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Optimal Allocations With Incomplete Record-Keeping and No Commitment

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ABSTRACT

We study a random-matching, absence-of-double-coincidence environment in which people cannot precommit and in which there are two imperfect ways to keep track of what other people have done in the past: money and a public record of all past actions that is updated with an average lag. We study how the magnitude of that lag affects the allocations that are optimal from among allocations that are stationary and feasible and that satisfy incentive constraints which arise from the absence of commitment and the imperfect ways of keeping track of what others have done in the past.

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Technological developments, especially during the past two or three decades, have greatly enhanced our ability to keep up-to-date records of transactions. Although most observers of such developments would agree that they have the potential to improve well-being through their effects on trading arrangements, there do not seem to be any models of the potential benefits. We provide and study one such model.

A model of the benefits of enhanced record-keeping ability requires a setting or environment in which trading is inherently difficult and in which such an enhancement reduces the difficulties. The environment we use is one in which trading is extremely difficult. People meet each other at random in pairs and in such meetings there are no double coincidences in terms of produced goods, all of which are perishable. Moreover, people cannot precommit themselves to future actions. In such an environment, any current utility loss, as would be implied by production by a person in a single-coincidence meeting, must be accompanied by some offsetting future utility gain. Moreover, people must be able to demonstrate in the future that they are entitled to that gain. In our environment, there are two imperfect devices that serve that purpose: money, which is durable, and a public record of all past actions which is updated only with some frequency. The updating, which is determined by a random process, is characterized by an exogenous average lag between updatings. We interpret better record-keeping ability as a lower average lag.

For this environment, we define a set of *incentive feasible allocations* and study how the average lag between updatings affects the optimum in that set. In particular, we are able to give some description of how the mix of monetary transactions and nonmonetary transactions varies with that average lag, where monetary transactions are those in which production is accompanied by the acquisition of money and nonmonetary transactions are those in which production is not accompanied by the acquisition of money and therefore involves the use of some form of credit.

When the lag with which the record of past transactions is updated is infinite so that the history of each person's actions is private information to the person, our environment is a prototypical absence-of-double-coincidence environment which has been used for a variety of purposes, all related to the role in exchange of outside assets and, in particular, outside money.[1] When the lag is zero so that the history of each person's actions is public information, results in Kocherlakota (1996) imply that outside money is inessential. Our formulation bridges the gap between those two extremes. For us, part of the history of each person's actions is public, all but that pertaining to some number of most recent periods—a number which on average is the average lag with which the public part of history is updated. In general, when that average lag is neither infinite nor zero, there is a role for both monetary transactions and some form of credit in the optimum.

1. The Model

Here we describe the environment, define and characterize incentive feasible allocations, and set out the optimum problem.

1.1 The Environment

Time is discrete and the horizon is infinite. There are N distinct divisible and perishable goods at each date and there is a $[0,1]$ continuum of each of N types of people. Each type is specialized in consumption and production in the following way: a type n person consumes good n and produces good $n+1$ (modulo N), for $n = 1, 2, \ldots, N$, where $N \geq 3$. Each type n person maximizes expected discounted utility with discount factor $\beta \in (0,1)$. Utility in a period is given by $u(\cdot) - y$, where the argument of u is the amount consumed and y is the amount produced (see Figure 1).[2] The function u is defined on $[0,\infty)$, is increasing and twice differentiable, and satisfies $u(0) = 0$, $u'' < 0$, and $u'(0) = \infty$.

There is an additional good called money which is neither produced nor consumed. Money is indivisible and perfectly durable and each person can store at most one unit of it from one date to the next. The amount of money per type is strictly between zero and one and the initial distribution is symmetric across types so that a fraction of each type has no money and the remaining fraction has one unit.

In each period, people are randomly matched in pairs. Meetings are of two sorts: single-coincidence meetings, those between a type n person (the *producer*) and a type $n + 1$ person (the *consumer*) for some n; and no-coincidence meetings, those in which neither person produces what the other consumes. (Because the number of types, N, exceeds two, there are no double-coincidence meetings.)

People in a meeting have the following common information. They know each other's type and holding of money. They also know the actions of all agents that occurred prior to the last period in which the record of all such actions was updated. In our environment, that record is not updated immediately after every action. Instead, at the beginning of each period, prior to people being randomly matched in pairs, the updating occurs with probability ρ, while no updating occurs with probability $1 - \rho$. This way of updating implies an average lag of $1/\rho$ periods between updates of the public record of all past actions.

Finally, as emphasized in Aiyagari and Wallace (1991), Huggett and Krasa (1996), and Kocherlakota (1996), an absence of commitment is a crucial feature of such models. That is, there is no planner at each meeting who can dictate the actions of the participants in the meeting or punish the participants for not carrying out some implicit or explicit promise that they made earlier. In our model, if there were such outside enforcement, then the allocation problem would be trivial: the planner could simply direct each producer in a single-coincidence meeting to produce and give y^* to the consumer, where y^* is the unique solution of $u'(y^*) = 1$ (see Figure 1). In particular, the use of outside money would not be essential if there were such outside enforcement. We embed a limited form of no-commitment in the form of *sequential individual rationality* (SIR). That is, we assume that an individual can always not produce in a meeting and leave the meeting with whatever money he or she brought into the meeting.

