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A Model of a Currency Shortage

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ABSTRACT

Until the mid-19th century, shortages of currency were sometimes serious problems. One common response was to prohibit the export of coins. We use a random matching model with indivisible money to explain a shortage and to judge the desirability of a prohibition on the export of coins. The model, although extreme in many regards, represents better than earlier models a demand for outside money and the problems that arise when that money is indivisible. It can also rationalize a prohibition on the export of coins.

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In 1810, Marquerie, the newly appointed Governor General of Australia, made a number of monetary proposals to the English Colonial Office.[1] He proposed that a bank with note-issuing power be created and he asked for £5,000 in copper to be issued at double value. Despite Marquerie's claim that there was "no other circulating medium in this colony than the notes of hand of private individuals [which had] already been productive of infinite frauds, abuses and litigation," both proposals were rejected (see Butlin (1968, pp. 78-81)).

> By way of consolation, [Marquerie] was promised a shipment of £10,000 in Spanish dollars...[and was] urged...to take the necessary steps to prevent re-export....
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> By November 1812 Marquerie had received the "seasonable supply of specie", and proceeded to publish the "very strong Colonial Law" which he believed "absolutely necessary" to prevent export of dollars. The device of the "holey" dollar was adapted from the practice of "cutting" coins which was very widespread at the period. At least two other colonies used "ring" dollars before New South Wales-- Dominica in 1798 and Trinidad in 1811. ...For retaining the coin in the colony he relied upon stiff penalties. (Butlin, 1968, pp. 80, 81.)

Until at least the middle of the 19th century, the kinds of difficulties Marquerie describes seemed to occur in many parts of the world. Moreover, as hinted at in the above passage, a prohibition on the export of coin was a fairly common policy response. Although some economic historians have suggested that significant coinage indivisibility was the source of the currency shortages and associated difficulties (see Hanson (1979) and Glassman and Redish (1988)), no one has produced a model with indivisible money, showed that it can give rise to a shortage of money, and used it to appraise the policy of prohibiting the export of money. That is what we do in this paper.

Our model is closely related to Shi (1995), Trejos and Wright (1995) and Aiyagari, Wallace, and Wright (1996). Those are models with pairwise meetings, absence of double coincidence, privacy of individual trading histories, and indivisible outside money. In those models, the assumption that money is indivisible is adopted for tractability. Here, in contrast, we view the indivisibility as descriptive of the situations we wish to model. The version we use has diverse groups of people and a single world currency. The diversity, which takes the form of different disutilities of production, makes some people willing to

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[1]We are indebted to Peter Hartley of Rice University for calling our attention to early Australian monetary history.

give up less goods than others in order to acquire a unit of indivisible outside money. The single currency is meant to approximate the situation of a world on a common specie standard.

For some parameters and for initial conditions in which *almost all* money is held by those willing to give up large amounts of goods in exchange for money, the model has an equilibrium under free trade in which *all* money ends up in the hands of those willing to give up a large amount of goods in exchange for a unit of money. Thus, if a country has none of such people and has almost no money, as is roughly descriptive of the situation in Australia and in other relatively poor areas at the beginning of the 19th century and earlier, then the money that the country starts with as an initial condition is eventually exported. The consequence in the simple model we use is that eventually all trade, domestic and international, disappears from that country. For the same parameters and initial conditions, our model implies that a prohibition on the export of money from such a country is equivalent to imposing (international) autarky for that country and that such autarky is noncomparable to free trade; residents who start with money prefer free-trade, while those who start without money prefer autarky. However, as we demonstrate numerically, for some such parameters and some initial conditions, a representative resident of such a country prefers autarky. That is, the expected utility of a resident of the country, an expected value taken with respect to the initial distribution of money for residents, can be higher under autarky than under free trade. In that sense, our model rationalizes a prohibition on the export of coin.

1. The Model

Time is discrete, with integer dates $t \geq 0$, and the horizon is infinite. There are N distinct divisible and perishable goods at each date and there is a [0,1] continuum of each of N types of people. Each type is specialized in consumption and production in the following way: a type i person consumes good i and produces good i+1 (modulo N), for i = 1,2,...,N, where $N \geq 3$. Each type i person maximizes expected discounted utility with discount factor $\beta \in (0,1)$.

A fraction p_j of each type have utility in a period given by $u(x) - y/a_j$, where x is the amount of good consumed and y is the amount of good produced, and where j = 1,2, $a_2 >$ $a_1 > 0$, and $p_1 + p_2 = 1$. The function u is defined on $[0,\infty)$, is increasing and twice differentiable, and satisfies $u(0) = 0$, $u'' < 0$, and $u'(0) = \infty$. The a_2 people are called high productivity people and the a_1 people the low productivity people because the former experience less disutility per unit produced than the latter.

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People meet pairwise at random and each person's trading history is private information to the person. Together, these assumptions rule out all but *quid pro quo* trade for optimizing people. In particular, they rule out private credit. The only storable objects are indivisible units of (fiat) money and each person has a storage capacity of one unit. The amount of money per type is exogenous and constant and is denoted M. We assume that M $\leq .5\min(p_1,p_2)$. In a meeting, each person sees the trading partner's type, productivity, and amount of money held.

The sequence of actions within a period is as follows. Each person begins a period holding either one unit of money or nothing. Then people meet pairwise at random. Because of the unit upper bound on individual holdings of money and the indivisibility of money, there is a potential for trade only when a type i person meets a type i+1 person and the type i+1 person, the potential consumer, has money and the type i person, the potential producer, does not. We call such meetings *trade meetings*. People in trade meetings bargain. If the outcome of bargaining implies exchange, then production and consumption occurs. Then people begin the next period.

