Robert Morris Townsend
511 Words

ESSAYS ON MARKETS UNDER UNCERTAINTY

ABSTRACT

This thesis consists of a series of essays on the
theory of exchange under uncertainty. The first essay
examines the welfare implications of futures markets in the
context of complete markets for contingent claims. It is
shown that in a C-good, S-state world the equilibrium allocations
resulting from the operation of pre-state noncontingent
futures markets and post-state spot markets may be Pareto
optimal. This proposition turns on the fact that a futures
contract can be interpreted as a security whose state-specific
return is the post-state spot price. If the matrix of spot
prices has rank S, then, with futures and spot markets,
agents can achieve the same allocations over states as with
complete markets for contingent claims.

The second essay examines the Keynes-Hicks theory of
futures markets. In that theory, risk averse hedgers seek
to reduce price risk by selling forward that portion of
supply which is planned or fixed in advance. On the other
side of the market are speculators who seek a profit by
purchasing forward when the futures price is below the spot
price expected to prevail at the time of maturity. The
second essay argues that this partial equilibrium terminology
is not useful. It presents an example in which risk averse

agents purchase forward the good with which they are endowed; in a general equilibrium context, futures contracts may be viewed as a hedge against the random valuation of stochastic endowments. In general, the direction of trade would seem to be determined by tastes and endowments as in standard theory.

The third essay examines the feasibility of fixing a relative spot price in a two commodity world in which risk averse agents with stochastic endowments maximize expected utility over an infinite horizon. It is shown that a price-fixing scheme will fail with probability one, regardless of the price set by the government and regardless of the initial level of buffer stocks. The proof of this proposition turns on the properties of random walks.

The final essay explores a model in which exchange is costly and the market structure is endogenous. There is a set of risk averse agents each of whom is endowed with a quantity of a capital good and with a stochastic technology which transforms the capital good into a distribution of the single consumption good of the model. The distributions for different agents are such that there are gains to portfolio diversification. But there are also costs; for each bilateral deal between agents there is a fixed cost in terms of the capital good. In the model, agents adopt strategies under which they are willing to act as intermediaries, buying shares in investment projects and selling shares in the resulting portfolio. It is shown that there exist noncooperative

equilibria for the model, and in those equilibria there are markets which separate agents into disjoint groups. Subject to some qualifications, the allocations of a core can be supported as noncooperative equilibria and all noncooperative equilibria yield allocations in that core. It is argued that free entry is crucial in determining the allocation of resources.

ESSAYS ON MARKETS UNDER UNCERTAINTY

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INTRODUCTION TO THE ESSAYS

Kenneth Arrow [2] and Gerard Debreu [4] have formulated a theory of exchange under uncertainty. In that theory a set of states of the world is an exhaustive listing of all possible outcomes or events. A contingent commodity claim is a claim on a particular commodity which is binding only in specified states. By increasing the dimensionality of the commodity space in this way, classical propositions on the existence and optimality of a competitive equilibrium were made applicable to risky environments. Yet many authors, including Arrow [1], have noted the seeming absence of markets in which contingent claims are traded, and it has been suggested that the dearth of such active markets is inconsistent with the Arrow-Debreu theory. There are markets in which commodity futures are traded, yet these would seem to be a far cry from complete markets for contingent claims in which claims could be traded for any commodity in any state. Also, there appear to be active spot markets at every date, but with complete markets spot trading would not seem to be needed. Nor does the theory explain financial intermediaries and other trade facilitating institutions. In short, the theory seems inadequate in the descriptive sense.

The absence of markets in which contingent claims are traded also has normative implications; it suggests that some mutually beneficial exchange does not take place. Hence it would seem worthwhile to examine government policy in such an environment. As suggested by Buchanan [3], the government may have a role in setting up institutions to facilitate exchange. Alternatively, in the absence of complete markets, one may wish to find a second best policy. Various policies have been suggested; these include limiting the movements of spot and futures prices and the regulation of various financial institutions.

This thesis consists of a series of essays which examines the descriptive deficiency of the Arrow-Debreu theory and which deals with the policy implications of the absence of contingent contracts. In the first essay, "On the Optimality of Futures Markets," I examine the welfare implications of futures markets in the context of complete markets for contingent claims. It is shown that in a C-good, S-state world the equilibrium allocations resulting from the operation of pre-state noncontingent futures markets and post-state spot markets may be Pareto optimal. This proposition turns on the fact that a futures contract can be interpreted as a security whose state specific return is the post-state spot price. If the matrix of spot prices has rank S, then, with futures and spot markets, agents can achieve the same allocations over states as with complete markets for contingent claims.

The interpretation of futures contracts as securities leads me to examine the standard theory of futures markets. In the Keynes-Hicks theory, risk averse hedgers seek to reduce price risk by selling forward that portion of supply which is planned or fixed in advance. On the other side of the market are speculators who seek a profit by purchasing forward when the futures price is below the spot price expected to prevail at the time of maturity. The second essay argues that this partial equilibrium terminology is not useful. It presents an example in which risk averse agents purchase forward the good with which they are endowed; in a general equilibrium context, futures contracts may be viewed as a hedge against the random valuation of stochastic endowments. In general, the direction of trade would seem to be determined by tastes and endowments as in standard theory.

The third essay of the thesis examines government policy in the absence of complete markets. Jacques Drèze [5] has suggested that it is the absence of markets in which farmers can trade crops contingent on the determinants of supply and demand which had led the government to consider buffer stock programs. The essay, "Price Fixing Schemes and Optimal Buffer Stock Policies," examines the feasibility of fixing a relative spot price in a two commodity world in which risk averse agents with stochastic endowments maximize expected utility over an infinite horizon. It is shown that a price fixing scheme will fail with probability one regardless

of the price set by the government and regardless of the initial level of buffer stocks. The proof of this proposition turns on the properties of random walks.

The final essay, "Intermediation with a Nonconvex Transactions Technology," explores a model in which exchange is costly and the market structure is endogenous. There is a set of risk averse agents each of which is endowed with a quantity of a capital good and with a stochastic technology which transforms the capital good into a distribution of the single consumption of the model. The distributions for different agents are such that there are gains to portfolio diversification. But there are also costs; for each bilateral deal between agents there is a fixed cost in terms of the capital good. This nonconvexity makes standard results on the existence and optimality of a competitive equilibrium inapplicable. In the model agents adopt strategies under which they are willing to act as intermediaries, buying shares in investment projects and selling shares in the resulting portfolio. It is shown that there exist noncooperative equilibria for the model and in those equilibria there are markets which separate agents into disjoint groups. In this sense markets may be said to be incomplete. Subject to some qualifications, the allocations of a core can be supported as noncooperative equilibria and all noncooperative equilibria yield allocations in that core. It is argued that free entry is crucial in determining the allocation of resources.

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ESSAY I

ON THE OPTIMALITY OF FUTURES MARKETS

I.   Introduction

In his seminal article, "The Role of Securities in the
Optimal Allocation of Risk Bearing," Arrow [2] provided a
convenient framework in which problems involving choice
under uncertainty could be analyzed. By extending the
dimensionality of commodity space to include the number of
states, classic results on the existence and optimality of a
competitive equilibrium were made applicable to uncertain
situations. Yet many authors have commented on the existence
of only a small number of markets in which contingent claims
are actively traded. In particular, the existence of futures
markets rather than markets for contingent claims is often
taken as prima-facie evidence of some inefficiency.

The purpose of this essay is to show that in some cases
the equilibrium allocation resulting from the operation of
pre-state noncontingent futures markets and of post-state
spot markets may be Pareto optimal. This proposition turns
on the fact that a futures contract can be viewed as a
security whose state specific return is the post-state spot
price. If the rank of the matrix of spot prices is equal to
the number of states, then a restriction to the trading of
futures contracts will not be binding in equilibrium.

Agents can achieve precisely the same allocation across
states as they could with complete markets for contingent
claims in which claims could be traded for any commodity in
any state; all individual budget constraints will be satisfied
and all futures markets will clear at appropriately selected
futures prices.

In his article Arrow [2] stressed that risk bearing is
not allocated by the sale of claims against specific commodities
but rather by the sale of securities payable in money. In
light of such remarks, the interpretation of commodity futures
contracts as securities deserves clarification. An attempt
is made to relate the results of this essay to those of
Arrow.[1]/

II. Assumptions and Technology of the Model with Contingent Claims

The model is a pure exchange economy with random endowments.
Before the random endowments are realized, each individual
can decide on the quantity and type of contingent claims to
purchase or issue. After endowments are realized, contingent
claims are executed and trade and consumption take place in
spot markets. In the model no actual transfer or consumption
of resources takes place in forward markets; only claims are
traded. In spot markets there is no uncertainty; every
individual knows his endowment and the market price. An
individual's consumption is not limited to claims acquired
in forward markets--retrading is possible in spot markets.
Such retrading is crucial to the propositions of this paper.

In the model there are C commodities, I agents, and S states. Let $Z_{is} \epsilon R_+^C$ denote the exogenous endowment of agent i in state s with element Z_{isc} for commodity c. By assumption for each state s and each agent i, $Z_{isc} > 0$ for some c. Also $\sum_{i=1}^{I} Z_{isc} > 0$ for each state s and each commodity c. Let $C_{is} \epsilon R_+^C$ denote the consumption of agent i in state s. Let π_{is} denote the subjective probability for agent i that state s will occur. It is assumed that $\pi_{is} > 0$ for each agent i and each state s.

Prior to the realization of the state, each agent maximizes $\sum_{s=1}^{S} \pi_{is} U^{is}(C_{is})$. It is assumed that

(i) $U^{is}(\cdot)$ is a function of class C^2 from R_+^C to R^1 with strictly positive first partial derivatives; $U_C^{is}(0) = \infty$

(ii) $U^{is}(\cdot)$ is strictly concave.

By construction spot markets are mutually exclusive, and hence commodity C can be taken as the numeriare in each spot market. Then, from (i) the marginal utility of the numeraire is infinite at the origin. Under property (ii) each agent is said to be risk averse. These assumptions on preferences and endowments are sufficient to ensure the existence of a competitive equilibrium with complete markets for contingent claims and the existence of a competitive equilibrium in each state s.[2]

Let $P_s \epsilon R^C$ denote the spot prices in terms of commodity C in state s with element P_{sc} for commodity c. Let $Y_{is} \epsilon R_+^1$ denote the expenditures of agent i in terms of commodity C in state s. Then $Y_{is} = P_s \cdot C_{is}$. Let $h_{is}(Y_{is}, P_s) \epsilon R_+^C$

denote the commodity demands of agent i in state s. That is, $h_{is}(Y_{is}, P_s)$ is a maximizing choice of C_{is} for the function $U^{is}(C_{is})$ subject to the constraint that $Y_{is} = P_s \cdot C_{is}$. An indirect utility function $V^{is}(\cdot, \cdot)$ is then defined as $V^{is}(Y_{is}, P_s) = U^{is}[h_{is}(Y_{is}, P_s)]$. It is important to note that there is no uncertainty about the price vector P_s which will prevail in state s, though there may be disagreement over the probability that state s will occur.

Complete markets for contingent claims are described as follows. Each agent can issue or purchase contingent claims, each of which entitles the holder to one unit of the specified commodity if a particular state occurs. Let X_{isc} denote the claims on commodity c in state s held by agent i after trading in forward markets, with $X_{is} \epsilon R^C$. Then, given the endowments, $(X_{isc} - Z_{isc})$ is the excess demand for such claims in forward markets. Equilibrium in the market for claims exists when $\sum_{i=1}^{I} (X_{isc} - Z_{isc}) = 0$ for each state s and each commodity c. Let r_{sc} denote the price of one claim on commodity c in state s in terms of some abstract unit of account. The budget constraint for agent i in the market for claims is that the net value of his excess demands for claims be zero, or

$$\sum_{s=1}^{S} \sum_{c=1}^{C} r_{sc}(X_{isc} - Z_{isc}) = 0.$$

With contingent claims an agent's income in state s is the value of his commodity claims in state s. That is,

$Y_{is} = P_s \cdot X_{is}$. By property (i) of $U^{is}(\cdot)$, $V_1^{is}(0, P_s) = \infty$ for each state s, and therefore in equilibrium $Y_{is} > 0$.[3/]

In summary the objective of agent i is to find a critical point of the function

$$\sum_{s=1}^{S} \pi_{is} V^{is}(\sum_{c=1}^{C} P_{sc} X_{isc}, P_s) - \lambda \sum_{s=1}^{S} \sum_{c=1}^{C} r_{sc}(X_{isc} - Z_{isc})$$

where λ is a Lagrange multiplier. This yields first order conditions

$$\pi_{is} P_{sc} V_1^{is}(Y_{is}^*, P_s) - \lambda r_{sc} = 0 \quad s=1,\ldots S \quad c=1,\ldots C$$

$$\sum_{s=1}^{S} \sum_{c=1}^{C} r_{sc}(X_{isc}^* - Z_{isc}) = 0$$

where here and below the superscript * denotes maximizing quantities. As noted, $Y_{is}^* > 0$ for each state s.

As $P_{sC} = 1$ for each state s, in equilibrium $r_{sc}/r_{sC} = P_{sc}$. This condition can be derived from the first order conditions above.

In general a maximizing choice $\{X_{is}^*; s=1,\ldots S\}$ will not be unique. For suppose $\{X_{is}^*; s=1,\ldots S\}$ are maximizing quantities. Then with spot markets agent i will be indifferent among all choices $\{X_{is}^{**}; s=1,\ldots S\}$ such that $P_s \cdot X_{is}^{**} = P_s \cdot X_{is}^* = Y_{is}^*$. That is, given contingent commodity prices and future spot prices, agents act as if selecting expenditures over states. In subsequent spot markets agents can use those expenditures to acquire maximizing commodity bundles.

An additional restriction consistent with a maximizing choice of expenditures is that there be no spot markets,

i.e., that each agent consume the commodities acquired with claims. This paper examines the constraints associated with futures contracts. For each agent i and each commodity c, these constraints are of the form $(X_{isc}-Z_{isc}) = (X_{iwc}-Z_{iwc})$ for all states w and s.

III. The Welfare Implications of Futures Markets

The principal result of this section is that if there are at least as many goods as states, then futures markets with subsequent spot markets may achieve an optimal allocation of risk-bearing.

In the propositions which follow it will be supposed that there exists a competitive equilibrium with complete contingent commodity markets and with no trade in subsequent spot markets. Given the market clearing claim prices $\{r_{sc}; s=1,\ldots S; c=1,\ldots C\}$, an S by C matrix P'' can be defined with entries $P_{sc} = r_{sc}/r_{sC}$. This matrix can be interpreted as the matrix of spot prices which is implicit in the initial equilibrium. That is, $P_{sc} = U^{is}_c(X^*_{is})/U^{is}_C(X^*_{is})$ for each commodity c, each state s, and each agent i. It will be shown that the matrix P'' may also be the matrix of equilibrium spot prices given a restriction to futures trading.

Proposition I[4/]: Suppose that a Pareto optimal allocation $\{X^*_{is}; s=1,\ldots S; i=1,\ldots I\}$ can be supported as a competitive equilibrium with complete markets for contingent claims and with no trade in spot markets with endowments $\{Z_{is}; s=1,\ldots S; i=1,\ldots I\}$ and claim prices

$\{r_{sc}; s=1,\ldots S; c=1,\ldots C\}$. Suppose also that the rank of

the matrix P'' is S. Then with the same endowments,

$\{x^*_{is}; s=1,\ldots S; i=1,\ldots I\}$ can also be supported as a competitive

equilibrium with futures contracts and subsequent spot

markets.

Proof: Given that the S by C matrix P'' is of rank S, attention

is limited to cases in which $C \geq S$. And as P'' is of rank S,

it is possible to delete C-S columns from P'' and leave a

square matrix P of rank S. Then without loss of generality

commodities can be numbered in such a way that the S commodities

with prices in P are the first S commodities.

Given the endowments $\{Z_{is}; s=1,\ldots S; i=1,\ldots I\}$, define

implicit endowments of futures contracts

$\{E_{ic}; c=1,\ldots S; i=1,\ldots I\}$ to satisfy the following equations:

$$
\begin{bmatrix}
P_{11} & P_{12} & \cdots & P_{1S} \\
\cdot & & & \cdot \\
\cdot & & & \cdot \\
\cdot & & & \cdot \\
P_{S1} & & & P_{SS}
\end{bmatrix}
\begin{bmatrix}
E_{i1} \\
\cdot \\
\cdot \\
\cdot \\
E_{iS}
\end{bmatrix}
=
\begin{bmatrix}
\sum_{c=1}^{C} Z_{i1c} P_{1c} \\
\cdot \\
\cdot \\
\cdot \\
\sum_{c=1}^{C} Z_{iSc} P_{Sc}
\end{bmatrix}
\quad i=1,\ldots I
$$

or $PE_i = Y_i$. As P is of full rank, this equation is well

defined and $E_i = P^{-1}Y_i$. That is, agent i can issue futures

contracts on commodities one through S up to the limits of

his ability to honor those claims in spot markets as given

by the incomes $P_s \cdot Z_{is}$ for each state s. If the optimal

allocation $\{X^*_{is}; s=1,\ldots S; i=1,\ldots I\}$ is to be achieved,

agent i must enter the spot markets holding futures contracts

$\{Q_{ic}; c=1,\ldots S\}$ where

$$
\begin{bmatrix}
P_{11} & P_{12} & \cdots & P_{1S} \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
\cdot & \cdot & & \cdot \\
P_{S1} & & & P_{SS}
\end{bmatrix}
\begin{bmatrix}
Q_{i1} \\
\cdot \\
\cdot \\
\cdot \\
Q_{iS}
\end{bmatrix}
=
\begin{bmatrix}
\sum\limits_{c=1}^{C} X^{*}_{i1c} P_{1c} \\
\cdot \\
\cdot \\
\cdot \\
\sum\limits_{c=1}^{C} X^{*}_{iSc} P_{Sc}
\end{bmatrix}
$$

or $PQ_i = Y^{*}_i$ and $Q_i = P^{-1}Y^{*}_i$. It is claimed that such an

allocation of futures contracts can be achieved in futures

markets at appropriately selected futures prices.

First it must be shown that individual budget constraints

are satisfied. Define a diagonal matrix D as

$$
D =
\begin{bmatrix}
r_{1C} & & & & 0 \\
& r_{2C} & & & \\
& & \cdot & & \\
& & & \cdot & \\
& & & & \cdot \\
0 & & & & r_{SC}
\end{bmatrix}
$$

Recall that $r_{sc}/r_{sC} = P_{sC}$. Then from above

$$
DP(Q_i - E_i) = D(Y^{*}_i - \hat{Y}_i)
$$

with typical row s,

$$
\sum_{c=1}^{S} r_{sc}(Q_{ic} - E_{ic}) = \sum_{c=1}^{C} r_{sc}(X^{*}_{isc} - Z_{isc}).
$$

Summing down rows

$$
\sum_{c=1}^{S} (Q_{ic} - E_{ic})\left(\sum_{s=1}^{S} r_{sc}\right) = \sum_{s=1}^{S} \sum_{c=1}^{C} r_{sc}(X^{*}_{isc} - Z_{isc}).
$$

But by hypothesis

$$
\sum_{s=1}^{S} \sum_{c=1}^{C} r_{sc}(X^{*}_{isc} - Z_{isc}) = 0.
$$

Hence

$$
\sum_{c=1}^{S} (Q_{ic} - E_{ic})f_c = 0.
$$

Here $f_c = \sum_{s=1}^{S} r_{sc}$ can be interpreted as a futures price of

commodity c. It remains to show that all such actions by

individuals are consistent in the sense that futures markets

clear. From above,

$$(Q_i - E_i) = P^{-1}(Y_i^* - \hat{Y}_i) \Rightarrow \sum_{i=1}^{I} (Q_i - E_i) = P^{-1} \sum_{i=1}^{I} (Y_i^* - \hat{Y}_i).$$

But

$$\sum_{i=1}^{I} (Y_i^* - \hat{Y}_i) = \sum_{i=1}^{I} \begin{bmatrix} \vdots \\ \sum_{c=1}^{C} P_{sc}(X_{isc}^* - Z_{isc}) \\ \vdots \end{bmatrix} =$$

$$\begin{bmatrix} \vdots \\ \sum_{c=1}^{C} P_{sc} \sum_{i=1}^{I} (X_{isc}^* - Z_{isc}) \\ \vdots \end{bmatrix}.$$

By hypothesis

$$\sum_{i=1}^{I} (X_{isc}^* - Z_{isc}) = 0.$$

Therefore

$$\sum_{i=1}^{I} (Q_i - E_i) = 0,$$

and futures markets clear.