The only new aspect of the above environment is the random updating of the public record of transactions. As noted above, the special cases of our environment with ρ set at zero (no updating and therefore no public record of past actions), and ρ set at unity have been studied before. As emphasized in Aiyagari and Wallace (1991), setting ρ at zero precludes *any form* of credit. And, if ρ is unity, then Kocherlakota (1996) shows that money is inessential. Our random updating scheme is a simple one-dimensional way to specify intermediate situations, those with partial or incomplete public knowledge of past actions. As we will explain later, it is even simpler than a specification in which history up to $t - T$ is public at each date t for a fixed parameter $T \geq 0$.

1.2 Incentive Feasible Allocations: A Definition

In order to define *incentive feasibility* we use some of the formalism of mechanism design. We first define trading mechanisms that are suitable for our environment. Then we define symmetric and stationary equilibrium outcomes of these mechanisms. For us, an allocation is incentive feasible if it is one of these equilibrium outcomes.[3]

A trading mechanism has two components. The first component specifies a sequential action set for an agent in a match. Within the period, there are Π stages to the mechanism. We assume that in stage $\pi \in \{1,2,\ldots,\Pi\}$ an agent in a match has an action set that depends only on whether he or she is a producer or consumer in a single-coincidence meeting or is in a no-coincidence meeting and on the agent's amount of money, m, and that of the agent's trading partner, m'. Therefore, the action set at stage π of an agent of type n can be denoted $A(\delta_{nn'},m,m',\pi)$, where $\delta_{nn'} = 1$ if $n = n'$

$+ 1$ (the type n agent is the producer in a single-coincidence meeting), $\delta_{nn'} = -1$ if $n = n' - 1$ (the type n agent is a consumer in a single-coincidence meeting), and $\delta_{nn'} = 0$ otherwise (the type n agent is in a no-coincidence meeting). At each stage and simultaneously, each agent makes a choice out of his or her action set, choices which are observable. The second component of the trading mechanism is an outcome function $f(a^n, a^{n'}, \delta_{nn'}, m, m')$, where a^n and $a^{n'}$ are the sequences of actions of n and n', respectively. The function f specifies physically feasible levels of production and consumption for each agent and physically feasible end-of-period money holdings for each; by physically feasible, we mean that no resources from outside the meeting are available and that the unit upper bound on money holdings is satisfied. We assume that f and A also satisfy SIR: for any allowable sequence of actions by the trading partner (that is, any sequence consistent with A), there exists an allowable sequence of actions for n such that f implies zero production by n and end-of-period money holdings m for n.

We next define an equilibrium strategy of such a trading mechanism. In doing this, we restrict ourselves to strategies that are symmetric and stationary. Loosely speaking, such a strategy specifies an agent's action as a time invariant function of the kind of meeting the agent is in and the agent's information regarding whether anyone has "cheated" in the past.

DEFINITION 1. A symmetric and stationary equilibrium (SSE) for a given trading mechanism, A and f, is a triplet of actions $\{a_1(\delta_{nn'}, m, m', \pi),\ a_2(\delta_{nn'}, m, m', \pi),\ a_3(\delta_{nn'}, m, m', \pi)\}_{\pi=1}^{\Pi}$ with $a_i \in A$, such that if all other agents follow this strategy currently and in the future, then it is weakly optimal for an agent to (i) play a_1 if the agent's information is consistent with everyone having played a_1 in the past, (ii) play a_2 if the public record reveals that not everyone has played a_1 in the past, and (iii) play a_3 if only the agent's private information reveals that not everyone played a_1 in the past.

Thus, in an SSE, a_1 is always played because no defection from it ever occurs. The equilibrium also specifies "off-equilibrium" actions a_2 and a_3 as weakly optimal responses to knowledge of past defections, where the response to past defections is allowed to differ depending on whether the defection is public information or private information. As noted above, an incentive feasible allocation is an SSE outcome for some trading mechanism satisfying the SIR restriction. Formally, we have

DEFINITION 2. Let f and A satisfy SIR and let (a_1,a_2,a_3) be an SSE for A and f. An *incentive feasible allocation* is $f[\{a_1(\delta_{nn'},m,m',\pi)\}_{\pi=1}^{\Pi},\ \delta_{nn'},\ \{a_1(\delta_{n'n},m',m,\pi)\}_{\pi=1}^{\Pi},m,m']$.

Because of the restrictions on a_1 and f and because $m \in \{0,1\}$, an incentive feasible allocation is a simple object. For each of the four kinds of single-coincidence meetings determined by the four possible combinations of money holdings, it specifies physically feasible production and consumption and end-of-period money holdings of each person in the meeting; and it does the same for each of the three kinds of no-coincidence meetings determined by the three possible combinations of money holdings. We next present a simple characterization of the set of incentive feasible allocations.

1.3 Incentive Feasible Allocations: A Characterization

In this section, we first describe and characterize a subset of the SSE outcomes of a mechanism that we call the coordination mechanism. We then show in Proposition 1 that this set of allocations is identical to the set of SSE outcomes to all possible mechanisms. The result is a characterization of the incentive feasible allocations that we use subsequently to describe optima.

The mechanism described here is called the coordination mechanism because people play a coordination game in each meeting. In a single-coincidence meeting, the two agents simultaneously choose an ordered four-tuple, (y,c,x,d), where y, c, and x are in R_+, $c \leq y$, and $d \in \{0,1\}$. If the two chosen four-tuples are mutually consistent (the same), then the following occurs: the *producer* produces y, the *consumer* consumes c and produces x; holdings of money are exchanged if and only if $d = 1$. If the two chosen four-tuples are different, then nothing happens and each goes on to the next period. We distinguish between two kinds of no-coincidence meetings. If the two agents have identical money holdings, then the two agents simultaneously choose an ordered pair (z,z') in R_+^2, where the first component is own production and the second is the trading partner's production. If their choices are mutually consistent, then both produce; if their choices are inconsistent, then nothing happens and they go on to the next period. If the two agents have different money holdings, then they simultaneously choose an ordered triple (z_{01},z_{10},h) where z_{01} and z_{10} are each in R_+ and where $h \in \{0,1\}$. If their choices are mutually consistent (identical), then the person without money produces z_{01}, the person with money produces z_{10}, and money holdings are exchanged if and only if $h = 1$; if their choices are different, then nothing happens and they go on to the next period.

