WP-73

AN INQUIRY INTO THE CONDITIONS UNDER WHICH A SINGLE ENDOWMENT TRUST FUND WILL PROVIDE PERPETUAL FUNDING FOR AN EXPENSE STREAM GROWING AT A COMPOUND ANNUAL RATE

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THE PROBLEM

A single contribution trust fund is to be established. Determine the minimum combination of endowment and rate of return required to provide a specified annual income stream. The annual income must be sufficient to cover the fund managers fee and a given annual expense stream which grows at a constant annual rate. The fund and expense stream last forever (in perpetuity).

NOTATION

t ≡ time, measured in years

X ≡ endowment made at $t = 0$

r ≡ annual rate of return earned by the fund

f ≡ percentage rate used to determine fund managers fee

F_t ≡ management fee due at the end of the t^{th} year

E_1 ≡ expense amount due at the end of the first year

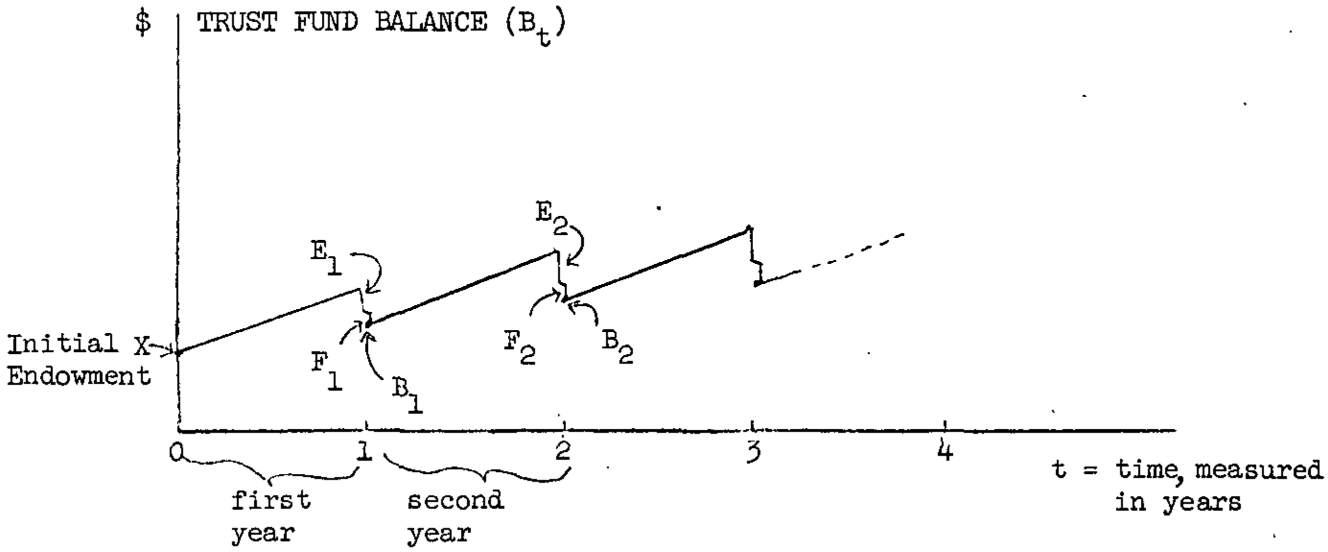
E_t ≡ expense amount due at the end of the t^{th} year

i ≡ annual rate of growth of the expense stream

B_t ≡ balance in the trust fund at the end of the t^{th} year, after paying the expense amount (E_t) and management fee (F_t)

FURTHER DESCRIPTION OF THE PROBLEM

Graphically we're dealing with a world that looks like this . . .



At t = 0 (the beginning of the first year) an initial endowment of X dollars is provided. At the end of the first year the fund has earned income and capital appreciation at the rate of r%. The first year expense amount (E_1 dollars) and management fee (F_1 dollars) are paid leaving a fund balance of B_1 dollars. The second year begins (with balance \$$B_1$). Income and appreciation accrue at r% during the year and at t = 2 the annual expense (\$$E_2$) and the management fee (\$$F_2$) are paid leaving a balance of \$$B_2$. The process is repeated year after year. One of two outcomes will occur:

Outcome A: Income and appreciation of the fund are adequate to cover the expense stream and management fee in perpetuity.

Outcome B: The rate of return and/or the endowment is not adequate and eventually the fund balance will deplete to zero.

