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Bounded Learning from Incumbent Firms

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Abstract

Social learning plays an important role in models of productivity dispersion and long-run growth. In economies with a continuum of producers and unbounded productivity distributions, social learning can sometimes leave long-run growth rates completely indeterminate. This paper modifies a model in which potential entrants attempt to imitate randomly selected incumbent firms by introducing an upper bound on how much entrants can learn from incumbents. When this upper bound is taken to infinity, a unique long-run growth rate emerges, even though the economy without upper bound has an unbounded continuum of balanced growth rates.

1 Introduction

Many models of long-run growth rely on some form of learning from others. In two relatively recent strands of the literature, productivity levels are widely dispersed and learning from more productive agents can only happen in random meetings—potential entrants meeting random incumbent firms in Luttmer [2007], and random meetings between different producers in Alvarez, Buera and Lucas [2008], Lucas [2009], Lucas and Moll [2014], and Perla and Tonetti [2014]. Although designed to be tractable and explain both long-run growth and persistent heterogeneity in productivity, these models have steady state predictions that are highly indeterminate. Initial productivity distributions with a thick right tail lead to faster growth than initial productivity distributions with a thin right tail. And for any positive growth rate, no matter how high, there is an initial productivity distribution with a sufficiently thick right tail that makes that growth rate part of a balanced growth path. More dispersion in productivities implies more significant social learning opportunities, and this naturally makes faster growth possible. But intuition suggests that this should be a transitory phenomenon that goes away as the left tail catches up with the right tail. Instead, somewhat surprisingly, high levels of dispersion and growth can persist. It is as if that right tail of the productivity distribution is an inexhaustible fount of knowledge.

This paper shows that this type of severe dependence on initial conditions can be viewed as an artefact of an assumption that agents have unbounded capacities to learn from others. In the case of Luttmer [2007], firms attempting to enter can imitate any randomly sampled incumbent firm, no matter how productive the incumbent. In the current paper, potential entrants can only copy incumbents that are not too advanced relative to the baseline of the least productive firms in the economy. The least productive firms are exiting all the time, and the assumption is that the productivity of exiting firms is freely available to every potential entrant. Potential entrants can improve this minimal level of productivity to the productivity of a randomly sampled incumbent, provided the improvement is not too large.[1]

This social learning mechanism introduces a cap on learning by potential entrants. The type of phenomena that can occur in such an economy turns out to depend in interesting ways on how tight this cap is. If it is sufficiently tight, the economy may not have a balanced growth path because the value of attempting entry is too low. Entry is necessary for balanced growth because low-productivity firms choose exit rather than pay fixed

[1] In Luttmer [2012], entrants can improve upon exiting firms only by a fixed step, without learning from the more productive continuing firms. This mechanism cannot be used to investigate the multiplicity associated with unbounded social learning.

continuation costs. A larger cap may make entry possible, but the incentives to enter need not be monotone in the growth rate of the economy. This introduces the possibility of high- and low-growth steady states. Also, for still relatively low caps on learning, the economy may have steady state equilibria in which there is technological regress in the aggregate, even though individual firm productivities tend to drift upwards. The reason: the economy has a growing population, and it is hard to replicate a population of incumbent firms when the opportunities to imitate particularly successful incumbent firms are limited.

When the cap on learning is large enough, the existence of a balanced growth path is guaranteed, and the economy will actually grow. Moreover, when the cap on learning is taken to infinity, the equilibria of economies with a finite cap on learning all converge to a single balanced growth path. That is, the model produces a definite equilibrium prediction for long-run growth, one that does not depend on assumptions about initial conditions established in a distant past. The resulting balanced growth path is one of the continuum of balanced growth paths that are possible in the economy with no cap on learning at all. Among all those possible balanced growth paths, it is the one with the slowest growth rate and with a productivity distribution that has the thinnest right tail.

Solving for and characterizing an endogenously trending distribution of productivities in an economy in which households and firms make forward-looking decisions can be hard. What greatly simplifies the analysis in this paper is the fact that, for any candidate growth rate, it is possible to analytically solve for the unique de-trended distribution of productivities. The uniqueness is a consequence of the cap on learning. For any particular finite cap on learning, the resulting distribution of log productivities below the cap is the product of a sine function and an exponential function. Above the cap the distribution of log productivities is exponential. In terms of productivity levels, this means that the distribution above the cap is a Pareto distribution. As the cap becomes large, the distribution of log productivities converges to the gamma distribution selected from a continuum of possible equilibrium distributions in Luttmer [2007]. As was pointed out in Luttmer [2007], this gamma distribution emerges when the social learning mechanism of the model is initialized with a productivity distribution that has a bounded support. But that selection argument was incomplete: it did not take into account the general equilibrium feedback of the out-of-steady state productivity dynamics on household and firm decision rules.

Related Literature Kortum [1997] generates long-run growth using draws from a fixed "source distribution" of productivities. The two key elements of the model are: the source distribution must be a power law, and the average number of draws must grow exponentially over time. The resulting distribution of de-trended productivities converges to a Fréchet distribution. In a recent paper, Buera and Oberfield [2020] combine these two key elements with social learning in a model of global knowledge diffusion and growth.

Emphasizing environments that are consistent with Fréchet productivity distributions, Buera and Lucas [2018] survey models in which productive agents meet randomly and learn from each other. In particular, they suggest considering an upper bound on how much individuals can gain in a random meeting and characterize an approximate solution to the resulting delay-differential equation. Instead of an upper bound, Luttmer [2007] examined the possibility that what entrants actually learn is a depreciated version of what randomly sampled incumbents can do. This also leads to a delay-differential equation, but one that can be solved explicitly. Importantly, such a setback in learning still generates a continuum of stationary productivity distributions and balanced growth rates.

Lucas and Moll [2014] give an example that shows how initial conditions become unimportant when producers can learn not only from each other, but also from an outside source with a sufficiently thick right tail. Instead, Staley [2011] adds Brownian log productivity shocks (that is, an outside source with thin tails) to a version of Alvarez, Buera and Lucas [2008]. As in Luttmer [2007], this makes it possible to give an explanation for thick-tailed productivity distributions that goes beyond an assumption about initial conditions or an outside source. The resulting Kolmogorov forward equation also has many solutions, implying a continuum of possible long-run growth rates. Building on truncation arguments developed in Brunet and Derrida [1997], Staley [2011] argues that letting a finite population of agents become large results in growth rates that approach the lowest of these long-run growth rates from below. In the current paper, the Kolmogorov forward equation for an economy with a continuum of agents and a cap on learning is very similar to the differential equations used in the approximations of Brunet and Derrida [1997] and Staley [2011]. But here the productivity distributions and equilibria obtained for a large economy with a cap on learning are exact.

Models with a continuum of firms have the practical virtue that they allow for the labor and product market interactions between firms to be competitive or monopolistically competitive. But they must be viewed as approximations for the bounded productivity distributions and the imperfectly competitive conditions that rule in the finite economies we observe. An interesting alternative approach is examined by Benhabib, Perla and Tonetti [2019], who construct a model with a continuum of firms and productivity growth rates that follow a Markov switching processes on a finite state space. Such processes automatically produce bounded productivity distributions from bounded initial conditions (see also Le [2014]).

Outline of the Paper The economy with bounded learning is described in Section 2. The conditions for a balanced growth path are described in Section 3. The upper bound on learning is taken to infinity in Section 4, and the unique balanced growth path is characterized. In Section 5, this unique limit is related to the continuum of long-run growth rates that are possible in an economy without an upper bound on learning.

2 The Economy

There is a unit measure of dynastic households. The size of a household is $H_t = He^{\eta t}$, where $\eta > 0$. Dynastic household preferences over consumption flows $C = \{C_t\}_{t\geq 0}$ are given by the utility function

$$\mathcal{U}(C) = \int_0^\infty e^{-\rho t} H_t \ln(C_t/H_t)\mathrm{d}t,$$

where $\rho > \eta$. There is a complete set of markets that allows households to trade consumption across time and states of the world. State prices are $\{e^{-\rho t}/(C_t/H_t)\}_{t\geq 0}$.

2.1 Factor Supplies

Everyone can choose to be a worker or an entrepreneur at any point in time. Workers supply labor and entrepreneurs create opportunities to set up a firm. Individuals differ in their abilities to do these tasks. Specifically, there is a time-invariant distribution $\mathcal{P}$ over abilities $(x, y) \in \mathbb{R}_+^2$, where x is the flow of labor services someone can supply, and y is the Poisson arrival rate at which the same individual can generate opportunities to set up a new firm. Throughout, it is assumed that the population means of both abilities are positive and finite. The price of labor services is w_t and the value of an opportunity to set up a new firm is q_t, both measured in some arbitrary unit of account. As in Roy [1951], a type-(x, y) individual chooses an occupation that attains $\max\{w_t x, q_t y\}$. If $\mathcal{P}$ is such that ties can be ignored, then this implies that the per-capita supply of labor is

$$\mathcal{L}(q_t/w_t) = \int x\iota\left\{w_t x > q_t y\right\} \mathrm{d}\mathcal{P}(x, y),$$

and that the per-capita flow of entry opportunities is

$$\mathcal{E}(q_t/w_t) = \int y\iota\left\{w_t x < q_t y\right\} \mathrm{d}\mathcal{P}(x,y).$$

The supply of labor is bounded above by the mean of x and weakly decreasing in q_t/w_t. The supply of entry opportunities is bounded above by the mean of y and weakly increasing in q_t/w_t. If everyone is either of a type $(x, 0)$ or of a type a $(0, y)$, then these per-capita factor supplies are inelastic. On the other hand, if there is a $(\widehat{x}, \widehat{y}) \in \mathbb{R}^2_{++}$ so that $x/y = \widehat{x}/\widehat{y}$ with $\mathcal{P}$-probability 1, then the per-capita supplies of labor and entry opportunities are perfectly elastic at $q_t/w_t = \widehat{x}/\widehat{y}$.