We denote by s mutually consistent actions in the above game that are symmetric across agent types and stationary. In particular, we permit the actions to depend only on the kind of meeting (single-coincidence or no-coincidence) and on the current money holdings of the person and the person's trading partner. Those restrictions and the above specification of the game imply that s is a 19-dimensional vector that we can represent as follows: $s \equiv ((y_{ij}, c_{ij}, x_{ij}, z_{ij})_{i,j=0}^{1}, (d_{01}, d_{10}, h))$. Here y_{ij} is production by a producer in a single-coincidence meeting when the producer has i units of money, the first subscript, and the consumer has j units of money, while c_{ij} and x_{ij} are consumption and production, respectively, by the consumer in such a meeting; z_{ij} is production by a person in a no-coincidence meeting when the person has i units of money, the first subscript, and the trading partner has j units of money, the second subscript; d_{ij} is money-exchange action in single-coincidence meetings when the producer has i units of money and the consumer has j units, while h is the money-exchange action in no-coincidence meetings when the two agents have different amounts of money.

We next describe restrictions on s implied by the condition that a *discovered defection* from s results in complete and permanent autarky, which under our assumptions implies a utility of 0 in every period. However, since all previous actions are publicly revealed at each date only with probability ρ, we also have to describe the consequences to a person of being an *undiscovered defector,* a person who has defected in the past but whose defection has not been publicly revealed. To do that, we let v_j denote the expected discounted utility of beginning a period with j units of money for someone who has not defected (and has not seen a defection by a trading partner) and let v_j' be the same utility for an *undiscovered defector* (or for someone who has seen a defection by a trading partner), all under the assumption that there has been no public revelation of a defection. The v_j and the v_j' pertain to the beginning of a period, before the probabilistic updating of the record of past actions occurs.

This timing and our other assumptions imply that for $j = 0, 1$ and $i = 0, 1$ and $i \neq j$, v_j satisfies

$$
\begin{aligned}
(1) \qquad v_j = \beta v_j &+ m_j[u(c_{jj}) - x_{jj}] + m_j[-y_{jj}] + m_i[u(c_{ij}) - x_{ij} - \beta d_{ij}(v_j - v_i)] \\
&+ m_i[-y_{ji} - \beta d_{ji}(v_j - v_i)] + m_j(N-2)[-z_{jj}] + m_i(N-2)[-z_{ji} - \beta h(v_j - v_i)]
\end{aligned}
$$

where m_0 is the fraction of each type who start with no money divided by N and m_1 is the fraction of each type who start with one unit of money divided by N, so that $m_0 + m_1 = 1/N$. The right-side of (1) is the payoff, assuming no defection, for someone who begins with j units of money. There are four single-coincidence situations. With probability m_j, this person meets a *producer* with the same amount of money and realizes current utility $u(c_{jj}) - x_{jj}$. Also with probability m_j, this person meets a *consumer* with the same amount of money and realizes current utility $-y_{jj}$. In those meetings, exchanges of money holdings do not matter. With probability m_i, this person meets a *producer* with $i \neq j$ units of money and realizes current utility $u(c_{ij}) - x_{ij}$. Also with probability m_i, this person meets a *consumer* with $i \neq j$ units of money and realizes current utility $-y_{ji}$. In these latter meetings, exchanges of money holdings matter. In a meeting with a *producer* with i units of money, money holdings are exchanged if $d_{ij} = 1$; in a meeting with a consumer with i units of money, money holdings are exchanged if $d_{ji} = 1$. If money is exchanged, then the person begins the next period with i units of money rather than j units. There are two no-coincidence situations. With probability $m_j(N-2)$, this person is in a no-coincidence meeting with someone with the same amount of money and realizes current utility $-z_{jj}$. In this meeting, an exchange of money holdings does not matter. With probability $m_i(N-2)$, this person is in a no-coincidence meeting with someone with $i \neq j$ units of money and realizes current utility $-z_{ji}$. In such a meeting, an exchange of money holdings occurs if $h = 1$.

If a person begins a period having defected in the past, then with probability ρ, the deviation is discovered which leads to autarky and a zero expected discounted utility, while with probability $1 - \rho$, the deviation is not discovered that period. The defector's expected discounted utility v_j' is, therefore, the product of $1 - \rho$ and the payoff of being an undiscovered defector. For $j = 0, 1$ and $i = 0, 1$ and $i \neq j$, that payoff, given no defection by anyone else, is the right-side of equation (2).

$$
\begin{aligned}
(2) \qquad v_j'/(1-\rho) = {} & \beta v_j' + m_j \max\{[u(c_{jj}) - x_{jj}],0\} + m_j \max\{[-y_{jj}],0\} \\
& + m_i \max\{[u(c_{ij}) - x_{ij} - \beta d_{ij}(v_j'-v_i')],0\} + m_i \max\{[-y_{ji} - \beta d_{ji}(v_j'-v_i')],0\} \\
& + m_j(N-2)\max\{[-z_{jj}],0\} + m_i(N-2)\max\{[-z_{ji} - \beta h(v_j'-v_i')],0\}
\end{aligned}
$$

The right-side of (2), aside from having defection expected discounted values appear instead of equilibrium values, differs from the right-side of (1) in only one respect: a defector can always choose to defect again. Those choices are reflected by letting the defector choose between getting the nondefector payoff and not doing anything.