As regards bargaining, we make the division of the gains from trade a parameter and examine outcomes for essentially all values of that parameter. A special case of our model is the Nash bargaining solution and limiting cases are take-it-leave-it offers by the consumer or the producer. Conditional on a way of dividing up the gains from trade, the equilibrium concept, spelled out in detail below, is Nash equilibrium with rational expectations.

Several comments are in order about the above specification. First, the introduction of different disutilities is the slight innovation to the background model in Aiyagari, Wallace, and Wright (1996), which has $a_j \equiv 1$.[2] Second, a country in our model is an arbitrary collection of people defined by its size and the fraction who are low productivity people. We do not, as in Matsuyama, Kiyotaki, and Matsui (1993) and Trejos and Wright (1996), add parameters for country-specific meeting patterns. Third, the assumption that M $\leq$ .5$\min(p_1,p_2)$ limits the amount of money so that even if all of it is held by people of one productivity, there is not so much money that the probability of a trade meeting is decreasing in the amount of money. Because that possibility is due to the upper bound on individual money holdings which is imposed only for tractability, it is sensible to restrict the amount of money to rule out that possibility. Finally, although money in the model is

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[2] That model, in turn, is based closely on the models in Shi (1995) and Trejos and Wright (1995). The absence-of-double coincidence aspect is based on Kiyotaki and Wright (1989).

intrinsically useless, we want to be able to interpret the results as applying to a commodity-money world. We can if each unit of the monetary object is a commodity by virtue of having a sufficiently small and exogenous utility value when consumed. An equilibrium of the fiat-money model is also an equilibrium of such a commodity-money model if the value of money in trade in that equilibrium is always higher than its exogenous consumption value. Because the value of money in trade in any of the equilibria described below is bounded away from zero, that condition can be met.[3]

2. Definitions of a free-trade equilibrium and a free-trade steady state

The above model is specified so that there is symmetry across types, types defined by the good consumed and the good produced. Throughout, we consider only equilibria which are symmetric across such types. To permit there to be such equilibria, the initial distribution of money holdings is assumed to be symmetric in that sense. If m_{jt} denotes the fraction of each type who are a_j people and who begin period t with a unit of money, then the initial condition is (m_{10}, m_{20}), where $m_{10} + m_{20} = M$.

For symmetric equilibria, the "state" of the system at date t is the distribution of money holdings, the pair (m_{1t}, m_{2t}), denoted m_t. An equilibrium is a sequence of such distributions and a sequence of trades in trade meetings which together satisfy some conditions. As regards trades, we have to describe whether money is offered and how much is produced in meetings distinguished by the productivities of the consumer and the producer. The equilibrium conditions, spelled out in detail below, are that trades solve a bargaining problem and that the sequence of money distributions satisfy a law of motion.

We begin with the law of motion for money holdings for given trading activity. Written in the form that the change in holdings of the a_j people is the inflow into their holdings minus the outflow from their holdings and assuming no disposal of money, the law of motion is

(1) $m_{jt+1} - m_{jt} = \lambda_{kjt} m_{kt} (p_j - m_{jt})/N - \lambda_{jkt} m_{jt} (p_k - m_{kt})/N$

for j,k = 1,2 and j $\neq$ k. Here $m_{kt}(p_j - m_{jt})/N$ is the per type fraction of meetings which are trade meetings between an a_k consumer and an a_j producer, and λ_{kjt} is the fraction of these in which trade occurs (the first subscript is the productivity of the consumer and the second

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[3] Such a commodity-money specification would have the added bonus of ruling out equilibria in which money has no value or has only its low consumption value.

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is that of the producer). That gives the inflow into holdings of money by a_j people. The outflow term, $\lambda_{jkt} m_{jt}(p_k - m_{kt})/N$, is analogous.

We next describe the bargaining problem in a trade meeting at date t between an a_j consumer (with money) and an a_k producer (without money). Let c_{jkt} be the amount produced in such a meeting (the first subscript is the productivity of the consumer and the second is that of the producer). Also, let $v_{jt}(h)$ be the expected discounted utility for an a_j person of beginning date t with h units of money where h is either 0 (no money), or 1 (one unit of money) and let v_t be the 4-element vector of these expected discounted utilities. We describe the bargaining problem in a trade meeting at t for an arbitrary vector of valuations of money holdings at t+1, v_{t+1}.

Bargaining problem. Let $\Delta_{jt+1} \equiv \beta[v_{jt+1}(1) - v_{jt+1}(0)]$ for j = 1,2. For a given v_{t+1} and a given $\alpha \in [0,1]$, x_{jkt} is a solution to the bargaining problem in a trade meeting at date t between an a_j consumer and an a_k producer if $x_{jkt} = \text{argmax} H(x_{jkt})$ where

(2) $H(x_{jkt}) \equiv [u(x_{jkt}) - \Delta_{jt+1}]^{\alpha} [\Delta_{kt+1} - x_{jkt}/a_k]^{1-\alpha}$

subject to

(3a) $u(x_{jkt}) - \Delta_{jt+1} \geq 0$

and

(3b) $\Delta_{kt+1} - x_{jkt}/a_k \geq 0.$ •

Notice that date t+1 expected discounted utilities appear in the bargaining problem between an a_j consumer and an a_k producer only by way of Δ_{jt+1} and and Δ_{kt+1}, where the former is the cost for the a_j consumer of giving up money and the latter is the benefit for the a_k producer of acquiring money, a cost and a benefit measured in terms of discounted future utility. The constraints, (3a) and (3b), say that neither participant in a trade meeting can be hurt by trading relative to refusing to trade. For the consumer, that means that the gain from trading in terms of current period utility, $u(x_{jkt})$, must be no less than Δ_{jt+1}. For the producer, that means that the cost from trading in terms of current period utility, x_{jkt}/a_k, must be no greater than Δ_{kt+1}. The objective H is a Cobb-Douglas function of the gains from trade for the consumer and the producer; for the consumer the threat point (not trading) is $\beta v_{jt}(1)$ and the payoff from trading is $u(x_{jkt}) + \beta v_{jt}(0)$, while for the producer

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the threat point (not trading) is $\beta v_{kjt}(0)$ and the payoff from trading is $-x_{jkt}/a_k + \beta v_{kt}(1)$. The Cobb-Douglas weight α splits the gains from trade between the consumer and the producer; for example, α equal to 1/2 is the Nash bargaining solution and $\alpha = 1$ gives all the gains to the consumer.