The expense fund grows at an annual rate of i% . . .

$$E_t = E_1 \ (1 + i)^{(t-1)}$$

The management fee is f% of the sum of the balance at the beginning of the year plus income and appreciation earned during the year . . .

$$F_t = f\,(B_{t-1} + r\,B_{t-1})$$

SOLUTION

Given particular values for the management fee rate (f), the annual growth rate of the expense stream (i) and the beginning expense amount (E_1), two conditions must be satisfied in order to bring about outcome A (i.e. provide perpetual funding for the expense stream). The two conditions are:[1]

1. $r > \frac{f + i}{1 - f}$

2. $X \geq \frac{E_1}{(1 + r)\ (1 - f) - (1 + i)}$

The tables[2] on the following pages provide minimum combinations of rate of return and endowment required to provide perpetual funding.

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[1] Proof of these conditions is given in Appendix A

[2] The computer program use to derive the tables is given in Appendix B.

TABLES OF MINIMUM ENDOWMENT PER ONE DOLLAR
OF EXPENSE IN YEAR ONE

FUND MANAGERS FEE RATE = 0.2%

Annual Rate of Return of the Fund (r%)	Annual Rate of Growth of Expense Stream (i%) → 6%	7%	8%	9%	10%	11%
5%						
6%						
7%	127.23					
8%	56.05	127.55				
9%	35.95	56.12	127.88			
10%	26.46	35.97	56.18	128.21		
11%	20.93	26.47	36.00	56.24	128.53	
12%	17.31	20.94	26.48	36.02	56.31	128.87
13%	14.76	17.32	20.95	26.50	36.05	56.37
14%	12.87	14.77	17.33	20.96	26.51	36.08
15%	11.40	12.87	14.77	17.33	20.96	26.53

TABLES OF MINIMUM ENDOWMENT PER ONE DOLLAR
OF EXPENSE IN YEAR ONE

FUND MANAGERS FEE RATE = 0.3%

		Annual Rate of Growth of Expense Stream (i%)					
		6%	7%	8%	9%	10%	11%
Annual Rate of Return of the Fund (r%)	5%						
	6%						
	7%	147.28					
	8%	59.67	147.93				
	9%	37.41	59.77	148.59			
	10%	27.25	37.45	59.88	149.25		
	11%	21.43	27.27	37.50	59.99	149.93	
	12%	17.66	21.44	27.29	37.54	60.10	150.60
	13%	15.01	17.66	21.45	27.31	37.58	60.20
	14%	13.06	15.02	17.67	21.47	27.34	37.62
	15%	11.55	13.06	15.03	17.68	21.48	27.36

TABLES OF MINIMUM ENDOWMENT PER ONE DOLLAR
OF EXPENSE IN YEAR ONE

FUND MANAGERS FEE RATE = 0.4%

		Annual Rate of Growth of Expense Stream (i%)					
		6%	7%	8%	9%	10%	11%
Annual Rate of Return of the Fund (r%)	5%						
	6%						
	7%	174.83					
	8%	63.78	176.06				
	9%	39.00	63.94	177.30			
	10%	28.09	39.06	64.10	178.57		
	11%	21.95	28.12	39.12	64.27	179.86	
	12%	18.01	21.97	28.15	39.18	64.43	181.16
	13%	15.27	18.02	21.99	28.18	39.25	64.60
	14%	13.26	15.28	18.04	22.01	28.22	39.31
	15%	11.71	13.26	15.29	18.05	22.03	28.25

TABLES OF MINIMUM ENDOWMENT PER ONE DOLLAR
OF EXPENSE IN YEAR ONE

FUND MANAGERS FEE RATE = 0.5%

		Annual Rate of Growth of Expense Stream (i%)					
		6%	7%	8%	9%	10%	11%
Annual Rate of Return of the Fund (r%)	5%						
	6%						
	7%	215.05					
	8%	68.49	217.39				
	9%	40.73	68.73	219.78			
	10%	28.99	40.82	68.97	222.22		
	11%	22.50	29.03	40.90	69.20	224.72	
	12%	18.38	22.52	29.07	40.98	69.44	227.27
	13%	15.54	18.40	22.55	29.11	41.07	69.69
	14%	13.46	15.55	18.42	22.57	29.15	41.15
	15%	11.87	13.47	15.56	18.43	22.60	29.20

