Notes on Continuous Time Prediction With an Abortive Application to Macaulay's Test of the Expectations Theory of the Term Structure

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Introduction

One of the first applications of the rational expectations theory of the term structure of interest rates was Frederick Macaulay's study of rates on call loans and three-month time loans during the period 1890-1914. Since the time loan rate at a given moment could be viewed as an average of call loan rates expected over the next three months, Macaulay reasoned that time rates should lead call rates insofar as expectations are accurate. In particular, Macaulay asserted that rates on 90-day time loans should lead rates on call loans by 45 days. To see what underlies this assertion, let $R_m(t)$ be the rate on m-period time loans and r(t) be the call rate. Let $\hat{r}(t+\tau)$ be speculators' forecast of $r(t+\tau)$ made at time t. According to the expectations theory of the term structure, we have

$$R_m(t) = \frac{1}{m}\int_0^m \hat{r}(t+\tau)d\tau \ .$$

Assume that call loans are governed by a deterministic stochastic process and hence are perfectly predictable from observations on their own past. In particular, assume that

$$r(t) = \cos wt$$

and

$$\hat{r}(t+\tau) = \cos w(t+\tau) \ .$$

Substituting into the term structure formula gives

$$R_m(t) = \frac{1}{m}\int_0^m \cos w(t+\tau)d\tau$$

$$= \frac{1}{wm} [\sin w(t+m) - \sin wt] \ ,$$

which with the aid of standard trigonometric formulas can be written[1/]

$$(0) \qquad R_m(t) = A \cos[wt + \frac{m}{2} w] \ .$$

Formula (0) gives the response of m-period time rates to a perfectly forecastable call rate of cos wt. The time rate is predicted to lead the call rate by $\frac{m}{2}$ units of time, just as Macaulay asserted.[2/]

In the period 1890-1914, money market rates were characterized by a pronounced seasonal. Macaulay studied the seasonal components of time rates and call rates, and found a lead of time rates over call rates, as predicted, though he claimed that the magnitude of the lead was only about two-thirds of that predicted by the theory.

While Macaulay's estimate of the theoretical lead of time rates over call rates is correct for a deterministic stochastic process (i.e., one perfectly predictable from its own past), most econometricians agree that economic variables should be modeled as processes containing indeterministic (imperfectly predictable) elements. For indeterministic processes Macaulay's theoretical estimate of the lead of the time rate over the call rate is not a correct one, though it does serve as an interesting limiting value for the phase shift at frequencies at which r(t) displays a very sizable buildup of spectral power. This paper calculates the theoretical lead of time rates over call rates under the assumption that the call rate is an indeterministic process. This provides a correct benchmark against which to judge Macaulay's empirical results.

In performing these calculations it is useful to exploit some results on the effects of aggregation over time. I have chosen to make the natural assumption that speculators had data on the call rate at a much finer interval over time than is given by the monthly data that were analyzed by Macaulay. I approximate this situation by assuming that speculators in effect had continuous observations on $r(t)$ over the past. By positing a suitable continuous time stochastic model for $r(t)$, it is possible to use the rational expectations theory of the term structure to deduce the relationship between the continuous time $R_m(t)$ and $r(t)$ processes. Once this relation (projection) has been deduced, a formula of Sims can be applied to obtain the corresponding discrete-time model. Once this discrete-time model is available, the theoretical lead (spectral phase) of the discrete-time rate over the discrete-time call rate can be calculated at each frequency.

Rational Expectations with r(t) an Exogenous, Continuous Time Process

Let $R_m(t)$ be the yield to maturity on an m-period bond and let $r(t)$ be the call loan rate or the instantaneous rate of interest (also sometimes called the force of interest). Let $P_t r(t+\tau)$ be the linear least squares forecast of $r(t+\tau)$ based on information available at time t. (I use P_t to denote "projection onto the space spanned by information available at time t.") Then the rational expectations theory of the term structure asserts

$$(1) \qquad R_m(t) = \frac{1}{m}\int_0^m P_t r(t+\tau)d\tau \ .$$

I initially assume that speculators have available a continuous record $r(s)$, $-\infty \leq s \leq t$ which they use to form the linear least squares forecast of $r(t+\tau)$ based on this record. However, economists like Macaulay do not have available for analysis the same continuous record. Instead, to economists observations on $R_m(t)$ and $r(t)$ at only discrete points in time are available. In particular, the economist only has observations on the series

$$R_{mt} = R_m(t\cdot I) \qquad t = 0, \pm 1, \pm 2, \ldots$$

$$r_t = r(t\cdot I) \qquad t = 0, \pm 1, \pm 2, \ldots$$

where I is the sampling interval, which in the work below will be one month. My purpose here is to characterize the relationship between the discrete, sampled (or monthly) data R_{mt} and r_t that will prevail where (1) is the correct continuous time model.

I assume that the call rate r(t) is governed by the linear stochastic differential equation

$$(2) \qquad (D^n + b_{n-1} D^{n-1} + \ldots + b_0)\, r(t) = \xi(t)$$

where D = d/dt denotes mean-square stochastic differentiation and where $\xi(t)$ is a continuous time white noise with intensity (scale) parameter σ^2. It is convenient to write equation (2) in the form

$$(3) \qquad (\alpha_1 + D)(\alpha_2 + D) \ldots (\alpha_n + D)\, r(t) = \xi(t)$$

where the α_j's are the negatives of the roots of the characteristic polynomial

$$x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0 = 0 .$$

For r(t) to be a stationary stochastic process, it is necessary that the real part of each α_j be positive. The solution of the stochastic differential equation (3) is a stochastic process r(t) that is (n-1) times mean square differentiable[3/] and whose spectral density is given by[4/]

$$(4) \qquad Sr(w) = \frac{\sigma^2}{\prod_{j=1}^{n} (\alpha_j^2 + w^2)} \; .$$

Where (3) is the stochastic process governing $r(t)$, we seek the linear least squares predictor of $r(t+\tau)$ based on current and past $r(t)$'s. To motivate our expression for this predictor, write (3) for $(t+\tau)$:

$$(5) \qquad (\alpha_1 + D)(\alpha_2 + D) \; \ldots \; (\alpha_n + D) r(t+\tau) = \xi(t+\tau) \; .$$

Now since $\xi(t)$ is a white noise process, it obeys $0 = P_t[\xi(t+\tau)]$ $(\equiv P[\xi(t+\tau) | r(s), s \leq t])$. Noting that projection is a linear operator and writing $\hat{r}(t+\tau) \equiv P_t r(t+\tau)$ then permits us to write

$$(6) \qquad (\alpha_1 + D)(\alpha_2 + D) \; \ldots \; (\alpha_n + D) \hat{r}(t+\tau) = 0 \; ,$$

which is a deterministic differential equation in $\hat{r}(t+\tau)$ that we solve subject to the natural boundary conditions supplied by the record $r(s)$, $s \leq t$. In particular, the boundary conditions are

$$(7) \qquad \begin{aligned} \hat{r}(t) &= r(t) \\ D\hat{r}(t) &= Dr(t) \\ D\hat{r}(t) &= D^{n-1} r(t) \; . \end{aligned}$$

By the (n-1) times mean square differentiability of the process, the random variables $r(t)$, $Dr(t)$, ..., $D^{n-1}r(t)$ are known from the record $r(s)$, $s \leq t$. The solution of the deterministic boundary value problem (6), (7), is (see Whittle [ ])

$$(8) \qquad \hat{r}(t+\tau) = \left[\sum_{j=1}^{n} {}_j \; e^{-\alpha_j \tau} \prod_{k \neq j} \left(\frac{\alpha_k + D}{\alpha_k - \alpha_j} \right) \right] r(t) \; ,$$

which gives the linear least squares projection of $r(t+\tau)$ on $r(s)$, $s \leq t$ as a linear combination of values of $r(t)$ and its first $(n-1)$ time derivatives evaluated at time t. Equation (13) is a version of a classic formula for continuous time prediction due to Wiener.