It is easy to see that the constraint set, as given by (3a) and (3b), is not empty if and only if $\Delta_{jt+1} \le u(a_k \Delta_{kt+1})$; that is, if the maximum amount the producer is willing to produce, $a_k \Delta_k$, implies utility for the consumer that is no less than the cost to the consumer of giving up money, Δ_{jt+1}. If $\Delta_{jt+1} = u(a_k \Delta_{kt+1})$, then both constraints are binding and the solution is given uniquely by either one. If $\alpha = 1$ or $\alpha = 0$, then, again, the solution is given by one of the constraints at equality and is unique. If $0 < \alpha < 1$ and $\Delta_{jt+1} < u(a_k \Delta_{kt+1})$, then it is feasible to have neither constraint binding and, therefore, to have $H > 0$. Because a constraint at equality implies $H = 0$, it follows that in this case neither constraint is binding at a solution. Therefore, in this case the solution satisfies the first-order condition, $H' = 0$. Since H' is strictly decreasing, it follows that the solution in this case is also unique. We summarize all this in the following lemma, which we will use later.

Lemma 1. The constraint set in the bargaining problem is not empty if and only if $\Delta_{jt+1} \le u(a_k \Delta_{kt+1})$. If it is not empty, then the bargaining problem has a unique solution. If $0 < \alpha < 1$ and $\Delta_{jt+1} < u(a_k \Delta_{kt+1})$, then the solution is the unique solution to $H' = 0$. Otherwise, the solution is given by one of the constraints at equality. •

We next describe c_t and λ_t in terms of the bargaining problem. In doing this, we allow for the possibility, which will turn out to be relevant, that the constraint set in the bargaining problem is empty. We have,

(4) $\qquad c_{jkt}(v_{t+1}) = \begin{cases} x_{jkt} \text{ if } \Delta_{jt+1} \le u(a_k \Delta_{kt+1}) \\ \\ 0 \text{ if } \Delta_{jt+1} > u(a_k \Delta_{kt+1}) \end{cases}$

and

(5) $\qquad \lambda_{jkt}(v_{t+1}) = \begin{cases} 1 \text{ if } \Delta_{jt+1} < u(a_k \Delta_{kt+1}) \\ \\ [0,1] \text{ if } \Delta_{jt+1} = u(a_k \Delta_{kt+1}) \\ \\ 0 \text{ if } \Delta_{jt+1} > u(a_k \Delta_{kt+1}) \end{cases}$

In (4) we say that production is given by the solution to the bargaining problem if a solution exists (the constraint set is not empty); otherwise, production is zero. Equation (5) describes our assumptions about trade: if at least one partner gains from trade, then trade occurs; if both are indifferent ((3a) and (3b) hold with equality), then the fraction who trade can be anything; if there is no solution to the bargaining problem, then there is no trade. Now we can give definitions of an equilibrium and a steady state.

Definition of equilibrium. Let $\lambda_t \equiv (\lambda_{11t}, \lambda_{12t}, \lambda_{21t}, \lambda_{22t})$ and $c_t \equiv (c_{11t}, c_{12t}, c_{21t}, c_{22t})$. Given m_0, a sequence $(m_{t+1}, c_t, \lambda_t)$, $t \geq 0$, is an equilibrium if there exists a bounded v_t sequence such that (1), (4), and (5) hold and such that for $k = 1$ and $k = 2$,

(6a) $\qquad v_{kt}(0) = \Sigma_{j=1}^{2} (m_{jt}/N) \{ \lambda_{jkt}[- c_{jkt}/a_k + \beta v_{kt+1}(1)] + (1 - \lambda_{jkt}) \beta v_{kt+1}(0) \}$

$\qquad\qquad + [1 - \Sigma_{j=1}^{2} (m_{jt}/N)] \beta v_{kt+1}(0)$

and

(6b) $\qquad v_{kt}(1) = \Sigma_{j=1}^{2} [(p_j - m_{jt})/N] \{ \lambda_{kjt}[u(c_{kjt}) + \beta v_{kt+1}(0)] + (1 - \lambda_{kjt}) \beta v_{kt+1}(1) \}$

$\qquad\qquad + [1 - \Sigma_{j=1}^{2} (p_j - m_{jt})/N] \beta v_{kt+1}(1). \quad \bullet$

In equation (6a), which describes expected discounted utility for an a_k person who begins date t without money, (m_{jt}/N) is the probability of meeting an a_j consumer with money. In equation (6b), which describes expected discounted utility for an a_k person who begins date t with a unit of money, $(p_j - m_{jt})/N$ is the probability of meeting an a_j producer without money.

Definition of a steady state. A steady state is (m, c, λ) such that $(m_{t+1}, c_t, \lambda_t) = (m, c, \lambda)$ is an equilibrium for $m_0 = m$. $\bullet$

It is evident from the bargaining problem and (6b) that $\alpha = 0$ is inconsistent with an equilibrium with valued money. For that reason and because nothing special is learned from $\alpha = 1$, from now on we assume $0 < \alpha < 1$.