TABLES OF MINIMUM ENDOWMENT PER ONE DOLLAR
OF EXPENSE IN YEAR ONE

FUND MANAGERS FEE RATE = 0.6%

		Annual Rate of Growth of Expense Stream (i%)					
		6%	7%	8%	9%	10%	11%
Annual Rate of Return of the Fund (r%)	5%						
	6%						
	7%	279.33					
	8%	73.96	284.09				
	9%	42.63	74.29	289.02			
	10%	29.94	42.74	74.63	294.12		
	11%	23.07	29.99	42.84	74.96	299.40	
	12%	18.77	23.11	30.05	42.96	75.30	304.88
	13%	15.82	18.79	23.14	30.10	43.07	75.64
	14%	13.67	15.83	18.81	23.17	30.16	43.18
	15%	12.03	13.68	15.85	18.83	23.20	30.21

APPENDIX A

THE MATHEMATICS

The Process

(P-1) $B_o=X$

(P-2) $E_t=E_1(1+i)^{t-1}$ $t=1,2,\ldots$

(P-3) $B_t=B_{t-1}(1+r)(1-f)-E_t$ $t=1,2,\ldots$

Conditions on the Variables and Parameters

(C-1) $t=0,1,2,3,\ldots$

(C-2) $r\geq 0$

(C-3) $0\leq f<1$

(C-4) $0\leq E_1<\infty$

(C-5) $i\geq 0$

(C-6) $0<X<\infty$

Theorem 1: Given (P-1),...,(P-3) and (C-1),...,(C-6) the fund balance at the end of the t^{th} year is...

$$B_t=X[(1+r)(1-f)]^t-E_1\sum_{k=1}^{t}[(1+r)(1-f)]^{(t-k)}(1+i)^{(k-1)}$$

for $t=1,2,\ldots$

Proof (by induction):

A. By (P-1), (P-3) and inspection it is true for $t=1$

B. Assume it is true for $t=M$.

Then by assumption...

$$B_M=X[(1+r)(1-f)]^M-E_1\sum_{k=1}^{M}[(1+r)(1-f)]^{(M-k)}(1+i)^{(k-1)}$$

by (P-3)

by (P-3)

$$B_{M+1}=B_M(1+r)(1-f)-E_{M+1}$$

by (P-2)

$$B_{M+1}=B_M(1+r)(1-f)-E_1(1+i)^M$$

by the induction assumption...

$$B_{M+1}=X[(1+r)(1-f)]^M(1+r)(1-f)-E_1(1+r)(1-f)\sum_{k=1}^{M}[(1+r)(1-f)]^{(M-k)}(1+i)^{(k-1)}$$
$$-E_1(1+i)^M$$

$$B_{M+1}=X[(1+r)(1-f)]^{M+1}-E_1\left\{\sum_{k=1}^{M}[(1+r)(1-f)]^{[(M+1)-k]}(1+i)^{[k-1]}+(1+i)^M\right\}=X[(1+r)(1-f)]^{M+1}-E_1\left\{\sum_{k=1}^{M+1}[(1+r)(1-f)]^{[M+1-k]}(1+i)^{k-1}-[(1+r)(1-f)]^{o}(1+i)^M+(1+i)^M\right\}$$

$$B_{M+1}=X[(1+r)(1-f)]^{M+1}-E_1\sum_{k=1}^{M+1}[(1+r)(1-f)]^{[(M+1)-k]}(1+i)^{[k-1]}\quad \text{q.e.d.}$$

<u>Theorem 2</u>: Outcome A will occur if and only if $B_t>0 \quad \forall_t=1,2,\ldots$

Proof:

1. <u>Necessary.</u> Outcome A asserts $B_t\,(1+r)(1-f)\geq E_{t+1} \quad \forall_{t=1,2,\ldots}$

   Where $0\leq E_{t+1}$ by (C-4) and (P-2)

   $0\leq f<1$ by (C-3)

   $0\leq r$ by (C-2)

   so $B_t\geq 0$.

   if $B_t=0$ then $B_{t+1}=-E_t<0$

   So Outcome A $\Rightarrow B_t>0 \quad t=1,2,\ldots$

2. <u>Sufficient.</u>

   to show that

   $$B_p>0\forall_p\Rightarrow B_q(1+r)(1-f)\geq E_{q+1} \quad \forall_q$$

   is equivalent to showing

   $$B_q(1+r)(1-f)<E_{q+1} \text{ for some } q \Rightarrow B_p<0 \text{ for some } p\,.$$

from (P-3)

$$B_{q+1}=B_q(1+r)(1-f)-E_{q+1}$$

therefore $B_q(1+r)(1-f)<E_{q+1} \Rightarrow B_{q+1}<0$ q.e.d.