Performing the integration indicated in (1), we combine (8) and (1) to arrive at

$$(9) \qquad R_m(t) = \left\{ \frac{1}{m} \sum_{j=1}^{n} \frac{1}{\alpha_j} (1 - e^{-\alpha_j m}) \prod_{k \neq j} \left(\frac{\alpha_k + D}{\alpha_k - \alpha_j} \right) \right\} r(t)$$

which can be written compactly as

$$(10) \qquad R_m(t) = \left(\sum_{k=0}^{n-1} h_k D^k \right) r(t)$$

where the h_k's are determined by matching the coefficients on powers of D^k in (9) and (10). The frequency response function of R_m to r is given by

$$(11) \qquad \begin{aligned} h(w) &= \sum_{k=0}^{n-1} h_k (i\ w)^k \\ &= \frac{1}{m} \sum_{j=1}^{n} \frac{1}{\alpha_j} (1 - e^{-\alpha_j m}) \prod_{k \neq j} \left(\frac{\alpha_k + iw}{\alpha_k - \alpha_j} \right) . \end{aligned}$$

The spectrum of $R_m(t)$ is linked to the spectrum of $r(t)$ by

$$S_R(w) = |h(w)|^2 \ s_r(w) \ .$$

The discrete data studied by Macaulay were monthly averages. Such data can be thought of as being formed by the two-step procedure of first taking a moving average of the original continuous time data, and then second sampling the resulting continuous time monthly average series once a month. The continuous time moving average processes are defined by

$$R_m^a(t) = \int_{-1/2}^{1/2} R_m(t+\tau)d\tau$$

$$= \int_{-1/2}^{1/2} b_{1/2}(\tau)\ R_m(t+\tau)d\tau$$

$$r^a(t) = \int_{-1/2}^{1/2} r(t+\tau)d\tau$$

$$= \int_{-1/2}^{1/2} b_{1/2}(\tau)r(t+\tau)d\tau$$

where $b_{1/2}(\tau) = \begin{cases} 1 & 1/2 \leq \tau \leq 1/2 \\ 0 & |\tau| < 1/2 \end{cases}$.

Using (*) to denote convolution, we can write the above compactly as

$$R_m^a(t) = b_{1/2} * R_m(t)$$

$$r^a(t) = b_{1/2} * r(t) \ .$$

The spectrum of the continuous time moving average process $r^a(t)$ is given by

(12) $$s_r^a(w) = \left| \frac{\sin \frac{w}{2}}{\frac{w}{2}} \right|^2 s_r(w)$$

where $\dfrac{\sin \frac{w}{2}}{\frac{w}{2}} = \int_{-1/2}^{1/2} e^{-iw\tau}d\tau$

is the Fourier transform of the "unit box" $b_{1/2}(\tau)$.

The spectrum of the process that is formed by sampling $r^a(t)$ at unit intervals is given by "folding" $s_r^a(w)$:

(13) $$S_r^a(w) = F[s_r^a(w)]$$

$$= \sum_{k=-\infty}^{\infty} s_r^a(w+2\pi k)$$

where F[ ] is the folding operator defined by

(14) $$F[f(w)] = \sum_{k=-\infty}^{\infty} f_r^a(w+2\pi k) .$$

Taking moving averages on both sides of (10), we have that the continuous-time $R_m^a(t)$ is related to $r^a(t)$ by (10) with R_m and r replaced by unit averaged versions of themselves:

(15) $$R_m^a(t) = \left(\sum_{k=0}^{n-1} h_k D^k\right) r^a(t) .$$

Equation (15) describes the relationship between the continuous time process $R_m^a(t)$ and $r^a(t)$. By applying a formula due to Sims, we can use (15) to derive the implied model linking the discrete time, sampled processes R_{mt}^a and r_t^a. These sampled processes are defined by

$$R_{mt}^a = R_m^a(t) \qquad t = 0, \pm 1, \pm 2, \ldots$$
$$r_t^a = r^a(t) \qquad t = 0, \pm 1, \pm 2, \ldots .$$

Sims's formula implies that R_{mt}^a has a representation

$$R_{mt}^a = \sum_{k=-\infty}^{\infty} H_k r_{t-k}^a + \varepsilon_t$$

where $E\,\varepsilon_t r_{t-k}^a = 0$ for all t and k, and where

$$H_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(w)e^{+iwk}dw .$$

The frequency response function H(w) is given by Sims's formula

(16) $$H(w) = F\left[h(w)\,\frac{s_r^a(w)}{S_r^a(w)}\right] .$$

Writing H(w) in polar form

$$H(w) = G(w)e^{i\theta(w)}$$

gives us a formula for the phase lead $\theta(w)$ in radians, which is the counterpart for our indeterministic model of the phase shift calculated by Macaulay.

Estimation

Given observations on the discrete time, monthly averaged data, we propose to estimate the parameters $\alpha_1, \ldots, \alpha_n$ of the underlying continuous time process. The spectral density of the monthly average, discrete time data is related to these parameters by

$$(17) \qquad S_r^a(w) = \sigma^2 F \left[\left| \frac{\sin \frac{w}{2}}{\frac{w}{2}} \right|^2 \frac{1}{\prod_{j=1}^{n} (\alpha_j^2 + w^2)} \right] .$$

The following frequency-domain procedure is asymptotically equivalent to maximum likelihood estimation.5/ First, define the Fourier transform of r_t^a , t=1, ..., T as

$$r(w_j) = \sum_{t=1}^{T} r_t^a e^{-iw_j t}$$

for $w_j = \frac{2\pi j}{T}$, j = 0, 1, ..., T .

The periodogram $I(w_j)$ is defined as

$$I(w_j) = |r(w_j)|^2 \quad .$$

We choose $\alpha_1, \ldots, \alpha_n$ to minimize the criterion (see Hannan [ ])

$$\sum_{j=0}^{T-1} \frac{I(w_j)}{S_r^a(w_j)} \; ,$$

holding σ^2 fixed.

That is, with respect to $\alpha_1, \ldots, \alpha_n$ we minimize

$$\sum_{j=0}^{T-1} \frac{I(w_j)}{F\left[\left|\frac{\sin \frac{w}{2}}{\frac{w}{2}}\right|^2 \cdot \frac{1}{\prod_{k=1}^{n} (\alpha_k^2 + w_j^2)}\right]}$$

where $w_j = \frac{2\pi_j}{T}$, $j = 0, 1, \ldots, T$.

Sample Empirical Results

Figures 1 and 2 show graphs of the spectra of monthly average call rate and time rate over the period 1890-1913, calculated using a Parzen window and a maximum lag of 48. Figure 3 shows the phase statistic between the call rate and time rate (a negative phase means that the time rate leads the call rate). At the seasonal frequency w = .52 (the 12-month periodicity) the phase indicates that the time rate leads the call rate by only .13/.52 $\simeq$ 1/4 months, which is much smaller than the 1 1/2-month lead predicted by Macaulay's calculations. It is also less than the lead of about one month that Macaulay had actually estimated empirically. This last discrepancy could be explained either by the "window bias" in our Parzen estimator (which in effect averages across the periodogram, thereby including a bias) or else by some defect in Macaulay's procedure. I wouldn't venture to guess which at this point.

For the period 1890-1913, I estimated the parameters of a thirteen-order differential equation using the method of the previous section. (Actually only twelve parameters were estimated, a single real root of $\alpha_1 = .5108$ was imposed in order to prewhiten the series). In place of the periodogram ordinates that appear in the minimization criterion $\sum_{j=0}^{T-1} \frac{I(w_j)}{S_r(w_j)}$, we actually used the estimates of the spectral density reported above, this in order to economize on computations.

Experimentation with the alternative procedure that actually uses the periodogram ordinates showed that these two procedures gave very similar parameter estimates, as one would expect.

Our parameter estimates were as follows:

	Real Part	Imaginary Part
α_1 =	.5108000000	0.0000000000
α_2 =	.0000000238	0.3879150461
α_3 =	.0000000238	-0.3879150461
α_4 =	.0000000355	1.2048578216
. =	.0000000355	-1.2048578216
. =	.0000000203	1.9119125566
. =	.0000000203	-1.9119125566
. =	.0000000091	2.4637453220
. =	.0000000091	-2.4637453220
. =	.0000000001	3.1087259138
. =	.0000000001	-3.1087259138
. =	.0000000001	2.8673912625
α_{13} =	.0000000001	-2.8673912625

Figure 5 reports the gain and phase discrete time H(w) function implied by these estimates in conjunction with formula (16). Here a positive phase means that the time rate leads[6/] (excuse the break with the convention used in our earlier empirical results, an accident of the order in which the series were entered in a computer program). The phase is approximately linear with slope very nearly the value of 1.5 months that is predicted by Macaulay's deterministic calculations. Notice that neither the phase nor the gain resembles the configuration displayed by our empirical phase and gain diagrams.