3. A free-trade equilibrium in which low productivity people never produce

For an initial distribution of money holdings with sufficiently small holdings by a_1 people, we describe the parameters for which there exists a valued money equilibrium in which a_1 people never produce. This equilibrium, which converges to a steady state in which no money is held by a_1 people, is our representation of a shortage of money among a group consisting of a_1 people. We begin by describing the parameters for which such a steady state exists.

Proposition 1. There exists a_1* with $0 < a_1* < a_2$ such that $a_1 \leq a_1*$ is necessary and sufficient for the existence of a valued money steady state with $(m_1,m_2) = (0,M)$ and $(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = (0,1,0,1)$. •

Proof. First, let $v_2* = (v_2(0)*, v_2(1)*)$ denote the non-zero stationary solution for $(v_{2t}(0), v_{2t}(1))$ in (6a) and (6b) for $k = 2$ when c_{jht} is given by (4), $m_t = (0,M)$, and $\lambda_t = (0,1,0,1)$. Lemma 2, in the appendix, shows that v_2* is unique, that $\beta[v_2(1)* - v_2(0)*] = \Delta_2* > 0$, that $\Delta_2* < u(a_2\Delta_2*)$ (constraint (3a) is not binding so that $\lambda_{22} = 1$ satisfies (5)), and that v_2* does *not* depend on a_1 (because no trade with a_1 producers is imposed).

Next, let $v_1* = (v_1(0)*, v_1(1)*)$ denote the stationary solution for $(v_{1t}(0), v_{1t}(1))$ in (6a) and (6b) for $k = 1$ when c_{jht} is given by (4), $m_t = (0,M)$, $\lambda_t = (0,1,0,1)$, and $v_{2t} = v_{2t+1} = v_2*$. (Although there are no a_1 people with money in the steady state, the value to an a_1 person of having a unit of money is well-defined.) Lemma 3 shows that v_1* is unique, that $v_1(0)* = 0$ (because, given the assumed λ, a_1 people without money never produce, and, therefore, never consume), that $\beta[v_1(1)* - v_1(0)*] = \Delta_1* < u(a_2\Delta_2*)$ (constraint (3a) is not binding so that $\lambda_{12} = 1$ satisfies (5)), that v_1* does *not* depend on a_1 (again, because no trading with a_1 producers is imposed).

The proof is completed by choosing a_1* so that $a_1 \leq a_1*$ is necessary and sufficient to insure that the constraint set in the bargaining problem is empty or has both constraints binding in meetings with a_1 producers. Because, as shown in lemma 4, $\Delta_1* \geq \Delta_2*$, lemma 1 implies that $\Delta_2* = u(a_1*\Delta_1*)$ is necessary and sufficient for that conclusion about the constraint set. Because Δ_1* and Δ_2* are positive and do not depend on a_1, there is a unique and positive a_1*, which does not depend on a_1, that satisfies $\Delta_2* = u(a_1*\Delta_1*)$. Therefore, $a_1 \leq a_1*$ is necessary and sufficient for $\lambda_{11} = \lambda_{21} = 0$ to satisfy (5). Finally, since $u(a_1*\Delta_1*) = \Delta_2* < u(a_2\Delta_2*) \leq u(a_2\Delta_1*)$, we have $a_1* < a_2$. •

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We now show that if $a_1 < a_1^*$, then there exists an equilibrium that converges to the proposition 1 steady state from nearby initial conditions-- an equilibrium in which a_1 people never produce.

Proposition 2. If $a_1 < a_1^*$, then there exists $\delta_1 > 0$ such that $m_{10} < \delta_1$ implies existence of an equilibrium that converges to the proposition 1 steady state. •

Proof. We begin by examining convergence of (m_t, v_t) given $\lambda_t = (0,1,0,1)$ for all t. With c_{jkt} given by (4), equations (1) and (6a) and (6b) constitute a first-order difference equation system in (m_t, v_t) with an initial condition for m_t. Because (1) contains only m_t, existence of a path for (m_t, v_t) that converges to the steady state values of (m_t, v_t) follows if (1) implies that m_t converges to $(0,M)$. That convergence is immediate (and global) because $\lambda_t = (0,1,0,1)$ implies an inflow into m_{2t} and no outflow, and an outflow from m_{1t} and no inflow.

To conclude that any such convergent path is an equilibrium, we have to verify that $\lambda_t = (0,1,0,1)$ for all t satsifies (5). Let ρ denote the Euclidean metric. Convergence for $\lambda_t = (0,1,0,1)$ implies that there exists $R_0 > 0$, such that for any $R < R_0$, there exists δ_1 with $0 < \delta_1 < R$ such that if $\rho(m_{10}, 0) < \delta_1$, then $\rho(v_t, v^*) < R$ where $v^* = (v_1^*, v_2^*)$ is the steady state v_t of proposition 1 (see Luenberger, p. 322). Because $u(a_2 \Delta_2^*) > \max(\Delta_1^*, \Delta_2^*)$, because $a_1 < a_1^*$ implies $u(a_1 \Delta_1^*) < \min(\Delta_1^*, \Delta_2^*)$, and because u is continuous, there exists an $R' > 0$ so that $\rho(v_t, v^*) < R'$ implies $u(a_2 \Delta_{2t}) > \max(\Delta_{1t}, \Delta_{2t})$ and $u(a_1 \Delta_{1t}) < \min(\Delta_{1t}, \Delta_{2t})$. Therefore, if $R < \min(R', R_0)$, then there exists $\delta_1 > 0$ such that $\rho(m_{10}, 0) < \delta_1$ implies $\lambda_t = (0,1,0,1)$ for all t satisfies (5). •

We next give a multi-country interpretation of the proposition 2 equilibrium and compare what happens to a country of low productivity people in that equilibrium with what happens to it under a prohibition on the export of money.