In terms of these expected discounted utilities and the other notation we have introduced, the constraints that express the requirement that there is no incentive to defect from an arbitrary s are as follows. For $j = 0, 1$ and $i = 0, 1$ and $i \neq j$,

(3) $\quad -y_{ii} + \beta v_i \geq \beta v_i'$

(4) $\quad -y_{ij} + \beta[d_{ij}v_j + (1-d_{ij})v_i] \geq \beta v_i'$

(5) $\quad u(y_{ij}) - x_{ij} + \beta[d_{ij}v_i + (1-d_{ij})v_j] \geq \beta v_j'$

(6) $\quad u(y_{ii}) - x_{jj} + \beta v_i \geq \beta v_i'$

(7) $\quad -z_{ii} + \beta v_i \geq \beta v_i'$

(8) $\quad -z_{ji} + \beta[hv_i + (1-h)v_j] \geq \beta v_j'$.

In each case, the left-side of the inequality is the payoff from not defecting and the right-side is the payoff from defecting. Constraints (3) and (4) pertain to meetings in which the person is the *producer* in a single coincidence meeting, while (5) and (6) pertain to those in which the person is the consumer in such a meeting. Constraints (7) and (8) pertain to no-coincidence meetings; (7) to such meetings when the two people both have i units of money, (8) when they have different amounts of money.

We now show that the set of s's consistent with (1)–(8) is the set of incentive feasible allocations in definition 2.

PROPOSITION 1. A necessary and sufficient condition for an allocation s to be incentive feasible (satisfy definition 2 for some A and f consistent with SIR) is that $s \equiv ((y_{ij}, c_{ij}, x_{ij}, z_{ij})^1_{i,j=0}, (d_{01}, d_{10}, h))$ satisfies (1)–(8) and $c_{ij} \leq y_{ij}$.

Proof. *Sufficiency* (if s satisfies (1)–(8), then it satisfies definition 2). Obviously, the coordination mechanism has an action set A and an outcome function f that satisfy SIR. In particular, a person who names zero own-production and no money-exchange in the coordination mechanism always assures zero own-production and end-of-period money holdings equal to initial holdings. To complete the argument, we specify (a_1,a_2,a_3) so that definition 1 is satisfied and so that s is implied by everyone playing a_1. Let a_1 be the play corresponding to s, let a_2 be autarky (produce zero and keep one's money), and let a_3 be the play in each meeting that maximizes the right-side of (2). Then, by (1)–(8), this (a_1,a_2,a_3) satisfies definition 1. In particular, responding to the public revelation of defection by playing autarky is optimal, because if everyone else responds that way, it is weakly optimal for each person to respond that way. In fact, zero production is uniquely optimal.

Necessity (if s is incentive feasible (satisfies definition 2), then it satisfies (1)–(8)). Let A and f satisfy SIR, let (a_1,a_2,a_3) be an SSE for A and f, and let $s \equiv f(a_1,a_1,.)$, so that s is an incentive feasible allocation. If s does not satisfy (1)–(8), then by selectively choosing when to play a_1 and when to play the element of A consistent with doing nothing, a person can do better by some defection from a_1 given that the agent's information is consistent with no past defection and given that all other agents play a_1 currently and in the future. That, of course, contradicts the assumption that (a_1,a_2,a_3) is an SSE for A and f and, therefore, that s is incentive feasible. □

Two comments are in order about the characterization of incentive feasibility provided by proposition 1. First, if we were only interested in complete, as opposed to partial, revelation of past actions, then we would find that s is incentive feasible if the threat of *immediate* and permanent autarky is no better for any person than carrying out s. We need a richer description of what happens off-equilibrium because we have to describe what happens to an *undiscovered* defector.

Proposition 1 shows that the coordination mechanism with reversion to complete autarky in the case of public revelation of defection gives us that description. Second, as just noted, we have assumed in (1)–(8) that the society reverts to complete autarky after public revelation of defection, rather than autarky only for the discovered defector. Proposition 1 also applies if it is assumed that autarky is imposed only on discovered defectors. Under that interpretation, (1)–(8) would continue to characterize all incentive feasible allocations, but v_i would apply to all nondefectors whether or not they had witnessed a defection by a trading partner and v_i' would apply only to defectors. The only advantage of reversion to complete autarky rather than punishment only of defectors is that reversion to complete autarky is robust to deviations by pairs in meetings; if others will play autarkically in the future, then there is no beneficial departure for two people in a meeting even if they cooperate. The same cannot so obviously be said about punishment of defectors, because a failure to punish a defector in a meeting is discovered only with some probability.

From now on, we use the characterization of incentive feasibility given by proposition 1. In particular, an incentive feasible allocation is an s that satisfies (1)–(8)—an s such that there is $v = (v_0, v_1)$ and $v' = (v_0', v_1')$ that satisfy (1)–(8) and $c_{ij} \leq y_{ij}$.