2.2 Differentiated Commodities

Consumption is a composite good that consists of a continuum of differentiated commodities of different types. The elasticity of substitution is constant and equal to $\varepsilon > 1$. The type of a commodity is indexed by a real variable z, and this type can change over time. Every differentiated commodity is produced by a different firm, using a linear labor-only technology with unit productivity. At time t, the measure of firms producing differentiated commodities of a type less than or equal to z is given by $N(t, z)$. Prices will be such that commodities of the same type are consumed at the same aggregate rate, denoted by $c_{z,t}$. Aggregate household consumption can then be defined as

$$C_t = \left(\int e^{z/\varepsilon} c_{z,t}^{1-1/\varepsilon} N(t, \mathrm{d}z)\right)^{1/(1-1/\varepsilon)}.$$

The type z of a differentiated commodity is reflected in its utility weight $e^{z/\varepsilon}$. One can interpret $e^{z/(\varepsilon-1)}$ as a measure of quality or productivity.

Given C_t, $N(t,\cdot)$, and type-z prices $p_{z,t}$, the demand curves are

$$c_{z,t} = \left(\frac{p_{z,t}}{P_t}\right)^{-\varepsilon} e^z C_t, \quad P_t = \left(\int e^z p_{z,t}^{1-\varepsilon} N(t, \mathrm{d}z)\right)^{1/(1-\varepsilon)}.$$

This indicates that e^z measures what type does for market size, holding prices fixed. Profit maximization implies the usual Lerner price,

$$p_{z,t} = \frac{w_t}{1 - 1/\varepsilon}.$$

Inserting this into the expression for the price index P_t and writing $N_t = N(t, \infty)$ gives

$$\frac{w_t}{P_t} = \left(1 - \frac{1}{\varepsilon}\right)\left(e^{Z_t}N_t\right)^{1/(\varepsilon-1)}, \qquad e^{Z_t} = \frac{1}{N_t}\int e^z N(t, \mathrm{d}z). \tag{1}$$

Given the supply of firms, this determines the real wage. The implied variable profits $v_{z,t}$ and use of labor $l_{z,t}$ are

$$\begin{bmatrix} v_{z,t} \\ w_t l_{z,t} \end{bmatrix} = \begin{bmatrix} 1/\varepsilon \\ 1 - 1/\varepsilon \end{bmatrix} e^{z-Z_t} \times \frac{P_t C_t}{N_t}.$$

So the distribution of variable profits and employment is governed by the distribution of e^{z-Z_t}. It follows that the aggregate amount of variable labor L_t needed to produce C_t is determined by $w_t L_t = (1 - 1/\varepsilon)P_t C_t$. Variable profits in units of labor can therefore be written as $v_{z,t}/w_t = e^{z-Z_t}(L_t/N_t)/(\varepsilon - 1)$. Average variable profits in units of labor are $(L_t/N_t)/(\varepsilon - 1)$.

2.3 Social Learning, Entry, and Exit

For a particular firm at time t, the type z_t of its differentiated commodity evolves according to

$$\mathrm{d}z_t = \theta \mathrm{d}t + \sigma \mathrm{d}W_t,$$

where $\theta \in (-\infty, \infty)$ and $\sigma > 0$ are constant parameters common to all firms, and W_t is a firm-specific standard Brownian motion. These Brownian motions are assumed to be independent across firms. To remain active, firms must pay a continuation cost of $\phi > 0$ units of labor. Nonpayment results in an immediate and irreversible exit. As will become clear, the fixed continuation cost implies that there is an exit threshold b_t so that only firms with $z_t \in (b_t, \infty)$ choose to continue.

Entry requires both an entry opportunity and learning from incumbents. An entrepreneur who succeeds in creating an entry opportunity draws a type at random from the population of continuing firms. If this type is in $[b_t, b_t + \Delta]$, where $\Delta > 0$ is a parameter, then the entrepreneur can introduce a new differentiated commodity of that type. Otherwise the entry attempt fails. A natural interpretation is that everyone has access to the information needed to introduce a commodity with the same quality as the commodities of exiting firms. Building on that information, the entrepreneur with an entry opportunity can improve its quality to that of a randomly selected incumbent firm, as long as the required quality improvement is not too large.[2]

Entrepreneurs are not needed to operate a firm. The fixed continuation cost of ϕ units of labor is enough. Since markets are complete, the value $V(t,z)$ of a firm of type z at time t is determined by

$$\frac{V(t,z)}{P_t} = \max_{\tau} \mathrm{E}_t \left[\int_t^{t+\tau} e^{-\rho(s-t)} \times \frac{C_t/H_t}{C_s/H_s} \left(\frac{v_{z_s,s}}{P_s} - \frac{\phi w_s}{P_s} \right) \mathrm{d}s \right],$$

where $z_s = z + \theta(s-t) + \sigma(W_s - W_t)$ for all $s \geq t$ and τ is a state-contingent exit time. Conjecture that this leads to $V(t,z) = 0$ for all $z \leq b_t$ and $V(t,z) > 0$ for all $z > b_t$, for some b_t to be determined. Only firms with $z > b_t$ choose to continue.

An entry opportunity is an opportunity to sample from $N(t,\cdot)$ and enter with z whenever the sampled firm has $z \in [b_t, b_t + \Delta]$. The value of an entry opportunity is therefore

$$q_t = \left(\int_{b_t}^{\infty} N(t,\mathrm{d}z) \right)^{-1} \int_{b_t}^{b_t+\Delta} V(t,z) N(t,\mathrm{d}z). \tag{2}$$

This is the expected market value of a randomly sampled firm, taking into account that entry attempts above $b_t + \Delta$ result in failure. This undirected sampling technology means that there are weak incentives to enter when Δ is small. Holding fixed $N(t,\cdot)$ and $V(t,\cdot)$, taking $\Delta \downarrow 0$ eliminates all incentives to attempt to enter.

2.4 Dynamics of the Firm Type Distribution

Restrict attention to situations in which $N(t,z)$ has a density $n(t,z)$. The flow of entry opportunities is $\mathcal{E}(q_t/w_t)H_t$. Entry opportunities lead to random draws from a distribution with probability density $n(t,z)/N_t$ on $[b_t,\infty)$ and zero otherwise. Draws from $(b_t + \Delta, \infty)$ result in failure to enter. At any $z \in (b_t, b_t + \Delta)$, instead, the flow of entrants is $\mathcal{E}(q_t/w_t)H_t \times n(t,z)/N_t$. Write this flow as $\alpha_t n(t,z)$, where

$$\alpha_t = \mathcal{E}(q_t/w_t)H_t/N_t.$$

This is the attempted entry rate per incumbent firm. With this definition, the Kolmogorov forward equation for $n(t,z)$ can then stated concisely as

$$\mathrm{D}_t n(t,z) = -\theta \mathrm{D}_z n(t,z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} n(t,z) + \alpha_t n(t,z),$$

[2] An alternative interpretation, outside the formal model presented here, is that only particularly productive firms find it worthwile to restrict imitation when there is a possibility that the imitator may introduce a perfect substitute.

for $z \in (b_t, b_t + \Delta)$ and

$$\mathrm{D}_t n(t,z) = -\theta \mathrm{D}_z n(t,z) + \frac{1}{2}\sigma^2 \mathrm{D}_{zz} n(t,z),$$

for $z \in (b_t + \Delta, \infty)$. Immediate exit at b_t means that $n(t, b_t) = 0$. For all $t \in (0, \infty)$, the density will be differentiable with respect to z at $b_t + \Delta$. The density at the initial date is given. The fact that forward-looking firms can exit instantaneously means that this initial density acts as an upper bound for $n(0, z)$.

3 Balanced Growth

A balanced growth path is taken to be any competitive equilibrium for this economy in which (i) per-capita consumption, real wages, and average real variable profits grow at a common rate, and (ii) the distribution of de-trended firm types is time invariant.