4. Free-trade versus a prohibition on the export of money

Consider a country that consists of a subset of the a_1 people in the world, a subset consisting of a fraction q of each type and a subset that includes all the a_1 people who initially hold money. We will call this subset of people country 1 and the rest-of-the-world country 2. We assume that $m_{10}/q < 1$. The only consequence of calling the above subsets of people countries is that we permit country 1 to impose a prohibition on the export of

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money. We interpret the prohibition on the export of money as preventing any trade in which a resident of country 1 gives money to a resident of country 2.

We begin with two results that hold for an initial condition in the neighborhood of the proposition 1 steady state. The first is that under a prohibition on the export of money and $a_1 < a_1{}^*$, there is a constant equilibrium in which there is no trade between the countries, but there is trade within each country. The second is that such a constant equilibrium is noncomparable for residents of country 1 to the proposition 2 free-trade equilibrium.

Proposition 3. If there is a prohibition on the export of money from country 1 and if $a_1 < a_1{}^*$, then there exists $\delta_2 > 0$ such that $m_{10} < \delta_2$ implies existence of a constant valued-money equilibrium with no trade between countries. •

Proof. The proof begins by constructing an autarkic equilibrium, which is constant and has valued money in each country. It is then shown that no resident of country 2 is willing to offer money to a resident of country 1.

First, consider country 2 under autarky. It differs from the whole world under free trade with $m_0 = (0, M)$ only in that M is replaced by $M - m_{10}$. Therefore, continuity of the proposition 1 solutions for $\Delta_1{}^*$ and $\Delta_2{}^*$ in M imply that there exists $\delta_2 > 0$ such that $m_{10} < \delta_2$ implies existence of a constant equilibrium for country 2 under autarky in which money is valuable and in which a_1 residents of country 2 never produce and never consume. Moreover, since, as shown in lemma 5, $\Delta_2{}^*$ is decreasing in M, it follows that

(7) $\Delta_2{}^{**} \geq \Delta_2{}^*$

where $\Delta_2{}^{**}$ denotes the autarky valued-money solution for Δ_2.

Next, consider country 1 under autarky. As regards trades, it is an economy of identical people. Hence, we can again apply the results of lemma 2. In particular, let $v_1{}^{**}$ denote the non-zero stationary solution for $(v_{1t}(0), v_{1t}(1))$ in (6a) and (6b) for k = 1 with c_{jht} given by (4), $m_t = (m_{10}, 0)$, $p_1 = q$, and with $\lambda_{11t} = 1$ and autarky ($\lambda_{12t} = \lambda_{21t} = 0$). It follows from lemma 2 that $v_1{}^{**}$ is unique, that $\beta[v_1(1)^{**} - v_1(0)^{**}] = \Delta_1{}^{**} > 0$, and that $\Delta_1{}^{**} < u(a_1\Delta_1{}^{**})$ (constraint (3a) is not binding so that $\lambda_{11} = 1$ satisfies (5)).

From proposition 1, we have $\Delta_1{}^* > u(a_1\Delta_1{}^*)$. This and $\Delta_1{}^{**} < u(a_1\Delta_1{}^{**})$ imply

(8) $\Delta_1{}^{**} < \Delta_1{}^*$

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By proposition 1, we also have $\Delta_2^* \geq u(a_1\Delta_1^*)$. This and (7) and (8) imply $\Delta_2^{**} > u(a_1\Delta_1^{**})$, which implies that an a_2 consumer does not offer money to an a_1 producer who is a resident of country 1. •

Now we establish noncomparability for residents of country 1 between a proposition 2 equilibrium and a proposition 3 equilibrium. That is done by comparing date 0 expected discounted utilities for those who start with money and those who do not across those equilibria.

Proposition 4. If $a_1 < a_1^*$, then there exists $\delta > 0$ such that $0 < m_{10} < \delta$ implies existence of a proposition 2 (free-trade) equilibrium and a proposition 3 (prohibition-on-the-export-of-money) equilibrium satisfying the condition that residents of country 1 who begin without money are better off in the latter than in the former, while residents who begin with money are better off in the former than in the latter. •

Proof. The comparison for those who do not have money is immediate. Because they never trade in the proposition 2 equilibrium, they have zero expected discounted utility in that equilibrium. Because $\Delta_1^{**} < u(a_1\Delta_1^{**})$ in the proposition 3 equilibrium, they have positive expected discounted utility in that equilibrium. It remains to establish that those who start with money prefer the proposition 2 equilibrium.

Because a_1 people without money have zero discounted expected utility in the free-trade steady state, inequality (8) is equivalent to

(9) $v_1(1)^* > v_1(1)^{**}(m_{10}) - v_1(0)^{**}(m_{10}) \equiv g(m_{10})$

where, as we indicate, $v_1(1)^*$, the proposition 1 steady-state expected disounted utility for an a_1 person with money, does not depend on m_{10}. Because (9) holds for all m_{10} satisfying $m_{10} < \delta_2$ and, in particular, for $m_{10} = 0$ and because $v_1(0)^{**}(0) = 0$ (expected discounted utility of not having money is zero if there is no money),

(10) $v_1(1)^* - v_1(1)^{**}(0) = v_1(1)^* - g(0) \equiv \varepsilon > 0$

By proposition 2, there exists δ_3 such that $m_{10} < \delta_3$ implies

(11) $|v_{10}(1)^*(m_{10}) - v_1(1)^*| < \varepsilon/2$

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and, by the continuity of v_1^{**} in m_{10}, also implies