1.3 The Optimum Problem

We interpret the optimum problem in the following way. Immediately prior to period 0, before people are randomly and symmetrically assigned their money holdings, everyone is together for the last time. Whether or not types are preassigned, everyone at that time is identical. We, therefore, assume that everyone is treated the same. That is, we assume that the objective is a weighted average of the expected discounted utilities of those who have money and those who do not, with weights given by the proportions holding and not holding money. Thus, the optimum problem is to choose s from the set of incentive feasible allocations as characterized in proposition 1 to maximize $w \equiv N(m_0v_0+m_1v_1)$, where Nm_0 is the probability of starting without money and Nm_1 is that of starting with money.

It follows from multiplying (1) by m_j and summing over j that

(9) $$w = [N/(1-\beta)]\{\textstyle\sum_{i=0}^{1}\sum_{j=0}^{1}m_im_j[u(c_{ij}) - x_{ij} - y_{ij} - (N-2)z_{ij}]\}.$$

Thus, the objective depends on production and consumption and not on the money transfers variables, (d_{01},d_{10},h).

It is immediate from (9) that the unconstrained optimum, the optimum if (3)-(8) are not binding, is $c_{ij} \equiv y_{ij} \equiv y^*$ where y^* is the unique solution to $u'(y^*) = 1$ (see Figure 1) and $x_{ij} \equiv z_{ij} \equiv 0$. It also follows that the objective and (1)-(8) are symmetric in the money holdings in the following sense. If s is a solution for $(m_0,m_1) = (a,b)$, then s with each 0 subscript replaced by 1 and each 1 subscript replaced by 0 is a solution for $(m_0,m_1) = (b,a)$. This symmetry reflects the fact that holding or not holding money in this model is a binary label and nothing more. Therefore, if we impose, as we do, the normalization $v_1 \geq v_0$, and characterize the optimum under that restriction for each (m_0,m_1), then we are not missing any optima. We should not, however, use the results under that normalization to study how the optimum varies with (m_0,m_1), because the overall optimum when $(m_0,m_1) = (a,b)$ is the larger of the optima under our restriction for $(m_0,m_1) = (a,b)$ and that for $(m_0,m_1) = (b,a)$.

2. Results

We first present some general results about the optimum and then a further characterization for the case in which the discount factor, β, is sufficiently large. We begin with three technical results that provide the ingredients for the other results. The first shows that any optimum is in a compact set. It allows us to invoke the Theorem of the Maximum.

Lemma 1. There exists $y_{\max} > 0$ such that a necessary condition for s to be optimal is $s \in S \equiv [0, y_{\max}]^{16} \times \{0,1\}^3$, a compact subset of R^{19}.

Proof. What has to be proved is that a necessary condition for s to be optimal is $\max(y_{ij}, x_{ij}, z_{ij}) \leq y_{\max}$. Let $m_- = \min(m_0, m_1)$ and $m_+ = \max(m_0, m_1)$. Let $y_{\max}$ be the unique solution to $(m_-)^2[u(y_{\max}) - y_{\max}] + [(m_+)^2 + 2m_0m_1][u(y^*) - y^*] = 0$, where the left-hand side is the value of w when all outputs and consumption are set equal to their unconstrained best magnitudes except for the y_{ij} and c_{ij} which has the smallest weight in (9); those are set equal to $y_{\max}$. Evidently $y_{\max}$ is positive (see Figure 1) and unique. Whenever $\max(y_{ij}, x_{ij}, z_{ij}) > y_{\max}$ for some i and j, it follows from (9) that $w < 0$. Because an s with $c_{ij} \equiv y_{ij} \equiv x_{ij} \equiv z_{ij} \equiv 0$ is incentive feasible (by (1) and (2), such an s implies $v_j \equiv v_{j'} \equiv 0$ and therefore satisfies (3)–(8)) and implies $w = 0$, a necessary condition for a solution to the optimum problem is $w \geq 0$ and, therefore, $\max(y_{ij}, x_{ij}, z_{ij}) \leq y_{\max}$ for all i and j. □

The next result describes properties of the constraint set for a given s. Its proof appears in the Appendix. The main ingredient in the proof of lemma 2 is the fact that the right-side of (1) and the right-side of (2) are contractions with modulus β, the discount factor, when they are viewed as mappings from next-period expected discounted values to current period expected discounted values for a given s.

Lemma 2. Let $s \in S$ be given. (i) Equation (1) has a unique solution for $v \equiv (v_0, v_1)$ and equation (2) has a unique solution for $v' \equiv (v_0', v_1')$ and each is continuous in s. (ii) The solution to (2) for v' is continuous in ρ and weakly decreasing in ρ. Moreover, if each component of v' is positive, then each component of v' is strictly decreasing in ρ. (iii) If s is incentive feasible, then $v \geq v'$. Moreover, if s is incentive feasible and $\rho = 0$, then $v = v'$.

The next result is a consequence of the Theorem of the Maximum; its proof also appears in the Appendix.

Lemma 3. The maximized objective, denoted $w^*(\rho)$, is continuous in ρ and the set of optimal allocations, denoted $s^*(\rho)$, is upper-hemicontinuous in ρ.

Our first descriptive result compares two societies with different updating probabilities. It shows that the one with the higher probability is at least as well off. This is not surprising since it would seem as if the potential gain from defecting from any given s is lower the higher is ρ.

PROPOSITION 2. If $\rho_2 > \rho_1$, then $w^*(\rho_2) \geq w^*(\rho_1)$.