Finally it is argued that each spot market is in equilibrium

at prices P_s. All agents act as if the matrix P'' is the

matrix of spot prices given a restriction to futures markets.

It has been shown that at those prices agents can achieve

the same distribution of incomes across states as in the

initial allocation. That is, each agent is on the same

budget line in each state as in the initial allocation. As

spot markets were in equilibrium initially at prices P_s,

spot markets of the restricted model are also in equilibrium

at those same prices. Q.E.D.

It should be noted that in the proof of Proposition I

it is <u>not</u> required that actual delivery be made in spot

markets of commodities which were sold forward. It is

supposed that agents accept delivery of all commodity bundles

which when valued at spot prices yield incomes equivalent to

the yield of the futures contract in question.

<u>Proposition II</u>: Suppose there exists an equilibrium in

complete markets for contingent claims and no trade in spot

markets in which the matrix P" is of rank S. Then at least

one equilibrium to the model with complete markets for

contingent claims and subsequent spot markets has the property

that all contracts are unconditional. The allocation of

that equilibrium will be Pareto optimal.

Proof: The proof follows immediately from Proposition I and

the optimality of the initial equilibrium. Q.E.D.

An attempt is now made to relate these results to those

of Arrow [2]. His principal conclusion is that an optimal

allocation of risk-bearing can be achieved by competitive

securities markets with subsequent spot markets.[5] He

emphasizes that a security is a claim payable in money in

contrast to claims against specific commodities. Yet in the

context of a pure exchange economy, money is no more than a

numeraire. For example, if the Cth good is selected as a

numeraire in each of the mutually exclusive spot markets, a pure security yielding one monetary unit if state s occurs and zero otherwise is nothing other than a contingent claim on commodity C in state s. It can be shown by a proof quite similar to that of Proposition I than any optimal allocation can be achieved with S securities whose returns are linearly independent; in a pure exchange economy, securities are linear combinations of contingent commodity claims, and security returns are in terms of equilibrium spot prices. Hence futures contracts are securities, and Proposition I may be viewed as an extension of Arrow's initial results.

The hypothesis that the matrix P'' be of rank S should be subject to some scrutiny. In what follows an example is given in which P'' is not of rank S and in which an optimal allocation would be attained though trade be restricted to futures contracts with subsequent spot markets. In a second example in which P'' is not of rank S, an optimal allocation would not be attained given such a restriction. The hypothesis that P'' be of rank S in Proposition II cannot in general be weakened though there are special cases in which it is not necessary.

Example One

Consider an economy in which I agents have identical, homothetic, state independent utility functions $U(\cdot)$ over two goods, (X) and (Y). Suppose in addition that all agents have identical endowments over states as in Table I, with $\pi_{is} = \pi_{js}$ for all agents i and j for each state s.

Table I

Endowments of [(X),(Y)]	
state one	(x,y)
state two	$(\theta x, \theta y)$

Here $0 < \theta$. The equilibrium relative spot price will depend only on the ratio of the aggregate endowment of (X) to the aggregate endowment of (Y) and will be constant over states. Hence P" is of rank one. Yet, trivially, agents will consume their endowment across states; optimal allocations will be attained in a competitive equilibrium with no active markets of any kind.

Example Two

Consider an economy in which I agents have identical, homothetic, state independent utility functions U(·) over two goods, (X) and (Y). Agents can be divided equally into two groups, A and B, the endowments of which are listed in Table II.

Table II

Endowments of [(X),(Y)]		
	group A	group B
state one	$(x+e,y)$	$(x-e,y)$
state two	$(x-e,y)$	$(x+e,y)$

It is assumed that $\pi_{i1} = \pi_{i2} = 1/2$ for each agent i. In a competitive equilibrium with complete markets, agents of group A will agree to give up e units of (X) contingent on state one for e units of (X) contingent on state two, and conversely for agents of group B. In such an equilibrium each agent will have the same consumption bundle across states. However the equilibrium spot price will again be constant over states. If trading were restricted to futures contracts with subsequent spot markets, no futures contracts would be actively traded; to avoid arbitrage the relative futures price must be the same as the relative spot price, and a futures contract would not alter the distribution of income across states. An optimal allocation would not be achieved.

As noted in the proof of Proposition I, if S > C, P" cannot be of rank S. It is useful to consider the way in which the proof of Proposition I fails for such economies. From Proposition I there is need of an equation of the form $PQ_i = Y_i^*$ where P is an S x C matrix, Q_i is a C x 1 column vector, and Y_i^* is an S x 1 column vector. With S-C more equations than unknowns, such systems will in general be inconsistent. A necessary condition for allocations to be Pareto optimal is that the rate of substitution of contingent commodities be equal for all individuals. With complete markets all agents face the same price ratios for all pairs of commodities and equality of rates is ensured. The

inconsistency of the above equation suggests that with
$S > C$, there are not enough prices or trading instruments to
achieve such an equality of rates.

This can be illustrated in rather heuristic fashion by
consideration of a special case--three states and two commodities.
Suppose the rates of commodity substitution were equal for
all individuals. Let RCS' denote the rate of substitution
of the first commodity in state one for the second commodity
in state two, and let RCS" denote the rate of substitution
of the first commodity in state one for the second commodity
in state three. Let f denote the forward price of the first
commodity in terms of forward units of the second commodity.
Let X^{iF} denote the _net_ forward purchase by individual i of
the first commodity in the market for claims. Then the
following equations must hold:

$$RCS' = \frac{\pi_{i1} P_{11} V_1^{is} [P_{11} Z_{i11} + Z_{i12} + X^{iF}(P_{11}-f), P_{11}]}{\pi_{i2} V_1^{is} [P_{21} Z_{i21} + Z_{i22} + X^{iF}(P_{21}-f), P_{21}]}$$

$$RCS'' = \frac{\pi_{i1} P_{11} V_1^{is} [P_{11} Z_{i11} + Z_{i12} + X^{iF}(P_{11}-f), P_{11}]}{\pi_{i3} V_1^{is} [P_{31} Z_{i31} + Z_{i32} + X^{iF}(P_{31}-f), P_{31}]}$$

$V_1^{is}(\cdot, P_{s1})$ is continuous with respect to X^{iF} with

$V_1^{is}(0, P_{s1}) = \infty$, and so the existence of a solution for each
equation follows from the intermediate value theorem. Yet
in general there is no X_1^{iF} which solves _both_ equations. In
general then, with $S > C$, the equilibrium of the restricted
model is not Pareto optimal. It also follows that there
exists in general an allocation which is Pareto superior and

which could be supported with a state dependent redistribution program by a competitive equilibrium in complete markets for contingent claims.[6]

If tastes were identical, state independent, and homothetic, it would be possible to make sharper welfare comparisons. As noted above, spot prices would then be independent of the direction and type of forward contracts. If there were active forward markets, and if an individual chose not to participate in such markets, then his consumption possibility set would be as it would have been had there been no forward markets; the possibility of forward transactions can only make him better off. Hence the allocation of the model restricted to futures contracts with subsequent spot markets would be Pareto noninferior and possibly Pareto superior to the allocation with all forward markets prohibited.

IV. Concluding Remarks

Jacques Drèze [3] has stressed the need for research into the functions and shortcoming of existing institutions and for the application of standard welfare economics based on Pareto optimality to limited exchange opportunities for risk-bearing. The objective of this essay was to examine the workings and welfare implications of futures markets and to place those markets in the context of complete markets for contingent claims. It was found that with at least as many goods as states, pre-state futures markets with post-state spot markets may support Pareto optimal allocations.

A future contract can be viewed as a security whose state specific return is the endogenous market-clearing spot price.

It is assumed in Propositions I and II that agents know the distribution of spot prices which is consistent with their actions in forward markets. If the full set of markets is available prior to the realization of the state, then the market-clearing spot prices are known to everyone. If only futures markets are permitted, then the prices of forward markets do not convey information as to the future spot prices. Hence agents are assumed to have common if not rational expectations.[7] This assumption may be more palatable if for example tastes are identical and homothetic, for then spot prices are independent of the direction and type of contracts made by agents in forward markets. Without such an assumption, the distribution of spot prices will in general depend on the actions of all agents in the forward markets. It is this type of simultaneity that has led Radner [5] to question the existence and optimality of a competitive equilibrium if there are futures markets with subsequent spot markets. Jerry Green [4] has dealt with the question of existence of a temporary equilibrium given exogenous subjective distributions over future spot prices, yet he does not propose a mechanism of expectation formation. Though rational expectations have been assumed by many authors, the objections raised by Radner are deserving of

further analysis. However such an analysis goes well beyond
the scope of this paper.

The ultimate intent of a essay of this sort is to
explain why futures contracts with subsequent spot markets
is a prominent institutional configuration. If agents were
indifferent between complete markets for contingent claims
and futures contracts with subsequent spot markets, and if
there were a cost associated with the former contracts which
is not associated with the latter, then one structure would
emerge endogenously. A cost which might be associated with
contingent but not with futures contracts could be the cost
of state verification. It is in this sense that the requirement
that P'' be of rank S is fundamentally disappointing. If P''
is of rank S, then no two rows of P'' can be identical.
Agents will be fully informed by the spot market prices of
which state has occurred. State verification is costless
and is no obstacle to the making of contingent contracts.
Futures contracts with subsequent spot markets may allow
agents to do just as well, but there is nothing in the model
to lead them to choose one structure over the other. Making
the choice of market structure endogenous is fundamental to
coherent economic models. This is being pursued in further
research.

FOOTNOTES

[1] To facilitate comparison some of Arrow's [2] notation is retained.

[2] See Arrow-Hahn [1]. As each agent has strictly positive marginal utility for each good, all agents are resource related.

[3] In equilibrium in state s, P_s is strictly positive by virtue of property (i).

[4] I am indebted to Paul Anderson who initially suggested the condition that P be of full rank.

[5] This conclusion is based on the premise that the spot price which will prevail in each state is known to all agents.

[6] I am indebted to Steve Salant for this comment.

[7] For a distinction between common and rational expectations see Radner [6, p. 53].

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ESSAY II

A NOTE ON THE DIRECTION OF FUTURES TRADING
IN A GENERAL EQUILIBRIUM MODEL

I. Introduction

A standard dichotomy for traders in futures markets
originated with the work of Keynes [8] and Hicks [4]. In
their theory, risk averse hedgers seek to reduce price risk
by selling forward that portion of supply which is planned
or fixed in advance. On the other side of the market are
speculators who seek a profit by buying forward when the
futures price is below the spot price expected to prevail at
the time of maturity. It is the essence of speculation that
the speculator puts himself into a more risky position as a
result of his forward trading.

This essay argues that such a partial equilibrium
terminology for traders in futures markets is not useful.
It is shown that in a pure exchange economy with random
endowments, risk averse agents use futures contracts as a
hedge against the random valuation of those endowments. In
such a model, agents do not seek to reduce risk by purchasing
comsumption bundles forward; this is illustrated by an
example in which agents buy forward the good with which they
are endowed. It is argued that in general the direction of
trade in futures markets depends on tastes and endowments as

in standard theory. It is also argued that there need be no special relationship between a futures price and the price expected to prevail in spot markets; as with the direction of trade, the relationship of a futures price to spot prices depends on all the parameters of the model.

II. Assumptions and Technology of the Model

The model is a pure exchange economy with random endowments. Before the random endowments are realized, each agent can decide on the quantity and type of futures contract to purchase or issue. After endowments are realized, futures contracts are executed, and trade and consumption take place in spot markets. In the model no actual transfer or consumption of resources takes place in futures markets; only claims are traded. In spot markets each agent knows his endowment and the market price. An agent's consumption in spot markets is not limited to initial endowments and forward purchases—retrading is possible. Such retrading is crucial to the central propositions of this paper.

In the model there are two goods, denoted (x) and (y), and m agents. Let Ω denote a set of n states with typical element w. Let $Z^i(w)' = [Z^i_x(w), Z^i_y(w)]$ where $Z^i_j(w)$ denotes the exogenous endowment of commodity j in state w for agent i. For each state w and for each agent i, $Z^i_j(w) > 0$ for some commodity j. Also $\sum_{i=1}^m Z^i_j(w) > 0$ for each state w and each commodity j.

Each agent is assumed to maximize expected utility. Let $c^i_j(w)$ denote the consumption of commodity j in state w

by agent i. Let $g(w)$ denote the probability that state w
will occur. It is assumed that $g(w) > 0$ for each state w.
Then agent i acts as if to maximize $\sum_{w \varepsilon \Omega} g(w) \, U^i[C_x^i(w), \, C_y^i(w)]$
with respect to $C_j^i(w)$, $j = x,y$. Note that the function $U^i(\cdot,\cdot)$
does not depend on the state w. The following properties
are assumed:

   (i). $U^i(\cdot,\cdot) : R_+^2 \to R^1$ is of class C^2 with first partial
   derivatives $U_1^i(\cdot,\cdot)$, $U_2^i(\cdot,\cdot)$ strictly positive. Also
   $U_1^i(0,0) = +\infty.$

   (ii). The Hessian of $U^i(\cdot,\cdot)$ is negative definite.

Under property (ii), $U^i(\cdot,\cdot)$ is strictly concave, and all
agents are said to be risk averse.

   By construction spot markets are mutually exclusive,
and hence (x) can be taken as the numeraire in each spot
market. Then let $P(w)$ denote the price of (y) in terms of
(x) in state w. Let $p(w) = [1,P(w)]$. Let $I^i(w)$ denote the
expenditures in terms of (x) in state w by agent i. Then
$I^i(w) = P(w)C_y^i(w) + C_x^i(w)$. Let $h_j^i[I^i(w), \, P(w)]$ denote the
demand for commodity j in state w by agent i as a function
of $I^i(w)$ and $P(w)$. Then, taking $I^i(w)$ and $P(w)$ as parameters,
$h_j^i[I^i(w), \, P(w)]$ is a maximizing choice of $C_j^i(w)$ for the
function $U^i[C_x^i(w), \, C_y^i(w)]$ subject to the constraint that
$P(w)C_y^i(w) + C_x^i(w) = I^i(w)$. An indirect utility function
$V^i(\cdot,\cdot)$ is defined as

$$V^i[I^i(w), \, P(w)] = U^i\{h_x^i[I^i(w), \, P(w)], \, h_y^i[I^i(w), \, P(w)]\}$$

   In pre-state futures markets each agent can make commitments
to purchase or sell in post-state spot markets specified

amounts of specified commodities at specified prices independent of the state which occurs. More formally, let X^i and Y^i denote the forward purchases of (x) and (y) respectively by agent i. As futures markets and spot markets are mutually exclusive, (x) may also be taken as the numeraire of futures markets. Let F denote the price of (y) in terms of (x) in futures markets. The budget constraint for agent i in futures markets is then of the form

(1) $X^i + FY^i = 0$

Then from equation (1)

$$I^i(w) = p(w) \, Z^i(w) + Y^i[P(w)-F]$$

In summary, taking spot and futures prices as parameters, agent i acts as if to maximize

(2) $H^i(F,Y^i) = \sum_{w \in \Omega} g(w) V^i \{ p(w) \, Z^i(w) + Y^i[P(w)-F], \, P(w) \}$

with respect to Y^i subject to constraints of the form $I^i(w) \geq 0$ for each state w. By property (i) of $U^i(\cdot,\cdot)$, $V_1^i[0,P(w)] = +\infty$, and therefore in equilibrium in each state w, $I^i(w) > 0$. (Note also that by property (i), $0 < P(w) < \infty$ in a competitive equilibrium for each spot market.)

In a competitive equilibrium with spot and futures markets, the spot prices which each agent takes as parameters in choosing futures contracts must also be prices for which spot markets are in equilibrium. More formally, equilibrium spot prices are those for which $\sum_{i=1}^m \{ h_j^i[I^i(w), \, P(w)] - Z_j^i(w) \} = 0$ for each commodity j and each state w. In general, equilibrium spot prices will depend on the existence and direction of

futures trading. For purposes of simplification the following property is assumed:

(iii). Preferences of all agents are identical and homothetic. The superscript i on utility functions is now deleted. Under property (iii) the equilibrium spot price of state w will be a function of the ratio of the aggregate endowment of (x) in state w to the aggregate endowment of (y) in state w. That is,

$$P(w) = \frac{U_2[\sum_{i=1}^{m} Z_x^i(w), \sum_{i=1}^{m} Z_y^i(w)]}{U_1[\sum_{i=1}^{m} Z_x^i(w), \sum_{i=1}^{m} Z_y^i(w)]} = \theta\left\{\frac{\sum_{i=1}^{m} Z_x^i(w)}{\sum_{i=1}^{m} Z_y^i(w)}\right\}$$

with $\theta(\cdot)$ strictly monotone increasing. Hence under (iii), equilibrium spot prices will be independent of the existence and direction of futures trading. Note also that the assumptions on preferences and endowments made above do ensure the existence of a unique competitive equilibrium in each state w.[1] There remains the task of verifying the existence of a futures price F for which futures markets are also in equilibrium, i.e., [2]

$$\sum_{i=1}^{m} X^i = \sum_{i=1}^{m} Y^i = 0.$$

III. Competitive Equilibrium with Futures Markets

The purpose of this section is to establish the existence of a competitive equilibrium with spot and futures trading with the technology and preferences assumed thus far. In the process some characteristics of the demand for futures contracts are derived.

Let $P' = \underset{w \in \Omega}{\text{Min}}\, P(w)$ and $P'' = \underset{w \in \Omega}{\text{Max}}\, P(w)$. If $P' = P''$,

then in an equilibrium with futures markets $F = P''$; if

$F \neq P''$, riskless arbitrage would make $H^i(F, Y^i)$ unbounded.

For $F = P' = P''$, futures contracts will not alter the distribution

of expenditures across states, and therefore futures markets

will not be active. It is assumed subsequently that $P' < P''$.

Let $E = (P', P'')$. From the above remarks the search for a

market clearing futures price can be limited to $F \in E$.

<u>Lemma 1</u>: For each $F \in E$, $V_{11}[\cdot, P(w)] < 0$ and $H^i(F, \cdot)$ is

strictly concave.

Lemma 1 follows from property (ii) and the proof is not

given here.

By homotheticity $h^i_j[I^i(w), P(w)]$ is a linear and hence

continuous function of $I^i(w)$. Then $V[\cdot, P(w)]$ is continuous,

and therefore $H^i(F, \cdot)$ is continuous. Given some fixed $F \in E$

let $S_1 = \{w: F - P(w) > 0\}$ and let $S_2 = \{w: F - P(w) < 0\}$.

Then from the constraints that $I^i(w) \geq 0$ for each state w

$$\underset{w \in S_2}{\text{Max}} \left[\frac{p(w)\, Z^i(w)}{F - P(w)} \right] \leq Y^i \leq \underset{w \in S_1}{\text{Min}} \left[\frac{p(w)\, Z^i(w)}{F - P(w)} \right].$$

Hence for each $F \in E$ a maximizing choice of Y^i does exist as a

continuous function on a compact set achieves a maximum on

that set. By lemma 1 the choice must be unique. This

optimum value of Y^i will be denoted $Y^{i*} = \phi^i(F)$. A necessary

and sufficient condition for a maximum is the first order

condition:

(3) $\quad \underset{w \in \Omega}{\sum} g(w)[P(w) - F]V_1\{p(w)\, Z^i(w) + Y^{i*}[P(w) - F], P(w)\} = 0$

Let $I^i(w)^* = p(w) Z^i(w) + Y^{i*}[P(w)-F]$. As noted $I^i(w)^* > 0$

for each state w. Hence Y^{i*} will be an interior solution.

