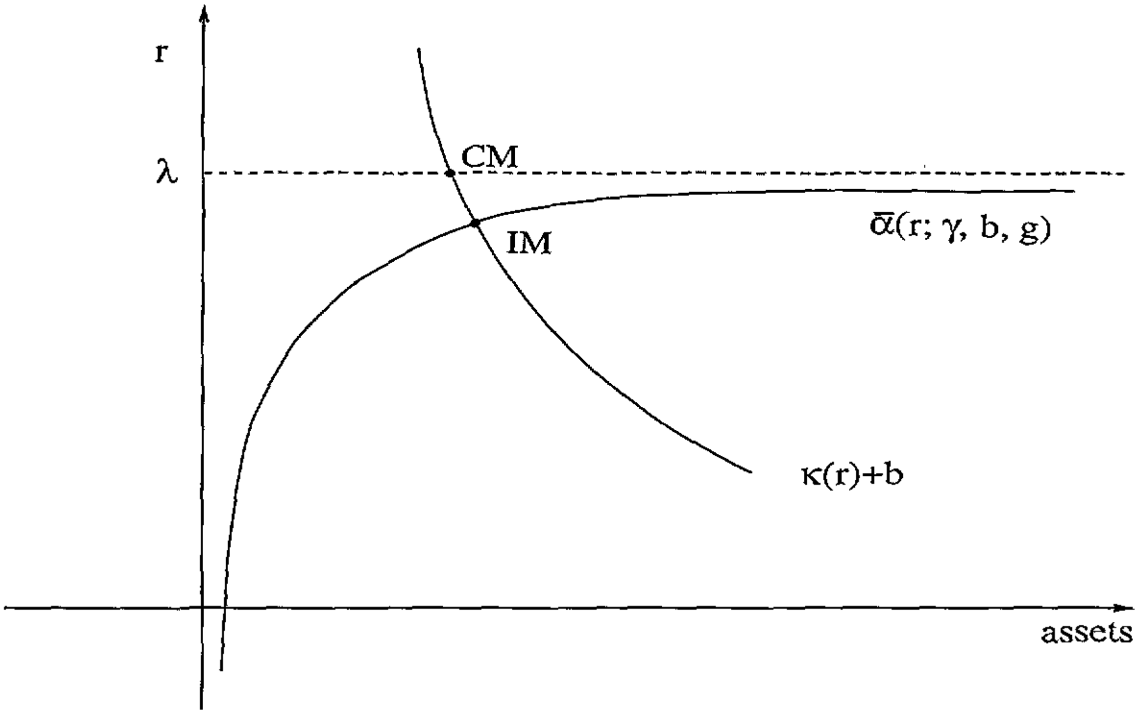

Figure 1. Interest Rate determination.