3.1 Minimal Requirements

Because $w_t L_t = (1 - 1/\varepsilon) P_t C_t$, any equilibrium in which real wages and per-capita consumption grow at a common constant rate must have a constant L_t/H_t. Average variable profits in units of labor are $(L_t/N_t)/(\varepsilon - 1)$. If these are also constant, then N_t/H_t must be constant as well. The equilibrium solution for real wages (1) then implies

$$\left[\frac{L_t}{H_t}, \frac{N_t}{H_t}\right] = \left[\frac{L}{H}, \frac{N}{H}\right], \qquad Z_t = Z + (\theta - \mu)t,$$

for some L/H, N/H, and μ to be determined. The equilibrium variable μ is the drift of $z_t - Z_t$ for the typical firm. At the same time, $-\mu$ measures the aggregate growth over and above θ that comes from replacing exiting firms by new firms that are more productive. From (1), the resulting real wages and per-capita consumption can be written in the form

$$\left[\frac{w_t}{P_t}, \frac{C_t}{H_t}\right] = \left[\frac{w}{P}, \frac{C}{H}\right] e^{\kappa t}, \qquad \kappa = \frac{\theta - \mu + \eta}{\varepsilon - 1},$$

with levels related via

$$\frac{w}{P} = \left(1 - \frac{1}{\varepsilon}\right)\left(e^Z N\right)^{1/(\varepsilon-1)}, \qquad \frac{C}{H} = \left(e^Z N\right)^{1/(\varepsilon-1)} \times \frac{L}{H}.$$

Improvements in the firm type distribution and gains from variety both contribute to the growth rate κ of real wages and per-capita consumption.

3.2 The Value Function

These basic features of a balanced growth path are enough to fully characterize firm values and exit decisions. To see this, recall that $v_{z,t}/w_t = e^{z-Z_t}(L_t/N_t)/(\varepsilon - 1)$ and use the fact that C_t/H_t and w_t/P_t grow at a common rate to write

$$\frac{V(t,z)}{P_t} = \frac{\phi w_t}{P_t} \times \max_{\tau} \mathrm{E}_t \left[\int_t^{t+\tau} e^{-\rho(s-t)} \left(\frac{e^{z_s - Z_s} L}{(\varepsilon - 1)\phi N} - 1 \right) \mathrm{d}s \right],$$

where $z_s - Z_s = z - Z_t + \mu(s-t) + \sigma(W_s - W_t)$ for all $s \geq t$. This is now a textbook stopping problem (Dixit and Pindyck [1994]). A calculation given in Luttmer [2007] shows that

$$\frac{V(t,z)}{P_t} = \frac{\phi w_t}{P_t} \times U(y), \qquad e^y = \frac{e^{z-Z_t} L}{(\varepsilon - 1)\phi N}, \tag{3}$$

where $U(y) = 0$ for all y at or below some exit threshold a, and

$$U(y) = \frac{1}{\rho} \frac{\xi}{1+\xi} \left(e^{y-a} - 1 - \frac{1 - e^{-\xi(y-a)}}{\xi} \right), \tag{4}$$

for all $y > a$. The exit threshold a is determined by

$$e^a = \frac{\xi}{1+\xi} \left(1 - \frac{1}{\rho} \left(\mu + \frac{1}{2}\sigma^2 \right) \right), \quad \xi = \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} \right)^2 + \frac{\rho}{\sigma^2/2}}. \tag{5}$$

Observe that $U(\cdot)$ only depends on μ and exogenous parameters. Lemma 1 describes the most important properties of $U(\cdot)$. The proof is in Appendix A.

Lemma 1 *The value function (4)-(5) is well defined if and only if* $\mu + \frac{1}{2}\sigma^2 < \rho$. *Given this restriction, it has the following properties:*

(i) *The value function is strictly increasing and unbounded in* $y > a$.

(ii) *The exit threshold is strictly decreasing in* μ, $\lim_{\mu \to -\infty} a = 0$, *and* $\lim_{\mu \uparrow \rho - \sigma^2/2} a = -\infty$.

(iii) *For any* $u \in (0, \infty)$ *or* $y \in (-\infty, \infty)$,

$$\lim_{\mu \to -\infty} U(a+u) = 0, \qquad \lim_{\mu \uparrow \rho - \sigma^2/2} U(a+u) \in (0, \infty), \qquad \lim_{\mu \uparrow \rho - \sigma^2/2} U(y) = \infty,$$

and $U(a+u)$ *is increasing in* μ.

An immediate implication of the characterization (3)-(5) of $V(t,z)$ is that the time-t exit threshold for firm of type z must be $b_t = b + (\theta - \mu)t$, where b solves

$$e^a = \frac{e^{b-Z}L}{(\varepsilon - 1)\phi N}. \tag{6}$$

In other words, the gap $Z_t - b_t = Z - b$ must be constant over time.

3.3 Imposing Stationarity

Now conjecture that, for appropriate initial conditions, there is an equilibrium in which not just $Z_t - b_t$ but the entire cross-sectional distribution of $z - b_t$ is time invariant. That is, conjecture that for some probability density $f(\cdot)$,

$$n(t,z) = N_t f(z - b_t),$$

for all t and all $z \geq b_t$. Given $f(\cdot)$, the definition (1) of Z_t implies a consistency condition

$$e^{Z-b} = \int_0^\infty e^u f(u)\mathrm{d}u. \tag{7}$$

And the value q_t/w_t of an entry opportunity, defined in (2), becomes

$$\frac{q_t}{w_t} = \phi \int_0^\Delta U\left(a+u\right) f(u)\mathrm{d}u. \tag{8}$$

It follows that $q_t/w_t = q/w$, and so the per-capita factor supplies $\mathcal{E}(q_t/w_t)$ and $\mathcal{L}(q_t/w_t)$ are constant over time. Since N_t/H_t is constant, this means that the arrival rate of entry opportunities is also constant over time, at some level $\alpha_t = \alpha$. The definition of α_t gives

$$\frac{N}{H} = \frac{1}{\alpha} \times \mathcal{E}\left(\frac{q}{w}\right). \tag{9}$$

This can be viewed as a steady state supply of firms.

Recall that μ is the drift of $z_t - b_t$. The Kolmogorov forward equation for $n(t,z)$ therefore reduces to

$$\eta f(u) = -\mu \mathrm{D} f(u) + \frac{1}{2}\sigma^2 \mathrm{D}^2 f(u) + \alpha f(u), \tag{10}$$

for $u \in (0, \Delta)$ and

$$\eta f(u) = -\mu \mathrm{D} f(u) + \frac{1}{2}\sigma^2 \mathrm{D}^2 f(u), \tag{11}$$

for $u \in (\Delta, \infty)$. The boundary conditions are $f(0) = 0$ together with continuity and differentiability at Δ. The definition of $f(\cdot)$ requires it to be a probability density on $(0, \infty)$.

3.3.1 An Accounting Identity

With these boundary conditions, integrating the differential equation (10)-(11) over $(0, \infty)$ yields

$$\alpha \int_0^\Delta f(u)\mathrm{d}u = \eta + \frac{1}{2}\sigma^2 \mathrm{D}f(0).$$

Since the exit rate is given by $\frac{1}{2}\sigma^2 \mathrm{D}f(0)$, this result confirms a basic accounting condition: net of exit, the successful entry rate must be just enough to supply the new firms needed for the number of firms to grow at the population growth rate η. This condition would fail if $f(\cdot)$ were not differentiable at Δ.[3]

3.4 Clearing the Labor Market

The employment size of firms scales with $e^u = e^{z-b}$. The relation (6) between the thresholds a and b together with the consistency condition (7) for $Z - b$ imply that the labor market clearing condition $\mathcal{L}(q/w) H = \phi N + L$ can be written as

$$\frac{N}{H} = \frac{1}{\phi} \frac{\mathcal{L}(q/w)}{1 + (\varepsilon - 1) \int_0^\infty e^{a+u} f(u)\mathrm{d}u}. \tag{12}$$

The right-hand side is simply the ratio of the per-capita supply of labor over the amount of labor used by an average firm, including the fixed cost. The resulting N/H can be viewed as the steady state demand for firms, complementing the supply (9).

3.5 Solving the Kolmogorov Forward Equation

Any stationary density $f(\cdot)$ must satisfy the Kolmogorov forward equation (10)-(11). This equation is parameterized by μ and α, and both μ and α are endogenous variables in this economy. The key fact is that, for any $\mu \in (-\infty, \infty)$, there is precisely one attempted entry rate $\alpha > 0$ for which it is possible to construct a stationary density. So α and $f(\cdot)$ are pinned down jointly as a function of μ. As described further in Section 5, this key fact is wrong when $\Delta = \infty$.

To show this, hold fixed some $\mu \in (-\infty, \infty)$. The two pieces (10) and (11) of the Kolmogorov forward equation each define a characteristic equation. On $(0, \Delta)$, a solution

[3] If $f(\cdot)$ has a concave kink at Δ, then positing an additional flow of entrants at Δ can restore a version of this accounting identity, as in Luttmer [2012].

of the type $e^{-\chi z}$ implies $\chi \in \{\chi_-, \chi_+\}$, where

$$\chi_\pm = -\frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 - \frac{\alpha - \eta}{\sigma^2/2}}. \tag{13}$$

On (Δ, ∞), a solution of the form $e^{-\zeta z}$ gives $\zeta \in \{\zeta_-, \zeta_+\}$ and

$$\zeta_\pm = -\frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\eta}{\sigma^2/2}}. \tag{14}$$

Observe that $\eta > 0$ implies $\zeta_- < 0 < \zeta_+$, irrespective of the sign of μ. Since $f(u) \to 0$ as $u \to \infty$, this forces $f(u) \propto e^{-\zeta_+ u}$ on (Δ, ∞), with a scale to be determined.