Lemma 2: Given properties (i)-(iii) of $U(\cdot,\cdot)$, $\phi^i(F)$ has a

continuous derivative.

Proof: Let

$$G^i(F,Y^i) = \sum_{w \varepsilon \Omega} g(w)[P(w)-F] V_1\{p(w) Z^i(w) + Y^i[P(w)-F], P(w)\}$$

Then for each $F \varepsilon E$, $G^i[F, \phi^i(F)] = 0$; $G^i_1(\cdot,\cdot)$, $G^i_2(\cdot,\cdot)$ are

continuous with respect to F and Y^i; and $G^i_2(\cdot,\cdot) < 0$ by

lemma 1. Hence the implicit function theorem applies. Q.E.D.

Lemma 3: Under properties (i)-(iii) of $U(\cdot,\cdot)$, $\phi^i(F) \to +\infty$ as

$F \to P'$ and $\phi^i(F) \to -\infty$ as $F \to P''$.

Proof: Suppose it were not the case that $\phi^i(F) \to +\infty$ as

$F \to P'$.

Then one can construct a sequence $\{F_n\}$ in E such that

$F_n \to P'$ as $n \to \infty$ and the corresponding sequence $\phi^i(F_n)$ is always

less than some positive number K. From the constraints that

$I^i(w) \geq 0$, there exists some N_1 and some B such that for all

$n > N_1$, $\phi^i(F_n) > B$. Hence $\phi^i(F_n)$ is bounded in some left-

deleted neighborhood of P'. Let $W = \{w:P(w) = P'\}$. Then

there exists some N_2 such that for all $n > N_2$ and for each

$w \notin W$, $P(w) > F_n$. Then

$$\lim_{F_n \to P'} G^i[F_n, \phi^i(F_n)] =$$

$$\lim_{F_n \to P'} \sum_{w \varepsilon W} g(w)V_1\{p(w) Z^i(w) + \phi^i(F_n) [P(w)-F_n], P(w)\}[P(w)-F_n] +$$

$$\lim_{F_n \to P'} \sum_{w \notin W} g(w)V_1\{p(w) Z^i(w) + \phi^i(F_n) [P(w)-F_n], P(w)\}[P(w)-F_n]$$

$$\geq 0 + \sum_{w \in W} g(w)V_1\{p(w) \ Z^i(w) + K[P(w)-P'], \ P(w)\}[P(w)-P'] > 0$$

since $V_1[\cdot,P(w)]$ is continuous and strictly positive. This

is the desired contradiction as for each $F \epsilon E$, $G^i[F,\phi^i(F)] = 0$.

The proof that $\phi^i(F) \rightarrow -\infty$ as $F \rightarrow P''$ follows similarly.

Q.E.D.

Proposition I: Under properties (i)-(iii) of $U(\cdot,\cdot)$, the

model with futures contracts and subsequent spot markets

possesses a competitive equilibrium.

Proof: Define $\phi(F) = \sum_{i=1}^{m}\phi^i(F)$. By lemma 2, $\phi^i(F)$ is

continuous on E. By lemma 3 there exist F' and F'' in E such

that $\phi(F') > 0$ and $\phi(F'') < 0$. Therefore by the intermediate

value theorem there exists some F* such that $\phi(F*) = 0$.

This establishes existence. Q.E.D.

IV. On the Direction of Futures Trading

The purpose of this section is to examine the essential

workings of futures markets in the stochastic general equilibrium

framework of the model. It is shown that risk averse agents

use futures contracts as a hedge against the random valuation

of initial endowments. In general the direction of trade

and the equilibrium futures price depend on tastes and

endowments as in standard theory.

Some additional restrictions on preferences and endowments

are needed. It is assumed that all agents are endowed with

either (x), agents of group A, or (y), group B.

Hence

$$Z_x^i(w) = \begin{cases} S_x^i X(w), & i \in A \\ 0 & i \in B \end{cases}$$

$$Z_y^i = \begin{cases} S_y^i Y, & i \in B \\ 0 & i \in A \end{cases}$$

$$\sum_{i \in A} S_x^i = 1; \quad \sum_{i \in B} S_y^i = 1; \quad 0 < S_x^i < 1; \quad 0 < S_y^i < 1.$$

$X(w)$ represents the aggregate over all individuals of the economy's endowment of (x) in state w. Y is nonstochastic and represents the aggregate endowment of (y). One can interpret the model as consisting of two groups. One group, farmers, produces only wheat, the output of which is exogenous and stochastic. Bad weather is assumed to diminish the crop to the same extent for all farmers. The other group produces only manufactured goods, the output of which is certain. In terms of the technology of section II, individuals of group B have degenerate distributions of holdings of (y), and members of group A have identically distributed holdings of (x) up to a constant of proportionality. Without loss of generality it is assumed that $X(w)$ is strictly increasing over the total of n states. Hence $P(w)$ is also strictly increasing.

An additional property is assumed:

(iv). All individuals have constant relative risk aversion.

A coefficient of relative risk aversion, C^i, is defined in terms of the indirect utility function:

$$C^i = C^i[I^i(w), P(w)] = -V_{11}[I^i(w), P(w)]I^i(w)/V_1[I^i(w), P(w)]$$

Constancy means that C^i is independent of both prices and incomes.

A coefficient of absolute risk aversion, D^i, is defined as follows:

$$D^i[I^i(w), P(w)] = -V_{11}[I^i(w), P(w)]/V_1[I^i(w), P(w)]$$

Then $D^i[I^i(w), P(w)] = C^i/I^i(w)$ and D^i is strictly decreasing in $I^i(w)$ and independent of $P(w)$. (For notational purposes D^i will be taken as a function of $I^i(w)$ alone.)

Lemma 4: Given properties (i)-(iv) of $U(\cdot,\cdot)$, $\phi^i(F)$ is strictly monotone decreasing.[3/]

Proof: By lemma 2 $\phi^i(F)$ is differentiable, and

$$\frac{d\phi^i(F)}{dF} = \frac{-G_1^i(F,Y^{i*})}{G_2^i(F,Y^{i*})}.$$

$$G_1^i(F,Y^{i*}) = -Y^{i*}\sum_{w\in\Omega} g(w)[P(w)-F]V_{11}[I^i(w)^*, P(w)]-$$

$$\sum_{w\in\Omega} g(w)V_1[I^i(w)^*, P(w)]$$

$$G_2^i(F,Y^{i*}) = \sum_{w\in\Omega} g(w)[P(w)-F]^2 V_{11}[I^i(w)^*, P(w)] < 0$$

It remains to establish that $G_1^i(F,Y^{i*}) < 0$. For suppose $[P(w)-F] \geq 0$ and that $Y^{i*} \geq 0$. Then

$$D^i\{p(w)Z^i(w) + Y^{i*}[P(w)-F]\} \leq D^i[p(w) Z^i(w)]$$

Hence

$$[P(w)-F]V_{11}\{p(w)\ Z^i(w)\ +\ Y^{i*}[P(w)-F],\ P(w)\}\ \geq$$

$$-V_1\{p(w)\ Z^i(w)\ +\ Y^{i*}[P(w)-F],\ P(w)\}\ D^i[p(w)\ Z^i(w)][P(w)-F]$$

If $[P(w)-F]\leq 0$, this relationship still holds. Hence,

$$\sum_{w\epsilon\Omega}\ g(w)[P(w)-F]V_{11}[I^i(w)^*,P(w)]\ \geq$$

$$-\sum_{w\epsilon\Omega}\ g(w)[P(w)-F]D^i[p(w)\ Z^i(w)]V_1[I^i(w)^*,P(w)]$$

For members of group A, $D^i[p(w)\ Z^i(w)] = D^i[S_x^iX(w)]$ is strictly monotone decreasing with respect to w. For individuals of group B, $D^i[S_y^iY\ P(w)]$ is strictly monotone decreasing with respect to w. Hence, from the first order condition (3),

$$\sum_{w\epsilon\Omega}\ g(w)[P(w)-F]V_{11}[I^i(w)^*,\ P(w)]\ >\ 0$$

and $d\phi^i(F)/dF < 0$. The case of $Y^{i*} < 0$ is similar with appropriate changes in sign. Q.E.D.

Following Arrow [2], if relative risk aversion is constant, the willingness to accept a bet should remain unchanged as the bet and income are increased proportionately. Formally, this leads to lemma 5.

Lemma 5: In this example, with properties (i)-(iv) of $U(\cdot,\cdot)$, let Y^{i*} denote an optimal choice given exogenous endowment $p(w)\ Z^i(w)$, and some fixed $F\epsilon E$. Then, if the endowment changes to $k[p(w)\ Z^i(w)]$, the optimizing choice will change to kY^{i*} for the same fixed F.

Proof: Following Stiglitz [9], constant relative risk averse functions are of one of the following forms where $a(\cdot)$ and $d(\cdot)$ are functions of $P(w)$:

$$V[I^i(w), P(w)] = a[P(w)] \ln [I^i(w)] + d[P(w)]$$

$$V[I^i(w), P(w)] = a[P(w)][I^i(w)]^{c+1} + d[P(w)] \quad -1 < c < 0$$

The proof follows immediately from the first order conditions (3). Q.E.D.

<u>Corollary to Lemma 5:</u> If $Y^{i*} = 0$ initially, then as $p(w) Z^i(w)$ changes to $k[p(w) Z^i(w)]$, Y^{i*} will remain zero.

From lemma 5 one may treat the economy as if consisting of two individuals in which A has all of (x) in all states and B has all of (y). Each individual treats prices as parameters.

The following property is also assumed:

(v). $U(\cdot,\cdot)$ displays constant elasticity of substitution σ. Here

$$\sigma = \frac{d\{\log[C_x^i/C_y^i]\}}{d\{\log[U_2(C_x^i,C_y^i)/U_1(C_x^i,C_y^i)]\}}$$

Let $R(w) = X(w)/P(w)Y$. The findings are summarized in the following proposition.

<u>Proposition II:</u> Under the assumptions of the example with properties (i)-(v) of $U(\cdot,\cdot)$,

(i).   if $\sigma > 1$ then $Y^{A*} < 0$,

(ii).   if $\sigma = 1$ then $Y^{A*} = Y^{B*} = 0$,

(iii).   if $0 < \sigma < 1$ then $Y^{A*} > 0$.

Proof: Case (i) will be considered in detail. In the argument which follows, it is useful to refer to Figure 1. Let

$$G^A(F, Y^A) =$$

$$\sum_{w \in \Omega} g(w)[P(w)-F]V_1\{X(w) + Y^A[P(w)-F], P(w)\}.$$

By lemma 2 and lemma 3 there exists some F'' such that $G^A(F'', 0) = 0$. Define $s \in \Omega$ such that $P(s) \leq F'' < P(s + 1)$. Let

$$G^B(F, Y^B) =$$

$$\sum_{w \in \Omega} g(w)[P(w)-F]V_1\{P(w)Y + Y^B[P(w)-F], P(w)\}.$$

By homotheticity, with $\sigma > 1$, $R(w)$ is a strictly increasing function of w. Let $k = 1/R(s)$. Then $X(w)/R(w) > kX(w)$, $w = s + 1, \ldots n$. By lemma 1, $V_{11}(\cdot, \cdot) < 0$, and by construction $P(w)-F'' > 0$ for $w = s + 1, \ldots n$. Therefore,

$$\sum_{w=s+1}^{n} g(w)[P(w)-F'']V_1[X(w)/R(w), P(w)] >$$

$$\sum_{w=s+1}^{n} g(w)[P(w)-F'']V_1[kX(w), P(w)].$$

Similarly, for $w = 1, 2, \ldots s$, $X(w)/R(w) \geq kX(w)$ and $P(w) - F'' \leq 0$. Therefore,

$$G^B(F'', 0) > \sum_{w=1}^{n} g(w)[P(w)-F'']V_1[kX(w), P(w)].$$

But by the corollary to lemma 5,

$$G^A(F'', 0) = 0 \Rightarrow \sum_{w=1}^{n} g(w)[P(w)-F'']V_1[kX(w), P(w)] = 0.$$

Therefore, $G^B(F'', 0) > 0$. By a similar argument there exists an F' such that $G^B(F', 0) = 0$ and $G^A(F', 0) < 0$. By lemma 1,

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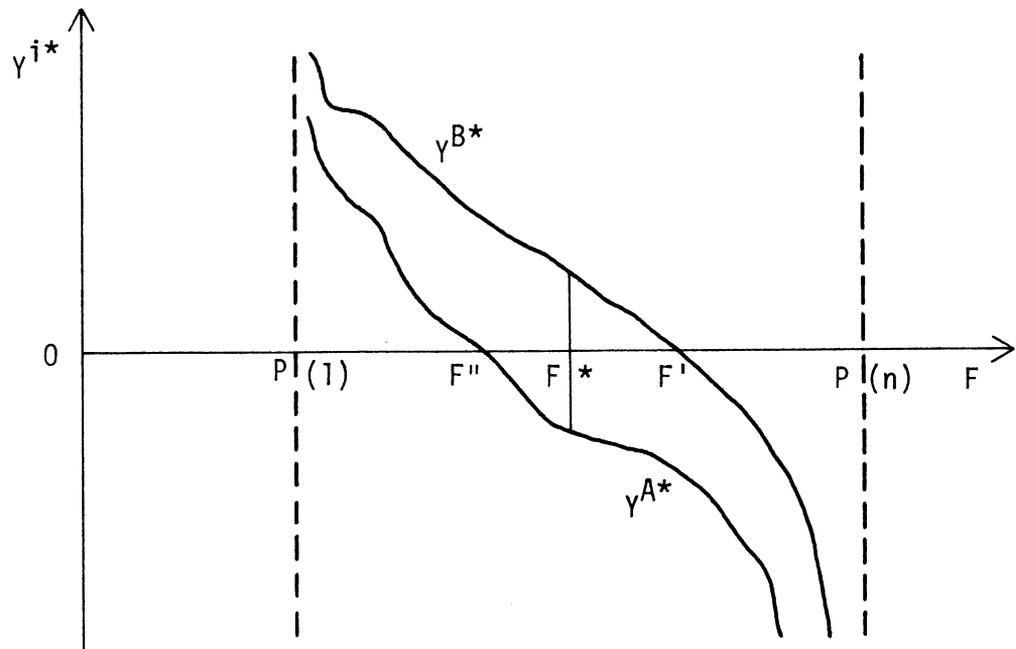
Figure 1

$G^B(F'',0) > 0 => \phi^B(F'') > 0$ and $G^A(F',0) < 0 => \phi^A(F') < 0$.

Define $\phi(F) = \phi^A(F) + \phi^B(F)$. By lemma 2, $\phi(F)$ is continuous.

As $\phi(F'') > 0$, $\phi(F') < 0$, there exists an F* such that

$F'' < F* < F'$ and $\phi(F*) = 0$. F* is an equilibrium forward

rate with $\phi^A(F*) < 0$. Uniqueness follows immediately from

montonicity of lemma 4.

Case (ii). If $\sigma = 1$, then $P(w)Y/X(w) = k$ and by lemma 5,

A and B will always be on the same side of the market. The

equilibrium solution must be $Y^{A*} = Y^{B*} = 0$.

Case (iii) follows from case (i) with appropriate

changes in sign. Q.E.D.

Roughly speaking, if $\sigma > 1$, the value in terms of (x)

of the exogenous endowment of B increases less as w increases

than does the exogenous endowment of A. In the terminology

of Hildreth [6], for a given forward rate, B is less anxious

to engage in a venture, the outcome of which is negatively

correlated with w. This difference in insurance values

creates a forward market despite agreement on the probability

distribution of future spot rates. With $\sigma = 1$, A's and B's

exogenous endowments in terms of (x) are perfectly correlated,

and a forward contract has the same insurance value for

each. It should be noted that with $\sigma > 1$, A will buy (x)

forward even though A is endowed with (x) only and can

anticipate consuming (y) in all states. With active spot

markets, optimal behavior under risk aversion does not

necessarily involve the "elimination of risk" by purchasing

the consumption bundle forward. From the proof of Proposition II

it is clear that the equilibrium futures price F* depends on all the parameters of the model.

V.   Concluding Remarks

This essay has argued that a distinction between speculators and hedgers is not useful in a general equilibrium context. In a pure exchange economy the direction of trade would seem to depend on preferences and endowments as in standard theory. Nor need there be some special relationship between the futures price F* and the expected spot price $\sum_{w \in \Omega} g(w) P(w)$. The expectations hypothesis that $F^* = \sum_{w \in \Omega} g(w) P(w)$ would seem to have measure zero among all possible configurations of prices. Attempts to find a suitable definition of speculation in general equilibrium models may lead to terminological debates which serve no useful purpose,[4] while a more general theory may explain phenomena inconsistent with the Keynes-Hicks dichotomy.[5]

[1] See Arrow-Hahn [1].

[2] Jerry Green [3] has analyzed conditions necessary for the existence of a temporary equilibrium in a model in which subjective price distributions may differ. Though more general than the model of this paper in many respects, the Green model assumes that endowments are certain and that utility functions are bounded. The purpose of this paper is to reveal the essential workings of futures markets by way of some particular examples.

[3] This proof was suggested by a reading of Arrow [2].

[4] Recently Hirshleifer [7] has proposed a definition of speculation in the context of a pure exchange economy with random endowments.

[5] Hieronymus [5] cites a USDA report on the holding of corn futures on January 27, 1967 which revealed that farmers held more than a third of the net long speculative position and were also hedging in the wrong direction.

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ESSAY III

PRICE FIXING SCHEMES AND
OPTIMAL BUFFER STOCK POLICIES

I. <u>Introduction</u>

In actual economies uncertainty as to the direction and extent of future price movements is pervasive. This has led some government policy makers to argue that government buffer stock programs which attempt to fix prices could reduce uncertainty and increase economic welfare. J. M. Keynes argued for the establishment of an ever-normal granary to eliminate the violence of individual price fluctuations associated with an unregulated competitive system [8]. Other economists respond with the long-held belief that prices play a crucial role in allocating resources efficiently. Waugh [11] and Oi [9] have even argued that price instability may be beneficial, though Samuelson [10] showed that the fluctuations proposed in a partial equilibrium setting were not really feasible in a closed model.

To some extent confusion on these issues stems from the failure of economists to provide coherent stochastic models. Most economic models are deterministic and hence give the impression that uncertainty is an anomaly rather than a fact of life. As for stochastic models, most are partial equilibrium and hence leave open the question of whether the uncertainty

can be eliminated. This essay takes a modest step toward
the formulation of a coherent general equilibrium model in
which one can analyze price-fixing schemes.

The model adopted is a pure exchange economy in which
the endowments of two goods, (x) and (y), are stochastic.
In each period the aggregate endowments of (x) and (y) are
independently distributed, and each series is assumed to be
independent and identically distributed over time. Both
goods can be stored with no storage costs and no depreciation.
Acting competitively, each of m risk averse individuals
maximizes expected utility over an infinite horizon by
choice of the amount of each good to consume in each period
and the amount to carry over to the following period. The
rate at which (x) exchanges for (y) in each period is the
price upon which the analysis focuses.

Presumably the simplicity of the model is one of its
desirable features. One would suspect that the government
is more apt to set the "right" price if the economy and the
stochastic processes of the economy are not complicated.
For the specific model examined it turns out that regardless
of the price set by the government and regardless of the
initial level of buffer stocks, a price-fixing scheme eventually
fails with probability one. The proof of this proposition
turns on a well-known theorem that random walks with no
drift in R^2 pass any boundary with probability one.