Figure 6 graphs the discrete time H_k distributed lag coefficients. Notice that the lag distribution is two-sided, though most of the weight is placed on current and the first three past values. It is interesting that the phase statistic is approximately linear despite this not being a fixed delay system.

In all, these results indicate that the deterministic calculations of phase made by Macaulay seem to provide a quite good approximation to our continuous time indeterministic calculations, if not to the actual date. That this is so surprised me.

Rational Expectations with Endogenous Call Rate

The preceding calculations proceed on the assumption that the call rate $r(t)$ in continuous time is a process that is strictly econometrically exogenous with respect to the m period rate $R_m(t)$. That is, linear least squares forecasts of $r(t)$ based on current and past $r(s)$ and $R_m(s)$ are posited to depend only on the available record on $r(s)$, so that $R_m(s)$, $s \leq t$, is assumed not to aid in predicting r. Here that restrictive assumption is relaxed, as I take up the general case in which past observations on R_m do help predict r.

I assume that the continuous time vector process $(r(t), R_m(t))$ is generated by the stochastic differential equation system

$$DZ(t) = AZ(t) + Bu(t) \tag{18}$$

where

$$
Z(t) = \begin{bmatrix} r(t) \\ D_r(t) \\ \cdot \\ \cdot \\ \cdot \\ D^{n-1}r(t) \\ R_m(t) \\ DR_m(t) \\ \cdot \\ \cdot \\ \cdot \\ D^{n-1}R_m(t) \end{bmatrix}, u(t) = \begin{array}{l} \left[\begin{array}{c} u_1(t) \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ u_2(t) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \\ \\ \\ \leftarrow (n+1)^{st} \text{ row} \\ \\ \\ \\ \\ \\ \end{array}
$$

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
\cdot & & & & & & & & & & \\
\cdot & & & & & & & & & & \\
\cdot & & & & & & & & & & \\
0 & 0 & 0 & & 0 & 1 & & 0 & & & 0 \\
a_1 & a_2 & a_3 & & & a_n & a_{n+1} & & & & a_{2n} \\
0 & 0 & 0 & & & 0 & 0 & 1 & 0 & & \\
0 & 0 & 0 & & & 0 & 0 & 0 & 1 & & \\
\cdot & & & & & & & & & & \\
\cdot & & & & & & & & & & \\
0 & 0 & 0 & & & 0 & 0 & 0 & & 0 & 1 \\
c_1 & c_2 & c_3 & & & c_n & c_{n+1} & & & & c_{2n}
\end{bmatrix}
$$

$$
B = \begin{array}{l}
\left[\begin{array}{cccccccc}
0 & 0 & \cdot & \cdot & \cdot & \cdot & & 0 \\
\cdot & & & & & & & \\
\cdot & & & & & & & \\
\cdot & & & & & & & \\
b_{11} & \cdot & \cdot\,\cdot & 0 & b_{12} & 0 & \cdot\,\cdot & 0 \\
& & & & & & & \\
b_{21} & 0 & \cdot\,\cdot & 0 & b_{22} & 0 & \cdot\,\cdot & 0
\end{array}\right] \\
\begin{array}{cccccccc}
\uparrow & & & & \uparrow & & & \\
\text{1st column} & & & & (n+1)^{st} \text{ column} & & &
\end{array}
\end{array}
\begin{array}{l} \\ \\ \\ \\ \leftarrow n^{th} \text{ row} \\ \\ \leftarrow 2n^{th} \text{ row} \\ \\ \\ \end{array}
$$

Here $u_1(t)$ and $u_2(t)$ are mutually orthogonal white noises that are assumed to satisfy $P_{t-s}u_1(t) = P_{t-s}u_2(t) = 0$ for all $s > 0$. Writing out (18) we have the pair of stochastic differential equations

$$D^n r(t) = a_1 r(t) + a_2 Dr(t) + \ldots + a_n D^{n-1} r(t)$$
$$+ a_{n+1} R_m(t) + \ldots + a_{2n} D^{n-1} R_m(t) + b_{11}u_1(t) + b_{12}u_2(t)$$

$$D^n R_m(t) = c_1 r(t) + c_2 Dr(t) + \ldots + C_n D^{n-1} r(t)$$
$$= c_{n+1} R_m(t) + \ldots + c_{2n} D^{n-1} R_m(t) + b_{21}u_1(t) + b_{22}u_2(t).$$

The white noise vector $u(t)$ is assumed to have $2n \times 2n$ "intensity" matrix V.

From the vector $z(t)$ we can recover $r(t)$ and $R_m(t)$ according to

$$r(t) = \underset{(1x2n)}{f} \; z(t)$$

$$R_m(t) = \underset{(1x2n)}{d} \; z(t)$$

where $$f = (1 \quad 0 \quad . \; . \; . \quad 0)$$

$$d = (0 \quad 0 \; . \; . \; . \; 0 \; \underset{\substack{\uparrow \\ (n+1)^{st} \text{ column}}}{1} \; 0 \; . \; . \; . \;)$$

The solution of the vector stochastic differential equation (18) can be written

$$z(t+\tau) = e^{A\tau} z(t) + \int_t^{t+\tau} e^{A(t+\tau-s)} Bu(s)ds$$

where $e^A \equiv \sum_{n=0}^{\infty} \frac{A^n}{n!}$. Since $P_t \; u(t+s) = 0$ for $s > 0$, we have

$$(19) \qquad \hat{Z}(t+\tau) \equiv P_t z(t+\tau) = e^{A\tau} z(t) \; .$$

Thus, our formula for the linear least squares predictors of $R_m(t+\tau)$ and $r(t+\tau)$ are

$$\text{(20)} \qquad \begin{aligned} \hat{R}_m(t+\tau) &= d\, e^{A\tau} z(t) \\ \hat{r}(t+\tau) &= f\, e^{A\tau} z(t) \ . \end{aligned}$$

Let $A = P\Lambda P^{-1}$ where Λ is a diagonal matrix of the eigenvalues of A and the columns of P are the eigenvectors of A. Then (20) can be written

$$\begin{aligned} \hat{R}_m(t+\tau) &= d\, Pe^{\Lambda\tau} P^{-1} z(t) \\ \hat{r}(t+\tau) &= f\, Pe^{\Lambda\tau} P^{-1} z(t) \ . \end{aligned}$$

These formulae are extensions of (9) and express the optimal τ-ahead predictor as a linear combination of (mean-square) derivatives of $R_m(t)$ and of $r(t)$ of orders from zero to (n-1). For the univariate problem analyzed in (9), the eigenvalues in Λ would equal the roots of the characteristic polynomial, the α_j's, that appear in (9). Under the rational expectations theory of the term structure, we want

$$R_m(t) = \frac{1}{n} \int_0^n \hat{r}(t+\tau) d\tau$$

or

$$\text{(21)} \qquad R_m(t) = f \cdot [\frac{1}{m} \int_0^m e^{A\tau} d\tau] z(t) \ .$$

From (20) we have that for $s > 0$

$$\text{(22)} \qquad P_{t-s}\, R_m(t+\tau) = de^{A(\tau+s)} z(t-s) \ .$$

But from (21) we also have

$$\begin{aligned} P_{t-s} R_m(t+\tau) &= f[\frac{1}{m} \int_0^m e^{A\tau} d\tau] e^{As}\, z(t-s) \\ \text{(23)} \qquad &= f[\frac{1}{m} \int_0^m e^{A(\tau+s)} d\tau] z(t-s) \ . \end{aligned}$$

Comparing (22) and (23), we see that the rational expectations theory of the term structure imposes the following restrictions across rows of A:

$$f\,\frac{1}{n}\int_0^n e^{A(\tau+s)}d\tau = de^{As}\,, \qquad s > 0\,. \tag{24}$$

It is instructive to consider the following algorithm for synthesizing an A matrix that satisfies the restriction (24). Given a fixed n^{th} row of A, the algorithm calculates the $2n^{th}$ row that satisfies (24).

First, fill in the n^{th} row of A and set the $2n^{th}$ row equal to a row of zeroes. Call this initial A matrix A_0.