But the roots (13) may be real or complex, depending on the magnitude of the attempted entry rate α. Suppose that α is large enough so that the roots $\chi_\pm$ are, in fact, complex. To streamline the calculations, let $\psi = \mathrm{Re}(\chi_+)$ and $\omega = \mathrm{Im}(\chi_+)$, so that

$$\psi = -\frac{\mu}{\sigma^2}, \quad \omega = \sqrt{\frac{\alpha - \eta}{\sigma^2/2} - \psi^2}. \tag{15}$$

Any linear combination of $e^{-\chi_+ u}$ and $e^{-\chi_- u}$ is a solution to the differential equation (10) on $(0, \Delta)$. Requiring these linear combinations to be real forces solutions to be of the form $f(u) = [A\cos(\omega u) + B\sin(\omega u)]\, e^{-\psi u}$, for real coefficients A and B. Imposing the boundary condition $f(0) = 0$ implies $A = 0$. Imposing continuity at $u = \Delta$ then implies a solution of the form

$$f(u) = B \begin{cases} \sin(\omega u) e^{-\psi u}, & u \in [0, \Delta], \\ \sin(\omega\Delta) e^{-\psi\Delta} e^{-\zeta_+(u-\Delta)}, & u \in [\Delta, \infty). \end{cases} \tag{16}$$

Taking $B > 0$, this solution will be positive on $(0, \Delta)$ if and only if $\omega\Delta \in (0, \pi)$. Imposing differentiability at $u = \Delta$ gives $\omega\cos(\omega\Delta)e^{-\psi\Delta} - \psi\sin(\omega\Delta)e^{-\psi\Delta} = -\zeta_+ \sin(\omega\Delta)e^{-\psi\Delta}$. Together with $\zeta_+ = \psi + \sqrt{\psi^2 + \eta/(\sigma^2/2)}$, this implies the following restriction on ω,

$$-\frac{\cos(\omega\Delta)}{\sin(\omega\Delta)/(\omega\Delta)} = \Delta\sqrt{\psi^2 + \frac{\eta}{\sigma^2/2}}. \tag{17}$$

As a function of ω, the left-hand side of (17) increases monotonically from -1 to ∞ on the domain $[0, \pi/\Delta)$, crossing zero at $\pi/(2\Delta)$. So there will be precisely one value of $\omega \in (0, \pi/\Delta)$ for which the density (16) is positive throughout $(0, \infty)$ and differentiable at Δ. Observe that this ω is in $(\pi/(2\Delta), \pi/\Delta)$, is increasing in ψ^2, decreasing in Δ, and that $\omega\Delta \in (\pi/2, \pi)$ is increasing in Δ.

Using $\sin(\omega u)e^{-\psi u} = \left(e^{-(\psi-i\omega)u} - e^{-(\psi+i\omega)u}\right)/(2i)$ to integrate (16) shows that the cumulative distribution must be of the form

$$F(u) = B \begin{cases} \frac{\omega-[\omega\cos(\omega u)+\psi\sin(\omega u)]e^{-\psi u}}{\omega^2+\psi^2}, & u \in [0,\Delta], \\ \frac{\omega-[\omega\cos(\omega\Delta)+\psi\sin(\omega\Delta)]e^{-\psi\Delta}}{\omega^2+\psi^2} + \sin(\omega\Delta)e^{-\psi\Delta}\left(\frac{1-e^{-\zeta_+(u-\Delta)}}{\zeta_+}\right), & u \in [\Delta,\infty). \end{cases} \tag{18}$$

The normalizing constant $1/B$ is therefore

$$\frac{1}{B} = \frac{\omega - [\omega\cos(\omega\Delta) + \psi\sin(\omega\Delta)]\,e^{-\psi\Delta}}{\omega^2+\psi^2} + \frac{\sin(\omega\Delta)e^{-\psi\Delta}}{\zeta_+}. \tag{19}$$

Given the ω implied by (17), inverting (15) says that the attempted entry rate must be

$$\alpha = \eta + \frac{1}{2}\sigma^2\left(\omega^2+\psi^2\right). \tag{20}$$

So the requirement that the stationary density must be differentiable at Δ pins down both the density $f(\cdot)$ and the attempted entry rate α. Since ω is decreasing in Δ, so is α. The larger the range of types that potential entrants can imitate, the fewer entry attempts are needed for stationarity. Observe that ω and α only depend on μ via ψ^2, and so the sign of μ does not affect ω or α. Since the right-hand side of (17) is increasing in ψ^2, both ω and α are increasing in ψ^2.

The sign of μ does very much affect the shape of the stationary distribution. For example, it becomes concentrated at 0 as $\mu \to -\infty$, and certainly not as $\mu \to \infty$. On (Δ,∞), the distribution of $e^{u-\Delta}$ is a standard Pareto distribution with a tail index ζ_+. Along any balanced growth path, this tail index must satisfy $\zeta_+ > 1$, or else the mean of e^u, and hence aggregate employment, will not be finite. This implies an upper bound on μ. These results can be summarized as follows.

Proposition 1 *Given some $\mu \in (-\infty,\infty)$ and any $\Delta \in (0,\infty)$, the Kolmogorov forward equation (10)-(11) has a solution if and only if the attempted entry rate α satisfies (17) and (20), where $\psi = -\mu/\sigma^2$. The solution is given by (16) and (19), where ω is determined by (17) and $\zeta_+ = \psi + \sqrt{\psi^2 + \eta/(\sigma^2/2)}$. The attempted entry rate is strictly increasing in μ^2, and $\mu^2 \to \infty$ implies $\alpha \to \infty$. The mean of e^u is finite if and only if μ satisfies $\eta > \mu + \sigma^2/2$. It converges to 1 as $\mu \to -\infty$.*

The construction (15)-(20) takes as given that the characteristic roots $\chi_\pm$ are complex. Appendix B shows that real $\chi_\pm$ lead to $\lim_{z\uparrow\Delta} \mathrm{D}f(z) > \lim_{z\downarrow\Delta} \mathrm{D}f(z)$, violating the need for differentiability at Δ. A somewhat intricate calculation (available upon request) of the derivative of $F(u)$ with respect to Δ shows that an increase in Δ causes $F(\cdot)$ to shift to the right, in the sense of first-order stochastic dominance. A larger range of types that potential entrants can imitate pulls the stationary distribution away from the exit threshold.

3.6 The Equations for Balanced Growth

Equating the supply (9) and demand (12) for N/H gives the market clearing condition

$$\frac{\mathcal{E}\left(q/w\right)}{\mathcal{L}\left(q/w\right)} = \frac{\alpha/\phi}{1+(\varepsilon-1)\int_0^\infty e^{a+u}f(u)\mathrm{d}u}. \tag{21}$$

By definition, the relative price q/w must also satisfy

$$\frac{q}{w} = \phi\int_0^\Delta U\left(a+u\right)f(u)\mathrm{d}u. \tag{22}$$

The left-hand sides of (21)-(22) are functions only of q/w, and the right-hand sides are functions only of μ, via the value function $U(\cdot)$ and the implied exit threshold a defined in (4)-(5), and via the stationary density $f(\cdot)$ and the associated attempted entry rate α defined in (16)-(20). It is clear that one can immediately eliminate q/w and reduce (21)-(22) to a single equilibrium condition in μ.

Given a solution for μ and the implied α, $f(\cdot)$ and $U(\cdot)$, the number of firms per capita N/H can be inferred from either the supply curve (9) or the demand curve (12). The economy is on a balanced growth path if there is an exit threshold b for which the initial type density satisfies $n(0,z)/H = (N/H)f(z-b)$. The economy has a unique balanced growth path if and only if the equations (21)-(22) have precisely one solution for μ. The possibilities can be illustrated using two polar case assumptions about the relative factor supplies $\mathcal{E}(\cdot)/\mathcal{L}(\cdot)$.

3.6.1 The Perfectly Elastic Case

If there is no heterogeneity in abilities, then the relative supply $\mathcal{E}\left(q/w\right)/\mathcal{L}\left(q/w\right)$ of entrepreneurial services and labor is perfectly elastic at some q/w that is implied by abilities. In equilibrium, everyone must be indifferent between supplying entrepreneurial services and supplying labor, and the equilibrium condition for μ is simply (22). The factor supplies will adjust to match (21). This will be a proper balanced growth path only if the mean of e^u is finite, which corresponds to $\eta > \mu + \frac{1}{2}\sigma^2$. Figure 1 shows the right-hand side of (22) for μ up to this bound. In this example, $\eta < \sigma^2/2$, and so μ can only be negative.

Since $\rho > \eta$, the value function is well defined and bounded above on $[0, \Delta]$ for all μ that satisfy $\eta > \mu + \frac{1}{2}\sigma^2$. The right-hand side of (22) therefore has a finite upper bound. If the relative price q/w implied by abilities is greater than this upper bound, then (22) will have no solution. Hence there will be no balanced growth path. This has nothing to do with the possibility of finding a stationary density $f(\cdot)$ that solves the Kolmogorov forward equation. That can be done for any $\Delta \in (0, \infty)$, and it will imply a certain amount of entry. The problem is that the value of the most profitable firm that can be imitated is $U(a+\Delta)$, and this can be made arbitrarily small by taking Δ small. With a perfectly elastic $\mathcal{E}(\cdot)/\mathcal{L}(\cdot)$, this shuts down the entry needed for stationarity.