Proof. Because ρ does not appear in the objective, it is enough to show that the constraint set is weakly increasing in ρ. The parameter ρ appears in the constraint set only by way of v'. From the form of the constraint set, the constraint set is weakly increasing in ρ if v' is nonincreasing in ρ for a given s. That was established in part (ii) of lemma 2. □

We now present some further description of the optimum, but only for discount factors sufficiently close to one. Specifically, we assume that $\beta > \beta^*$, where β^* is the unique solution to

(10) $$y^*(1-\beta^*+\beta^* m_0)/(\beta^* m_0) = u(y^*).$$

Because $(1-\beta+\beta m_0)/(\beta m_0) \rightarrow \infty$ as $\beta \rightarrow 0$, is decreasing in β, and $(1-\beta+\beta m_0)/(\beta m_0) \rightarrow 1$ as $\beta \rightarrow 1$ and because $u(y^*)/y^* > 1$ (see Figure 1), it follows that β^* is unique and satisfies $0 < \beta^* < 1$. It also follows that $\beta > \beta^*$ implies $y^*(1-\beta+\beta m_0)/(\beta m_0) < u(y^*)$.

We say that a solution to the optimum problem is *entirely monetary* if production is positive if and only if the consumer has money and the producer does not ($y_{01} > 0$ and all other y's and all the x's and z's are zero); we say that a solution has *no role for money* if production does not depend on money holdings; and we say that a solution has *some role for money* if production does depend on money holdings.[4] The following result shows that the role of money in this sense depends on ρ.

PROPOSITION 3. If $\beta > \beta^*$, then the optimum is *entirely monetary* if and only if $\rho = 0$, has *no role for money* if $\rho = 1$, and has *some role for money* for ρ in a neighborhood of $\rho = 0$.

Proof. This proof has several parts. We begin by constructing, in turn, the optima at $\rho = 0$ and at $\rho = 1$. Then we complete the argument.

The optimum when $\rho = 0$. Given $v = v'$ (see part (iii) of lemma 2), $y_{00} = y_{11} = 0$ is immediate from constraints (3). Constraint (4) for $i = 1$ becomes $-y_{10} - \beta d_{10}(v_1 - v_0) \geq 0$. Because $v_1 \geq v_0$, it follows that $y_{10} = 0$. Conditional on $y_{00} = y_{10} = y_{11} = 0$, it follows by inspection of (9) that an upper bound on the objective is given by $y_{01} = c_{01} = y^*$ and $x_{ij} \equiv z_{ij} \equiv 0$. We now show that this constrained upper bound is incentive feasible.

Obviously, (6) and (7) are satisfied. Also, (8) is satisfied provided $h = 0$, and (5) for $i = 1$ is satisfied provided $d_{10} = 0$. The remaining constraints, (4) and (5) for $i = 0$, evaluated at the above constrained upper bound are $y^* \leq \Delta$ and $u(y^*) \geq \Delta$, where $\Delta \equiv d_{01}\beta(v_1 - v_0)$. By (1), Δ at the above constrained upper bound is given by,

$$(11) \qquad \Delta = d_{01}[m_0 u(y^*) + m_1 y^*]\beta/(1-\beta+\beta/N) < u(y^*)$$

where the inequality follows from $u(y^*) > y^*$ and $m_0 + m_1 = 1/N$. Therefore, constraint (5) for $i = 0$ is not binding. Using the equality part of (11), constraint (4) for $i = 0$ and $d_{01} = 1$ is

$$(12) \qquad y^*(1-\beta+\beta m_0)/(\beta m_0) \leq u(y^*).$$

Because $\beta > \beta^*$, (12) is satisfied. Therefore, the above constrained optimum is incentive feasible. It also follows from (11) that $d_{01} = 1$ is the unique optimum, because (4) for $i = 0$ would otherwise imply $y_{01} = 0$. Also, it follows from $v_1 - v_0 > 0$ that $d_{10} = h = 0$ are unique. Finally, because (12) holds with strict inequality, it follows that constraint (4) for $i = 0$ holds with strict inequality. By (1), that, in turn, implies that $v_0 > 0$. Because $v = v'$, we, therefore, have the conclusion, to be used later, that both components of v' are positive.

The optimum when $\rho = 1$. In this case, $v_0' = v_1' = 0$. That and our assumption about β imply, as we now show, that the unconstrained optimum is incentive feasible. First, notice that at the unconstrained optimum, $v_0 = v_1 = [u(y^*) - y^*]/[(1-\beta)N] \equiv v^*$. It follows that constraints (5)–(8) are not binding at the unconstrained optimum. It also follows that constraints (3) and (4) at $y_{ij} \equiv y^*$ reduce to $y^* \leq \beta v^*$, which is equivalent to $y^*(1-\beta+\beta/N)/(\beta/N) \leq u(y^*)$. Because $m_0 < 1/N$, it follows that $(1-\beta+\beta/N)/(\beta/N) < (1-\beta+\beta m_0)/(\beta m_0)$. Therefore, $\beta > \beta^*$ implies that the unconstrained optimum is incentive feasible.