Second, form a matrix e^{A_1} by setting all of its rows except the $2n^{th}$ equal to the corresponding rows of e^{A_0}, and setting the $2n^{th}$ row according to

$$d\cdot e^{A_1} = f\left[\frac{1}{n}\int_0^n e^{A_0(\tau+1)}d\tau\right].$$

Once e^{A_1} is available, A_1 can be calculated as $A_1 = \log e^{A_1}$ where

$$\log e^A \equiv \sum_{n=1}^{\infty} \frac{1}{n}\,(I-e^A)^n\,. \tag{25}$$

Then iterate on the second step until the algorithm converges. Convergence is assured if the eigenvalues of A have negative real parts.

Actually, there is no need to calculate the sum in (25). Instead, assuming that the eigenvalues of e^A are distinct, we can write e^A as

$$e^A = P\,\Delta\,P^{-1}$$

where the columns of P are the eigenvectors of e^A and Δ is a diagonal matrix consisting of the eigenvalues of e^A. Then we have

$$\log e^{A} = P \log \Delta \; P^{-1}$$

where $\log \Delta$ is the diagonal matrix consisting of natural logarithms of the corresponding elements of Δ. This last equation follows from noting that

$$\log e^{A} = \sum_{n=1}^{\infty} \frac{1}{n} (I-e^{A})^{n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} (I-P\Delta P^{-1})^{n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} P(I-\Delta)^{n}P^{-1}$$

$$= P[\sum_{n=1}^{\infty} \frac{1}{n} (I-\Delta)^{n}]P^{-1}$$

$$= P \log \Delta \; P^{-1} .$$

Similarly, there is no need to calculate e^{A} via an infinite sum. Instead, write

$$A = P\Lambda P^{-1}$$

where Λ are the eigenvalues of A and the columns of P are the eigenvectors of A. Then

$$e^{A} = Pe^{\Lambda}P^{-1} .$$

This formula is useful in calculating the right-hand side of (21) as

$$f \, . \, P[\frac{1}{n} \int_{0}^{n} e^{\Lambda\tau} d\tau]P^{-1} .$$

In particular, to use our algorithm we need to calculate

$$\int_0^n e^{A(\tau+1)} d\tau = \int_0^n Pe^{\Lambda(\tau+1)} P^{-1} d\tau$$

$$= P\left\{\frac{1}{\lambda_i} [e^{\lambda i(n+1)} - e^{\lambda i}]\right\} P^{-1} .$$

The Discrete-Time Model When the Call Rate is Econometrically Endogenous

Since the solution of the stochastic differential equation (2) is

$$z(t+\tau) = e^{A\tau} z(t) + \int_t^{t+\tau} e^{A(t+\tau-s)} Bu(s)ds$$

where $E\ u(s) \cdot z(s-v) = 0$ for all $v > 0$, we have that

$$C_z(\tau) = Ez(t+\tau)z(t)' = e^{A\tau} Ez(t)z(t)'$$

or

(26) $$C_z(\tau) = e^{A\tau} C_z(0) ,$$

so that the covariogram matrix $C_x(\tau)$ obeys the systematic part of the matrix differential equation (18) with initial condition given by $C_z(0)$. It can be shown that $C_z(0)$ is the solution of the equation

(27) $$0 = AC_z(0) + C_z(0)A' + BVB'$$

where V is the "intensity" matrix of the vector white noise u(t). (See Kwakernaak and Sivan, p. 104.)

Writing out $C_z(\tau)$, we have that

$$C_z(\tau) = \begin{bmatrix} C_r(\tau) & C_{r,Dr}(\tau) & C_{r,D^2r}(\tau) & \cdots & C_{rR}(\tau) & C_{r,DR}(\tau) & \cdots & C_{-r}{}_{D^{n-1}R}(\tau) \\ C_{Dr,r}(\tau) & C_{Dr,Dr}(\tau) & & & & & & C_{Dr,D^{n-1}R}(\tau) \\ C_{D^2r,r}(\tau) & & & & & & & C_{D^2r,D^{n-1}R}(\tau) \\ \vdots & & & & & & & \\ C_{Rr}(\tau) & & & & & & & \\ C_{DR,r}(\tau) & & & & & & & \\ \vdots & & & & & & & \\ C_{D^{n-1}R,r}(\tau) & & & & & & & C_{D^{n-1}r,D^{n-1}R}(\tau) \end{bmatrix}$$

where $C_{v,y}(\tau) = E[v(t+\tau)y(t)]$, and $v(t)$ and $y(t)$ are two scalar stochastic processes.

The spectral density matrix of the vector z process is given by

(28) $$\sum_z (w) = (jwI-A)^{-1}\ BVB'(-jwI-A')^{-1}\ .$$

The spectral densities of the sampled moving average processes R_m^a and r_t^a and their cross-spectrum are derived by folding the corresponding elements of $\sum_z (w)$, after multiplying them by $|\sin(^w/2)/(^w/2)|^2$ to account for the moving average. In particular, we have

$$S_R(w) = F\left[\left|\frac{\sin\frac{w}{2}}{\frac{w}{2}}\right|^2 S_R(w)\right]$$

$$S_r(w) = F\left[\left|\frac{\sin\frac{w}{2}}{\frac{w}{2}}\right|^2 S_r(w)\right]$$

$$S_{Rr}(w) = F\left[\left|\frac{\sin\frac{w}{2}}{\frac{w}{2}}\right|^2 S_{Rr}(w)\right]$$

where $S_{Rr}(w)$ is the cross-spectrum of the moving average, sampled data and $S_{Rr}(w)$ is the cross-spectrum between the continuous processes. Writing $S_{Rr}(w)$ in polar form

$$S_{Rr}(w) = J(w)e^{i\theta(w)}$$

gives the phase lead $\theta(w)$ at each frequency. The parameter $\theta(w)$ is the counterpart for this stochastic model of the theoretical phase stochastic that Macaulay calculated for the deterministic model.

Estimation

Define the cross-spectral matrix

$$S(w) = \begin{bmatrix} S^a_r(w) & S^a_{rR}(w) \\ S^a_{Rr}(w) & S^a_R(w) \end{bmatrix}$$

whose elements obey (24) and (28); also define the periodogram matrix and

$$I(w_j) = \begin{bmatrix} |r(w_j)|^2 & r(w_j)R^*(w_j) \\ R(w_j)r^*(w_j) & |R(w_j)|^2 \end{bmatrix}$$

where $R(w_j) = \sum_{t=1}^{T} R^a_{mt} e^{iw_j t}$, $w_j = \frac{2\pi j}{T}$,

$j=0,1, \ldots, T$. Then estimates of the parameters of the continuous time model (18) under the restrictions (24) can be obtained by minimizing

$$\sum_{j=0}^{T-1} tr(I(w_j)S(w_j)^{-1}) ,$$

which leads to estimates that are asymptotically equivalent to maximum likelihood estimates (see Hannan).[7/]

Footnotes

1/ Write sin w(t+m) - sin wt =

$$\frac{1}{2i}\left[e^{iw(t+m)} - e^{-iw(t+m)} - e^{iwt} + e^{-iwt}\right]$$

$$= \frac{1}{2i}\left[(e^{iw(t+m)} + e^{-iwt}) - (e^{-iw(t+m)} + e^{iwt})\right]$$

$$= \frac{1}{2i}\left[e^{iw\frac{m}{2}}(e^{iw(t+\frac{m}{2})} + e^{-iw(t+\frac{m}{2})}) - e^{-iw\frac{m}{2}}(e^{-iw(t+\frac{m}{2})} + e^{iw(t+\frac{m}{2})})\right]$$

$$= \frac{1}{2i}\left[e^{iw\frac{m}{2}} - e^{-iw\frac{m}{2}}\right].2\cos[wt + \tfrac{m}{2}w]$$

$$= 2\sin w\,\frac{m}{2}\,.\,\cos[wt + \frac{m}{2}\,w]$$

$$= A\cos[wt + \frac{m}{2}\,w]$$

where $A = 2\sin w\,\frac{m}{2}$.

2/ To convert the phase lead of $w\,\frac{m}{2}$ radians to time units, we divide it by angular frequency w to get $\frac{m}{2}$. The term cos wt peaks at t=0, while the term $A\cos[wt + \frac{m}{2}\,w]$ peaks at t given by $wt + \frac{m}{2}\,w = 0$ or $t = \frac{m}{2}$. Thus, the lead of $\frac{m}{2}$ time units of cos wt over $A\cos(wt + \frac{m}{2}\,w)$.