As Figure 1 illustrates, the right-hand side of (22) need not to be monotone in μ, and so multiple balanced growth paths are also possible. By Lemma 1, it is true that an increase in μ does always raise $U(a + u)$ for any given $u \in [0, \Delta]$. But more rapid firm growth also reduces the probability of sampling a firm in $[0, \Delta]$. This makes entry attempts fail more often, which tends to lower the value of attempting entry. In Figure 1, this second effect dominates when μ is close to its upper bound.

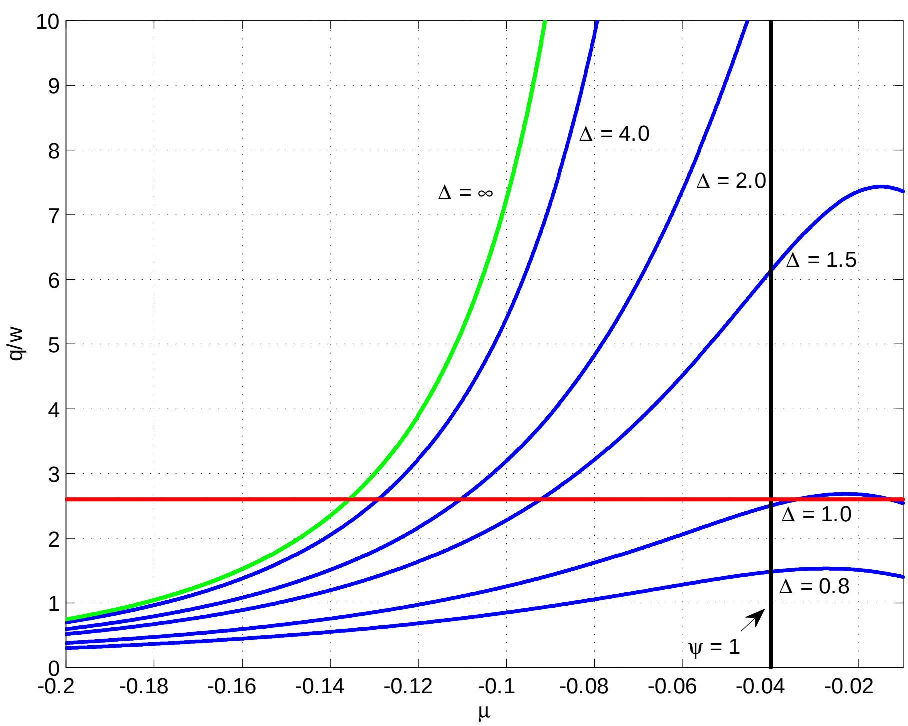

FIGURE 1 The value of attempting entry

3.6.2 The Perfectly Inelastic Case

Alternatively, if every individual can do only one thing, then the aggregate factor supplies are given. The condition that determines μ is then (21). This condition depends on the value function only via its exit threshold a. With μ determined by (21), the relative price q/w follows from (22). The condition (21) is shown in Figure 2 for various Δ. The ranking across Δ of this equilibrium condition is explained by the fact that α is decreasing in Δ, together with the fact that an increase in Δ moves $F(\cdot)$ to the right in the sense of first-order stochastic dominance.

In contrast to the perfectly elastic case, this economy always has a balanced growth path, provided that there is a positive measure of entrepreneurs. This is because the right-hand side of (21), while not necessarily monotone, varies throughout $(0, \infty)$. To check this, observe that, by Proposition 1, $\mu \to -\infty$ sends the attempted entry rate α to infinity and shrinks the mean of e^u down to 1. Using part (ii) of Lemma 1 as well, the mean of e^{a+u} converges to 1. So the right-hand side of (21) can be made arbitrarily large. On the other hand, taking $\mu + \frac{1}{2}\sigma^2 \uparrow \eta$ sends the mean of e^{a+u} to infinity and produces a positive limit for α. This means that the right-hand side of (21) can be made small.

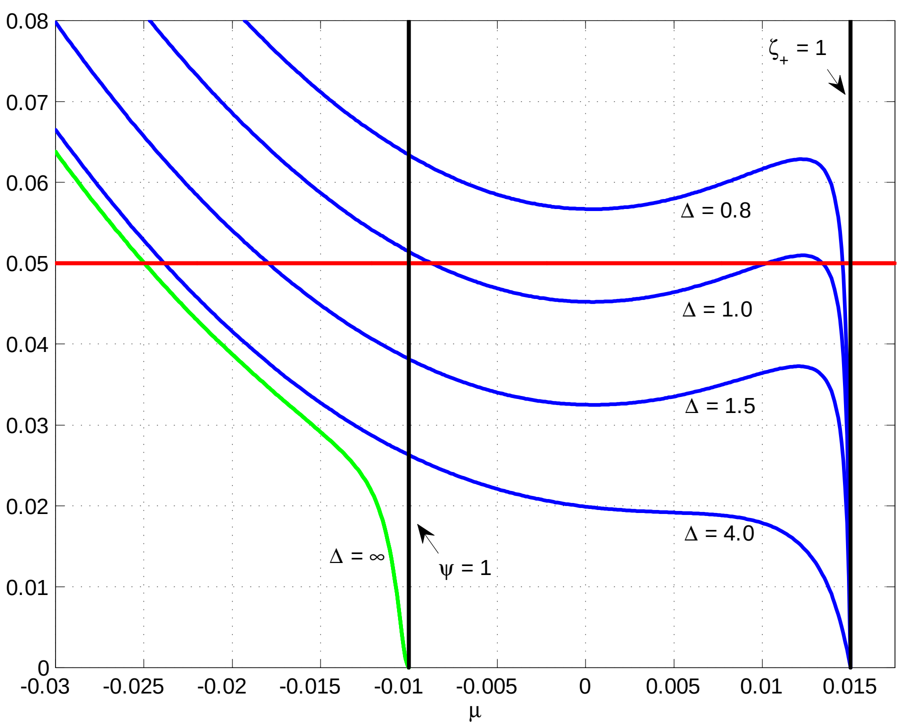

FIGURE 2 Relative factor demands

It is easy, though, to construct examples of economies with more than one balanced growth path. Specifically, take $\eta > \sigma^2/2$, so that $\zeta_+ > 1$ is consistent with positive values of μ. By taking $\varepsilon - 1 > 0$ to be very small, one can ensure that the right-hand side of (21) is dominated by the properties of α, except when $\mu + \frac{1}{2}\sigma^2$ gets very close to its upper bound η. And α only depends on μ^2, making the right-hand side of (21) non-monotonic in μ. In Figure 2, $\eta = 0.02$, $\sigma = 0.1$, $\varepsilon = 1.01$, and the $\Delta = 1$ case shows an example with three distinct equilibria.

As Figure 2 shows, two of these equilibria have $\mu > 0$, and so the trend $\theta - \mu$ of the aggregate type index $\ln(Z_t)$ is below the drift θ that governs the types of individual firms. In particular, there could be aggregate technological regress even if the types of individual firms drift upwards. In view of the constraint $\eta > \mu + \frac{1}{2}\sigma^2$, this is a possibility only because there is population growth. To understand what is happening, consider that population growth implies a need to expand the population of firms over time. Although entrants are more productive than exiting firms, replicating the entire incumbent population of firms is inherently difficult because entrants are in $[b_t, b_t + \Delta]$ while incumbents are spread throughout $[b_t, \infty)$.[4]

4 Balanced Growth When $\Delta < \infty$ is Large

The possibilities of non-existence, multiplicity, and technical regress disappear when the range of incumbents who can be imitated becomes large. A first step in showing this is to understand how the attempted entry rate α and the stationary density $f(\cdot)$ depend on Δ.

4.1 The Attempted Entry Rate

Fix any $\psi = -\mu/\sigma^2$ and observe that $\Delta \to \infty$ increases the right-hand side of (17) without bound. This implies that $\omega\Delta \uparrow \pi$ and $\omega \downarrow 0$, both monotonically, as $\Delta \to \infty$. It then follows from (20) that the attempted entry rate α converges from above to $\alpha = \eta + \frac{1}{2}\sigma^2\psi^2$.