The essay proceeds as follows. Section II gives the
assumptions and technology of the model. Section III presents

a proof that price-fixing schemes will fail eventually with probability one. The proof is by contradiction; properties of individual storage decisions are derived on the assumption that the price will remain fixed, and these properties are then shown to be inconsistent with the initial level of government stocks. Section IV discusses the properties of the model without government and shows that the class of economies which possess competitive equilibria without government is nonempty. Section V offers some suggestions for further research.

II. <u>Assumptions and Technology of the Model</u>

The stochastic nature of the endowments is now made precise. In each time period there is a finite set of states Ω. Let F be the set of all subsets of Ω, and let P be a probability measure with $0 < P(\omega)$ for each $\omega \varepsilon \Omega$. Then (Ω, F, P) is a discrete probability space. Let X_t and Y_t denote the economy's total endowment of (x) and (y) respectively at time t. X_t and Y_t are assumed to be independent, non-negative, Borel measurable functions on (Ω, F). Further, X_t and Y_t are each independent and identically distributed over time. It is also assumed that there exists some $\overline{\omega} \varepsilon \Omega$ such that $X_t(\overline{\omega}) = Y_t(\overline{\omega}) = 0$. Let $X_t = E(X_t) + \varepsilon_{xt}$ and $Y_t = E(Y_t) + \varepsilon_{yt}$ where ε_{xt} and ε_{yt} are symmetric and take on values in the integer lattice. There will also be need of an infinite product space. Let $\Omega_t = \Omega$ and $F_t = F$ for all t. Then $(\Pi_{t=1}^{\infty} \Omega_t, \Pi_{t=1}^{\infty} F_t, \mu)$ is the desired space where $\Pi_{t=1}^{\infty} \Omega_t$

is the set of all sequences $(\omega_1, \omega_2, \ldots)$ such that $\omega_t \varepsilon \Omega_t$ for all t and $\Pi_{t=1}^{\infty} F_t$ is the smallest sigma-field containing the cylinder sets.

The maximization problem of each of the m individuals of the economy is now stated formally. Each individual is assumed to have a fixed and constant proportion of the aggregate endowment of each good, δ_x^j and δ_y^j being the jth individual's share of X_t and Y_t respectively. Let $z_t^{j\prime} = [\delta_x^j X_t, \delta_y^j Y_t]$. Let $K_{x,t}^j$ and $K_{y,t}^j$ denote the jth individual's stocks of (x) and (y) respectively at the end of the period t, to be chosen during period t. Let X_t^j and Y_t^j denote the jth individual's consumption of (x) and (y) respectively at time t. Let R_t denote the relative price of (y) in terms of (x) at time t. Let $r_t^{\prime} = [1, R_t]$. Then, regarding as parameters current and futures prices $\{R_t; t=1,2,\ldots\}$ and initial stocks $\{K_{x,0}^j \geq 0, K_{y,0}^j \geq 0\}$, the objective of individual j is to maximize $E_0 \sum_{t=1}^{\infty} \beta^{t-1} U^j(X_t^j, Y_t^j)$ with respect to $\{K_{x,t}^j, K_{y,t}^j, X_t^j, Y_t^j; t=1,2,\ldots\}$ subject to the following constraints:

(i). $\qquad X_t^j + R_t Y_t^j \leq K_{x,t-1}^j + R_t K_{y,t-1}^j +$

$\qquad\qquad r_t^{\prime} z_t^j - K_{x,t}^j - R_t K_{y,t}^j \qquad\qquad t=1,2,\ldots$

(ii). $\qquad X_t^j + R_t Y_t^j \geq 0 \qquad\qquad\qquad t=1,2,\ldots$

(iii). $\qquad K_{x,t}^j \geq 0; \; K_{y,t}^j \geq 0 \qquad\qquad t=1,2,\ldots$

Here $0 < \beta < 1$, and $E_t(\cdot)$ denotes the expectation conditioned on all realizations up to and including time t. Constraint (i) is the budget constraint for individual j at time t; to

be noted is the imposed absence of borrowing possibilities
and forward contracts. Constraint (ii) states that expenditures
at time t must be non-negative. Constraint (iii) states
that the stocks of each good must be non-negative.

$U^j(\cdot,\cdot)$ is assumed to have the following properties:

(i). $U^j(\cdot,\cdot) = g^j[H^j(\cdot,\cdot)]$ where $g^j(\cdot)$ is of class C^2
with first derivative $g_1^j(\cdot) > 0$ and second derivative
$g_{11}^j(\cdot) < 0$. Also

$g^j(0) = 0$, $g_1^j(H) \to 0$ as $H \to \infty$, and $g_1^j(H) \to \infty$ as $H \to 0$.

(ii). $H^j(\cdot,\cdot): R_+^2 \to R_+^1$ is of class C^2 with strictly
positive first partial derivatives. Also $H_{11}^j(\cdot,\cdot) < 0$,

$$\begin{vmatrix} H_{11}^j & H_{12}^j & H_1^j \\ H_{21}^j & H_{22}^j & H_2^j \\ H_1^j & H_2^j & 0 \end{vmatrix} > 0$$

(iii). $H^j(\cdot,\cdot)$ is homogeneous of degree one.

(iv). $H_1^j(X_t^j,Y_t^j) \to \infty$ as $(X_t^j,Y_t^j) \to (0,\overline{Y})$ for each $\overline{Y} > 0$

$H_2^j(X_t^j,Y_t^j) \to \infty$ as $(X_t^j,Y_t^j) \to (\overline{X},0)$ for each $\overline{X} > 0$

Thus $U^j(\cdot,\cdot)$ is a positive monotonic transformation of a
homogenous of degree one function and is said to be homothetic.
$U^j(\cdot,\cdot)$ possesses properties (ii) and (iv), among others.

III. The Eventual Failure of Price Fixing

The analysis of this section consists of a proof by
contradiction. It is supposed that a government maintains a fixed
price by its willingness to exchange (x) for (y) with the public
at the specified price. In general such a policy requires

that the government maintain stocks of both goods. Assuming

that the policy is feasible, the behavior relations of

agents are derived; much of the analysis of the paper consists

of solving the stochastic dynamic programming problem with

which each agent would be confronted. The crucial property

which emerges is that individual stocks are bounded. Then,

using the properties of random walks in R^2, a contradiction

is obtained; with probability one the government's stocks

will be insufficient to maintain the fixed price.

The assumption that price will remain fixed for all

time does not eliminate uncertainty from the decision problem

of the individual; endowments are still stochastic. Yet,

each individual need be concerned only with the total value

of stocks of (x) and (y) in terms of one of the goods, say

(x). The exchange of (y) for a predetermined amount of (x)

in any future period is guaranteed by the government. The

individual's decision problem may be viewed in two stages.

In a given period and state, the individual has available a

known amount of savings in terms of (x) of the previous

period and the realized value of his endowment in the specified

state. He decides on his optimal amount of savings in terms

of (x) and consequently on his current expenditures in terms

of (x). With this timing problem solved, the individual

then chooses his current consumption of (x) and (y) at the

specified price.

To state this more formally the following notation is

needed. Let $\bar{R}$ denote the relative price of (y) in terms of

(x) fixed by the government. Then, by assumption, $R_t = \bar{R}$ for all $t \geq 1$. Let $r' = [1,R]$. Let $K_t^j = K_{x,t}^j + \bar{R}K_{y,t}^j$. Let $I_t^j = X_t^j + \bar{R}Y_t^j$. Then, by the strict monotonicity of preferences, $I_t^j = K_{t-1}^j + r'z_t^j - K_t^j$. By virtue of the separability of the objective function over time, maximizing choices of X_t^j and Y_t^j can be expressed as functions of I_t^j and $\bar{R}$ only. Let $h_x^j(I_t^j,\bar{R})$ and $h_y^j(I_t^j,\bar{R})$ denote maximizing choices of X_t^j and Y_t^j respectively for the function $U^j(X_t^j,Y_t^j)$ subject to the constraint that $X_t^j + \bar{R}Y_t^j = I_t^j$. Then an indirect utility function $V^j(\cdot,\cdot)$ is defined by $V^j(I_t^j,\bar{R}) = U^j[h_x^j(I_t^j,\bar{R}),h_y^j(I_t^j,\bar{R})]$. Upon substitution, given $K_0^j \geq 0$, the objective of individual j is to maximize $E_0 \sum_{t=1}^{\infty} \beta^{t-1}V^j(K_{t-1}^j+r'z_t^j-K_t^j,\bar{R})$ with respect to $\{K_t^j; t=1,2,\ldots\}$ subject to the following constraints:

(i). $K_{t-1}^j + r'z_t^j - K_t^j \geq 0$ $t=1,2\ldots$

(ii). $K_t^j \geq 0$ $t=1,2,\ldots$

Constraint (ii) is apparently weaker than the restriction that $K_{x,t}^j$ and $K_{y,t}^j$ each be nonnegative. However, the imposition of the latter constraint would not alter the optimal storage rules for $\{K_t^j; t=1,2,\ldots\}$.

From the homogeneity of $H^j(\cdot,\cdot)$ the demand functions $h_x^j(\cdot,\bar{R})$ and $h_y^j(\cdot,\bar{R})$ are linear. It follows that $V^j(\cdot,\bar{R})$ is strictly concave.[1] Clearly $V^j(\cdot,\bar{R})$ is of class C^1 and strictly increasing. From property (i) of $U^j(\cdot,\cdot)$, $V_1^j(I,\bar{R}) \to \infty$ as $I \to 0$ and $V_1^j(I,\bar{R}) \to 0$ as $I \to \infty$.

First to be established are the existence and properties of the solution to the individual's optimization problem.[2]

Let $W_t^j = K_{t-1}^j + r'Z_t^j$ denote the wealth of individual j at time t in terms of (x). The idea is to find a function $f^j(\cdot,\cdot)$ such that

(1) $\quad f^j(W_t^j,\overline{R}) = \underset{0 \leq K_t^j \leq W_t^j}{\text{Sup}} [V^j(W_t^j-K_t^j,\overline{R})+\beta E_t f^j(r'Z_{t+1}^j+K_t^j;\overline{R})]$.

$f^j(\cdot,\cdot)$ is found by assuming the horizon is finite and taking the appropriate limit. Let

(2) $\quad f^{j,N}(W_N^j,\overline{R}) = \underset{0 \leq K_N^j \leq W_N^j}{\text{Max}} [V^j(W_N^j-K_N^j,\overline{R})+\beta E_N f^{j,N-1}(r'Z_{N-1}^j+K_N^j,\overline{R})]$

where $f^{j,N}(\cdot,\cdot)$ denotes the maximum of expected utility of individual j with N periods left in the individual's planning horizon. (Note also that in equation (2), time is measured with respect to the number of periods left in the agent's planning horizon.)

From the properties of $V^j(\cdot,\overline{R})$ and equation (2) it is established by induction that $f^{j,N}(W,\overline{R})$ is strictly concave in W for each N and monotone increasing with respect to N for each W. (See lemmas 1-5 in the Appendix.) From the discount rate β and the property that X_t and Y_t are bounded from above, it follows that $f^{j,N}(W,\overline{R})$ is uniformly bounded with respect to N for each W. (See lemma 6.) Hence for each $W \geq 0$, the limit of $f^{j,N}(W,\overline{R})$ exists as $N \to \infty$, and this limit is denoted $f^j(W,\overline{R})$. (See lemma 7.) It follows that $f^j(\cdot,\overline{R})$ is concave on $[0,\infty)$ and continuous on $(0,\infty)$ with finite partial derivatives from the left, $f_{1-}^j(W,\overline{R})$, and from the right, $f_{1+}^j(W,\overline{R})$; $f_{1-}^j(W,\overline{R}) \geq f_{1+}^j(W,\overline{R})$. (See lemma 3.) Also $f^j(W,\overline{R})$ satisfies the functional equation (1). (See

lemma 8.) Though continuity of $f^j(\cdot,\overline{R})$ at zero is not established, a maximum to the right side of equation (1) is attained on $[0,W_t^j]$. (See lemma 9.) As $f^j(\cdot,\overline{R})$ and $V^j(\cdot,\overline{R})$ are strictly concave, this maximizer is unique and is denoted $\tilde{K}_t^j = \phi^j(W_t^j,\overline{R})$. The maximizing level of consumption in terms of (x) is denoted $\psi^j(W_t^j,\overline{R})$.

Given that $\{\tilde{K}_t^j; \ t=1,2,\ldots\}$ is a sequence of current actions and contingent plans which maximize

$$V^j(W_t^j-K_t^j,\overline{R}) + \beta\, E_t f^j(K_t^j+r'z_{t+1}^j,\overline{R}),$$

it follows that $\{K_t^j\}$ maximize

$$V^j(W_t^j-K_t^j,\overline{R}) + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} V^j(r'z_\tau^j+K_{\tau-1}^j-K_\tau^j,\overline{R}).$$

(See lemma 11.) This justifies the functional equation approach.

Next to be established are the properties of the Markov processes $\tilde{K}_t^j = \phi^j(K_{t-1}^j+r'z_t^j,\overline{R})$, $j=1,2,\ldots m$. First, $0 < \tilde{K}_t^j < W_t^j$. For suppose that $\tilde{K}_t^j = 0$. Then $I_{t+1}^j(\overline{\omega}) = 0$. But $V_1^j(0,\overline{R}) = \infty$, and hence this could not be optimal; individual stocks are never zero. Similarly, if $\tilde{K}_t^j = W_t^j$, then $I_t^j = 0$, and this could not be optimal.

The differentiability of $f^j(\cdot,\overline{R})$ everywhere has not been established, but necessary and sufficient conditions for an optimum with respect to K_t^j are

$$-V_1^j(W_t^j-\tilde{K}_t^j,\overline{R}) + \beta\, E_t f_{1-}^j(r'z_{t+1}^j+\tilde{K}_t^j,\overline{R}) \geq 0$$

$$-V_1^j(W_t^j-\tilde{K}_t^j,\overline{R}) + \beta\, E_t f_{1+}^j(r'z_{t+1}^j+\tilde{K}_t^j,\overline{R}) \leq 0.$$

It follows that $\phi^j(\cdot,\overline{R})$ is nondecreasing and $\psi^j(\cdot,\overline{R})$ is strictly increasing. (See lemma 12.) Both are continuous. (See lemma 13.) Also $f^j_{1+}(W,\overline{R}) \to 0$ as $W \to \infty$. (See lemma 14.)

The important results on the boundedness of individual stocks are contained in Proposition I.

<u>Proposition I</u>: Given $\overline{R} > 0$, let $M^j = \text{Max}_{w \varepsilon \Omega} r'z^j_t(\omega) > 0$.

(i). For every $\overline{R} > 0$, there exists a unique K^j_*, possibly depending on $\overline{R}$, such that $\phi^j(K^j_*+M^j,\overline{R}) = K^j_* > 0$. Also $\phi^j(0,\overline{R}) = 0$.

(ii). For every $\overline{R} > 0$, let $A^j = [0,K^j_*]$. Then if $K^j_t \varepsilon A^j$, $K^j_\tau \varepsilon A^j$ for all $\tau > t$.

(iii). For all $K^j_0 \geq 0$, $\mu\{K^j_t \notin A^j \text{ i.o.}\} = 0$.

Proof of (i):[3/]

It is useful to refer to Figure 1. As $V^j_1(0,\overline{R}) = \infty$, $\phi^j(M^j,\overline{R}) > 0$. This establishes a positive intercept in Figure 1. Now suppose that $\phi^j(K^j_{t-1}+M^j,\overline{R}) > K^j_{t-1}$ for all $K^j_{t-1} \geq 0$ so that $\phi^j(\cdot,\overline{R})$ stays above the 45° line of Figure 1. Then it would be the case that for all $K^j_{t-1} \geq 0$

$$-V^j_1(M^j,\overline{R}) + E_t\beta f^j_{1+}(r'z^j_{t+1}+K^j_{t-1},\overline{R}) > 0.$$

Then

$$\lim_{K^j_{t-1} \to \infty} E_t\beta f^j_{1+}(r'z^j_{t+1}+K^j_{t-1},\overline{R}) \geq V^j_1(M^j,\overline{R}) > 0.$$

But $f^j_{1+}(W,\overline{R}) \to 0$ as $W \to \infty$ and this establishes a contradiction. Therefore there exists some K^j such that $\phi^j(K^j+M^j,\overline{R}) \leq K^j$. By continuity and the intermediate value theorem there

A Fixed Point For $\phi(\cdot,\bar{R})$
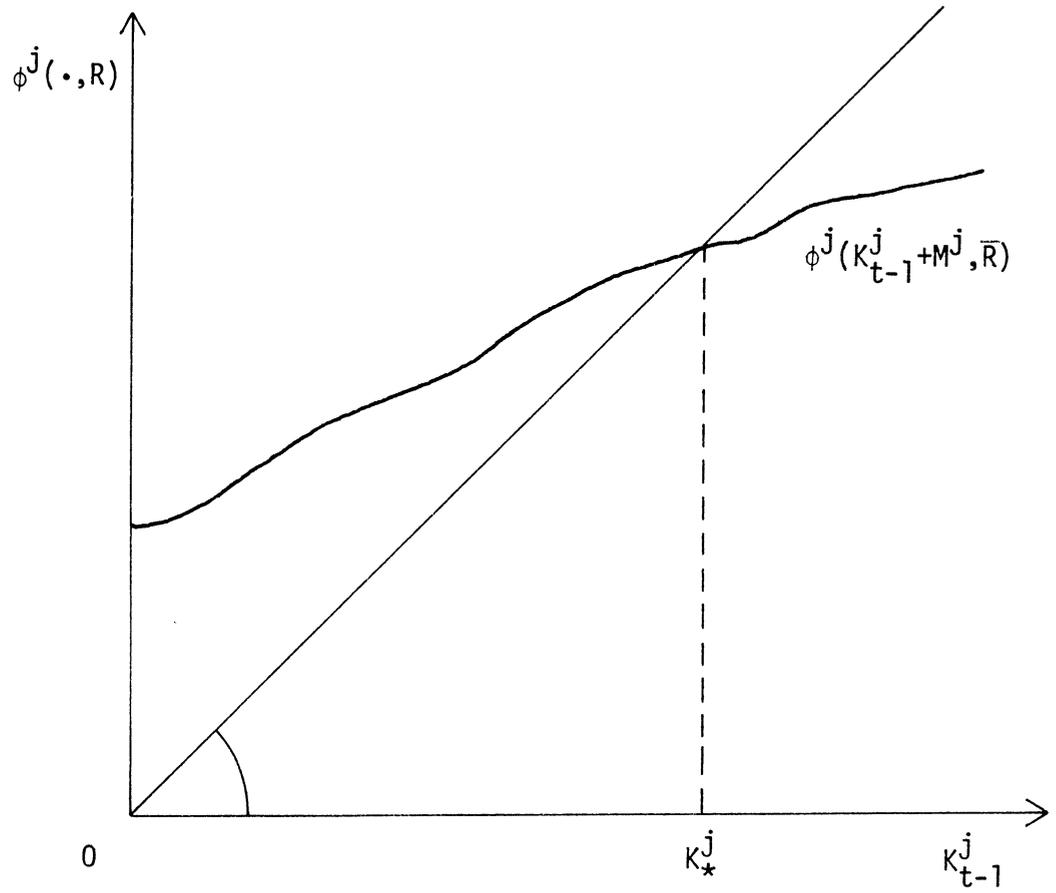
Figure 1

exists some K_*^j such that $\phi^j(K_*^j + M^j, \overline{R}) = K_*^j$. Uniqueness

follows from the strict monotonicity of $\psi^j(\cdot, \overline{R})$. As $V_1^j(0, \overline{R}) = \infty$,

$K_*^j > 0$. Clearly $\phi^j(0, \overline{R}) = 0$.