The rest of the argument. We have shown that the optima at $\rho = 0$ and at $\rho = 1$ are unique (in the outputs) and that the former is *entirely monetary* and that the latter has *no role for money*. The conclusion that there is some role for money for ρ in the neighborhood of $\rho = 0$ is immediate from the characterization of the unique $\rho = 0$ optimum and upper-hemicontinuity of the optimum. It remains to establish that the optimum is not *entirely monetary* when $\rho > 0$. Because both components of v' are positive at $\rho = 0$, it follows from part (ii) of lemma 2 that no constraint is binding at $s = s^*(0)$ and $\rho > 0$. Because $s^*(0)$ is not the unconstrained optimum, it follows that the objective is strictly increasing in ρ at $\rho = 0$. Therefore, by proposition 2, $w^*(\rho) > w^*(0)$ for all $\rho > 0$. Because $s^*(0)$ is the best possible *entirely monetary* solution, it follows that if $\rho > 0$, then no solution is *entirely monetary*. □

An implication of the argument in the proof is that if $\beta > \beta^*$, then the optimum has *no role for money* when ρ is sufficiently near 1, because, by continuity, the unconstrained optimum is incentive feasible in a neighborhood of $\rho = 1$. Thus, for $\beta > \beta^*$, a society with sufficiently frequent updating of the record of actions has *no role for money.*

Although our proof of proposition 3 relies on the assumption that $\beta > \beta^*$, we suspect that the conclusions hold for all β. If $\rho = 0$, then it follows from the argument in the above proof that the optimum is unique and purely monetary. However, if $\beta < \beta^*$, then the optimal y_{01} is less than y^*. Although our simple argument showing that $\rho > 0$ implies some nonmonetary production does not apply in that case, we think the conclusion is true because $\rho > 0$ permits some detection of defection and $u'(0) = \infty$ makes some production in all single-coincidence meetings very valuable. We also suspect, but have not been able to show, that if $\rho = 1$, then the optimum has $y_{ij} \equiv y$ for some y even if β is too low to permit the common value of y to be as high as y^*.

3. Conclusion

We have described and studied an environment in which trading is difficult because of absence-of-double-coincidence, absence of commitment, and incomplete knowledge of past actions of others. The environment has two imperfect ways of keeping track of past actions of others, (outside) money and a record of the actions of others that is updated with an average lag. In general, this setting makes it desirable to use both (outside) money and some form of credit to accomplish trade.

Although we have used the term *credit* as a label for nonmonetary trade, other terms such as insurance or mutual charity (gift-giving) could as well have been used. Because the precise form taken by the nonmonetary trade is determined by the details of our specification, not much significance should be attached to its precise form. Our framework determines allocations that are optimal in the set that we have defined to be incentive feasible. That set, and, in general, the optimum, depends on the environment and on the stationarity that we have imposed. For example, the nonmonetary trade would be quite different if the environment included private information about tastes or technologies. Also, we have not shown that the limitation to stationary allocations in our sense is innocuous. Therefore, it is conceivable that the inclusion of nonstationary allocations would affect the results concerning the optimum.[5]

Our formulation and results would be of limited interest if they could not be extended to environments somewhat different from the one we have chosen to study. We have chosen one particular way to model incomplete public knowledge of individual trading histories. The virtue of our way is its simplicity. Almost identical is a specification in which knowledge of what happened T periods ago and earlier becomes public at each date. For such a specification, defection actions would have to be described for T distinct periods.

We have assumed that money is indivisible and that there is a unit upper bound on individual holdings of money. It is obvious that monetary trade is limited by that assumption relative to what would be possible with more general individual holdings of money.[6] However, it is easy to see that trade that has any role for money cannot achieve the unconstrained optimum no matter what holdings of money are allowed. In a general sense, money has a role if expected discounted utility depends on money holdings. Since the unconstrained solution is independent of money holdings, it follows that money has no role when ρ and the discount factor are sufficiently near one.[7] In that sense, our conclusions about monetary trade are robust to allowing more general individual holdings of money.

We have also assumed that meetings are pairwise at random. One simple way to bridge the gap between that extreme assumption and the alternative in which everyone is together at each date is a specification in which K people chosen at random meet at each date, where K is an arbitrary finite integer. Although that specification permits some double coincidence trade, as do some other pairwise meeting specifications (see, for example, Shi (1995)), if $\rho = 0$, then it will remain true that trade will have to be quid pro quo. If, instead, $\rho = 1$, then for sufficiently high discount factors the analogue of our unconstrained solution will again be achievable. Therefore, it seems that the general thrust of our results is robust to at least some ways of generalizing the assumption about meetings.

Finally, we have assumed that the degree of public information about individual histories is exogenous. A richer model would make the nature of the public information something that can be chosen with more revealing systems, those with higher ρ, being more costly in terms of resources. In the context of such a model, it may be possible to study whether the choice regarding public information, the choice of ρ, can be decentralized efficiently. We hope that our model of the benefits of greater knowledge about individual trading histories is a useful starting point for such a richer model.

Footnotes

[1]It is the environment in Aiyagari, Wallace, and Wright (1996), which, in turn, is based on Shi (1995), Trejos and Wright (1995), and Kiyotaki and Wright (1989).

[2]The assumption that the disutility of production is equal to the amount produced is without loss of generality. For details, see Aiyagari, Wallace, and Wright (1996).

[3]The discussion in this section closely parallels that in Kocherlakota (1996).

[4]This usage of "no-role-for-money" is different from that in Kocherlakota (1996). He shows, without qualification, that money is *inessential* if $\rho = 1$. Here, because we limit allocations to be stationary in the way described above, money is *essential* to support allocations that are not constant across holdings of money.

[5]In a version with private information about preferences or technologies, our kind of stationarity is *not* innocuous.

[6]With more general holdings of money, an arbitrary initial distribution, even if symmetric, is not a steady state distribution. One way to maintain the restriction to stationary strategies would be to allow the initial distribution of money to be chosen, subject to a given total amount of money.

[7]If money is divisible and $\rho = 0$, then one suspects that any nonautarkic steady state is such that each person always has some money. In that case, there seems to be no straightforward way to define *entirely monetary*. The degree of moneyness would have to be defined by the degree to which expected discounted utility depends on holdings of money. As we have been suggesting, *no role for money* is straightforwardly defined as no dependence on money holdings.