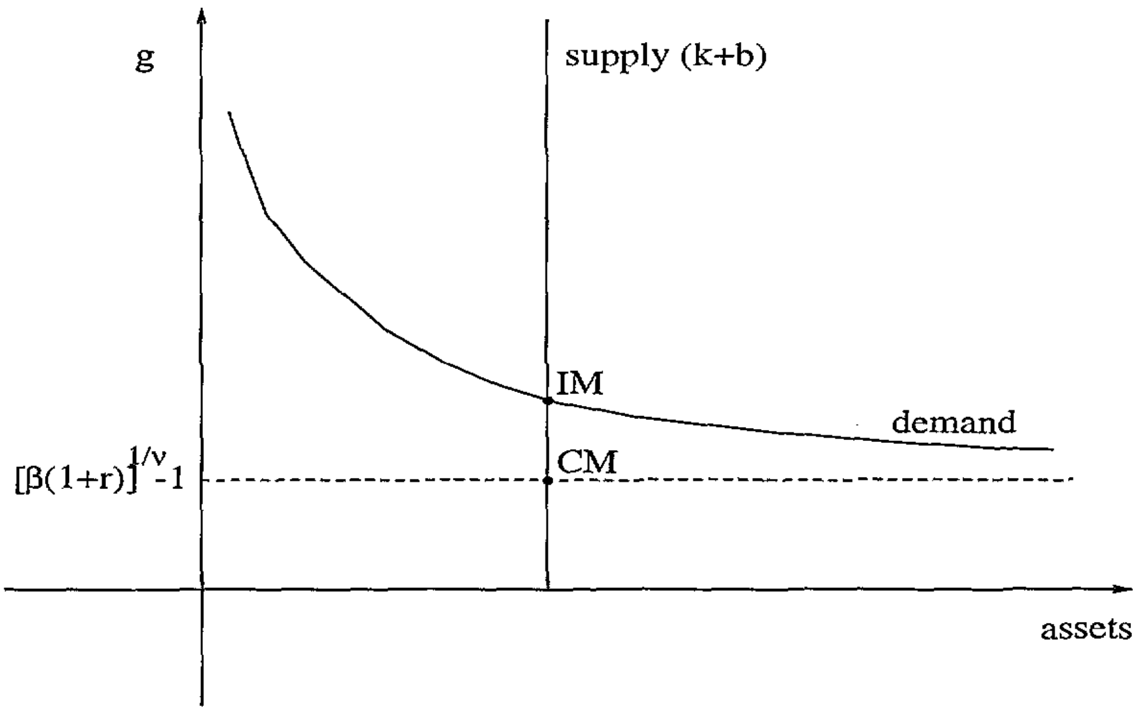

Figure 2. Growth rate determination.

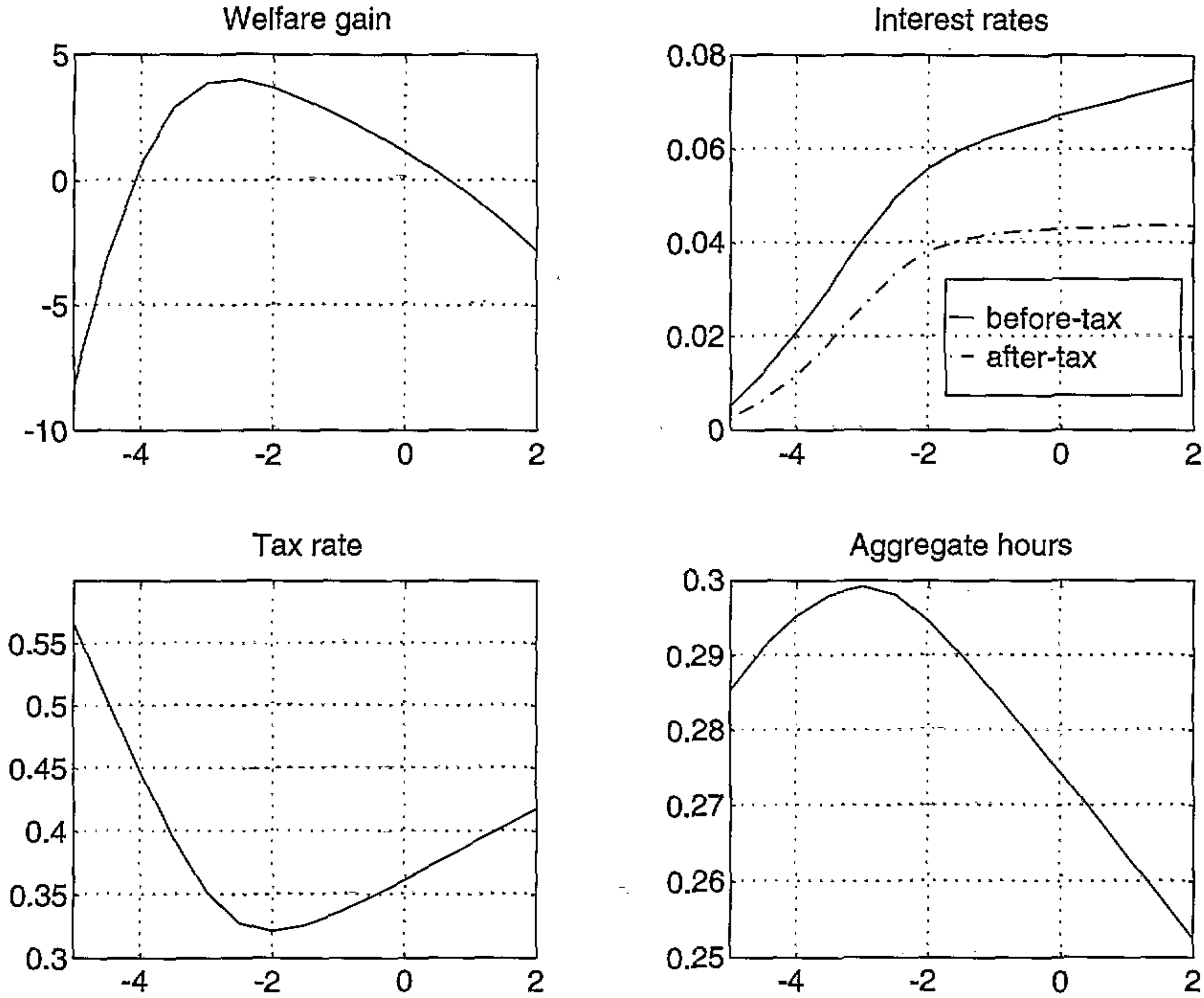

Figure 3. Welfare gain, interest rates, tax rate, and aggregate hours versus the debt/GNP ratio (x-axis) for the benchmark economy.