Proof of (ii):

Suppose $K_t^j \epsilon A^j$. Then by monotonicity:

$$\phi^j(K_t^j + r'z_{t+1}^j, \overline{R}) \leq \phi^j(K_*^j + M^j, \overline{R}) = K_*^j$$

$$\phi^j(K_t^j + r'z_{t+1}^j, \overline{R}) \geq \phi^j(0, \overline{R}) = 0.$$

Proof of (iii):

Suppose $K_{t-1}^j > K_*^j$. Then if for $\tau > t-1$, $\tilde{K}_\tau^j \leq K_*^j$,

case (ii) applies and (iii) is proved. Suppose that $\tilde{K}_t^j > K_*^j$

for all t. It can be shown $\tilde{K}_t^j \to K_*^j$ as $t \to \infty$. For suppose

that $\tilde{K}_t^j \geq K_{t-1}^j > K_*^j$. Then

$$-V_1^j(M^j, \overline{R}) + \beta E_t f_{1+}^j(K_*^j + r'z_{t+1}^j, \overline{R}) > 0.$$

But,

$$\phi^j(M^j + K_*^j, \overline{R}) = K_*^j \Rightarrow -V_1^j(M^j, \overline{R}) + E_t \beta f_{1+}^j(K_*^j + r'z_{t+1}^j, \overline{R}) \leq 0.$$

Therefore, by contradiction $\tilde{K}_t^j < K_{t-1}$ and $\tilde{K}_t^j \to B \geq K_*^j$ as

$t \to \infty$. By continuity $\phi^j(B + M^j, \overline{R}) = B$. By uniqueness $B = K_*^j$.

Now $\phi^j(K_*^j, \overline{R}) - K_*^j < 0$. By the continuity of $\phi^j(\cdot, \overline{R})$ there

exists some $\sigma > 0$ such that for all $K^j \epsilon (K_*^j, K_*^j + \sigma)$,

$\phi^j(K^j, \overline{R}) - K_*^j < 0$. As $\tilde{K}_t^j \to K_*^j$ as $t \to \infty$, there exists some T

such that for all $t > T$, $\tilde{K}_t^j < K_*^j + \sigma$. Recall that $r'z_{t+1}^j(\overline{\omega}) = 0$

and without loss of generality let $T = 0$. As $\sum_{t=1}^{\infty}[1 - P(\overline{\omega})]^t < \infty$,

the probability of never getting a realization of $\overline{\omega}$ is zero.

Hence $\mu\{\tilde{K}_t^j > K_*^j \text{ i.o.}\} = 0$. Q.E.D.

From Proposition I, it is clear that $\tilde{K}_t^j$ eventually enters the set A^j with probability one. Hence it is assumed that $K_0^j \varepsilon A^j$ and hence that K_t^j is bounded above by K_*^j for all t.

Having shown that there exists a solution to each individual's optimization problem given that the government is maintaining a fixed price, there will be determined some $\{I_t^j;\ j=1,2,\ldots m;\ t=1,2,\ldots\}$. Given I_t^j, individual j will purchase utility maximizing quantities of (x) and (y) at the fixed price $\bar{R}$. For the policy to be feasible it is specified that $\bar{R}$ be an equilibrium price in the sense that any excess demand for (x) by individuals be matched by an excess supply on the part of the government. It is now shown that no price can be maintained indefinitely far into the future regardless of the initial level of government stocks.

The proof of this proposition turns on associating with each fixed price $\bar{R}$ set by the government a pure exchange, nonstochastic, no-storage economy for which $\bar{R}$ is a competitive equilibrium price. In some sense the most difficult price for which to prove the infeasibility of price fixing is the competitive equilibrium price for the model with endowments $E(X_t)$ and $E(Y_t)$ for (x) and (y) respectively. Such a price corresponds to the competitive equilibrium price of the "average" economy in the model without storage but with random endowments. For this price, properties of a random walk in R^2 with zero drift are used. Roughly speaking, with probability one there will be runs over a number of periods

in which there are smaller than average endowments of (y) and higher than average endowments of (x). Total incomes will be at their average values, and agents will wish to consume average amounts of (y) at the fixed price. But (y) is in short supply, and eventual failure is inevitable. For all other prices, properties of random walks with nonzero drift are used; if the government sets a price of (y) which is in some sense too low, there will be an excess supply of (x).

More formally, consider a static pure exchange economy with fixed endowments. Each agent j has the same utility function $U^j(\cdot,\cdot)$ of the stochastic model with storage. Also let $Z^j = [\delta_x^j X, \delta_y^j Y]$ denote the endowment of agent j where the share parameters δ_x^j and δ_y^j are the same as in the stochastic model with storage. Then for $X > 0$ and $Y > 0$ there exists a unique competitive equilibrium price $R^* = \pi(X,Y)$ where $\pi(\cdot,\cdot):(0,\infty)^2 \to R_+^1$. It is shown by the implicit function theorem and properties (i) and (iv) of $U^j(\cdot,\cdot)$ that $\pi(\cdot,\cdot)$ is continuous. (See lemma 15 in the Appendix.) Also by property (iv), $\pi(X,\overline{Y}) \to 0$ as $X \to 0$ for each $\overline{Y} > 0$ and $\pi(\overline{X},Y) \to \infty$ as $Y \to 0$ for each $\overline{X} > 0$. (See lemma 16.)

<u>Proposition II</u>: Given any fixed price $\overline{R}\epsilon[0,\infty)$ and any initial level of government stocks of (x) and (y), there exists with probability one some T* at which those stocks will be insufficient to maintain the fixed price.

Proof: There are four cases to be considered.

Case (i): $\overline{R} = 0$

By property (ii) of $U^j(\cdot,\cdot)$ agents are never satiated with respect to consumption of (y). Hence if $\bar{R} = 0$, government stocks of (y) will be depleted in the first period of operation.

Case (ii): $\bar{R} = \pi[E(X_t), E(Y_t)]$.

Let G_0 denote the stock of (y) held by the government at $t = 0$, A_t denote the cumulative net amount of (y) sold to individuals by the government in exchange for (x) up through and including time t, and Y_t^* denote the aggregate of (y) over all individuals made available for consumption from private wealth in period t. Also let $S_{yT} = \sum_{t=1}^{T} \varepsilon_{yt}$, $S_{xT} = \sum_{t=1}^{T} \varepsilon_{xT}$, and $S_T = [S_{xT}, S_{yT}]$. Then S_T is a random walk in R^2 and has the property that for any integers B_x and B_y there exists some $T^* < \infty$ such that $S_{xT^*} = B_x$ and $S_{yT^*} = B_y$ with probability one.[4/] For the proof let B_y be the largest integer less than or equal to $-G_0 - 1 - \sum_{j=1}^{m} (K_*^j/\bar{R})$, and let B_x be such that $\sum_{j=1}^{m} h_y^j (\delta_x^j B_x + \delta_y^j B_y \bar{R} - K_*^j) \geq 0$. This last choice is possible by the linearity of $h_y^j(\cdot, \bar{R})$. In period t the government must sell $\sum_{j=1}^{m} h_y^j(I_t^j, \bar{R}) - Y_t^*$ units of (y) to the public. Then the cumulative net sales of (y) at time T* will be

$$A_{T^*} = \sum_{t=1}^{T^*} [\sum_{j=1}^{m} h_y^j(I_t^j, \bar{R}) - Y_t^*] = \sum_{j=1}^{m} h_y^j(\sum_{t=1}^{T^*} I_t^j, \bar{R}) - \sum_{t=1}^{T^*} Y_t^*$$

using the linearity of $h_y^j(\cdot, \bar{R})$. Also,
$\sum_{t=1}^{T^*} Y_t^* \leq \sum_{t=1}^{T^*} Y_t + \sum_{j=1}^{m} (K_*^j/\bar{R})$ and $\sum_{t=1}^{T^*} I_t^j \geq \sum_{t=1}^{T^*} (\delta_x^j X_t + \delta_y^j Y_t \bar{R}) - K_*^j$.
From these two inequalities and by substitution one obtains the following inequality:

$$A_{T*} \geq \sum_{j=1}^{m} \{h_y^j[\delta_x^j E(X_t)T* + \delta_x^j S_{xT*} + \delta_y^j E(Y_t)\overline{R}T* + \delta_y^j S_{yT*}\overline{R} - K_*^j, \overline{R}]\} -$$

$$E(Y_t)T* - S_{yT*} - \sum_{j=1}^{m}(K_*^j/\overline{R}) =$$

$$T*\{\sum_{j=1}^{m} h_y^j[\delta_x^j E(X_t) + \delta_y^j E(Y_t)\overline{R}, \overline{R}] - E(Y_t)\} +$$

$$\sum_{j=1}^{m} h_y^j(\delta_x^j B_x + \delta_y^j B_y \overline{R} - K_*^j, \overline{R}) - B_y - \sum_{j=1}^{m}(K_*^j/\overline{R}).$$

But by the choice of $\overline{R}$,

$$\sum_{j=1}^{m} h_y^j[\delta_x^j E(X_t) + \delta_y^j E(Y_t)\overline{R}, \overline{R}] - E(Y_t) = 0.$$

Hence by the choice of B_x and B_y, $A_{T*} \geq G_0 + 1 > G_0$.

Case (iii): $0 < \overline{R} < \pi[E(X_t), E(Y_t)]$.

$\pi(\cdot, \cdot)$ is continuous and $\pi[X, E(Y_t)] \to 0$ as $X \to 0$.

Hence by the intermediate value theorem there exists some

$X*$, $0 < X* < E(X_t)$, such that $\overline{R} = \pi[X*, E(Y_t)]$. Let $\gamma = E(X_t) - X*$.

Then $X_t = X* + (\gamma + \varepsilon_{xt})$, with $E(\varepsilon_{xt} + \gamma) > 0$. Let $S_{xT} = \sum_{t=1}^{T}(\varepsilon_{xt} + \gamma)$

and let $S_{yT} = \sum_{t=1}^{T} \varepsilon_{yt}$. As in case (ii) let B_y be the

largest integer less than or equal to $-G_0 - 1 - \sum_{j=1}^{m}(K_*^j/\overline{R})$.

Then by Feller [5, pp. 202-3], S_{xt} visits $(-\infty, a)$ a finite

number of time for all $a > 0$. Hence there exists some $T*$

such that $S_{yT*} = B_y$ and S_{xT*} is large enough that

$\sum_{j=1}^{m} h_y^j(\delta_x^j S_{xT*} + \delta_y^j S_{yT*}\overline{R} - K_*^j, \overline{R}) \geq 0$. Then as in case (ii)

$$A_{T*} \geq T*\{ \sum_{j=1}^{m} h_y^j [\delta_x^j X* + \delta_y^j E(Y_t)\bar{R},\bar{R}] - E(Y_t) \} +$$

$$\sum_{j=1}^{m} h_y^j (\delta_x^j S_{xT*} + \delta_y^j S_{yT*}\bar{R} - K_*^j,\bar{R}) - S_{yT*} - \sum_{j=1}^{m} (K_*^j/\bar{R}) > G_0.$$

Case (iv): $\quad \pi[E(X_t),E(Y_t)] < \bar{R} < \infty.$

By a proof quite similar to case (iii) initial stocks of (x) will be insufficient to maintain the fixed price. Q.E.D.

It might be argued that the expected time of failure is infinite, and that this mitigates the conclusions of Proposition II. However, Proposition II asserts that failure occurs with probability one in _finite_ time; it is inconsistent for agents to act as if the government would never run out of stocks at the fixed price.

IV. Competitive Equilibrium Without Government

The principal result of this paper is that a price-fixing policy is not feasible. This result would be vacuous if the model without government possessed no equilibria. Though the model without government is difficult to analyze in any generality, it is argued by way of an example that the set of economies which possess competitive equilibria is nonempty.

It is useful first to consider the source of the difficulty. In the absence of government, individuals will not act in such a way as to fix a price. For let $G_0 = 0$ and suppose that agents acted in such a way as to fix a price. By Proposition II a contradiction will be obtained. As the

relative price will not be constant over time in a competitive
equilibrium, agents may not be indifferent between maintaining
stocks of (x) and stocks of (y). Hence in the absence of
all types of forward contracts one might anticipate relations
of the form

$$[\tilde{K}^j_{y,t}, \tilde{K}^j_{x,t}] = \tilde{\phi}^j(K^j_{x,t-1}, K^j_{y,t-1}, \delta^j_x X_t, \delta^j_y Y_t, R_t, R_{t+1}\cdots)$$

where $\tilde{K}^j_x$ and $\tilde{K}^j_y$ denote the optimal choice of storage of (x)
and (y) respectively at time t by agent j and a function of
previous stocks of (x) and (y), current endowment, current
price, and all future prices in all states. Yet in an
equilibrium with rational expectations, the anticipated
distribution of future prices should be the distribution
consistent with current choices of stocks for each good by
all agents. This is the analytical difficulty mentioned
above. Assuming existence, one might anticipate that the
model could be described by reduced form equations

$$[K^j_{x,t}, K^j_{y,t}] =$$

$$\phi^j(K^1_{x,t-1}, K^1_{y,t-1}, K^2_{x,t-1}, \cdots K^m_{y,t-1}; X_t, Y_t) \quad j=1,2,\ldots m.$$

For the example it is assumed that agents have identical
preferences and endowments. Then each agent can be treated
in isolation; in a rational expectations equilibrium each
will act as if he would never trade with another agent. It
remains therefore to establish the existence of a solution
to each agent's dynamic programming problem in such a model.
The analysis is similar to that of the earlier programming
problem.

Let Z_t denote the endowment of agent j at time t, a vector in R^2. Then $Z_t = (X_t/m, Y_t/m)$. Also let $K_t = (K^j_{x,t}, K^j_{y,t})$ and $W_t = Z_t + K_{t-1}$. The idea is to find a functional equation $f(\cdot): R^2_+ \to R^1$ such that

$$(4) \quad f(W_t) = \sup_{0 \le K_t \le W_t} [U(W_t - K_t) + \beta E_t f(Z_{t+1} + K_t)].$$

The function $f(\cdot)$ is found by assuming the horizon is finite and taking the appropriate limit. Let

$$(5) \quad f^N(W_N) = \max_{0 \le K_N \le W_N} [U(W_N - K_N) + \beta E_N f^{N-1}(K_N + Z_{N-1})].$$

From the properties of $U(\cdot)$ it is established by induction that $f^N(W)$ is strictly concave for each N and monotone increasing with respect to N for each $W \ge 0$. (See lemmas 1'-5' in the Appendix.) From the discount rate β and the property that Z_t is bounded from above it follows that $f^N(W)$ is uniformly bounded with respect to N for each W. (See lemma 6'.) Hence for each $W \ge 0$, the limit of $f^N(W)$ exists as $N \to \infty$, and this limit is denoted $f(W)$. (See lemma 7'.) $f(W)$ is concave on $[0,\infty)$ and continuous on $(0,\infty)$. (See lemma 3'.) Also $f(W)$ satisfies the functional equation (4). (See lemma 8'.) It is argued that the right side of (4) will achieve a maximum on $[0,W_t]$ with respect to K_t as boundary values can be ruled out by properties (i) and (iv) of $U(\cdot)$. (See lemma 9'.) The unique maximizer is denoted $\tilde{K}_t = \phi(W_t)$. The sequence $\{\tilde{K}_t\}$ is shown to maximize $E_0 \sum_{t=1}^{\infty} \beta^{t-1} U(Z_t + K_{t-1} - K_t)$. (See lemma 11'.)

As one might expect, the distribution of prices of the competitive equilibrium of this model displays some "smoothness" as compared with the distribution which would prevail if storage by individuals were prohibited. Such a prohibition might be based on the belief that individual speculation is "destabilizing," a proposition which has been much discussed in the literature. As preferences are identical and homothetic,

$$(7) \quad R_t = \theta \left\{ \frac{(X_t/m) + K_{x,t-1} - K_{x,t}}{(Y_t/m) + K_{y,t-1} - K_{y,t}} \right\}$$

where $\theta(\cdot)$ is continuous and strictly monotone increasing. It is a property of the equilibrium with individual storage that consumption of each good is never zero. In contrast suppose no stocks were held by the government or individuals. From property (iv) of $U(\cdot)$ and the fact that there exist zero realizations of Y_t, it follows that R_t may be infinite with positive probability. The effect of individual storage is to introduce serial correlation into the price series; in some sense prices are stabilized.

V. Concluding Remarks

This essay takes a modest step toward the formulation of a coherent general equilibrium framework in which price-fixing schemes and other government buffer stock policies can be analyzed. In the model, attempts by a government to set a price fail with probability one. The next step might be to examine the sensitivity of these results to the various

restrictive assumptions employed. In particular the effect of more general utility functions and stochastic processes could be analyzed.

In the essay attention is limited to the feasibility of fixing a relative price, and this may give the unintended impression that such a rigid policy has been pursued in the past or proposed for the future. Historically, government buffer stock programs have been implemented to limit sharp reductions in prices and to "stabilize" farm incomes. As for the future, it is not clear what policy makers have in mind. An essay of this sort is useful if it only serves to make policy proposals more precise.

Jacques Drèze has suggested that one reason government buffer stock programs are considered is the absence of markets in which agricultural output could be traded contingent on the determinants of supply and demand. This suggests that one should examine the welfare implications of a class of feasible government buffer stock policies in a model with a limited number of markets. Yet, unlike the model of this paper, a restriction of the numbers and types of forward markets ought not to be exogenous. In the context of such a model one could examine the efficiency of competitive equilibria with individual storage and verify the existence or nonexistence of government policies which yield second-best allocations.

APPENDIX

Lemma 1: If $F^j(r'Z^j_{t+1}+K^j_t,\overline{R})$ is a concave function of K^j_t and $V^j(\cdot,\overline{R})$ is strictly concave, then $V^j(W^j_t-K^j_t,\overline{R})$ + $\beta E_t F^j(r'Z^j_{t+1}+K^j_t,\overline{R})$ is a strictly concave function of W^j_t and K^j_t.

Proof: The proof follows immediately from the definition of concavity.

Lemma 2: If $V^j(W^j_t-K^j_t,\overline{R})$ + $\beta E_t F^j(r'Z^j_{t+1}+K^j_t,\overline{R})$ possesses a maximum with respect to K^j_t on $[0,W^j_t]$, then that maximum is a strictly concave function of $W^j_t \varepsilon [0,\infty)$.

Proof: See Bellman [1, lemma 1, p. 21].

Lemma 3: If $F^j(W,\overline{R})$ is concave with respect to W on $[0,\infty)$, then $F^j(\cdot,\overline{R})$ is continuous on $(0,\infty)$ with finite partial derivatives from the left, $F^j_{1-}(W,\overline{R})$, and from the right, $F^j_{1+}(W,\overline{R})$; $F^j_{1-}(W,\overline{R}) \geq F^j_{1+}(W,\overline{R})$.

Proof: See Katzner [7, lemma B. 2-4, p. 187].

Lemma 4: For each $N \geq 1$, $f^{j,N}(\cdot,\overline{R})$ is strictly concave on $[0,\infty)$.

Proof: The proof is by induction.

$$f^{j,1}(W,\overline{R}) = \underset{0 \leq K^j_1 \leq W}{\text{Max}} V^j(W-K^j_1,\overline{R})$$

Clearly $f^{j,1}(\cdot,\overline{R})$ is strictly concave on $[0,\infty)$. Assume $f^{j,N}(\cdot,\overline{R})$ is strictly concave on $[0,\infty)$.

$$f^{j,N+1}(W,\overline{R}) =$$

$$\underset{\substack{0 \leq K^j_{N+1} \leq W}}{\text{Max}} [v^j(W-K^j_{N+1},\overline{R}) + \beta E_{N+1}f^{j,N}(K^j_{N+1}+r^{\prime}z^j_N,\overline{R})]$$

The strict concavity of $v^j(\cdot,\overline{R})$ and $f^{j,N}(\cdot,\overline{R})$ with $V^j_1(0,\overline{R}) = \infty$ ensures the existence and uniqueness of a maximum. By lemma 2, $f^{j,N+1}(\cdot,\overline{R})$ is strictly concave on $[0,\infty)$.