Appendix

Lemma 2. Let $s \in S$ be given. (i) Equation (1) has a unique solution for $v \equiv (v_0, v_1)$ and equation (2) has a unique solution for $v' \equiv (v_0', v_1')$ and each is continuous in s. (ii) The solution to (2) for v' is continuous in ρ and weakly decreasing in ρ. Moreover, if each component of v' is positive, then each component of v' is strictly decreasing in ρ. (iii) If s is incentive feasible, then $v \geq v'$. Moreover, if s is incentive feasible and $\rho = 0$, then $v = v'$.

Proof of (i). Fix s and let T denote the right-side of (1) and T' denote the right-side of (2) multiplied by $(1-\rho)$. It follows that $T\colon R^2 \to R^2$ and similarly for T' and that a solution for v to (1) is a fixed point of T and that a solution for v' to (2) is a fixed point of T'. Each of T and T' satisfies Blackwell's sufficient conditions (montonicity and discounting) for a contraction with modulus β, the discount factor. Therefore, each has a unique fixed point. We next show that the fixed point is continuous in s. For this purpose, let $H(s,v)$ denote the right-side of (1) and let $H'(s,v)$ denote the right-side of (2) multiplied by $(1-\rho)$, where $v \in R^2$. It is immediate that H and H' are continuous in s. Suppose now that $\{s_n\} \to s$, that $v_n = H(s_n, v_n)$, and that $v = H(s,v)$. We want to show that $\{v_n\} \to v$. We have

$$\begin{aligned} \|v_n - v\| &= \|H(s_n,v_n) - H(s,v)\| \leq \|H(s_n,v_n) - H(s_n,v)\| + \|H(s_n,v) - H(s,v)\| \\ &\leq \beta\|v_n - v\| + \|H(s_n,v) - H(s,v)\| \end{aligned}$$

where the first inequality is the triangle inequality and the second follows from the fact that for any $s \in S$, H is a contraction with modulus β. Therefore,

$$\|v_n - v\| \leq \|H(s_n,v) - H(s,v)\| / (1-\beta)$$

which by continuity of H in s implies that $\{v_n\} \to v$ as $\{s_n\} \to s$. The same argument applies to H'. □

Proof of (ii). Continuity of v' in ρ. This follows from the same argument used to establish continuity in s. Formally, one-can regard the s vector in the above continuity argument as including the parameter ρ.

Montonicity of v' in ρ. Assume $\rho_2 > \rho_1$ and let $T'(\cdot;s,\rho_k) \equiv f_k(\cdot)$: $R^2 \rightarrow R^2$ denote the right-side of (2) multiplied by $(1-\rho_k)$ and let v^k denote the unique fixed point of f_k. Obviously, $f_2(\cdot) \leq f_1(\cdot)$. Therefore, $f_2(v^1) \leq f_1(v^1) = v^1$. And because f_2 is increasing, $\{(f_2)^n(v^1)\}$ is a weakly decreasing sequence which converges to v^2. Therefore, $v^2 \leq v^1$. Notice that if $v^1 > 0$, then $f_2(v^1) < f^1(v^1)$ and $v^2 < v^1$, where strict inequality here means strictness for each component. □

Proof of (iii). By constraint (3), if s is incentive feasible, then $v \geq v'$. Now we show that if $\rho = 0$, then (1) and (2) imply $v \leq v'$, and therefore, $v = v'$. For a given s, let $T(\cdot;s) \equiv f(\cdot)$: $R^2 \rightarrow R^2$ denote the right-side of (1) and let $T'(\cdot;s) \equiv f'(\cdot)$: $R^2 \rightarrow R^2$ denote the right-side of (2) multiplied by $(1-\rho)$. Let v^* be the fixed point of f and v^{**} be that of f'. Because the right-side of (2) differs from the right-side of (1) only by the presence of the max functions, it follows that $f' \geq f$. Therefore, $f'(v^*) \geq v^*$; and because f' is weakly increasing, $\{(f')^n(v^*)\}$ is a weakly increasing sequence. Since $\{(f')^n(v^*)\}$ converges to v^{**}, it follows that $v^{**} \geq v^*$. Therefore, if s is incentive feasible and $\rho = 0$, then $v = v'$. □

Lemma 3. The maximized objective, denoted $w^*(\rho)$, is continuous in ρ and the set of optimal allocations, $s^*(\rho)$, is upper-hemicontinuous in ρ.

Proof. Because the objective given by (9) is continuous in s, in order to be able to invoke the Theorem of the Maximum it is sufficient to show that the constraint set is nonempty and compact. The constraint set is nonempty because $y_{ij} \equiv c_{ij} \equiv x_{ij} \equiv z_{ij} \equiv 0$ implies $v_j = v_j' = 0$, and, therefore, is feasible. The constraint set is compact because S is compact, (3)–(8) are defined by weak inequalities, and $\nu \equiv (\nu_0, \nu_1)$ and $\nu' \equiv (\nu_0', \nu_1')$ are continuous functions of s (see part (i) of lemma 2). Also, because ρ appears in the constraint set only by way of ν' and because ν' is continuous in ρ (see part (ii) of lemma 1), the constraint correspondence is continuous in ρ. It follows that the maximized objective is continuous in ρ and that the solution for s is upper-hemicontinuous in ρ. $\square$

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Figure 1
Utility of Consumption and Disutility of Production