4.2 The Limiting Distribution

The attempted entry rate converges to a well-defined limit that is linear in ψ^2. The distribution $F(u)$, on the other hand, only converges to a proper distribution when ψ is positive. To see this, note that $F(u)$ on $[0, \Delta]$ can be written as

$$F(u) = \frac{\omega B}{\omega^2 + \psi^2} \left(1 - \left(\cos(\omega u) + \psi u \times \frac{\sin(\omega u)}{\omega u} \right) e^{-\psi u} \right),$$

[4] These same considerations also apply when there is no population growth but firms are sometimes randomly forced to exit.

and that the formula (19) for the normalizing constant can be restated as

$$\begin{aligned}\frac{\omega^2+\psi^2}{\omega B} &= 1-\left(\cos(\omega\Delta)+\psi\times\frac{\sin(\omega\Delta)}{\omega}\right)e^{-\psi\Delta}+\frac{\omega^2+\psi^2}{\zeta_+}\frac{\sin(\omega\Delta)}{\omega}\times e^{-\psi\Delta}\\ &= 1+\left\{1-\frac{\psi}{\sqrt{\psi^2+\frac{\eta}{\sigma^2/2}}}+\frac{\omega^2+\psi^2}{\psi+\sqrt{\psi^2+\frac{\eta}{\sigma^2/2}}}\frac{1}{\sqrt{\psi^2+\frac{\eta}{\sigma^2/2}}}\right\}\left|\cos(\omega\Delta)\right|e^{-\psi\Delta}.\end{aligned}$$

The second equality uses the equilibrium condition (17) for $\omega\Delta\in(\pi/2,\pi)$ to replace $\sin(\omega\Delta)/\omega$, as well as the formula (14) for ζ_+. Given that $\omega\Delta\to\pi$ as $\Delta\to\infty$, it is now easy to see that $\omega B/(\omega^2+\psi^2)\to 0$ if $\psi<0$, and that $\omega B/(\omega^2+\psi^2)\to 1$ if $\psi>0$. In any case, the fact that $\omega\downarrow 0$ as $\Delta\to\infty$ implies

$$\lim_{\Delta\to\infty}\left(1-\left(\cos(\omega u)+\psi u\times\frac{\sin(\omega u)}{\omega u}\right)e^{-\psi u}\right)=1-(1+\psi u)\,e^{-\psi u}$$

for any fixed $u\in(0,\infty)$. Only if $\psi>0$ does this produce a proper limiting distribution, a gamma distribution with density $f(u)=\psi^2ue^{-\psi u}$. If $\psi<0$ instead, then $\omega B/(\omega^2+\psi^2)\to 0$ means that all probability escapes to infinity. Appendix C shows this is also true when $\psi=0$. Together, these observations deliver the first part of the following lemma.

Lemma 2 *The stationary distribution function converges to*

$$\lim_{\Delta\to\infty}F(u)=\begin{cases}0, & \psi\in(-\infty,0],\\ 1-(1+\psi u)e^{-\psi u}, & \psi\in(0,\infty),\end{cases}$$

for any $u\in[0,\infty)$. *Define* $G(\Delta)=\int_0^\Delta e^u f(u)\mathrm{d}u$ *and let* $\Psi=\{\psi:\psi\leq 1\leq\zeta_+\}$. *Then*

$$\lim_{\Delta\to\infty}G(\Delta)=\begin{cases}\infty, & \psi\in(-\infty,1],\\ \left(\frac{\psi}{\psi-1}\right)^2, & \psi\in(1,\infty),\end{cases}$$

and there is a positive constant k *so that* $\inf_{\psi\in\Psi}G(\Delta)\geq k\Delta^2$ *for all* Δ *large enough. The attempted entry rate converges to* $\alpha=\eta+\frac{1}{2}\sigma^2\psi^2$ *as* Δ *becomes large.*

The properties of the truncated mean $G(\Delta)$, which is a key determinant of the incentives to enter, are proved in Appendix C. Observe that this truncated mean explodes when $\psi\leq 0$, even though all probability escapes to infinity in that case. For $\psi>0$, the limit of the truncated mean is simply the mean of e^u under the limiting distribution.

For all finite Δ, the right tail index of the distribution $F(u)$ is given by $\zeta_+=\psi+$

$\sqrt{\psi^2 + \eta/(\sigma^2/2)}$. Here we learn that the large-Δ limiting distribution, as long as it is a proper distribution, has a tail index ψ. So the tail index is discontinuous at $\Delta = \infty$. Since $\psi > 0$ implies $\zeta_+ > 2\psi$, every proper limiting distribution has a distinctly thicker right tail than the finite-Δ distributions with the same $\psi > 0$. This is reflected in the gap between the asymptotes $\psi > 1$ and $\zeta_+ > 1$ shown in Figure 2. This gap corresponds to $\psi \in \Psi$, or $-\frac{1}{2}\sigma^2 \leq \mu + \frac{1}{2}\sigma^2 \leq \eta$ in terms of μ. On the interior of Ψ, the finite-Δ tail index ζ_+ is large enough for aggregate employment to be finite, while the large-Δ limit implies infinite aggregate employment.

4.3 When Δ is Large Enough

Lemma 2 can be used to rule out, for all large enough Δ, the possible non-existence of a balanced growth path shown in Figure 1 and the possibility that $\mu > 0$ shown in Figure 2.

For example, consider the perfectly elastic case shown in Figure 1, with relative supplies of managerial and labor services perfectly elastic at some $q/w \in (0, \infty)$. At $\psi = 1$, Lemma 2 implies that the value of attempting entry grows without bound as $\Delta \to \infty$. Fix any Δ large enough so that the value of attempting entry exceeds q/w at $\psi = 1$. Lemma 1 says that letting $\psi \to \infty$ implies $U(a + \Delta) \downarrow 0$, and hence, that the value of attempting entry converges to zero as $\psi \to \infty$. The value of entry is continuous in ψ, and so it must cross the threshold q/w at least once for some $\psi > 1$. Proposition 2 covers smooth factor supply curves.

Proposition 2 *Assume the relative factor supply curve $\mathcal{E}(\cdot)/\mathcal{L}(\cdot)$ is continuous. For all Δ large enough, finite-Δ economies must have at least one equilibrium ψ_Δ, and any equilibrium must have $\psi_\Delta > 1$.*

To prove existence, start again with $\psi = 1$. Lemma 2 then implies that the mean of e^u grows without bound as $\Delta \to \infty$. And the attempted entry rate converges to a finite and positive limit. Therefore, one can choose Δ large enough to ensure that the right-hand side of (21) is arbitrarily close to zero. On the other hand, Lemmas 1 and 2 imply that q/w becomes large when $\psi = 1$ and $\Delta \to \infty$. The fact that $\mathcal{E}(q/w)/\mathcal{L}(q/w)$ is weakly increasing then implies that the left-hand side of (21) will be positive and bounded away from 0 as $\Delta \to \infty$. So it is possible to choose Δ large enough so that the left-hand side of (21) dominates the right-hand side at $\psi = 1$. For any such Δ, take $\psi \to \infty$. Lemma 1 says that $U(a + \Delta) \to 0$ in that case, and so it must be that $q/w \to 0$ as $\psi \to \infty$. This minimizes $\mathcal{E}(q/w)/\mathcal{L}(q/w)$. But the attempted entry rate α goes to ∞ as $\psi \to \infty$ and the mean of e^{a+u} converges to 1. So then the right-hand side of (21) will grow without bound as $\psi \to \infty$.

The continuity of the relative factor supplies implies that there will be a $\psi > 1$ where (21) holds with equality.

To rule out equilibria with $\psi \leq 1$, recall that any finite-Δ equilibrium must have $\zeta_+ > 1$. So it suffices to rule out equilibria with $\psi \in \Psi = \{\psi : \psi \leq 1 \leq \zeta_+\}$. From Lemma 2, the mean of e^{a+u} is larger than $e^a k\Delta^2$ for all $\psi \in \Psi$ and all Δ large enough. And Lemma 1 ensures that e^a is bounded away from zero on Ψ. The right-hand side of (21) will therefore be small throughout Ψ when Δ is large. Lemmas 1 and 2 also imply that the value of attempting entry is larger than $(-1 + k\Delta^2\xi/(1+\xi))/\rho$ for all $\psi \in \Psi$ and all Δ large enough. The left-hand side of (21), taking into account (22), will therefore be larger than the right-hand side for all $\psi \in \Psi$ and all Δ large enough.

4.4 The Limiting Equations for Balanced Growth

For every $\psi > 1$, Lemma 2 implies that the right-hand sides of the equilibrium conditions (21) and (22) converge, as $\Delta \to \infty$, to limiting values that can be calculated using $\alpha = \eta + \frac{1}{2}\sigma^2\psi^2$, $f(u) = \psi^2 u e^{-u\psi}$, and $\Delta = \infty$ on the right-hand side of (22). For $\psi \leq 1$, the means of e^u and $U(a+u)$ are infinite, and this implies that (21) must be violated. These limiting equilibrium conditions are easy to characterize and the non-monotonicities that appear in Figures 1 and 2 are no longer possible.

Proposition 3 *Consider the equilibrium conditions (21)-(22) with $\psi > 1$, $\alpha = \eta + \frac{1}{2}\sigma^2\psi^2$, $f(u) = \psi^2 u e^{-\psi u}$, and $\Delta = \infty$. The right-hand side of (21) is strictly increasing in $\psi \in (1, \infty)$ and maps onto $(0, \infty)$. The right-hand side of (22) is strictly decreasing in $\psi \in (1, \infty)$ and maps onto $(0, \infty)$. When the relative factor supply curve $\mathcal{E}(\cdot)/\mathcal{L}(\cdot)$ is continuous, this implies a unique balanced growth path, denoted by $\psi_\infty \in (1, \infty)$.*

The mean of e^u is $(\psi/(\psi-1))^2$, which is decreasing in $\psi > 1$ and grows without bound as $\psi \downarrow 1$. It is intuitive that the means of e^{a+u} and $U(a+u)$ are also decreasing in $\psi = -\mu/\sigma^2$ and explode as $\psi \downarrow 1$. The proof is in Appendix D. The fundamental comparative static implied by Proposition 3 is that an increase in $\mathcal{E}(\cdot)/\mathcal{L}(\cdot)$ raises ψ and therefore increases $\theta - \mu = \theta + \psi\sigma^2$, the overall rate of technical progress in this economy. Individual firm revenues and employment grow more slowly, leading to more churn that is made possible by more entry, and this moves the distribution of firm types to the right at a faster pace.