Lemma 5: For each $W \geq 0$, $f^{j,N}(W,\overline{R})$ is monotone increasing with respect to N.

Proof: The proof is by induction.

$$f^{j,1}(W,\overline{R}) = \underset{\substack{0 \leq K^j_1 \leq W}}{\text{MAX}} \{v^j(W-K^j_1,\overline{R})\} = v^j(W,\overline{R})$$

$$f^{j,2}(W,\overline{R}) = \underset{\substack{0 \leq K^j_2 \leq W}}{\text{MAX}} \{v^j(W-K^j_2,\overline{R}) + \beta E_2 f^{j,1}(r^{\prime}z^j_1+K^j_2,\overline{R}) \geq$$

$$v^j(W,\overline{R}) + \beta E_2 f^{j,1}(r^{\prime}z^j_1,\overline{R}) > f^{j,1}(W,\overline{R})$$

Assume $f^{j,N}(W,\overline{R}) > f^{j,N-1}(W,\overline{R})$, and let the maximizing value of K^j_N be denoted $\tilde{K}^j_N$. Then

$$f^{j,N+1}(W,\overline{R}) =$$

$$\underset{\substack{0 \leq K^j_{N+1} \leq W}}{\text{MAX}} \{v^j(W-K^j_{N+1},\overline{R}) + \beta E_{N+1}f^{j,N}(r^{\prime}z^j_N+K^j_{N+1},\overline{R})\} \geq$$

$$v^j(W-\tilde{K}^j_N,\overline{R}) + \beta E_{N+1}f^{j,N}(r^{\prime}z^j_N+\tilde{K}^j_N,\overline{R}) \geq$$

$$v^j(W-\tilde{K}^j_N,\overline{R}) + \beta E_N f^{j,N-1}(r^{\prime}z^j_{N-1}+\tilde{K}^j_N,\overline{R}) = f^{j,N}(W,\overline{R}).$$

__Lemma 6:__ For each $W \geq 0$, $f^{j,N}(W,\overline{R})$ is uniformly bounded in N.

__Proof:__ The highest level of discounted expected utility

for agent j is dominated by that associated with agent j

receiving all of (x) and (y) of the economy in each state in

every period, the maximum of aggregate income over all

states being realized in each state in every period, and

agent j acting as if he knew in the current decision period

that he would receive such endowments in the future. Then

the endowment of individual j in each state in every period

would be $M = \text{Max}_{w \varepsilon \Omega}[X_t(\omega) + \overline{R}Y_t(\omega)]$. Assume that in addition

to this endowment agent j receives current wealth each

period. Then there would be no storage in any period and

for all N,

$$f^{j,N}(W,\overline{R}) < \sum_{t=1}^{N} \beta^{t-1} v^j(W+M,\overline{R}) < \sum_{t=1}^{\infty} \beta^{t-1} v^j(W+M,\overline{R}) =$$

$$v^j(W+M,\overline{R})/(1-\beta).$$

__Lemma 7:__ For each $W \geq 0$, the limit of $f^{j,N}(W,\overline{R})$ exists as

$N \to \infty$ and is concave with respect to W. This limit is

denoted $f^j(W,\overline{R})$.

__Proof:__ Existence follows from lemma 5 and lemma 6. Concavity

follows from lemma 4.

__Lemma 8:__ $f^j(W,\overline{R})$ satisfies the functional equation (1).

__Proof:__ See Bellman [1, pp. 12-13].

__Lemma 9:__ The objective function $v^j(W_t^j-K_t^j,\overline{R}) + \beta E_t f^j(r'Z_{t+1}^j+K_t^j,\overline{R})$

achieves a maximum with respect to K_t^j on $[0,W_t^j]$.

Proof: By concavity $f^j(\cdot,\overline{R})$ is continuous on $(0,\infty)$. If $f^j(\cdot,\overline{R})$ were continuous at zero then a maximum is achieved as the objective function is continuous with respect to K_t^j on the compact set $[0,W_t^j]$. Suppose $f^j(\cdot,\overline{R})$ possesses a discontinuity at zero and that no maximum is attained. Recall that $r'Z_t^j(\overline{\omega}) = 0$, $P(\overline{\omega}) > 0$. Then

$$\sup_{K_t^j \varepsilon [0,W_t^j]} \{V^j(W_t^j-K_t^j,\overline{R}) + \beta E_t f^j(r'Z_{t+1}^j+K_t^j,\overline{R})\} =$$

$$\lim_{K_t^j \to 0} \{V^j(W_t^j-K_t^j,\overline{R}) + \beta E_t f^j(r'Z_{t+1}^j+K_t^j,\overline{R})\}.$$

Also

$$f^j(0+K_t^j,\overline{R}) =$$

$$\sup_{K_{t+1}^j \varepsilon [0,K_t^j]} \{V^j(K_t^j-K_{t+1}^j,\overline{R}) + \beta E_{t+1} f^j(r'Z_{t+2}^j+K_{t+1}^j,\overline{R})\}.$$

Both sides of this equation are nonincreasing with respect to K_t^j. Taking the limit on both sides as $K_t^j \to 0$,

$$\lim_{K_t^j \to 0} f^j(K_t^j,\overline{R}) = V^j(0,\overline{R}) + \beta E_{t+1} \lim_{K_{t+1}^j \to 0} f^j(r'Z_{t+2}^j+K_{t+1}^j,\overline{R}).$$

Hence

$$\sup_{K_t^j \varepsilon [0,W_t^j]} \{V^j(W_t^j-K_t^j,\overline{R}) + \beta E_t f^j(r'Z_{t+1}^j+K_t^j,\overline{R})\} =$$

$$V^j(W_t^j,\overline{R}) + \beta P(\overline{\omega})[V^j(0,\overline{R}) + \beta E_{t+1} \lim_{K_{t+1} \to 0} f^j(r'Z_{t+2}^j+K_{t+1}^j,\overline{R})] +$$

$$\beta \sum_{\omega \neq \overline{\omega}} P(\omega) f^j[r'Z_{t+1}^j(\omega),\overline{R}].$$

But $V_1^j(0,\overline{R}) = \infty$, and this establishes a contradiction.

<u>Lemma 10</u>: $f^j(\cdot,\overline{R})$ is strictly concave.

Proof: This follows from lemma 2 and the functional equation (1).

<u>Lemma 11</u>: Given that $\{\tilde{K}_t^j,\ t=1,2,\ldots\}$ is a sequence of current actions and contingent plans which maximize

$$v^j(W_t^j-K_t^j,\overline{R}) + \beta E_t f^j(r^\prime z_{t+1}^j+K_t^j,\overline{R}),$$

then $\{\tilde{K}_t^j,\ t=1,2,\ldots\}$ also maximizes

$$v^j(W_t^j-K_t^j,\overline{R}) + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} v^j(r^\prime z_\tau^j+K_{\tau-1}^j-K_\tau^j,\overline{R}).$$

Proof: (The proof follows a rough outline of T. Muench.)

It is sufficient to argue the case at $t=1$.

$$E_1 \sum_{t=1}^{T} \beta^{t-1} v^j(r^\prime z_t^j+K_{t-1}^j-K_t^j,\overline{R}) =$$

$$E_1\{f^j(W_1^j,\overline{R}) + \sum_{t=1}^{T} \beta^{t-1}[v^j(W_t^j-K_t^j,\overline{R})+\beta f^j(W_{t+1}^j,\overline{R})-f^j(W_t^j,\overline{R})]\}-$$

$$E_1\beta^T f^j(W_{T+1}^j,\overline{R}) =$$

$$f^j(W_1^j,\overline{R}) + E_1 \sum_{t=1}^{T} \beta^{t-1} E_t[v^j(W_t^j-K_t^j,\overline{R})+\beta f^j(W_{t+1}^j,\overline{R})-f^j(W_t^j,\overline{R})] -$$

$$E_1\beta^T f^j(W_{T+1}^j,\overline{R})$$

Note that

$$f^j(W_t^j,\overline{R}) = \underset{0 \le K_t^j \le W_t^j}{\text{MAX}} \{v^j(W_t^j-K_t^j,\overline{R}) + \beta E_t f^j(W_{t+1}^j,\overline{R})\}.$$

Let $K_t^j = \phi^j(W_t^j,\overline{R})$. Then

$$v^j(W_t^j-\tilde{K}_t^j,\overline{R}) + \beta E_t f^j(W_{t+1}^j,\overline{R}) - f^j(W_t^j,\overline{R}) = 0.$$

For each $\omega \varepsilon \Pi_{t=1}^{\infty} \Omega_t$, $\sum_{t=1}^{\infty} \beta^{t-1} v^j(r'z_t^j + K_{t-1}^j - K_t^j, \overline{R}) < \infty$. By the

monotone convergence theorem,

$$E_1 \sum_{t=1}^{\infty} \beta^{t-1} v^j(r'z_t^j + \tilde{K}_{t-1}^j - \tilde{K}_t^j, \overline{R}) = f^j(W_1^j, \overline{R}) - \lim_{T \to \infty} E_1 \beta^T f^j(W_{T+1}^j, \overline{R}).$$

By the strict concavity of $f^j(\cdot, \overline{R})$, $E_1 \beta^T f^j(W_{T+1}^j, \overline{R}) \to 0$ as

$T \to \infty$. Therefore,

$$f^j(W_1^j, \overline{R}) = E_1 \sum_{t=1}^{\infty} \beta^{t-1} v^j(W_t^j - \tilde{K}_t^j, \overline{R}).$$

<u>Lemma 12</u>: $\phi^j(\cdot, \overline{R})$ is nondecreasing and $\psi^j(\cdot, \overline{R})$ is strictly

increasing.

Proof: $v^j(\cdot, \overline{R})$ and $f^j(\cdot, \overline{R})$ are strictly concave and strictly

increasing with $v^j(\cdot, \overline{R})$ differentiable.

<u>Lemma 13</u>: $\phi^j(\cdot, \overline{R})$ and $\psi^j(\cdot, \overline{R})$ are continuous.

Proof: See Brock and Mirman [2, p. 490].

<u>Lemma 14</u>: $f_{1+}^j(W, \overline{R}) \to 0$ as $W \to \infty$.

Proof: From lemma 6 and lemma 7, $f^j(W, \overline{R}) \leq v^j(W+M, \overline{R})/(1-\beta)$

and $v_1^j(W+M, \overline{R}) \to 0$ as $W \to \infty$.

<u>Lemma 15</u>: $\pi(\cdot, \cdot)$ is continuous.

Proof: R^* is chosen so that $\sum_{j=1}^{m} [h_y^j(\delta_x^j X + \delta_y^j Y R^*, R^*) - \delta_y^j Y] = 0$.

Let $J[X,Y,R] = \sum_{j=1}^{m} [h_y^j(\delta_x^j X + \delta_y^j Y R^*, R^*) - \delta_y^j Y]$. Then

$J[X,Y,\pi(X,Y)] = 0$. By properties (i) and (iv) of

$U^j(\cdot, \cdot)$ and the implicit function theorem, $h_y^j(\delta_x^j X + \delta_y^j Y R, R)$

is of class C^1 with respect to X,Y, and R. Hence the

first partial derivatives of $J(\cdot, \cdot, \cdot)$ are all continuous

with respect to X,Y, and R. By linearity, the income

effect in the standard Slutsky equation is nonnegative,

and therefore $J_3(\cdot,\cdot,\cdot) < 0$. Hence the implicit function theorem applies and $\pi(\cdot,\cdot)$ is continuous.

Lemma 16: $\pi(X,\overline{Y}) \to 0$ as $X \to 0$ for each $\overline{Y} > 0$. $\pi(\overline{X},Y) \to \infty$ as $Y \to 0$ for each $\overline{X} > 0$.

Proof: The proof is by contradiction. Suppose that for some $\overline{Y} > 0$, $\pi(X,\overline{Y}) \not\to 0$ as $X \to 0$. Then one can construct a sequence $\{X_n\}$ such that $X_n \to 0$ as $n \to \infty$ and for which the corresponding sequence of equilibrium prices, $\pi(X_n,\overline{Y})$ is bounded from below by some $R'' > 0$. For each agent j,

$$h_y^j[\delta_x^j X_n + \delta_y^{j\overline{Y}}\pi(X_n,\overline{Y}),\pi(X_n,\overline{Y})] - \delta_y^{j\overline{Y}} \leq$$

$$h_y^j[\delta_x^j X_n + \delta_y^{j\overline{Y}}R'',R''] - \delta_y^{j\overline{Y}}.$$

But

$$\lim_{n \to \infty} h_y^j[\delta_x^j X_n + \delta_y^{j\overline{Y}}R'',R''] - \delta_y^{j\overline{Y}} = h_y^j[\delta_y^{j\overline{Y}}R'',R''] - \delta_y^{j\overline{Y}} < 0.$$

Hence as $X_n \to 0$, the excess supply of (y) is bounded away from zero, and this establishes the desired contradiction.

The following lemmas are numbered in such a way as to establish a correspondence with the earlier lemmas of the first programming problem.

Lemma 1': If $G(Z_t + K_{t-1})$ is a concave function of K_{t-1} and $U(\cdot)$ is strictly concave, then $U(W_t - K_t) + \beta E_t G(Z_{t+1} + K_t)$ is a strictly concave function of W_t and K_t.

Proof: The proof follows immediately from the definition of concavity.

<u>Lemma 2</u>: If $U(W_t - K_t) + \beta E_t G(Z_{t+1} + K_t)$ possesses a maximum with respect to K_t with $0 \leq K_t \leq W_t$, then that maximum is a strictly concave function of W_t.

Proof: See Bellman [1, remarks p. 33].

<u>Lemma 3'</u>: If $G(W)$ is concave for $W \geq 0$, then $G(W)$ is continuous for $W > 0$.

Proof: See Katzner [7; Theorem B. 4-5, p. 196].

<u>Lemma 4'</u>: For each $N \geq 1$, $f^N(W)$ is strictly concave for $W \geq 0$.

Proof: See lemma 4 and the boundary properties (i) and (iv) of $U(\cdot)$.

<u>Lemma 5'</u>: For each $W \geq 0$, $f^N(W)$ is monotone increasing with respect to N.

Proof: See lemma 5.

<u>Lemma 6'</u>: For each $W \geq 0$, $f^N(W)$ is uniformly bounded in N.

Proof: Let $M \varepsilon R_+^2$ denote the maximum realizations of X_t and Y_t. Then as in lemma 6, it can be shown that $f^N(W) < U(W+M)/(1-\beta)$.

<u>Lemma 7'</u>: For each $W \geq 0$, the limit of $f^N(W)$ as $N \to \infty$ exists and is concave with respect to W. This limit is denoted $f(W)$.

Proof: Existence follows from lemmas 5' and 6'. Concavity follows from lemma 4'.

<u>Lemma 8'</u>: f(W) satisfies the functional equation (4).

Proof: See Bellman [1, pp. 12-13].

<u>Lemma 9'</u>: $U(W_t - K_t) + \beta E_t f(Z_{t+1} + K_t)$ achieves a maximum with respect to K_t with $0 \le K_t \le W_t$.

Proof: Analogous to the proof of lemma 9, suppose a maximum is not attained on $[0, W_t]$ and consider the following cases:

(i). $[K_{x,t}, K_{y,t}] \rightarrow [0, \overline{K}_y]$, $\overline{K}_y > 0$

(ii). $[K_{x,t}, K_{y,t}] \rightarrow [\overline{K}_x, 0]$, $\overline{K}_x > 0$

(iii). $[K_{x,t}, K_{y,t}] \rightarrow [0, 0]$

A contradiction can be obtained by the boundary conditions (i) and (iv) of $U(\cdot)$.

<u>Lemma 10'</u>: f(W) is strictly concave.

Proof: This follows from lemmas 7' and 2' and the functional equation (4).

<u>Lemma 11'</u>: Given that $\{\tilde{K}_t\}$ is a sequence of current actions and contingent plans which maximize $U(W_t - K_t) + \beta E_t f(Z_{t+1} + K_t)$,

then $\{\tilde{K}_t\}$ also maximizes $U(W_t - K_t) + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} U(Z_\tau + K_{\tau-1} - K_\tau)$.

Proof: See lemma 11.

FOOTNOTES

[1] $V^j(\cdot,\overline{R}) = g^j[\tilde{V}(\cdot,\overline{R})]$ where $\tilde{V}^j(\cdot,\overline{R})$, the indirect

utility function for $H(\cdot,\cdot)$, is linear. Strict concavity of

$V^j(\cdot,\overline{R})$ then follows from the strict concavity of $g^j(\cdot)$.

Note also that this implies that $U^j(\cdot,\cdot)$ is strictly

concave. Let D denote the determinant of the bordered

Hessian of $U^j(\cdot,\cdot)$. It can be shown that

$V^j_{11} = (U^j_{11}U^j_{22}-U^j_{12}U^j_{21})/(-D)$. By property (ii), D > 0 and so

the numerator is positive. Also $U^j_{11} = g^j_{11}(H^j_1)^2+g^j_1 H^j_{11} < 0$.

Hence the Hessian of $U(\cdot,\cdot)$ is negative definite.

[2] This stochastic dynamic programming problem is really

a choice between consumption and savings in one good model

in which future income is subject to random shocks. After

completing this paper it has come to my attention that Foley

and Hellwig [4] have solved a similar problem in the context

of a different model.

[3] Elements of this proof were first suggested by John

Danforth, but the author alone is responsible for any errors.

[4] See Jain and Orey [6, p. 796].

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ESSAY IV

INTERMEDIATION WITH A NONCONVEX
TRANSACTIONS TECHNOLOGY

I.   Introduction

In the existing literature on equilibrium with transactions costs, the marketing technology is assumed to be convex. See Foley [2], Hahn [4], and Kurz [6]. The imposition of such a technology allows standard theory to be applied. But, as is well known, the same convexities prevent the model from explaining specialization. In a model of the structure of trade, it prevents the endogenous emergence of specialized marketing activities or of what might be called intermediation.

This essay proposes a simple model of trade with a nonconvex transactions technology. There is a set of risk averse agents, each of whom is endowed with a quantity of a single capital good and with a stochastic technology that transforms the capital good into a distribution of the single consumption good of the model. The distributions for different agents are such that there are gains to portfolio diversification. But there are also costs; for each bilateral deal between agents there is a fixed cost in terms of the capital good.

As noted, these nonconvexities make classical results on the existence and optimality of a competitive equilibrium inapplicable. What is needed then is an alternative formulation in which optimal allocations are defined and in which the behavior of agents is consistent with some type of equilibrium. The essay proceeds as follows. Section II gives the technology of the model and describes the nature of a competitive equilibrium in the model without transactions costs. Section III describes optimal allocations in the model with a fixed cost of exchange. An allocation is said to be optimal if it is in the core, where the core is defined taking into consideration the costs of exchange. In Section IV it is shown that, subject to some qualifications, the allocations of the core can be supported as noncooperative equilibria and that all allocations of noncooperative equilibria are in the core. Agents adopt strategies under which they are willing to act as intermediaries, buying shares in investment projects and selling shares in the resulting portfolio. These strategies include prices and fixed fees, and in this way intermediaries act as auctioneers. The noncooperative equilibria are defined with respect to these strategies. In such equilibria, markets separate agents into disjoint groups, and in this sense markets may be said to be incomplete. It is argued that free entry is crucial in determining the allocations of resources. Section V offers some concluding remarks.