By rearranging the equation of Theorem 1

$$B_t=[(1+r)(1-f)]^t \left[X- \frac{E_1}{1+i} \sum_{k=1}^{t} \left[\frac{1+i}{(1+r)(1-f)} \right]^k \right]$$

Theorem 3: Let the vector $\langle f,E_1,i \rangle$ have Domain $\mathcal{L}$ given by (C-3), (C-4), (C-5).

For any $D=\langle f,E_1,i \rangle \in \mathcal{L}$

$$B_t>0 \;\forall_{t=1,2,3,\ldots} \Leftrightarrow \begin{cases} r> \frac{f+i}{1-f} \ldots \text{Condition 1} \\ \text{and} \\ X\geq \frac{E_1}{(1+r)(1-f)-(1+i)} \ldots \textit{Condition 2} \end{cases}$$

Proof:

<u>Necessary</u> A) $\frac{1+i}{(1+r)(1-f)} > 0$ by (C-2), (C-3), (C-5).

Given $D\in\mathcal{L}$, $B_t>0 \quad \forall_t=1,2,\ldots \Rightarrow$

$$\text{B) } X> \frac{E_1}{1+i} \sum_{k=1}^{t} \left[\frac{1+i}{(1+r)(1-f)}\right]^k \quad \forall_t=1,2,\ldots$$

A, B, and (C-6) $\Rightarrow$

$$\text{C) } \lim_{t\to\infty} \sum_{k=1}^{t} \left[\frac{1+i}{(1+r)(1-f)}\right]^k <\infty$$

$$C \Leftrightarrow -1< \frac{1+i}{(1+r)(1-f)} < 1$$

∴ since A holds by assumption,

$$C \Rightarrow \frac{1+i}{(1+r)(1-f)} < 1$$

or equivalently $r > \frac{f+i}{1-f}$ Condition 1

$$\text{Condition 1} \Rightarrow \lim_{t\to\infty} \sum_{k=1}^{t} \left[\frac{1+i}{(1+r)(1-f)}\right]^k = \frac{1+i}{(1+r)(1-f)-(1+i)}$$

Since this series is strictly monotonically increasing, it follows that the limit of the series is its least upper bound:

$$\forall_{t=1,2,3,\ldots} \sum_{k=1}^{t} \left[\frac{1+i}{(1+r)(1-f)}\right]^k < \frac{1+i}{(1+r)(1-f)-(1+i)}$$

and $\forall y,\ y < \frac{1+i}{(1+r)(1-f)-(1+i)} \Rightarrow \exists t^* \geq 1 \ni$

$$\forall_t \geq t^* \quad \sum_{k=1}^{t} \left[\frac{1+i}{(1+r)(1-f)}\right]^k > y$$

Note: $\{B_t > 0 \quad \forall_{t=1,2,\ldots} \Rightarrow \text{Condition 1}\}$ has been shown for any X

$\therefore B_t > 0 \quad \forall_{t=1,2,\ldots}$ and Condition 1 $\Rightarrow$

$$X \geq \frac{E_1}{1+i} \lim_{t\to\infty} \sum_{k=1}^{t} \left[\frac{1+i}{(1+r)(1-f)}\right]^k$$

or equivalently

$$X \geq \frac{E_1}{(1+r)(1-f)-(1+i)} \qquad \text{Condition 2}$$

<u>Sufficiency</u>

Suppose $r > \frac{f+i}{1-f}$ and $X \geq \frac{E_1}{(1+r)(1-f)-(1+i)}$

then $(1+r)(1-f) > 1$,

$$0 < \frac{1+i}{(1+r)(1-f)} < 1,$$

$$\lim_{t\to\infty} \sum_{k=1}^{t} \left[\frac{1+i}{(1+r)(1-f)}\right]^k = \frac{1+i}{(1+r)(1-f)-(1+i)},$$

and $X \geqq \frac{E_1}{(1+r)(1-f)-(1+i)} \Rightarrow X > \frac{E_1}{1+i} \sum_{k=1}^{t} \left[\frac{1+i}{(1+r)(1-f)}\right]^k \quad \forall_{t=1,2,\ldots}$

$$\therefore B_t = [(1+r)(1-f)]^t \left[X - \frac{E_1}{1+i} \sum_{k=1}^{t} \left[\frac{1+i}{(1+r)(1-f)}\right]^k\right] > 0 \quad \forall_{t=1,2,\ldots}$$

q.e.d.