(12) $|v_1(1)^{**}(m_{10}) - v_1(1)^{**}(0)| < \varepsilon/2$

Therefore, by (10) - (12), $m_{10} < \delta_3$ implies

(13) $v_{10}(1)^*(m_{10}) - v_1(1)^{**}(m_{10}) = [v_1(1)^{**}(m_{10}) - v_1(1)^*] + [v_1(1)^* - v_1(1)^{**}(0)] +$

$$[v_1(1)^{**}(0) - v_1(1)^{**}(m_{10})] > -\varepsilon/2 + \varepsilon - \varepsilon/2 = 0$$

Hence, if we let $\delta = \min(\delta_1, \delta_2, \delta_3)$, where δ_1 is given in proposition 2 and δ_2 in proposition 3, then all the asserted conclusions follow. •

The noncomparability established in proposition 4 has several limitations. First, we have established it only for initial conditions close to the steady state in which low productivity people hold no money. Second, although our model lends itself to a representative agent welfare measure for a country-- namely, a weighted average of the date 0 expected discounted utilities for residents who start with money and those who do not with weights given by the fractions who start with money and who do not, respectively-- our local analysis does not permit us to say anything about that measure across equilibria except in the trivial case in which $m_{10} = 0$.[4] Although we cannot address these limitations generally, we can address them numerically for particular parameterizations for the model. As we now explain, our numerical results say that country 1 can prefer the prohibition to free trade according to the representative agent measure of welfare.

Our computation procedure is straightforward. We start with a given parameterization for the model including m_{10}. The equilibrium under the prohibition is a constant equilibrium, and therefore, is easily computed. To approximate the equilibrium under free trade, we proceed as follows. We begin by using the law of motion, (1), to compute the m_t sequence implied by $\lambda_t = (0,1,0,1)$ for all t and $m_0 = (m_{10}, M - m_{10})$. This sequence, as we know, converges to $(0, M)$. Then, for a given positive integer T, we set $v_T = v^*$, the steady state v under free trade, and work backward, date by date, to t = 0 using the

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[4] The representative-agent welfare measure has the interpretation of expected utility for residents of a country before they know who starts with money and who does not. In effect, we regard a country as choosing free-trade or a prohibition on the export of money prior to knowing who among its residents will start with money.

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computed m_t sequence. In particular, given v_{t+1}, we compute v_t as follows. We first check the constraint sets in the date t bargaining problems for emptiness in meetings with a_1 producers and nonemptiness in meetings with a_2 producers. If those conditions are not satisfied by v_{t+1}, then we conclude that do not have an equilibrium with $\lambda_t = (0,1,0,1)$ for all t and we stop. If they are satisfied, then we solve the two unconstrained bargaining problems for meetings with a_2 producers. As asserted in lemma 1, those two bargaining problems have unique solutions for the amounts produced. Then we impose those solutions, the given v_{t+1} and $\lambda_t = (0,1,0,1)$ in (6a) and (6b) to get a unique v_t. Conditional on satisfying the checks on the constraint sets in the bargaining problem at each date, this procedure produces, in principle, date 0 expected utilities for each T, and, hence, a sequence, denoted $\{v_0^T\}$, of date 0 expected utilities. Since $\{v_0^T\}$ is a bounded sequence in R^4, it has a convergent subsequence which has a limit in R^4. That limit is the date 0 term of an equilibrium. For our computations, we first computed v_0^T for T = 100, 200, and 300. Because v_0^{100} and v_0^{200} are identical to at least 5 significant digits and because v_0^{200} and v_0^{300} are identical to at least 10 significant digits, we accept v_0^{300} as a good approximation to v_0^*.

We present results for two parameterizations that differ only with regard to a_1. We set N = 3 (three types), $u(x) = x^{1/2}$, $a_2 = 1$, $\beta = .999$ (which corresponds to an annual discount factor equal to .95 if the model is one of weekly meetings), $p_1 = p_2 = 1/2$, M = .25 (which maximizes the probability of a trade meeting if all the money is concentrated in the hands of the a_2 people) and $\alpha = 1/2$ (the Nash bargaining solution). That much of the parameterization implies a_1^*, which turned out to be .2176. We then picked two alternative a_1's that satisfy $a_1 < a_1^*$: $a_1 = .21$ and $a_1 = .15$. We chose m_{10} to satisfy $m_{10}/(M - m_{10}) = a_1/a_2$ for $a_1 = .15$. This gives $m_{10} = .0326$. Finally, to form country 1, we set q = 1/2, so that country 1 consists of all the a_1 people.

The results appear in Table 1. There, each w represents the relevant weighted average of the v's in the two preceding rows. Thus, w_{10}^* in the third row is the country-1 representative-agent date 0 welfare under free trade and w_1^{**} in the sixth row is such welfare under the prohibition. We see that free-trade is better for country 1 if $a_1 = .15$, but the prohibition is better for country 1 if $a_1 = .21$. For both magnitudes of a_1 we get the noncomparability that we found locally in proposition 4. Notice that only the country 1 results under the prohibition depend on a_1. It turned out that free-trade is uniformly preferred by residents of country 2.

The finding that country 1 can do better according to a representative-agent criterion under autarky than under free-trade should not seem paradoxical. Our model is one with severe market incompleteness and it is well-known that adding markets can hurt in a setting

with market incompleteness (see Hart 1975). Also, Zhou (1995) finds in a somewhat different setting with two distinct monies that autarky can be Pareto superior to an equilibrium with trade.