II. Competitive Equilibrium Without Transactions Costs

The model consists of a set of I traders. To each agent $j \varepsilon I$ there is associated a production possibility set $Y^j \varepsilon R^{S+1}$ where S is the number of states in a set Ω. For any element $y^j \varepsilon Y^j$, let the first element y_1^j denote the input of the single factor of production of the model. The remaining S elements consist of the output of the unique produced good over states. Hence, for each $y^j \varepsilon Y^j$, $y_1^j \leq 0$ and $y_\omega^j \geq 0$, $\omega \varepsilon \Omega$; $y_1^j = 0$ implies $y^j = 0$. It is further assumed that Y^j is a closed convex cone. The Y^j, $j \varepsilon I$, completely describe production possibilities in the model.

From the production possibility frontier of Y^j define a production function $-\lambda^j y_1^j$ where λ^j is a nonnegative Borel measurable function on a probability space (Ω^j, F^j, μ^j). Here F^j is the set of all subsets of Ω^j, and μ^j is a discrete measure of F^j. It is supposed that the λ^j, $j \varepsilon I$ are independent and identically distributed so that each is defined on (Ω, F, μ) where $\Omega = \Pi_{j \varepsilon I} \Omega^j$, $F = \Pi_{j \varepsilon I} F^j$, and $\mu \{\omega \varepsilon \Omega : \lambda^j \leq a\} = \mu^j \{\omega^j \varepsilon \Omega^j : \lambda^j \leq a\}$. Let $\mu(\omega) = \mu_\omega$. It is supposed that $\mu_\omega > 0$ for each $\omega \varepsilon \Omega$.

The consumption posssibility set X^j for each agent j is R^{S+1}. Each agent is assumed to maximize expected utility $V^j(x^j) = \sum_{\omega \varepsilon \Omega} \mu_\omega U^j(c_\omega^j)$ where $x^j \varepsilon X^j$, and $c_\omega^j \varepsilon R_+^1$ denotes the consumption of agent j in state ω. $U^j(\cdot)$ is strictly concave, continuous, and strictly increasing. It is further assumed that tastes as represented by $U^j(\cdot)$ are identical for all individuals and display constant relative risk aversion.

Each agent is endowed with some of the factor of production, hereafter called capital and denoted K^j. Let $\bar{x}^j \epsilon R^{s+1}$ be the initial endowment of agent j. Then $\bar{x}^j = (K^j, 0, 0, \ldots)$ with $K^j > 0$ for each $j \epsilon I$. K^j is perfectly divisible. It is further assumed that initial endowments are identical for all agents.

It was claimed earlier that convexities prevent models from explaining specialized marketing activities or intermediation. In view of such remarks, it is useful to consider the properties of a competitive equilibrium in this model without transactions costs. If the set I were finite there would exist such a competitive equilibrium; see Arrow-Hahn [1]. Given prices $p \epsilon \Delta^{s+1} = \{p \epsilon R^{s+1}; p_j \geq 0; \sum_{i=1}^{s+1} p_i = 1\}$, the income of agent j would be $p_1 K^j + \sup_{y^j \epsilon Y^j} p \cdot y^j$. As $K^j > 0$, all households are resource related. There exists a consumption allocation $x_*^j \epsilon X^j$, a production allocation $y_*^j \epsilon Y^j$, and a price $p_* \epsilon \Delta^{s+1}$ which constitutes a competitive equilibrium in that

(i). $\quad p_* > 0$

(ii). $\quad \sum_{j \epsilon I} x_*^j \leq \sum_{j \epsilon I} y_*^j + \sum_{j \epsilon I} \bar{x}^j$

(iii). $\quad y_*^j$ maximizes $p_* y^j$, $y^j \epsilon Y^j$

(iv). $\quad x_*^j$ maximizes $V^j(x^j)$ subject to $p_* \bar{x}^j + p_* y_*^j \geq p_* x^j$.

It is argued that in a competitive equilibrium of the model agents will act as if purchasing shares in investment projects, where one share in project j entitles the holder to λ^j units of the consumption good, a random variable. As

production functions are of constant returns to scale, in the competitive equilibrium all shares must sell for a price of one in terms of the capital good. As the λ^j are independent and identically distributed, if each agent is restricted to purchasing shares, each will purchase equal numbers in all projects, including his own; see Samuelson [8]. (Such a portfolio will be referred to subsequently as completely diversified with respect to the set of traders I.) As endowments are identical, this will give all agents the same amount of the consumption good in any state, and hence with identical tastes each will have the same rates of commodity substitution across states. Hence, this allocation constitutes competitive equilibrium. Such an allocation will be Pareto optimal.

It should also be noted that even if there were some initial reallocation of the capital good among agents, agents would still act as if trading shares. If each agent purchased an equal number of shares in all projects, each would have the same distribution of the consumption good over states up to a constant of proportionality. As the $U^j(\cdot)$ display constant relative risk aversion, it follows that each agent would have the same rates of commodity substitution across states.[1/]

For purposes of subsequent reference it is useful to consider the properties of the limiting allocation of the competitive equilibria as the number of agents in I increases. The portfolio of each agent will remain completely diversified

with respect to I, and by the law of large numbers the yield
on this portfolio will approach the mean of each λ^i. The
number of shares that each agent purchases in any one project
will approach zero.[2]

Implicit in the definition of a competitive equilibrium
is the existence of complete markets for contingent claims
in which agents can trade claims on the consumption good for
each state ω, a claim which is binding only if the specified
state occurs. Agents are assumed to have perfect information,
and exchange per se is costless. Also, there is no explicit
mechanism for the determination of the prices which agents
take as parameters; one can introduce an auctioneer as deus
ex machina who calls out prices until markets clear, but
such an agent is not endogenous to the model. Under these
implicit assumptions there can be no nontrivial role for an
intermediary. Each agent acting on his own in competitive
markets with full information can do just as well as in any
coalition.

III. Core Allocations with Transactions Costs--

A Cooperative Economy

It is assumed subsequently that the set of traders I is
countably infinite and that exchange is not costless. There
is a fixed cost of 2α in terms of the capital good for each
bilateral trade, α per agent. Suppose there were three
agents and each were making two bilateral exchanges as in
Figure 1(a). Total transactions costs would be 6α. Such a
situation may be inefficient. Transactions costs could be

reduced to 4α if agent h were to act as an intermediary for the market consisting of three agents; see Figure 1(b). To formalize this the following definitions are needed. Let N^j denote the set of agents with whom agent j deals directly. Then, a set of agents M is said to constitute a market if for each $j \varepsilon M$, $N^j \subseteq M$, and there exists no proper subset Λ of M, such that for each $j \varepsilon \Lambda$, $N^j \subseteq \Lambda$. A market is thus defined to be the smallest set of agents for which every agent of the set deals with at least one other agent of the set and with no agent outside the set.

An agent h is said to act as an intermediary for a market M if

$$
N^j = \begin{cases} h & \text{if } j \neq h, \ j \varepsilon M \\ \\ M - h & \text{if } j = h \varepsilon M. \end{cases}
$$

Let m denote the number of agents in M.

In the cooperative economy specified agents are designated as intermediaries. Each intermediary selects a group of agents for projects in his portfolio. These sets are assumed to be disjoint, so that in effect intermediaries are forming markets. Each agent in a market agrees to sell shares in his project to the intermediary for a price of one in terms of the capital good, and the intermediary sells shares in his portfolio for a price of one in terms of the capital good. A share in the portfolio of an intermediary in a market M with m agents entitles the holder to $\sum_{i \varepsilon M} \lambda^i / m$ units of the consumption good. All transactions costs are
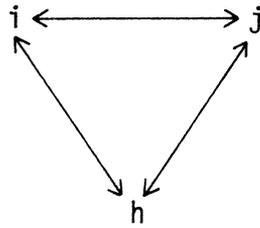
Exchange Without Intermediation

Figure 1(a)

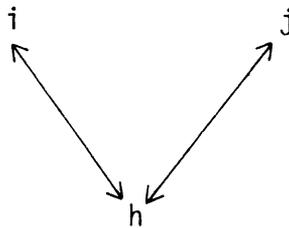Exchange With Intermediation

Figure 1(b)

shared equally by all agents in a given market. For m

agents in M these transactions costs will be $(m-1)(2\alpha)$. In

these circumstances agents in a market will trade shares

with the intermediary on a one-to-one basis up to the limits

of their initial endowments, less transactions costs. Each

intermediary determines the number of agents m in his market,

and each will act to maximize

(1) $E\ U\{[K^j - \frac{(2\alpha)(m-1)}{m}][\sum_{i \varepsilon M} \lambda^i/m]\}$

with respect to m. Here E denotes expectation with respect

to (Ω, F, μ).

Let Γ denote the set of positive integers which maximize

(1). As m varies over integers it is difficult, in general,

to characterize a maximum in terms of first-order conditions.

The set Γ need not be a singleton, nor must it be nonempty.

As the size of the market increases, marginal transactions

costs go to zero. But the marginal gain from increased

diversification also goes to zero as by the law of large

numbers $\sum_{i=1}^{n} \lambda^i/n \to E(\lambda^i)$ almost surely as $n \to \infty$. If $\{1\} \varepsilon \Gamma$

there can be an autarkic allocation for the cooperative

economy in which each agent claims the returns only on his

own project. If $\{2\} \varepsilon \Gamma$ there can be a sequence of bilateral

deals among pairs of agents. In general, for $n \varepsilon \Gamma$ there will

be a sequence of markets, each with n agents. If Γ is not a

singleton it is not required that each market have the same

number of agents.

Unfortunately, one cannot rule out $\Gamma = \emptyset$. In what follows it will be assumed that $\Gamma \neq \emptyset$, so it is important to examine the generality of this assumption. It is claimed that if $K^j \leq 2\alpha$, then $\Gamma \neq \emptyset$. For suppose $\Gamma = \emptyset$; then there exists a subsequence of integers $\{n_v\}$ such that (1) is strictly monotone increasing with respect to successive elements of the subsequence. But if $K^j < 2\alpha$, then there exists $n_v \epsilon \{n_v\}$ such that $K^j < (2\alpha)(n_v-1)/n_v$. Clearly this is not feasible. If $0 < K^j = 2\alpha$ then

$$\lim_{v \to \infty} E\ U\{[K^j - \frac{(2\alpha)(n_v-1)}{n_v}][\sum_{i=1}^{n_v} \lambda^i/n_v]\} = U(0),$$

and this also contradicts the strict monotonicity of (1) with respect to $\{n_v\}$.

It is also useful to show that with $\Gamma \neq \emptyset$, the allocation need not be autarkic. As an example, consider the function $U(w) = w^b$ with $b = .10$. Let each λ^i have a finite distribution with two realizations, m−e and m+e, occurring with probability one half. Let $K^j = 2\alpha$, m = 100, e = 100, and $\alpha = 1$. To show that the equilibrium is not autarkic it is sufficient to show that expected utility in (1) is greater for n = 2 than for n = 1. With the specified parameters, expected utility with autarky is .91 and expected utility with a bilateral trade is 1.21.

There also exist economies in which $K^j > 2\alpha > 0$ and $\Gamma \neq \emptyset$. As an example, let $U(w) = w^b$, $0 < b < 1$. Suppose that for each $b\epsilon(0,1)$, $\Gamma = \emptyset$, and as above consider the subsequence $\{n_v\}$. Let $\lambda^* = \text{Max}_\omega \lambda^i(\omega)$. For every $b\epsilon(0,1)$,

$\sum_{i=1}^{n_v} (\lambda^i/n_v)^b < (\lambda^*)^b$ with $(\lambda^*)^b < \infty$. Also by the strong law of large numbers and the continuity of $U(\cdot)$,

$(\sum_{i=1}^{n_v} \lambda^i/n_v)^b \to [E(\lambda^i)]^b$ almost surely as $v \to \infty$. Therefore, by the dominated convergence theorem,

$$\lim_{v \to \infty} E[(\sum_{i=1}^{n_v} \lambda^i/n_v)^b] = [E(\lambda^i)]^b.$$

Therefore, for each $b \varepsilon (0,1)$,

$$\lim_{v \to \infty} E\{ [K^j - \frac{(2\alpha)(n_v-1)}{n_v}]^b [\sum_{i=1}^{n_v} \lambda^i/n_v]^b \} = [K^j - 2\alpha]^b [E(\lambda^i)]^b.$$

It follows by the construction of the subsequence $\{n_v\}$ that for all integers n and for all $b \varepsilon (0,1)$

$$E\{ [K^j - \frac{(2\alpha)(n-1)}{n}]^b [\sum_{i=1}^{n} \lambda^i/n]^b \} < [K^j - 2\alpha]^b [E(\lambda^i)]^b.$$

Fix some n and take the limit on both sides as $b \to 1$. Then

$$K^j - \frac{(2\alpha)(n-1)}{n} \leq K^j - 2\alpha.$$

This establishes a contradiction. Hence, there exist values for the parameter 'b' such that $\Gamma \neq \emptyset$ with $K^j > 2\alpha > 0$.

If $\Gamma = \emptyset$, the objective function (1) does not possess a maximum with respect to the set of positive integers. As the number of agents m in a market M of the cooperative economy increases, transactions costs per agent have limit 2α, and the yield on the portfolio of the intermediary for M has limit $E(\lambda^i)$ almost surely. By the dominated convergence theorem, levels of expected utility are bounded above by a limit of $U\{[K-2\alpha][E(\lambda^i)]\}$.

It is now argued that for $\Gamma \neq \emptyset$, the allocations of the cooperative economy are optimal in that they are in an appropriately defined core. An allocation is said to be in the core if it can be achieved with the resources and technology of the model and cannot be blocked by any finite coalition. An allocation can be blocked by a coalition if there exists an alternative allocation that is feasible for that coalition, and in which at least one agent of the coalition is better off and no agent of the coalition is worse off than in the initial allocation. Here feasibility means that the coalition is able to achieve the alternative allocation with its own resources. It seems natural in the model to define feasibility in the context of the initial endowments K^j, the technology λ^j, and the transactions costs α. Hence, it is supposed that agents in a coalition take into consideration the transactions costs necessary to effect an exchange with other agents in the coalition.

Proposition I: If $\Gamma \neq \emptyset$, the allocations of the cooperative economy are in the core.

Proof: The proof is by contradiction. Suppose a set of agents B of number n were able to block an allocation of the cooperative economy and that those agents constitute a market. Consider exchange, production, and consumption decisions which result in Pareto optimal allocations for the set B; an allocation which is Pareto optimal for B is one which is feasible for B and for which no agent of B can be made better off without making some agent of B worse off.

As $U(\cdot)$ displays constant relative risk aversion, a necessary

condition for a consumption allocation to be optimal for B

is that $c_\omega^i = \theta^i \sum_{j \in B} c_\omega^j$ for each $\omega \epsilon \Omega$ and for each $i \epsilon B$, with

$0 \leq \theta^i \leq 1$, $\sum_{j \in B} \theta^j = 1$. Then to determine an optimal

production allocation it is sufficient to maximize

$$\sum_{\omega \epsilon \Omega} \mu_\omega U(c^j) = \theta^i \sum_{\omega \epsilon \Omega} \mu_\omega U(\sum_{j \epsilon B} c_\omega^j)$$

for any i. As $U(\cdot)$ is strictly concave, equal amounts

should be invested in all projects of B. With total transactions

costs minimized at $2\alpha(n-1)$, this yields

$$y_1^j = K^j - \frac{(2\alpha)(n-1)}{n}$$

for each $j \epsilon B$. Consider a symmetric allocation of the consumption

good over states so that $\theta^j = 1/n$ for each $j \epsilon B$. Then

$$c_\omega^j = [K^j - \frac{(2\alpha)(n-1)}{n}][\sum_{i \epsilon B} \lambda^i(\omega)/n].$$

By construction this allocation is Pareto optimal for B, and

all agents of B have a level of expected utility which is at

most equal to the level of the cooperative allocation.

Hence, the coalition B could not block the cooperative

allocation. If B did not constitute a market, then one can

consider separately the subsets of B which do constitute

markets. As above, no agent in any subset can be better off

than in the cooperative allocation without making some other

agent of the same subset (and hence of B) worse off. Hence,

no finite coalition can block, and the cooperative allocation

is in the core.[3/] Q.E.D.

<u>Proposition II</u>: If $\Gamma = \emptyset$, there do not exist core allocations with a finite number of agents in each market.

Proof: The proof is by contradiction. Suppose that there did exist a core allocation with a finite number of agent in each market. In any such market M with some number of agents m, exchange, production, and consumption decisions must result in an allocation which is Pareto optimal for M; otherwise M could block the core allocation, contradicting the definition of the core. Furthermore, in a core allocation all agents of I must have the same level of expected utility.[4/]

As all agents of M have the same level of expected utility, the allocation of the consumption good must be symmetric in M so that

$$c_\omega^j = [K^j - \frac{(2\alpha)(m-1)}{m}][\sum_{i \in M} \lambda^i(\omega)/m]$$

for each j in M. But as $\Gamma = \emptyset$ there exist some number of agents n > m which could form a market with efficient production and exchange and with symmetric consumption allocations in which all agents are better off than initially. This is the desired contradiction. Q.E.D.

<u>Proposition III</u>: If there exists a core allocation with a finite number of agents in each market, then $\Gamma \neq \emptyset$ and the core allocation is an allocation of the cooperative economy.

Proof: That $\Gamma \neq \emptyset$ follows from the contrapositive of Proposition II. As in the proof of Proposition II, in each market M of the core with m agents, production and exchange

must be efficient with a symmetric consumption allocation. As

there do not exist blocking coalitions, it follows that $m \varepsilon \Gamma$.

Q.E.D.

It has been shown that allocations of the core with a

finite number of agents in each market and allocations of

the cooperative economy with $\Gamma \neq \emptyset$ are equivalent. These

allocations are of the form:

$$y_1^j = K^j - \frac{(2\alpha)(m-1)}{m}; \quad c_\omega^j = [K^j - \frac{(2\alpha)(m-1)}{m}][\sum_{i \varepsilon M} \lambda^i(\omega)/m]$$

where M has some $m \varepsilon \Gamma$ agents. All such allocations may be

said to be optimal in the sense that there do not exist

alternative allocations with a finite number of agents in

each market which are Pareto superior for the set of all

agents I.

IV. Intermediation Strategies and Noncooperative Equilibria

It is now argued that the cooperative allocations can

be supported as equilibria of the model. However, the fixed

costs introduce a fundamental nonconvexity into the analysis,

so the equilibria referred to cannot be competitive in the

classic sense. For suppose agent j were to sell shares in

his project to agent h at cost. Then the production possibility

set for agent h acting as an intermediary for j would appear

as in Figure 2.

Let

$$\overline{Y} = \left\{ (y_1, z) \varepsilon R^2 : \begin{array}{l} z = 0 \text{ if } 0 \leq -y_1 \leq 2\alpha \\ 0 \leq z \leq -y_1 - 2\alpha \text{ if } -y_1 \geq 2\alpha \end{array} \right.$$

where z denotes the number of shares in project j. Clearly

the set $\overline{Y}$ is not convex. One approach to economies with

nonconvex sets is to replace those sets by their convex
hulls, to apply standard existence properties to the convexified
economies, and then to show that in some sense the excess
demands of the resulting social approximate equilibrium are
bounded and go to zero as the number of agents increases;
see Arrow-Hahn [1] and Starr [9]. The convex hull of $\overline{Y}$ is
the set below the forty-five degree line from the origin,
and that set is not closed. Also $\overline{Y}$ is not of finite inner
radii. As these properties violate two assumptions that are
used in the approach outlined above, an alternative approach
must be examined.

The resource allocation mechanism is now described as a
type of noncooperative game. Prior to the realization of
the state, each agent h adopts a strategy under which he is
willing to act as an intermediary for some specified market.
This strategy is denoted S^h with components:

S_0^h = the proposed market of agent h.

S_1^h = the yield in terms of the consumption good for one
share in the portfolio of agent h, a random variable
on (Ω, F, μ).

S_2^h = a price in terms of the capital good at which
agent h is willing to buy an unlimited number of
shares in any project j of S_0^h.