3/ A stochastic process x(t) is said to be mean-square continuous if $\lim_{\varepsilon\to 0} E\{|x(t+\varepsilon) - x(t)|^2\} = 0$ for all t. A process is mean-square continuous if its covariogram is continuous. A stochastic process x'(t) is its mean-square derivative if

$$\lim_{\varepsilon\to 0} E\{[\frac{x(t+\varepsilon) - x(t)}{\varepsilon} - x'(t)]^2\} = 0$$

for all t. A sufficient condition for a stationary stochastic process x(t) to have a mean-square derivative is for its covariogram $c(\tau)$ to be twice differentiable. A stationary process is n-times mean-square differentiable if its covariogram is 2n times differentiable. See Papoulis [ ]. In terms of its spectral density $s_x(w)$, a sufficient condition for a process x to be n times mean-square differentiable is

$$\int_{-\infty}^{\infty} w^{2n} s_x(w) < \infty\ .$$

4/ If r(t) follows an n^{th} order linear stochastic differential equation, then the discrete-time sampled processing r_t follows a mixed n^{th} order autoregressive, $(n-1)^{st}$ order moving average process. To see this, use the method of partial fractions to write $s_r(w)$ as

$$S_r(w) = \sum_{j=1}^{n} \frac{c_j}{(\alpha_j^2 + w^2)} = \sum_{j=1}^{n} \frac{\frac{c_j}{2\alpha_j}\,2\alpha_j}{(\alpha_j^2 + w^2)}$$

where $c_k = \left.\dfrac{\sigma^2}{\prod\limits_{j\neq k} (\alpha_j^2 + w^2)}\right|_{w^2=-\alpha_k^2}$

The covariogram of r, being the inverse Fourier transform of $S_r(w)$, then obeys

$$C_r(\tau) = \sum_{j=1}^{n} \left(\frac{c_j}{2\alpha_j}\right) e^{-\alpha_j|\tau|} \quad ,\tau \text{ real.}$$

Sampling $c_r(\tau)$ at the integers gives

$$c_r(\tau) = \sum_{j=1}^{n} \frac{k_j}{1-\lambda_j^2} \lambda_j^{|\tau|} \quad , \quad = 0, \pm 1, \pm 2, \ldots$$

where we have set $\lambda_j = e^{-\alpha_j}$, $\dfrac{c_j}{2\alpha_j} = \dfrac{k_j}{1-\lambda_j^2}$.

Then the spectral density of the discrete time sampled process r_t is the Fourier transform of the sampled covariogram

$$S_r(z) = \sum_{j=1}^{n} \frac{k_j}{(1-\lambda_j z)(1-\lambda_j z^{-1})}$$

where $z = e^{-iw}$.

Putting the above over a common denominator gives

$$S_r(z) = \frac{\sum\limits_{j=1}^{n} k_j \prod\limits_{k\neq j} (1-\lambda_k z)(1-\lambda_k z^{-1})}{\prod\limits_{j=1}^{n} (1-\lambda_j z)(1-\lambda_j z^{-1})} \quad ,$$

which is a rational spectral density, one characteristic of an $(n-1)^{st}$ order moving average, n^{th} order autoregressive process.

5/ The α's are only locally identifiable (see Phillips [ ]). In order to achieve global identification, I have imposed the condition that the imaginary parts of the α's are bounded by $\pm \Pi$. This condition is sufficient to achieve identification and has the effect of asserting that peaks in the spectrum on $[-\Pi, \Pi]$ aren't aliases of peaks at higher frequencies.

6/ To indicate what our conventions imply about the sign of phase, let $y_t = \int h(\tau)x(t-\tau)d\tau$ and set $x(t) = 2 \cos wt = (e^{iwt}+e^{-iwt})$. We then have

$$\begin{aligned} y_t &= \int h(\tau)[e^{iw(t-\tau)} + \int e^{-iw(t-\tau)}]d\tau \\ &= e^{iwt} \int h(\tau)e^{-iw\tau}d\tau + e^{-iwt} \int h(\tau)e^{+iw\tau}d\tau \\ &= e^{iwt}h(w) + e^{-iwt}h(-w) \end{aligned}$$

where $h(w) = \int h(\tau)e^{-iw\tau}$. Write $h(w) = |h(w)|e^{i\theta(w)}$, then

$$\begin{aligned} y_t &= e^{iwt}|h(w)|e^{i\theta(w)} + e^{-iwt}|h(w)|e^{-i\theta(w)} \\ &= |h(w)|(e^{i(wt+\theta(w))} + e^{-i(wt+\theta(w))}) \\ &= |h(w)|\cos(wt + \theta(w)). \end{aligned}$$

The variable y_t peaks at $wt + \theta(w) = 0$ or $t = -\theta(w)/w$. A positive phase statistic thus indicates that the output y_t leads the input x_t.

7/ Again, we are imposing the condition that the imaginary parts of the eigenvalues of A are between $-\Pi$ and $+\Pi$ in order to achieve global identification.

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Figure 1 Plot of Estimated Spectrum of Call Rate