5. Concluding remarks

Our model is not the first to deal with the consequences of money being indivisible. Earlier models include Marimon and Wallace (1987) and Cooley and Smith (1995). However, to make indivisibility matter, those and other models of indivisible assets simply assume that the markets that would accomplish the sharing of indivisible assets are missing. Here, in contrast, the inability to share indivisible money is implied by the never-meeting-again feature of meetings and the assumption that a person's trading history is private information. Those assumptions rule out any form of credit, including the sharing of indivisible assets. They are extreme in that they go beyond making "the notes of hand of private individuals... productive of infinite frauds, abuses and litigation", as was asserted by Marquerie to be the situation in Australia; they imply that "notes of hand" are never accepted. Although more extreme than we might like, those assumptions together with the other assumptions made constitute an internally consistent set that assures that all the trades that are incentive feasible are allowed to occur (see Aiyagari and Wallace (1991) for details). Previous models of indivisible assets cannot make that claim.

Perhaps the most extreme assumption we make is the unit upper bound on individual holdings. While it would be desirable to dispense with that bound or any exogenous bound, the nature of the results presented above seem robust to weakening the upper bound. With or without a bound, if money is indivisible, then small enough transactions will not be made using it. The upper bound does, however, make it awkward to consider the production of additional divisibility, as was done with the creation of "holey" or "ring" dollars in Australia in 1812. That is why we did not consider that aspect of the Australian policy.[5]

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[5] That aspect of the policy may not have been very important. "Ring dollars and dumps were too good for their purpose, since they were legal tender for any amount and could buy bills on England, which was nearly as good as being exportable. So they were treasured by those who could come by them or held by the Bank of New South Wales as a reserve..." (Butlin, p.84).

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Table 1. Date 0 discounted expected utilities for two examples. $(N = 3, u(x) = x^{1/2}, a_2 = 1, \beta = .999, p_1 = p_2 = q = 1/2, M = .25, \alpha = 1/2, m_{10} = .0326.)$

approximate expected utilities	$a_1 = .15$	$a_1 = .21$
$v_{10}(0)^*$	0	0
$v_{10}(1)^*$	0.5904	0.5904
w_{10}^*	0.0385	0.0385
$v_1(0)^{**}$	0.0230	0.0327
$v_1(1)^{**}$	0.1688	0.2363
w_1^{**}	0.0326	0.0456
$v_{20}(0)^*$	10.2157	10.2157
$v_{20}(1)^*$	10.5961	10.5961
w_{20}^*	10.3811	10.3811
$v_2(0)^{**}$	9.4597	9.4597
$v_2(1)^{**}$	9.9710	9.9710
w_2^{**}	9.6820	9.6820

Appendix

Lemma 2. If $v_2^* = (v_2(0)^*, v_2(1)^*)$ is the non-zero stationary solution for $(v_{2t}(0), v_{2t}(1))$ in (6a) and (6b) for $k = 2$ when c_{jht} is given by (4), $m_t = (0,M)$, and $\lambda_t = (0,1,0,1)$, then v_2^* is unique, $\beta[v_2(1)^* - v_2(0)^*] = \Delta_2^* > 0$, $\Delta_2^* < u(a_2 \Delta_2^*)$, and v_2^* does *not* depend on a_1. •

Proof. The condition $H'(c_{jkt}) = 0$ is equivalent to

(A1) $a_k u'(c_{jk})\gamma = [u(c_{jk}) - \Delta_j]/(\Delta_k - c_{jk}/a_k)$

where we have dropped the time subscript and where $\gamma = \alpha/(1 - \alpha)$. From (6a) and (6b) with $v_{kt}(h) = v_{kt+1}(h) = v_k(h)$, $m_t = (0,M)$, $\lambda_t = (0,1,0,1)$, and $k = 2$, we get, by subtracting (6a) from (6b),

(A2) $\Delta_2 = b_1 u(c_{22}) + b_2 c_{22}/a_k$

where $b_1 = (p_2 - M)/(N\rho + p_2) \in (0,1)$, $b_2 = M/(N\rho + p_2) \in (0,1)$, and $\rho = (1 - \beta)/\beta$. If we set $j = k = 2$ in (A1), substitute the right side of (A2) into (A1), and let x denote c_{22}, we get

(A3) $a_2 u'(x)\gamma = [(1 - b_1)u(x) - b_2 x/a_2]/[b_1 u(x) - (1 - b_2)x/a_2] \equiv F_{22}(x)$

Differentiation of F_{22} implies that

(A4) $\text{Sign}[F_{22}'(x)] = \text{Sign}\{(1 - b)[u(x) - u'(x)x]\} > 0$

where $b = b_1 + b_2$ and where the inequality follows from $b \in (0,1)$ and the strict concavity of u.

Now let $\hat{x}$ be the unique positive solution for x to $b_1 u(x) - (1 - b_2)x/a_2 = 0$. It follows that for all $x \in (0,\hat{x})$, the denominator of F_{22} is positive. And since $1 - b > 0$ implies that the numerator of F_{22} is positive for $x \in (0,\hat{x}]$, it follows from (A4) that F_{22} is positive and increasing for $x \in (0,\hat{x})$ and that $F_{22}(x) \to \infty$ as $x \to \hat{x}$. Since $a_2 u'(x)\gamma \to \infty$ as $x \to 0$ and is decreasing, (A3) has a unique solution with $x \in (0,\hat{x})$. Denote it c_{22}^* so that Δ_2^* is given by (A2) with $c_{22} = c_{22}^*$. Because $u(c_{22}^*) - \Delta_2^*$ is equal to the numerator of $F_{22}(c_{22}^*)$ and $\Delta_2^* - c_{22}^*/a_2$ is equal to the denominator of $F_{22}(c_{22}^*)$, both of which are

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positive, it follows that neither constraint in the bargaining problem for $j = k = 2$ is binding at $x_{22t} = c_{22}^*$ and $\Delta_2 = \Delta_2^*$. That confirms that c_{22}^* is a solution to the bargaining when $\Delta_2 = \Delta_2^*$ and implies that $\Delta_2^* < u(a_2\Delta_2^*)$.