<u>Corollary</u>

$$B_t > 0 \quad \forall_{t=1,2,3,\ldots} \Rightarrow \lim_{t\to\infty} B_t = \infty$$

Proof:

$$B_t > 0 \quad \forall_{t=1,2,3,\ldots} \Rightarrow r > \frac{f+i}{1-f} \text{ and } \{\text{by Theorem 3}\}\ X \geq \frac{E_1}{(1+r)(1-f)-(1+i)}$$

Case 1

Suppose $r > \frac{f+i}{1-f}$, $X > \frac{E_1}{(1+r)(1-f)-(1+i)}$, and let $\Delta = X - \frac{E_1}{(1+r)(1-f)-(1+i)} > 0$,

then $\lim_{t\to\infty} B_t = \lim_{t\to\infty} [(1+r)(1-f)]^t \lim_{t\to\infty} \left[X - \frac{E_1}{1+i} \sum_{k=1}^{t} \left[\frac{1+i}{(1+r)(1-f)} \right]^k \right] = \infty \cdot \Delta = \infty$

Case 2

Suppose $r > \frac{f+i}{1-f}$ and $X = \frac{E_1}{(1+r)(1-f)-(1+i)}$, then

$$B_t = (1+r)(1-f)^t \left[\frac{E_1}{(1+r)(1-f)-(1+i)} - \frac{E_1}{1+i} \sum_{k=1}^{t} \left[\frac{1+i}{(1+r)(1-f)} \right]^k \right]$$

$$\frac{B_t}{E_1} = \frac{[(1+r)(1-f)]^t}{(1+r)(1-f)-(1+i)} - \sum_{k=1}^{t} (1+i)^{k-1} [(1+r)(1-f)]^{t-k}$$

let $b=(1+r)(1-f)$

$d=(1+i)$

then $b>d>1$ by (C-5) and Condition 1

$$\frac{B_t}{E_1} = \frac{b^t}{b-d} - \sum_{k=1}^{t} d^{k-1} b^{t-k}$$

$$\frac{B_t}{E_1} = \frac{b^t - \sum_{k=1}^{t} d^{k-1} b^{t-(k-1)} + \sum_{k=1}^{t} d^k b^{t-k}}{b-d}$$

let $C = \frac{b-d}{E_1} > 0$, then $CB_t = b^t \left[1 - \sum_{k=1}^{t} \left(\frac{d}{b}\right)^{k-1} + \sum_{k=1}^{t} \left(\frac{d}{b}\right)^k \right]$

$$CB_t = b^t \left[1 + \left(\frac{d}{b}\right)^t - 1 \right]$$

$$CB_t = d^t$$

or $B_t = \frac{d^t}{C}$

$$\therefore \lim_{t \to \infty} B_t = \lim_{t \to \infty} \frac{d^t}{C} = \infty \quad . \qquad q.e.d.$$

APPENDIX B

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LIST
3 E1=1
5 FØR H = 1 TØ 5
10 F = (H+1)*(.001)
12 PRINT "FEE % =" F
13 WRITE (1,14)
14 FORMAT(11X,".06",7X,".07",7X,".08",7X,".09",7X,".10",7X,".11")
15 FØR Y = 1 TØ 11
20 R = .04 + Y*(.01)
25 FØR Z = 1 TØ 6
30 I = .05 + Z * (.01)
35 IF R > (I+F)/(1-F) THEN 45
40 X(Z) = 0
42 GØ TØ 50
45 X(Z) = E1/((1+R)*(1-F)-(1+I))
46 X(Z) = INT(X(Z)*100 +.5)/100
50 NEXT Z
55 WRITE (1,60) R,X(1),X(2),X(3),X(4),X(5),X(6)
60 FØRMAT(F5.2,6(4X,F7.2))
65 NEXT Y
70 PRINT
71 PRINT
72 PRINT
75 NEXT H
80 END
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