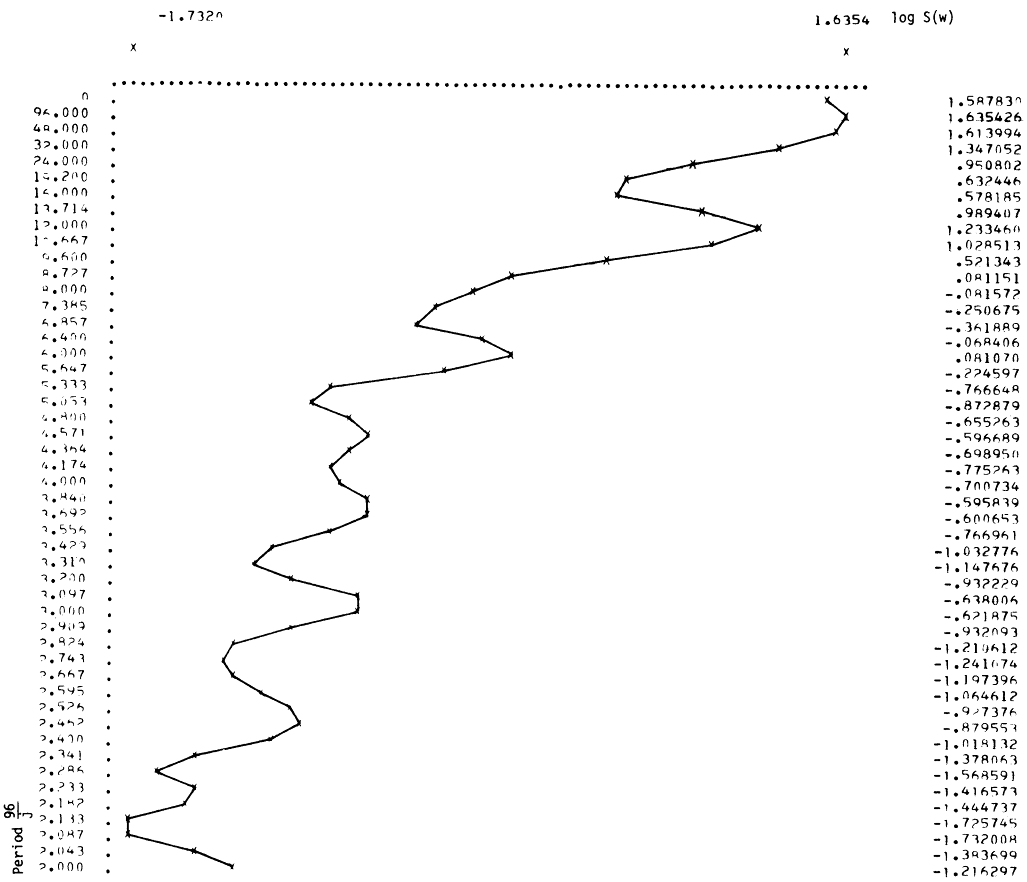


Figure 2 Plot of Estimated Spectrum of Time Rate

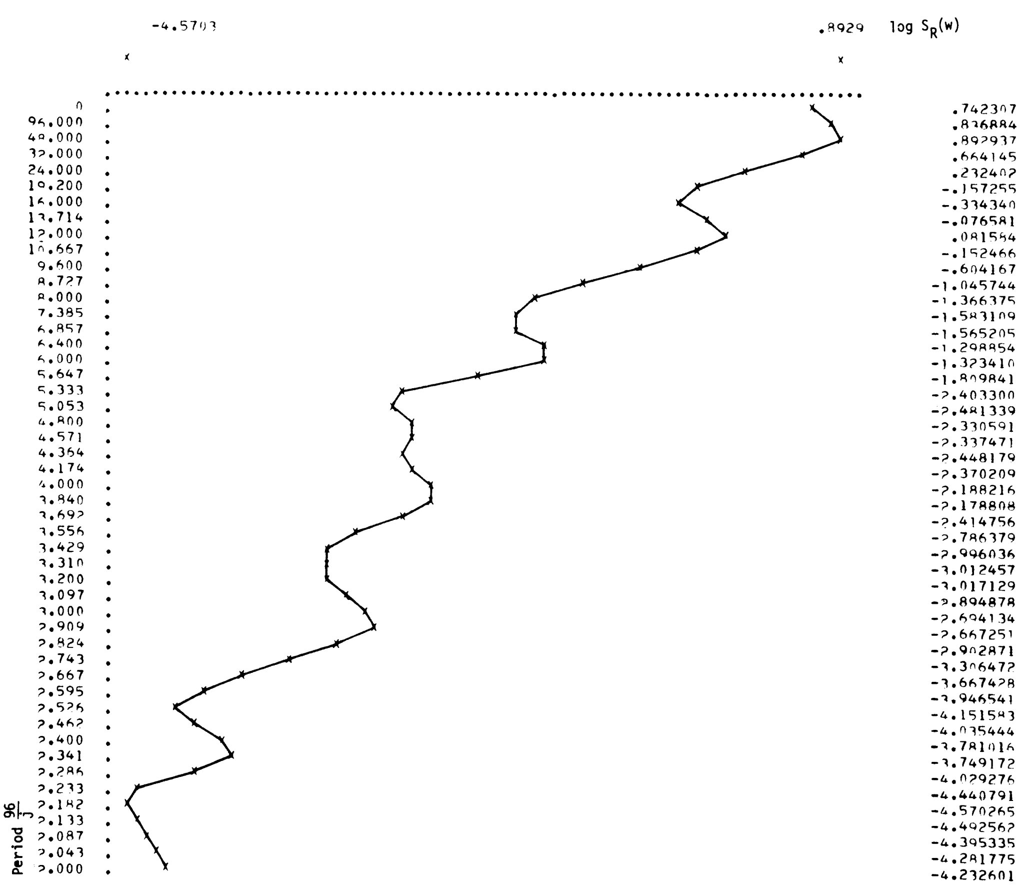


Figure 3 Plot of Phase

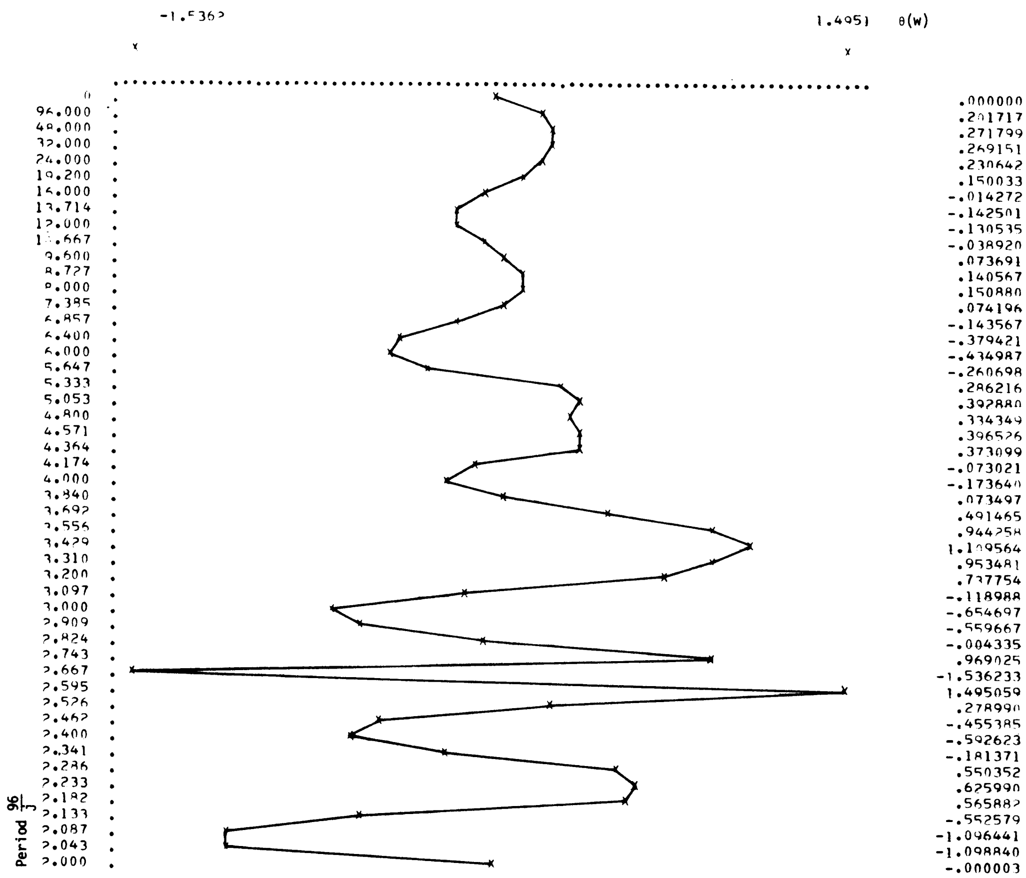


Figure 4--Cross-Spectrum Between Call and Time Rate

j $(w_j = \frac{2\pi j}{96})$	Period $= \frac{96}{j}$	Coherence	Phase $\theta(w)$	Asymptotic Standard Error of Phase	$\lvert H(w) \rvert$
0	.96000000D+02	.86601310D+00	.37865323D-28	.11394402D+00	.60976071D+00
1	.48000000D+02	.89164118D+00	.20171734D+00	.74046274D-01	.63342307D+00
2	.32000000D+02	.90318572D+00	.27179886D+00	.69542022D-01	.66269394D+00
3	.24000000D+02	.86978930D+00	.26915056D+00	.82183083D-01	.66285035D+00
4	.24000000D+02	.81725445D+00	.23064202D+00	.10044097D+00	.63121905D+00
5	.19200000D+02	.79608729D+00	.15003276D+00	.10749987D+00	.60117246D+00
6	.16000000D+02	.82245178D+00	-.14272133D-01	.98689072D-01	.57464946D+00
7	.13714286D+02	.88359766D+00	-.14250114D+00	.77093846D-01	.55163376D+00
8	.12000000D+02	.85931611D+00	-.13053455D+00	.85943263D-01	.52113449D+00
9	.10666667D+02	.78709176D+00	-.38920389D-01	.11047136D+00	.49154859D+00
10	.96000000D+01	.72294398D+00	.73691221D-01	.13149148D+00	.48434067D+00
11	.87272727D+01	.67398162D+00	.14056700D+00	.14772811D+00	.46732808D+00
12	.20000000D+01	.61563625D+00	.15087259D+00	.16783219D+00	.41273424D+00
13	.73846154D+01	.52971644D+00	.74195615D-01	.20013558D+00	.37384147D+00
14	.68571429D+01	.51206696D+00	-.14356701D+00	.20733989D+00	.39207274D+00
15	.64000000D+01	.63223417D+00	-.37942059D+00	.16199919D+00	.42978425D+00
16	.60000000D+01	.62661465D+00	-.43498702D+00	.16396249D+00	.39221228D+00
17	.56470588D+01	.44621985D+00	-.26069793D+00	.23662487D+00	.30237298D+00
18	.53333333D+01	.34720054D+00	.28621626D+00	.29124993D+00	.25995362D+00
19	.50526316D+01	.47499219D+00	.39288027D+00	.22330877D+00	.30836895D+00
20	.48000000D+01	.52781133D+00	.33434928D+00	.20090214D+00	.31437402D+00
21	.45714286D+01	.42038818D+00	.39652632D+00	.24940749D+00	.27153090D+00
22	.43636364D+01	.24677536D+00	.37309914D+00	.37108850D+00	.20716223D+00
23	.41739130D+01	.25285582D+00	-.73021170D-01	.36511684D+00	.22651576D+00
24	.40000000D+01	.45847072D+00	-.17363974D+00	.23084544D+00	.32184981D+00