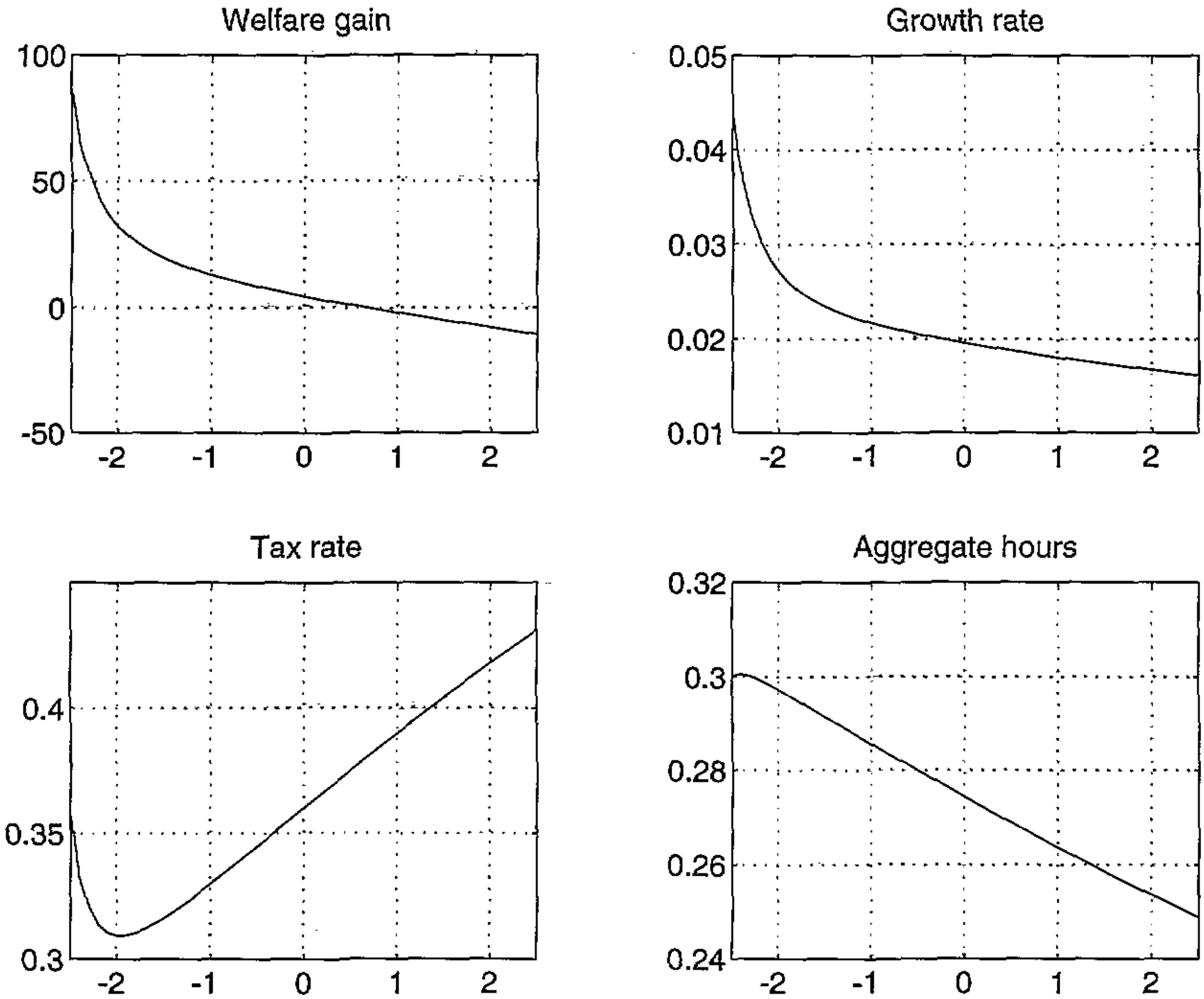

Figure 4. Welfare gain, growth rate, tax rate, and aggregate hours versus the debt/GNP ratio (x-axis) for the endogenous growth economy.