S_3^{hj} = a fixed fee in terms of the capital good for the
purchase of shares in the portfolio of agent h by
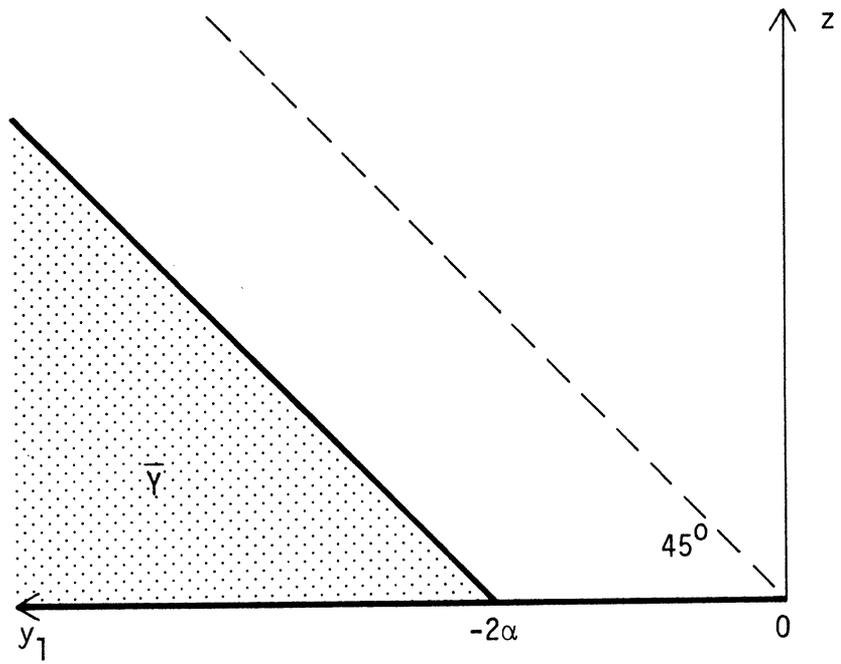agent j of S_0^h.

A Production Possibilities Set for Shares

Figure 2

S_4^h = a price in terms of the capital good at which

agent h is willing to sell shares in his portfolio

to agents j of S_0^h.

The following notation is also needed:

d^{jh} = the number of shares purchased by agent j in the

portfolio of agent h, h $\neq$ j.

D^{hj} = the number of shares purchased by agent h in

project j, j $\neq$ h.

n^j = the number of agents in N^j.

$$A^{jh} = \begin{cases} 1 \text{ if agent j purchases shares in the portfolio of} \\ \quad \text{agent h,} \\ 0 \text{ otherwise.} \end{cases}$$

Agents must act on the strategies S^h, hϵI prior to the

realization of the state. Once the state has occurred,

agents make the transfers of the consumption good required

to honor the claims issued under those strategies. Given

that intermediaries have been selected in some way, all

other agents regard the strategies S^h as parameters and

maximize expected utility. Hence, if agent j is not an

active intermediary so that d^{ij} = 0 for each iϵI-j, agent j

acts as if to maximize

(2) $\quad \sum\limits_{h:j\epsilon S_0^h} [A^{jh}][E\ U(d^{jh}S_1^h + y_1^j \lambda^j - D^{hj}\lambda^j)]$

with respect to $\{y_1^j, D^{hj}, d^{jh}, A^{jh}\}$ subject to the following

constraints:

(3) $\quad [\sum\limits_{h:j\epsilon S_0^h} (A^{jh})(K^j + D^{hj}S_2^h - d^{jh}S_4^h - S_3^{hj} - \alpha - y_1^j)] \geq 0$

(4) $\{ \sum_{h:j\in S_0^h} [A^{jh}] [d^{jh} S_1^h(\omega) + y_1^j \lambda^j(\omega) - D^{hj} \lambda^j(\omega)] \} \geq 0$ for each $\omega\in\Omega$

(5) $A^{jh} = 1 \Rightarrow A^{ji} = 0$ for each $i\in I-h$.

Constraint (3) is the budget constraint prior to the realization

of the state. Constraint (4) requires that agent j be able

to honor all claims on shares which he issues on his project.

Constraint (5) limits agent j to choosing one intermediary

in which to invest.

An intermediary who is active for his proposed market

S_0^h will have expected utility

(6) $E\ U(\sum_{j\in S_0^h} D^{hj} \lambda^j + y_1^h \lambda^h - \sum_{j\in S_0^h} d^{jh} S_1^h)$.

It is required that the strategy S^h be _feasible_ in that the

following constraints be satisfied:

(7) $(K^h + \sum_{j\in S_0^h} D^{jh} S_4^h + \sum_{j\in S_0^h} S_3^{hj} - \sum_{j\in S_0^h} D^{hj} S_2^h - \alpha m^h - y_1^h) \geq 0$

(8) $[\sum_{j\in S_0^h} D^{hj} \lambda^j(\omega) + y_1^h \lambda^h(\omega) - \sum_{j\in S_0^h} d^{jh} S_1^h(\omega)] \geq 0$ for each $\omega\in\Omega$.

Constraint (7) is the budget constraint and (8) ensures that

claims issued on the strategy S^h can be honored.

The determination of who is to act as an intermediary

is not exogenous to the model; each agent is free to announce

any feasible intermediation strategy, and it is assumed that

all agents select strategies in such a way as to maximize

expected utility. This leads to the following definition of

a blocking strategy.

<u>Definition</u>: A feasible strategy S^b for some agent $b\epsilon I$ with S_0^b finite is said to block a consumption allocation $\{x_*^j; j\epsilon I\}$ if when (2) is maximized for each agent $j\epsilon S_0^b - b$ subject to constraints (3)-(5) with $A^{jb} = 1$ for each $j\epsilon S_0^b - b$, the resulting consumption allocation $\{x_{**}^j; j\epsilon S^b\}$ yields $V^j(x_{**}^j) > V^j(x_*^j)$ for all $j\epsilon S^b$. S^b is a <u>blocking strategy</u>.

Agents do not cooperate as in the cooperative economy. All inactive agents take the strategies of intermediaries as given and maximize expected utility. See Nash [7]. Agent b above finds it in his own interest to announce a blocking strategy. Roughly speaking, agent b may be viewed as a firm who is aware of demand curves and seeks to enter profitable markets. This type of free entry will be crucial in determining the allocation of resources.

In what follows an equilibrium will be described in part as an allocation for which there exist no blocking strategies for any agent. A desirable characteristic of an equilibrium would also be that there does not exist an excess supply of intermediary services, i.e., that for an equilibrium consumption allocation $\{x_*^j, j\epsilon I\}$, an active intermediary h for a market M has expected utility $V(x_*^h) \leq V(x_*^j)$ for each $j\notin M$. If $V(x_*^h) > V(x_*^j)$ for some $j\notin M$, agent j would wish to act as an intermediary for the market M, and, by adopting slightly lower fees and prices, could attract all agents of M other than h to the newly proposed market. It

will be shown below that in an equilibrium the condition

that there be no blocking strategies implies that there is

no excess supply of intermediary services in this sense.

An equilibrium is now defined.

An <u>equilibrium</u> is a set of actions and strategies

$\{D_*^{ji}, d_*^{ji}, A_*^{ji}, S_*^j; i\varepsilon I\}$ for each agent $j\varepsilon I$, a set of allocations

$\{x_*^j \varepsilon X^j, y_{1*}^j \varepsilon Y^j; j\varepsilon I\}$, and a set of markets T such that:

(i).   If $d_*^{ij} = 0$ for each $i\varepsilon I-j$, then $\{y_{1*}^j, D_*^{hj}, d_*^{jh}, A_*^{jh}\}$

maximize (2) for agent j subject to constraints

(3)-(5) and to $S^h = S_*^h$ for each h such that $j\varepsilon S_{0*}^h$;

$$\sum_{h: j\varepsilon S_{0*}^h} d_*^{jh} S_{1*}^h(\omega) + y_{1*}^j \lambda^j(\omega) - \sum_{h: j\varepsilon S_{0*}^h} D_*^{hj} \lambda^j(\omega) = c_{\omega*}^j$$

for each $\omega\varepsilon\Omega$.

(ii). $\underset{M\varepsilon T}{U} M = I$; for each $M\varepsilon T$ there exists an $h\varepsilon M$ such that

$S_{0*}^h = M$; for each $j\varepsilon M-h$, $A^{jh} = 1$; for every such h

$$(K^h + \sum_{j\varepsilon S_{0*}^h} d_*^{jh} S_{4*}^h + \sum_{j\varepsilon S_{0*}^h} S_{3*}^{hj} - \sum_{j\varepsilon S_{0*}^h} D_*^{hj} S_{2*}^h - \alpha n^h - y_{1*}) \geq 0$$

$$\sum_{j\varepsilon S_{0*}^h} D_*^{hj} \lambda^j(\omega) + y_{1*}^h \lambda^h(\omega) - \sum_{j\varepsilon S_{0*}^h} d_*^{jh} S_{1*}^h(\omega) = c_{\omega*}^h$$

(iii).   There does not exist a blocking strategy for any

agent of I.

<u>Proposition IV</u>: If there exists an equilibrium, the equilibrium

allocation is in the core.

Proof: The proof is by contradiction. Suppose an equilibrium

allocation $\{x_*^j, y_{1*}^j; j\varepsilon I\}$ were not in the core. As in the

proof of Proposition I, suppose without loss of generality

that there exists a blocking coalition B of n agents which constitutes a market and in which exchange, production, and consumption decisions result in a Pareto optimal allocation for B. This yields an allocation

$$y_1^j = K^j - \frac{(2\alpha)(n-1)}{n}$$

$$c_{\omega^{**}}^j = \theta^j [nK^j - (2\alpha)(n-1)][\sum_{i \in B} \lambda^i / n]$$

for each $j \in B$ with $\sum_{j \in B} \theta^j = 1$.

By the definition of the core and the strict monotonicity of preferences, it is possible to choose $\{\theta^j; j \in B\}$ in such a way that $V^j(x_{**}^j) > V^j(x_*^j)$ for each $j \in B$. Let some agent $b \in B$ adopt the following strategy: $s_0^b = B$, $s_1^b = \sum_{i \in B} \lambda^i / n$, $s_2^b = s_4^b = 1$, $s_3^{bj} = \alpha[2(n-1)\theta^j - 1] + K^j[1 - n\theta^j]$ for each $j \in B-b$. If each agent $j \in B-b$ maximizes (2) subject to constraints (3)-(5) with $A^{jb} = 1$ for each $j \in B-b$, then the allocation $\{c_{\omega^{**}}^j j \in B\}$ would be achieved. This contradicts property (iii) of the equilibrium. Hence, the equilibrium allocation must be in the core. Q.E.D.

Proposition V: All core allocations in which there are a finite number of agents in each market can be supported as equilibria.

Proof: By Proposition III any core allocation with a finite number of agents in each market is also an allocation of the cooperative economy with $\Gamma \neq \emptyset$. It is claimed that the following actions, strategies, and markets support such an allocation.

(i). T is a set of markets with each $M \varepsilon T$ a market of

the core allocation. Hence, each $M \varepsilon T$ has some $m \varepsilon \Gamma$

agents, and $\underset{M \varepsilon T}{U} M = I.$

(ii). For each $M \varepsilon T$ there exist some agent $h \varepsilon M$ with

strategy:

$$S_0^h = M, \; S_1^h = \sum_{i \varepsilon M} \lambda^i / m, \; S_2^h = S_4^h = 1,$$

$$S_3^{hj} = \begin{cases} 0 \text{ if } m = 1 \\ (\alpha)(m-2)/m \text{ if } m \geq 2 \end{cases} \text{ for each } j \varepsilon S_0^h.$$

(iii). For each agent $j \varepsilon M - h$, $S_0^j = \emptyset$.

(iv). $d^{jh} = D^{hj} = K^j - \dfrac{(2\alpha)(m-1)}{m}$ for each $j \varepsilon M - h$

$A^{jh} = 1$ for each $j \varepsilon M - h$

$y_1^j = K^j - \dfrac{(2\alpha)(m-1)}{m}$ for each $j \varepsilon M$

$c_\omega^j = [K^j - \dfrac{(2\alpha)(m-1)}{m}][\sum_{i \varepsilon M} \lambda^i(\omega)/m]$ for each $\omega \varepsilon \Omega$ and

for each $j \varepsilon M$.

For each $j \varepsilon M - h$, $j \varepsilon S_0^h$ only. Given that j is not active, he

will maximize (2) subject to (3)-(5) regarding the strategy

of S^h under (ii) as fixed. With $S_2^h = S_4^h = 1$, such an agent

j will be willing to invest in h (so that $A^{jh} = 1$), trading

his own shares in project j for shares in the portfolio of

h up to the limit of his endowment less fixed fee; to satisfy

constraint (4),

$$y_1^j = D^{hj} = K^j - \dfrac{(2\alpha)(m-1)}{m}.$$

It should also be noted that agents of M will not collude in

their dealings with h. If one agent were to act as a conduit

for the funds of another to avoid the fixed fee for one of

the agents, an additional transactions cost of $\alpha-[(\alpha)(m-1)/m]$ would be incurred for the pair. Constraint (3) is also satisfied for each $j \epsilon M-h$. Hence, equilibrium condition (i) is satisfied.

Equilibrium condition (ii) follows from properties (i) and (ii) above with constraints (7) and (8) satisfied for each active intermediary h.

As the initial allocation is in the core, no finite set B can be made better off. Hence, there exist no blocking strategies for any agent, and equilibrium condition (iii) is satisfied. Q.E.D.

Several other aspects of the model are noted in passing.
1. In an equilibrium all agents have the same level of expected utility. Hence there is no excess supply of intermediary services, and all are indifferent as to who is acting as an intermediary with the equilibrium strategies.
2. There may be other schemes by which intermediaries can recoup fixed costs. Let s_5^h denote a minimum purchase requirement for shares in the portfolio of intermediary j for $j \epsilon s_0^h$. Then under condition (ii), for $m \geq 2$, consider the strategies:

$$s_1^h = \sum_{i \epsilon M} \lambda^i/m, \quad s_2^h = 1, \quad s_3^h = 0,$$

$$s_4^h = \frac{K^j - \alpha}{K^j - [(2\alpha)(m-1)/m]} \quad s_5^h = K^j - \frac{(2\alpha)(m-1)}{m}.$$

Each agent $j \varepsilon M$-h will act to maximize

$$(2)' \quad E \; U(d^{jh}s_1^h + y_1^j \lambda^j - D^{hj} \lambda^j)$$

subject to constraints

$$(3)' \quad (K^j + D^{hj} - d^{jh}s_4^h - \alpha y_1^j) \geq 0$$

$$(4)' \quad [d^{jh}s_1^h(\omega) + y_1^j \lambda^j(\omega) - D^{hj} \lambda^j(\omega)] \geq 0 \text{ for each } \omega \varepsilon \Omega.$$

$$(9) \quad d^{jh} \geq s_5^h.$$

Agent j would like to purchase less than s_5^h shares in the intermediary at prices s_4^h greater than one. However, he is constrained to purchase at least that amount and will not purchase more. Expenditures remain as before and the new strategies constitute an equilibrium for the model. The quantity $s_4^h - s_2^h$ can be interpreted as a type of bid-ask spread. It is on this margin that the intermediary covers transactions costs.

3. One may wish to determine whether the parametric behavior specified is incentive compatible. Following Hurwicz [5], a resource allocation mechanism is said to be individually incentive compatible if the behavior patterns specified are consistent with agents' natural inclinations. If an agent's project were essential to the portfolio offered by an intermediary, then it could be argued that such an agent might not sell passively at s_2^h but attempt to bargain. However, the model is constructed in such a way that the agent acting on his own has no bargaining power. As I is countably infinite, the removal of one agent from the model has no effect on the equilibrium allocation.

4. Finally it should be noted that if I were finite and α were zero, the noncooperative equilibria of the model would also be competitive equilibria. All agents would be in one market with completely diversified portfolios. Although one agent would act as an intermediary, at prices $s_2^h = s_4^h = 1$ and a fee $s_3^h = 0$, the role of that agent as intermediary would be inessential. In this sense, the noncooperative equilibrium as defined in this paper is a generalization of the definition of a competitive equilibrium, a generalization which can cope with nonconvexities.

V. Concluding Remarks

This essay has proposed a theory of intermediation. In the model a fixed cost of bilateral exchange creates a fundamental nonconvexity. Strictly speaking there do not exist competitive equilibria. However, there do exist noncooperative equilibria which are defined with respect to agent's strategies. These strategies include prices and fees at which agents are willing to act as intermediaries, buying shares in investment projects and selling shares in the resulting portfolio. These intermediaries emerge endogenously because they are able to economize on the fixed costs of exchange. Though agents take prices of intermediaries as given, all agents are free to choose their own intermediation strategies. This type of free entry is crucial in determining the allocation of resources. In this sense, the equilibria are competitive.

An allocation is said to be in the core if there are no finite coalitions which block, given the transactions costs of the model. It is shown that all allocations of the core, in which there are a finite number of agents in each market, can be supported as noncooperative equilibria and that all allocations of noncooperative equilibria are in the core. All such allocations are optimal in the sense that there do not exist Pareto superior allocations with a finite number of agents in each market. In this sense, incomplete markets may be efficient. These results are achieved by the levying of fixed fees, or, alternatively, by bid-ask spreads with minimum purchase criteria.

The model is a particularly elementary one. Agents are assumed to have identical tastes which display constant relative risk aversion and identical initial endowments. These facts are known to all agents. An effort to relax the assumption that tastes and endowments are identical would complicate the model. In particular, a restriction to the trading of shares might be binding and hence not satisfactory. Yet, in principle, the kind of analysis suggested above could be carried out. Intermediaries would take into consideration the desires of agents with diverse attitudes toward risk to trade contingent claims. Acting in their own interest, intermediaries may support efficient allocations.

However, if tastes and endowments are not known, there are new difficulties. Though an equilibrium may exist, it may not be optimal and incentive compatible. Agents may

understate their willingness to pay fixed fees to support a given market. This is a problem which has arisen in the context of models for the allocation of public goods. Though the work of Groves and Ledyard [3] suggests that the problem may not be insuperable, the effect may be to limit the number of agents in a market. A model with imperfect information may also be rich enough to account for the relatively limited number of types of assets which are traded in active markets. This is a subject of further research.

$\underline{1/}$Functions which display constant relative risk aversion
are of one of the following forms:

$$U(c_\omega^j) = \ln(c_\omega^j)$$

$$U(c_\omega^j) = (c_\omega^j)^b \quad 0 < b < 1.$$

Hence, if $c_\omega^i = \theta c_\omega^j$ for each $\omega\epsilon\Omega$, then

$$\mu_\omega U_1^i(c_\omega^i)/\mu_s U_1^i(c_s^i) = \mu_\omega U_1^i(\theta c_\omega^j)/\mu_s U_1^i(\theta c_s^j) = \mu_\omega U_1^j(c_\omega^j)/\mu_s U_1^j(c_s^j)$$

for each s and ω in Ω.

$\underline{2/}$This last point would be the source of some difficulty
if the set of traders were countably infinite, a case for
which the existence of a competitive equilibrium has not
been proved.

$\underline{3/}$It should be noted that if the set of agents I were
finite, it might not be possible for all agents to be in a
market with a number of agents in $\Gamma \neq \emptyset$. In such cases
there would not exist allocations in the core.

$\underline{4/}$Consider some market M of the core and its complement
with respect to I, M^c. As there are no blocking coalitions,
$V(x^i) \geq V(x^j)$ for each $j\epsilon M$ and each $i\epsilon M^c$. Suppose that for
some $i\epsilon M^c$ $V(x^i) > V(x^j)$. But then again, by the monotonicity
of preferences, the allocation could not be in the core.
Hence, $V(x^i) = V(x^j)$ for each $i\epsilon M^c$ and each $j\epsilon M$.

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