Continuation of Figure 4--Cross-Spectrum Between Call and Time Rate

j $(w_j = \frac{2\pi j}{96})$	Period $= \frac{96}{j}$	Coherence	Phase $\theta(w)$	Asymptotic Standard Error of Phase	$\lvert H(w) \rvert$
25	.38400000D+01	.50631693D+00	.73497442D-01	.20973892D+00	.32245851D+00
26	.36923077D+01	.44570452D+00	.49146537D+00	.23687176D+00	.26952309D+00
27	.35555556D+01	.43458548D+00	.94425783D+00	.24227690D+00	.24017431D+00
28	.34285714D+01	.41827613D+00	.11095638D+01	.25049153D+00	.24233446D+00
29	.33103448D+01	.30949152D+00	.95348090D+00	.31726799D+00	.21897397D+00
30	.22000000D+01	.14307445D+00	.73775360D+00	.51982436D+00	.13336775D+00
31	.30967742D+01	.70571571D-01	-.11898781D+00	.77083108D+00	.85949103D-01
32	.30000000D+01	.17229607D+00	-.65469714D+00	.46554960D+00	.14728298D+00
33	.29090909D+01	.24203796D+00	-.55966713D+00	.37587904D+00	.20661274D+00
34	.28235294D+01	.18199456D+00	-.43347891D-02	.45031356D+00	.18304578D+00
35	.27428571D+01	.15623245D+00	.96902501D+00	.49361906D+00	.14073112D+00
36	.26666667D+01	.19916098D+00	-.15362327D+01	.42592841D+00	.12979008D+00
37	.25935946D+01	.90206606D-01	.14950587D+01	.67455653D+00	.71091276D-01
38	.25263158D+01	.48716828D-01	.27899029D+00	.93860214D+00	.44026264D-01
39	.24615385D+01	.11110370D+00	-.45538523D+00	.60079655D+00	.68797267D-01
40	.24000000D+01	.10501124D+00	-.59262269D+00	.62009328D+00	.81407635D-01
41	.23414634D+01	.48140964D-01	-.18137088D+00	.94448499D+00	.67046731D-01
42	.22857143D+01	.68083843D-01	.55035207D+00	.78583709D+00	.76241426D-01
43	.22325581D+01	.10101404D+00	.62599024D+00	.63365333D+00	.70063275D-01
44	.21818182D+01	.48943094D-01	.56588220D+00	.93631865D+00	.46360252D-01
45	.21333333D+01	.15789268D-01	-.55257858D+00	.16769850D+01	.31504627D-01
46	.20869565D+01	.39697832D-01	-.10964413D+01	.10446882D+01	.52607697D-01
47	.20425532D+01	.12361265D-01	-.10988399D+01	.18986011D+01	.26104910D-01
48	.20000000D+01	.49067905D-03	-.34823714D-05	.95865160D+01	.49024885D-02

