Appendix A

The Johnson Model (Zero Proportional Transactions Costs)

Notation:

$[0,1]$ = time period

n = number of trips to the bond market in $[0,1]$

t_i = time of the ith trip to the bond market; $t_i \in [0,1]$,

$\quad i = 1,\ldots,n$.

$t_{n+1} \equiv 1$

$W(t)$ = wealth at time $t \in [0,1]$

$M(t)$ = money holdings at time $t \in [0,1]$

$B(t)$ = value of bond holdings at time $t \in [0,1]$

$P(t)$ = value of bonds purchased at time $t \in [0,1]$

$S(t)$ = value of bonds sold at time $t \in [0,1]$

r = rate of interest per period

c = consumption over the period

b = fixed cost per trip to the bond market

R = total interest return per period; $R=R(n)$

$I(t_i)$ = interest received at time t_i, $i = 1,\ldots,n$

Relations and restrictions:

(R.i)   All variables are non-negative

(R.ii) $n \geq 2$

(R.iii) $W(t) = M(t) + B(t)$; wealth is composed of money and bond

holdings. Interest becomes wealth only when it is paid out

as money.

(R.iv) $\bar{M} = \int_0^1 M(t)dt$

(R.v) $\bar{B} = \int_0^1 B(t)dt$

(R.vi) $\bar{W} = \int_0^1 W(t)dt$

(R.vii) $c = W(0) + R(n)$, consumption over the period equals initial wealth plus interest return.

(R.viii) $W(0) = M(0)$; initial wealth is in the form of money.

Axioms:

A.i) $M(t_i) \geq c \cdot (t_{i+1} - t_i)$ $i = 1, \ldots, n$; money on hand following a bond transaction must be sufficient to cover consumption expenditures until the next bond transaction.

A.ii) $M(t) = M(t_i) - c \cdot (t - t_i)$, where $i = \max \{j = 1, \ldots, m: t_j \leq t\}$ $(t \in [0,1])$.

A.iii) $R(n) = \sum_{i=1}^{n-1} B(t_i) \ r \cdot (t_{i+1} - t_i)$; simple interest definition

A.iv) $B(t_i) = \sum_{j=1}^{i} [P(t_j) - S(t_j) + I(t_j)]$, where

$I(t_j) \leq \min \{S(t_j), \sum_{k=1}^{j-1} [B(t_k) \cdot r \cdot (t_{k+1} - t_k) - I(t_k)]\}$ $j = 2, \ldots, n$

and $I(t_1) = 0$; interest payments are included in bond sales.

A.v) The individual's objective is to maximize $c - nb$ with respect to n, subject to $c = W(0) + R(n)$.

Comment: From Johnson's verbal description it would seem that (R.vii) should be $c = W(0) + R(n) - nb$ and that the objective (A.v) should be maximize c. However, from Johnson's equation (p.383) it appears that the individual spends at a rate of $W(0) + R$ per period and maximizes $W(0) + R(n) - nb$ with respect to n. This means that his interest return is applied to goods purchases, while transactions costs are paid at the end of the period from terminal wealth.

Optimality Conditions:

(C.i) $t_1 = 0$, $P(t_1) > 0$, $P(t_i) = 0$ $i = 2,\ldots,n$. The purchase of

all bonds occurs at $t = 0$.

(C.ii) $M(t_i) = c \cdot (t_{i+1}-t_i)$ $i = 1,\ldots,n$. Money on hand following a

bond transaction is the minimum needed to cover consumption

expenditures until the next bond transaction.

(C.iii) $I(t_j) = \min \{S(t_j); \sum_{k=1}^{j-1} [B(t_k) \cdot r(t_{k+1}-t_k) - I(t_k)]\}$ $j = 2,\ldots,n$;

when the individual converts bonds to money, he takes his interest

first and principal second.

Note: The first two conditions are essentially the same as Tobin's (p. 244).
The reason the three conditions are optimal is because with given transac-
tions costs, the greatest interest return is earned by keeping the largest
possible principal in bonds (easily proven): the implication of the three
conditions. The model simplifies as follows:

$$P(0) = M(0) - [M(0) + R(n)]t_2$$

$$S(t_i) = [M(0) + R(n)](t_{i+1}-t_i)   i = 2,\ldots,n$$

$$I(t_i) = r \cdot B_{t_{i-1}} \cdot (t_i-t_{i-1})   i = 2,\ldots,n$$

$$B(t_1) = M(0) - [M(0) + R(n)]t_2$$

$$B(t_2) = M(0) - [M(0) + R(n)]t_3 + I(t_2)$$

$$B(t_{n-1}) = M(0) - [M(0) + R(n)]t_n + \sum_{i=2}^{n-1} I(t_i)$$

$$B(t_n) = 0$$

$$R(n) = \sum_{i=2}^{n} I(t_i)$$

Johnson seems to be assuming two conditions which hold for Tobin's
model but are not consistent with his own specifications:

Unjustified assumptions:

1. $t_{i+1} - t_i = 1/n$ $i = 1,\ldots,n$. (Tobin's theorem on equal spacing of bond transactions.) It is this assumption which allows Johnson to write:

$\bar{M} = \dfrac{M(0) + R(n)}{2n}$. This is not the definition of $\bar{M}$ when

there is not equal spacing of bond transactions, and under Johnson's rules, equal spacing is not optimal. E.g.,

let $n = 2$. Then $M(0) + R(2) - 2b = M(0)\dfrac{(1+rt_2)}{1+rt_2^2} - 2b$

Maximizing with respect to t_2 yields

$\hat{t}_2 = \dfrac{\sqrt{1+r} - 1}{r} \neq 1/2$, and

$\bar{M} = [M(0) + R(2)] \left[1/2 - \dfrac{2(1+r) - (2+r)\sqrt{1+r}}{r^2} \right]$

$\neq \dfrac{M(0) + R(2)}{4}$.

2. $\bar{W} = \dfrac{M(0)}{2}$. Tobin writes:

$W(t) = M(0) - M(0)t$; or wealth at any point in time is equal to the wealth with which the individual started minus what he spent. Hence $\bar{W} = \dfrac{M(0)}{2}$ in Tobin's model. It is true in either model that:

$R(n) = r\bar{B}(n) = r[\bar{W}(n) - \bar{M}(n)]$. However, it is not true in Johnson's formulation that: $R(n) = r\left[\dfrac{M(0)}{2} - \bar{M}(n)\right]$, the reason being wealth at any point in time in his model equals what the individual started with minus what he

spent <u>plus</u> interest received.

Suppose, as Johnson does, that trips to the bond market are equally spaced in time. Then,

$$R(3) = M(0) \left[\frac{2r^2 + 9r}{r^2 + 9r + 27} \right]; \text{ but } r\left[\frac{M(0)}{2} - \bar{M} \right] =$$

$$r\left[\frac{M(0)}{2} - \frac{M(0) + R(3)}{6} \right] = M(0) \left[\frac{(3/2)r^2 + 9r}{r^2 + 9r + 27} \right]$$

$$\neq R(3).$$

These two assumptions make it possible for Johnson to write the objective function:

$$M(0) + R(n) - nb' = \left\{ M(0) + r\left[\frac{M(0)}{2} - \bar{M} \right] \right\} \left(1 - \frac{b}{2\bar{M}} \right),$$

where $\bar{M} = \frac{M(0) + R(n)}{2n}$, $R(n) = r\left[\frac{M(0)}{2} - \bar{M} \right]$.