Using the pointwise convergence of the equilibrium conditions (21)-(22), together with the fact that the limiting equilibrium conditions are monotone, it is easy to see that the unique equilibrium ψ_∞ obtained in Proposition 3 can be approached with equilibria ψ_Δ of a sequence of finite-Δ economies. Proposition 4 makes the stronger claim that all finite-Δ

equilibria converge to ψ_∞.

Proposition 4 *Assume the relative factor supply curves $\mathcal{E}(\cdot)/\mathcal{L}(\cdot)$ are continuous and let $E_\Delta \subset \{\psi : \zeta_+ > 1\}$ be the set of equilibria for the Δ economy. Then $\sup_{\psi \in E_\Delta} |\psi - \psi_\infty|$ converges to zero as Δ becomes large.*

The main part of the proof is showing that, when Δ is large enough, the equilibria $\psi_\Delta > 1$ for finite-Δ economies are bounded and bounded away from 1. The key steps are given in Appendix E.

5 Miraculous Growth in the $\Delta = \infty$ Economy

A solution to the large-Δ limit of the balanced growth conditions (21)-(22) also defines a balanced growth path for the $\Delta = \infty$ economy, in which entrants can imitate any incumbent firm. But in the $\Delta = \infty$ economy, there is, for each possible μ, a continuum of attempted entry rates and associated stationary densities. This means that there is a continuum of balanced growth paths. In fact, any high enough growth rate $\theta - \mu = \theta + \psi\sigma^2$ is consistent with balanced growth.

To see this, begin by observing that the Kolmogorov forward equation in the $\Delta = \infty$ economy is just (10) on the domain $(0, \infty)$. This equation is solved by linear combinations of $e^{-\chi_- u}$ and $e^{-\chi_+ u}$, with χ_- and χ_+ defined in (13). If $(\alpha - \eta)/(\sigma^2/2) > \psi^2$, then χ_- and χ_+ are complex conjugates, and from the construction of $f(u)$ on $[0, \Delta]$ for $\Delta < \infty$ we know that this can only produce a positive density on a bounded interval. On the other hand, if $(\alpha - \eta)/(\sigma^2/2) \le \psi^2$ and $\alpha \in [0, \eta]$, then $\chi_- \le 0 \le \chi_+$. This makes it impossible to have a linear combination of $e^{-\chi_- u}$ and $e^{-\chi_+ u}$ that converges to 0 at both $u = 0$ and $u = \infty$. This leaves attempted entry rates in the half-open interval $0 < (\alpha - \eta)/(\sigma^2/2) \le \psi^2$.This is a non-empty interval only if $\psi \neq 0$. If $\psi < 0$, then $\chi_- \le \chi_+ < 0$, which implies that linear combinations of $e^{-\chi_- u}$ and $e^{-\chi_+ u}$ are not integrable on $(0, \infty)$. So $\psi > 0$ is the only possibility that remains. Take $(\alpha - \eta)/(\sigma^2/2) < \psi^2$ so that χ_+ and χ_- are distinct. Then requiring $f(u)$ to be a probability density and imposing $f(0) = 0$ gives

$$f(u) = \frac{\chi_+ \chi_-}{\chi_+ - \chi_-} \times \left(e^{-\chi_- u} - e^{-\chi_+ u}\right), \tag{23}$$

for all $u \in [0, \infty)$. Letting $(\alpha - \eta)/(\sigma^2/2) \uparrow \psi^2$ yields $\chi_\pm \to \psi$, and hence $f(u) = \psi^2 u e^{-\psi u}$.[5] This is exactly the $\Delta \to \infty$ limit obtained for an economy in which entrants can only copy

[5]The differential equation (10) is solved by $e^{-\psi u}$ and $u e^{-\psi u}$ when $\alpha = \eta + \frac{1}{2}\sigma^2\psi^2$.

from $[0, \Delta]$. The right tail index of (23) is $\chi_- = \psi - \sqrt{\psi^2 - (\alpha - \eta)/(\sigma^2/2)}$, and $\chi_- > 1$ is needed to ensure that aggregate employment and the incentives to enter are finite. This condition shrinks the set of feasible α and ψ to

$$1 < \psi, \quad 2\psi - 1 < \frac{\alpha - \eta}{\sigma^2/2} \leq \psi^2. \tag{24}$$

This implies a continuum of possible attempted entry rates α, displayed in Figure 3.

In other words, for the same ψ, there are many stationary distributions, sustained by different levels of entry. The remaining conditions for a balanced growth path are (21)-(22), as before. By Proposition 3, there is a unique balanced growth path if we impose the condition $(\alpha - \eta)/(\sigma^2/2) = \psi^2$. Relaxing this condition to (24) generates a continuum of solutions to the two balanced growth conditions (21)-(22).

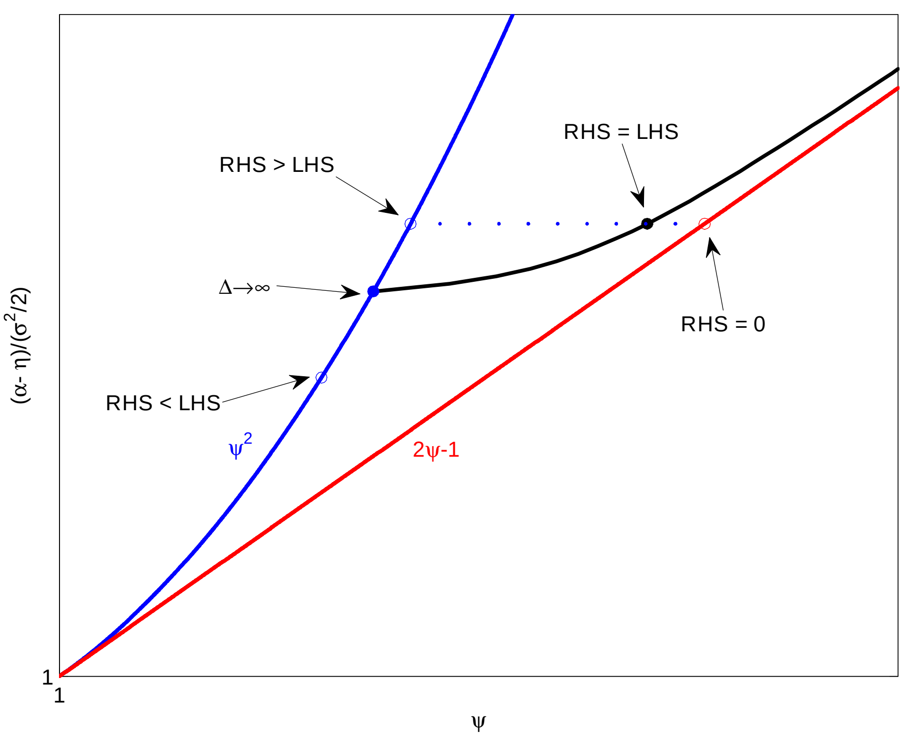

FIGURE 3 Constructing solutions to (21) with $\alpha < \eta + \frac{1}{2}\sigma^2\psi^2$

For example, consider the economy with inelastic factor supplies, so that solutions to (21) alone define balanced growth paths. To see how the right-hand side of (21) depends on α and ψ separately, subject only to the inequalities (24), note that (23) implies

$$\int_0^\infty e^u f(u) \mathrm{d}u = \frac{\chi_+ \chi_-}{\left(\chi_+ - 1\right)\left(\chi_- - 1\right)} = \frac{\frac{\alpha - \eta}{\sigma^2/2}}{\frac{\alpha - \eta}{\sigma^2/2} - (2\psi - 1)}.$$

First, observe that this integral is decreasing in α. That is, the mean of e^u is low for stationary distributions sustained by high levels of entry. It follows that the right-hand side of (21) is increasing in α on the domain (24). Next, consider ψ and note that this integral reduces to $(\psi/(\psi-1))^2$ at the boundary $(\alpha-\eta)/(\sigma^2/2) = \psi^2$. And $(\psi/(\psi-1))^2$ is decreasing in $\psi > 1$. But here, holding fixed α, the mean of e^u is increasing in ψ on the domain (24), with an asymptote: the mean of e^u grows without bound as $2\psi - 1$ approaches $(\alpha-\eta)/(\sigma^2/2)$ from below.[6] Moreover, the mean of e^{a+u} is also increasing in ψ, because part (ii) of Lemma 1 says that e^a is increasing in ψ. The right-hand side of (21) is therefore decreasing in ψ, holding fixed α.