Figure 5a Gain of Discrete Transfer Function | H(w) |

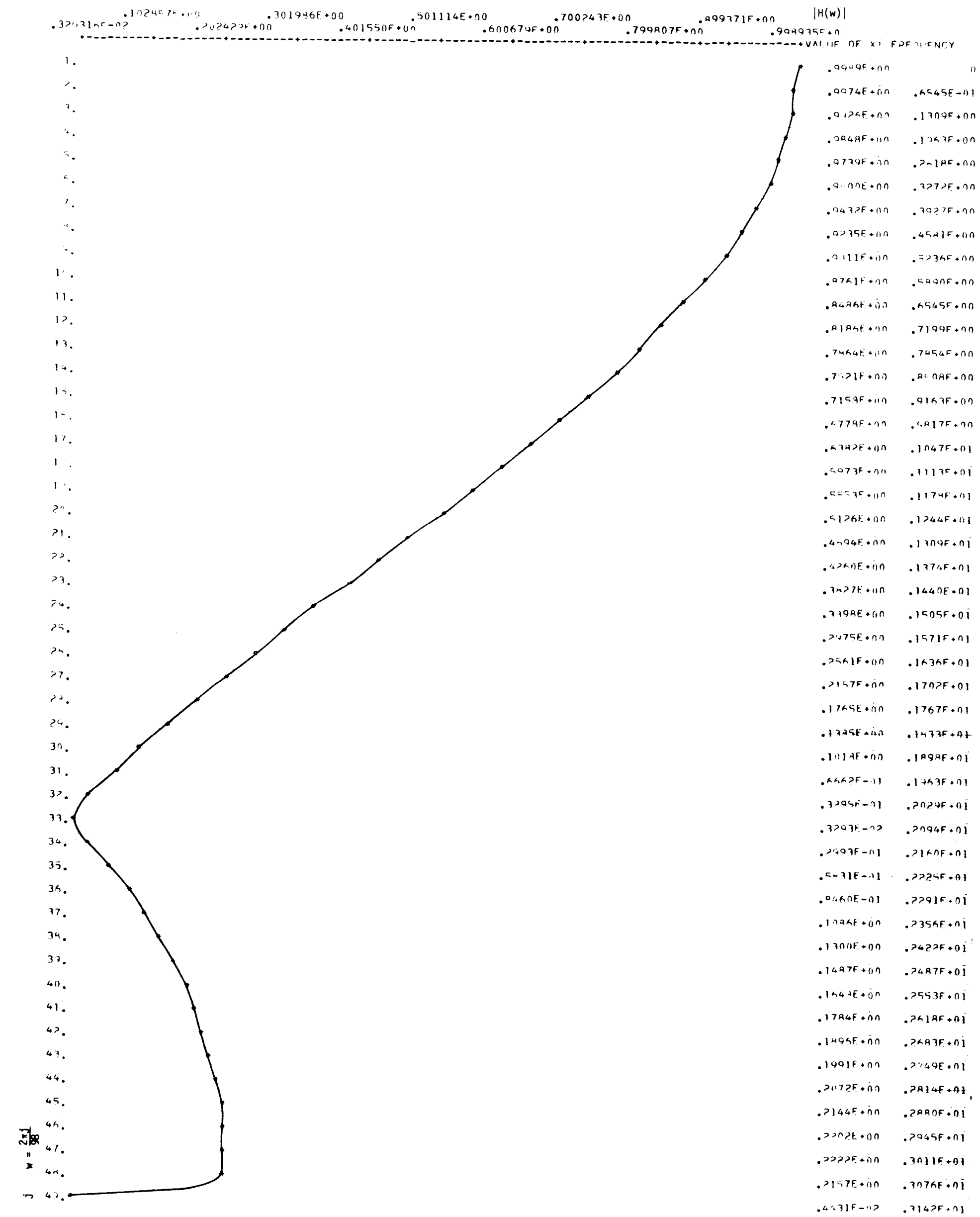


Figure 5b Phase of Discrete Transfer Function

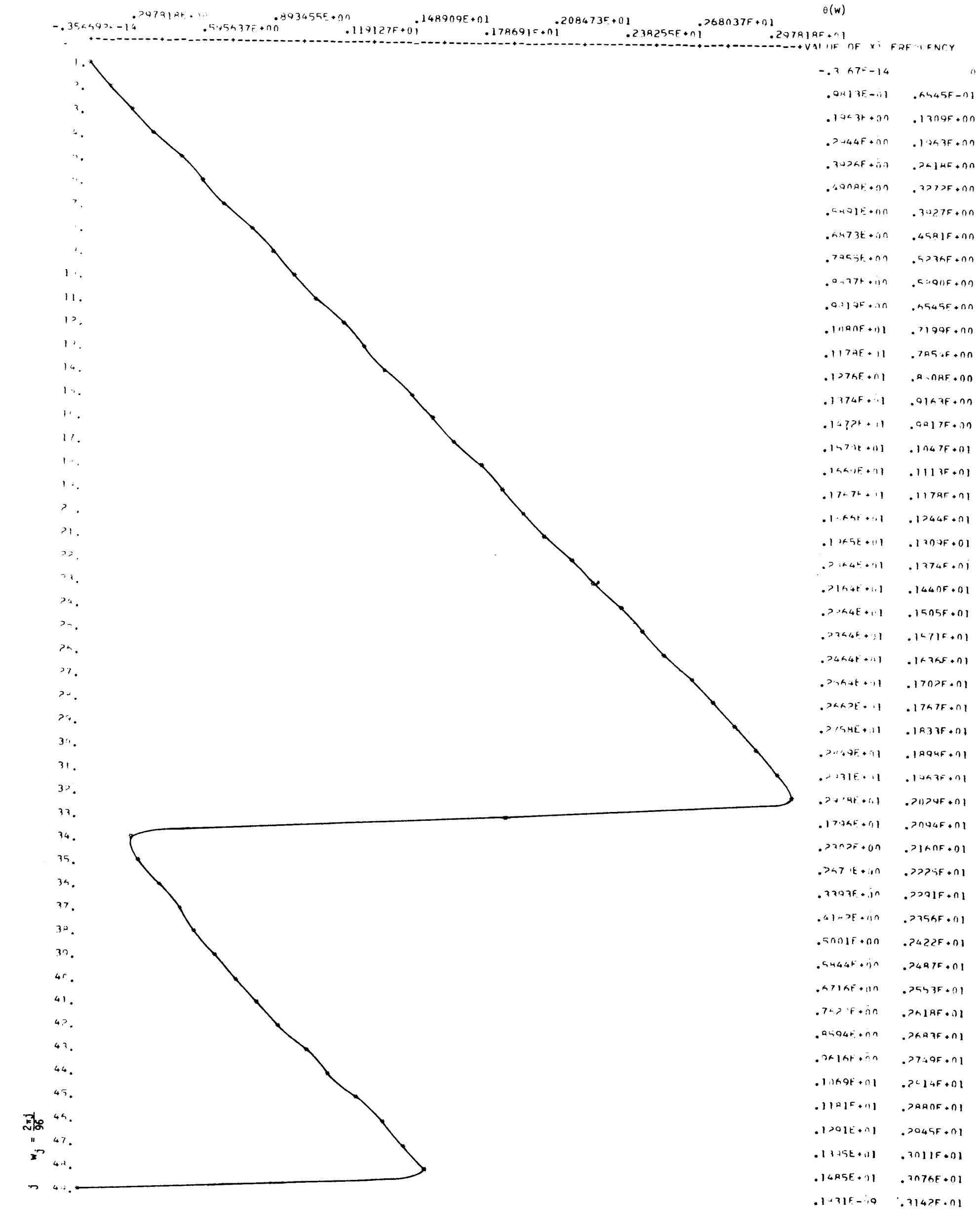
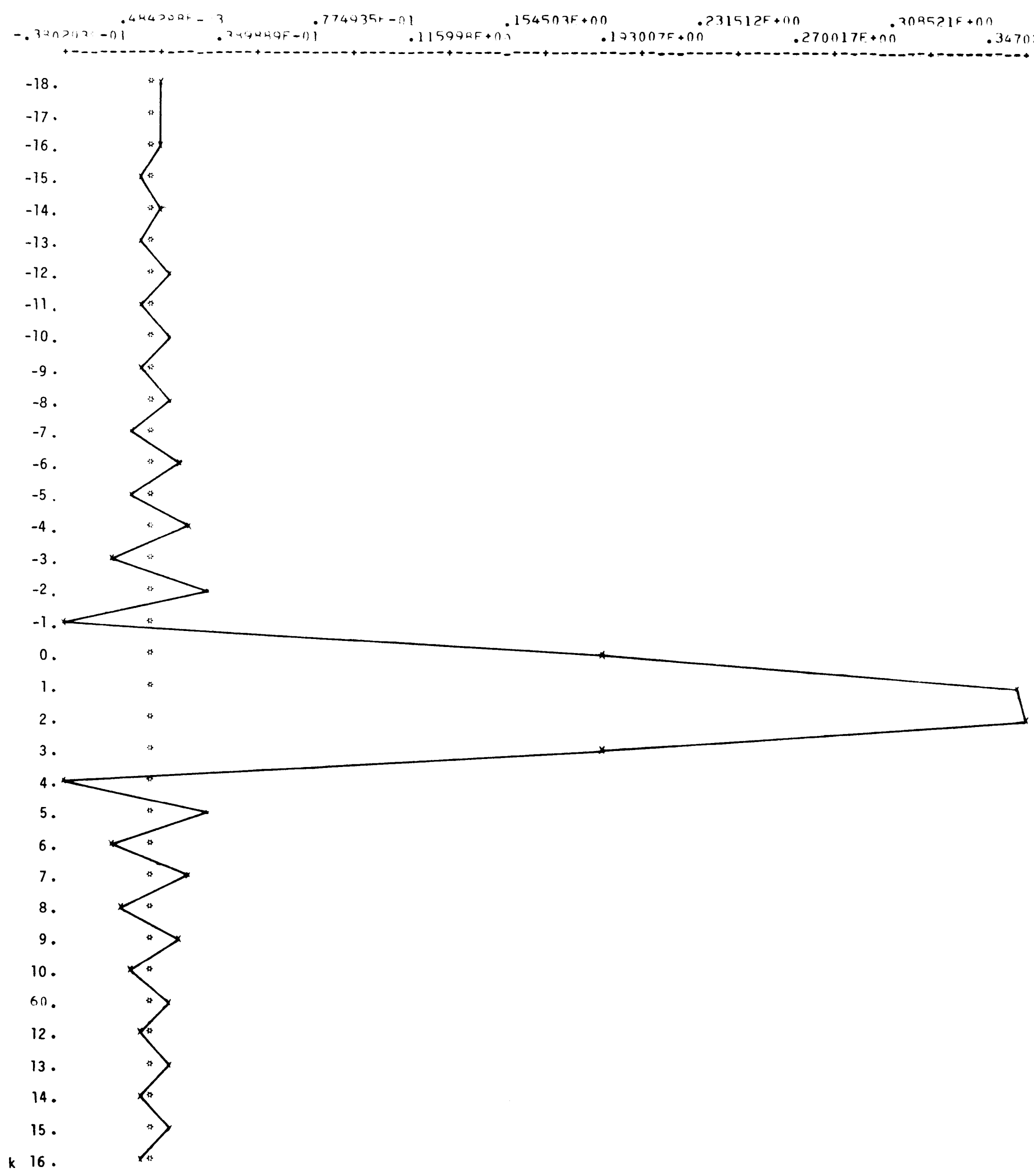


Figure 6 Discrete Distributed Lag Weights



k	VALUE OF X	VALUE OF Y
-18.	.31014E-02	-.84145E-14
-17.	-.33462E-02	.84609E-14
-16.	.36187E- 2	-.84237E-14
-15.	-.39236E- 2	.84332E-14
-14.	.42668E- 2	-.84145E-14
-13.	-.46551E-02	.83760E-14
-12.	.50963E-02	-.84237E-14
-11.	-.55935E- 2	.84315E-14
-10.	.61680E-02	-.84191E-14
-9.	-.68357E-02	.84385E-14
-8.	.74981E- 2	-.83821E-14
-7.	-.82047E-02	.85072E-14
-6.	.88685E- 2	-.84237E-14
-5.	-.96587E-02	.88650E-14
-4.	.12438E-01	-.84330E-14
-3.	-.17155E- 1	.87771E-14
-2.	.22591E-01	-.84330E-14
-1.	-.38420E-01	.87193E-14
0.	.17765E+00	-.84145E-14
1.	.34702E+00	.89143E-14
2.	.34703E+00	-.84145E-14
3.	.17798E+00	.84471E-14
4.	-.37952E- 1	-.84145E-14
5.	.21773E-01	.83083E-14
6.	-.16915E- 1	-.84145E-14
7.	.13825E- 1	.84424E-14
8.	-.11516E-01	-.84468E-14
9.	.97507E-02	.81417E-14
10.	-.83832E-02	-.84145E-14
60.	.73070E-02	.83483E-14
12.	-.64449E-02	-.83867E-14
13.	.57422E-02	.83760E-14
14.	-.51596E-02	-.84098E-14
15.	.46693E-02	.84477E-14
16.	-.42509E-02	-.84283E-14