It follows from (A2) that $\Delta_2^* > 0$. It is also straightforward to solve $\Delta_2^* = \beta[v_2(1)^* - v_2(0)^*]$ and the stationary version of either (6a) or (6b) evaluated at $c_{22} = c_{22}^*$ for a unique non zero v_2^*. Finally, because a_1 does not appear in (A3), v_2^* does not depend on a_1. •

Lemma 3. If $v_1^* = (v_1(0)^*, v_1(1)^*)$ is the stationary solution for $(v_{1t}(0), v_{1t}(1))$ in (6a) and (6b) for $k = 1$ when c_{jht} is given by (4), $m_t = (0,M)$, $\lambda_t = (0,1,0,1)$, and $v_{2t} = v_{2t+1} = v_2^*$, then v_1^* is unique, $v_1(0)^* = 0$, $\beta[v_1(1)^* - v_1(0)^*] = \Delta_1^* < u(a_2\Delta_2^*)$, and v_1^* does *not* depend on a_1. •

Proof. From (6a) and (6b) with $v_{kt}(h) = v_{kt+1}(h) = v_k(h)$, $m_t = (0,M)$, $\lambda_t = (0,1,0,1)$, and $k = 1$, we get $v_1(0)^* = 0$ and

(A5) $\Delta_1 = b_3 u(c_{12})$

where $b_3 = (p_2 - M)/(N\rho + p_2 - M) \in (0,1)$. If we set $j = 1$ and $k = 2$ in (A1), substitute the right side of (A5) and Δ_2^* into (A1), and let x denote c_{12}, then we get

(A6) $a_2 u'(x)\gamma = [(1 - b_3)u(x)]/(\Delta_2^* - x/a_2) \equiv F_{12}(x)$

Therefore, for all $x \in (0, a_2\Delta_2^*)$, F_{12} is positive and increasing and that $F_{12}(x) \to \infty$ as $x \to a_2\Delta_2^*$. It follows that (A6) has a unique solution with $x \in (0, a_2\Delta_2^*)$. Denote it c_{12}^* so that Δ_1^* is given by (A5) with $c_{12} = c_{12}^*$. Because $u(c_{12}^*) - \Delta_1^*$ is equal to the numerator of $F_{12}(c_{12}^*)$ and $\Delta_2^* - x/a_2$ is the denominator of $F_{12}(c_{12}^*)$, both of whch are positive, neither constraint in the bargaining problem for $j = 1$ and $k = 2$ is binding at $x_{12t} = c_{12}^*$, $\Delta_1 = \Delta_1^*$, and $\Delta_2 = \Delta_2^*$. That implies that c_{12}^* is the solution to the bargaining problem when $\Delta_1 = \Delta_1^*$ and $\Delta_2 = \Delta_2^*$ and implies that $\Delta_1^* < u(a_2\Delta_2^*)$. Finally, $v_1(1)^* = \Delta_1^*/\beta$ is unique and evidently does not depend on a_1. •

Lemma 4. $\Delta_1^* > \Delta_2^*$.

Proof. We first show that $c_{12}^* > c_{22}^*$. This follows if $F_{12}(c_{22}^*) < F_{22}(c_{22}^*)$. Because the denominators of $F_{12}(c_{22}^*)$ and $F_{22}(c_{22}^*)$ are equal,

(A7) $\text{Sign}[F_{22}(c_{22}^*) - F_{12}(c_{22}^*)\}] = \text{Sign}[(b_3 - b_1)u(c_{22}^*) - b_2 c_{22}^*/a_2]$

$$= \text{Sign}[b_1 u(c_{22}^*) - (1 - b_2)c_{22}^*/a_2] > 0$$

where the second equality follows from the definitions of the b_i and the inequality follows from the fact that $b_1 u(c_{22}^*) - (1 - b_2)c_{22}^*/a_2$ is the denominator of $F_{22}(c_{22}^*)$.

From (A2) and A(5),

(A8) $\Delta_1^* - \Delta_2^* = b_3 u(c_{12}^*) - b_1 u(c_{22}^*) - b_2 c_{22}^*/a_2 > (b_3 - b_1)u(c_{22}^*) - b_2 c_{22}^*/a_2 > 0$

where the first inequality follows from $c_{12}^* > c_{22}^*$ and where the second follows from (A7). •

Lemma 5. Δ_2^* is decreasing in M.

Proof. We first show that c_{22}^* is decreasing in M. Since c_{22}^* is the unique solution to (A3) and since the left side of (A3) does not depend on M, the claim follows if $F_{22}(x)$ is increasing in M at each x. If we let $\partial F_{22}/\partial M$ denote the partial derivative of $F_{22}(x)$ w.r.t. M at a given x, we have, from the definition of F_{22} in (A3),

(A9) $\text{Sign}[\partial F_{22}/\partial M] = \text{Sign}\{-[u(x) - x/a_2][u(x)(\partial b_1/\partial M) + (x/a_2)(\partial b_2/\partial M)]\} > 0$

where the inequality follows from $(\partial b_1/\partial M) = -(\partial b_2/\partial M) < 0$. From, (A2)

(A10) $\partial \Delta_2^*/\partial M = [u(c_{22}^*) - c_{22}^*/a_2](\partial b_1/\partial M) + [b_1 u'(c_{22}^*) + (b_2/a_2)](\partial c_{22}^*/\partial M)$

Because $u(c_{22}^*) > \Delta_2^* > c_{22}^*/a_2$, $u(c_{22}^*) - c_{22}^*/a_2 > 0$. Therefore, both terms on the right side of (A10) are negative. •