Since the mean of e^{a+u} moves in opposite directions with α and ψ on the domain (24), one expects balanced growth paths parameterized by an upward sloping curve $\psi \to \alpha$, as shown in Figure 3. To construct this curve, start from the equilibrium with $(\alpha - \eta)/(\sigma^2/2) = \psi^2$, increase α by some arbitrary amount and increase ψ^2 as well so that $(\alpha - \eta)/(\sigma^2/2) = \psi^2$ continues to hold. From the $\Delta \to \infty$ limit equilibrium condition, shown in Figure 2 and characterized in Proposition 3, we know that this raises the right-hand side of (21) above the left-hand side. To restore equilibrium while holding α fixed therefore requires an increase in ψ. The asymptote implies that equilibrium can be restored before $2\psi - 1$ reaches $(\alpha-\eta)/(\sigma^2/2)$ from below. As indicated in Figure 3, lowering α and ψ^2 from the equilibrium with $(\alpha-\eta)/(\sigma^2/2) = \psi^2$ will not work because this lowers the right-hand side of (21) below the left-hand side, and increasing ψ into the interior of (24) only makes this discrepancy worse.

In sum, the $\Delta = \infty$ economy has a continuum of balanced growth paths. The unique $\Delta \to \infty$ limiting equilibrium selects the balanced growth path with the lowest possible α and ψ. Since the growth rate of aggregate productivity is $\theta - \mu = \theta + \sigma^2\psi$, this is the balanced growth path with the lowest steady state growth rate. As Figure 3 shows, there are also balanced growth paths with arbitrarily high entry and growth rates. No such miracles are possible when $\Delta < \infty$, no matter how large.

A Proof of Lemma 1

If $\mu + \frac{1}{2}\sigma^2 \geq \rho$, then the value of continuing forever is infinite. For $\mu + \frac{1}{2}\sigma^2 < \rho$, the Bellman equation is solved in Luttmer [2007]. An easy derivative calculation shows that it is increasing for $y > a$, and it behaves like a multiple of e^y for y large. This proves part

[6] So the mean of e^u is decreasing in the firm employment growth rate $\mu = -\psi\sigma^2 < 0$. This unintuitive comparative static reflects the bootstrap nature of the multiplicity of stationary distributions. Stationarity with a lower firm growth rate and no change in the attempted entry rate requires a thicker right tail from which firms can shrink, on average, towards the exit threshold.

(i). To simplify the calculations for the other parts, recall $\psi = -\mu/\sigma^2$ and write

$$\delta = \frac{\rho}{\sigma^2/2}, \quad \xi = -\psi + \sqrt{\psi^2 + \delta}, \quad e^a = \frac{\xi}{1+\xi}\left(1 + \frac{2\psi - 1}{\delta}\right). \tag{25}$$

The expression for e^a is positive precisely when $\mu + \frac{1}{2}\sigma^2 < \rho$, which corresponds to $\psi > (1-\delta)/2$. It is easy to see that $\xi > 0$, $\partial\psi/\partial\xi < 0$, $\lim_{\psi\to\infty} \xi = 0$, and $\lim_{\psi\downarrow(1-\delta)/2} \xi = \delta$. One can also verify that

$$\frac{\partial e^a}{\partial \psi} = \frac{1}{\delta}\frac{-\psi + \sqrt{\psi^2+\delta}}{\sqrt{\psi^2+\delta}} > 0. \tag{26}$$

Letting $\psi \downarrow (1-\delta)/2$ gives $a \to -\infty$. Moreover,

$$\begin{aligned}\lim_{\psi\to\infty} e^a &= \lim_{\psi\to\infty} \frac{-\psi + \sqrt{\psi^2+\delta}}{1 - \psi + \sqrt{\psi^2+\delta}}\left(1 + \frac{2\psi-1}{\delta}\right) \\ &= \lim_{\psi\to\infty}\left(-\psi + \sqrt{\psi^2+\delta}\right)\frac{2\psi}{\delta} = \lim_{x\to\infty} 2x\left(-x + \sqrt{x^2+1}\right) = \lim_{x\to\infty}\frac{2x}{x + \sqrt{x^2+1}} = 1\end{aligned}$$

So $\mu \to -\infty$ implies $a \uparrow 0$. This proves part (ii). Part (iii) follows easily from the properties of ξ and, in the case of $U(y)$, the fact that $a \to -\infty$ as $\psi \downarrow (1-\delta)/2$.

B Proof of Proposition 1

A solution to the Kolmogorov forward equation must be positive, vanish at 0 and ∞, and be continuous and differentiable at Δ. The solution (16)-(20) satisfies these conditions for an attempted entry rate α that makes χ_+ and χ_- complex. Suppose instead that the attempted entry rate α satisfies $(\alpha-\eta)/\sigma^2/2 \leq \psi$ so that χ_+ and χ_- are real. The following argument shows that such a solution cannot be differentiable at Δ.

Suppose first that α is such that the roots coincide, so that $\chi_+ = \chi_- = \psi$. Then the differential equation (10) on $[0, \Delta]$ has the solutions $e^{-\psi u}$ and $ue^{-\psi u}$. Imposing $f(0) = 0$, $f(\infty) = 0$, and continuity at Δ forces

$$f(u) = B \begin{cases} ue^{-\psi u}, & u \in [0, \Delta], \\ \Delta e^{-\psi\Delta} e^{-\zeta_+(u-\Delta)}, & u \in [\Delta, \infty). \end{cases}$$

The slopes of this density are

$$\mathrm{D}f(u) = B \begin{cases} (1-\psi u)e^{-\psi u}, & u \in [0, \Delta), \\ -\Delta e^{-\psi\Delta}\zeta_+ e^{-\zeta_+(u-\Delta)}, & u \in (\Delta, \infty). \end{cases}$$

Then $\zeta_+ - \psi > 0$ implies

$$\lim_{u\uparrow\Delta} \mathrm{D}f(u) = (1-\psi\Delta)e^{-\psi\Delta} > -\zeta_+\Delta e^{-\psi\Delta} = \lim_{u\downarrow\Delta} \mathrm{D}f(u).$$

Alternatively, suppose that the roots are distinct, so that $\chi_+ > \chi_-$. Imposing $f(0) = 0$, $f(\infty) = 0$, and continuity at Δ yields

$$f(u) = B \begin{cases} e^{-\chi_- u} - e^{-\chi_+ u}, & u \in [0,\Delta], \\ \left(e^{-\chi_-\Delta} - e^{-\chi_+\Delta}\right) e^{-\zeta_+(u-\Delta)}, & u \in [\Delta,\infty). \end{cases}$$

This implies

$$\mathrm{D}f(u) = B \begin{cases} -\chi_- e^{-\chi_- u} + \chi_+ e^{-\chi_+ u}, & u \in [0,\Delta], \\ -\left(e^{-\chi_-\Delta} - e^{-\chi_+\Delta}\right) \zeta_+ e^{-\zeta_+(u-\Delta)}, & u \in [\Delta,\infty). \end{cases}$$

We want to verify that the following inequality is true,

$$\lim_{u\uparrow\Delta} \mathrm{D}f(u) = -\chi_- e^{-\chi_-\Delta} + \chi_+ e^{-\chi_+\Delta} > -\left(e^{-\chi_-\Delta} - e^{-\chi_+\Delta}\right)\zeta_+ = \lim_{u\downarrow\Delta} \mathrm{D}f(u).$$

This follows because the inequality can also be written as

$$(\zeta_+ - \chi_-)\left(e^{(\chi_+-\chi_-)\Delta} - 1\right) + \chi_+ - \chi_- > 0,$$

which is true because $\zeta_+ - \chi_- > 0$ and $\chi_+ - \chi_- > 0$.

C Proof of Lemma 2

For the distribution, only the case $\psi = 0$ remains. For all $u \in [0,\Delta]$, the density is then $f(u) = B\sin(\omega u)$, and hence

$$F(u) = \frac{\frac{1-\cos(\omega u)}{\omega}}{\frac{1-\cos(\omega\Delta)}{\omega} + \frac{\sin(\omega\Delta)}{\zeta_+}} = \frac{\frac{1-\cos(\omega u)}{\omega^2}}{\frac{1-\cos(\omega\Delta)}{\omega^2} - \frac{1}{\zeta_+}\frac{\cos(\omega\Delta)}{\sqrt{\eta/(\sigma^2/2)}}}.$$

The second equality uses the condition (17) that defines ω. That condition also implies that $\omega\Delta \to \pi$ and thus $\omega \to 0$ as $\Delta \to \infty$. By l'Hôpital, $\lim_{x\to 0}(1-\cos(x))/x^2 = \lim_{x\to 0}\sin(x)/(2x) = 1/2$. So the numerator converges, while the denominator goes to ∞ as $\Delta \to \infty$. So $F(u) \to 0$ as $\Delta \to \infty$ for every $u \in [0,\infty)$.

For the truncated mean of e^u, observe that

$$G(\Delta) = B\int_0^\Delta \sin(\omega u)e^{-(\psi-1)u}\mathrm{d}u = B \times \frac{\omega - [\omega\cos(\omega\Delta) + (\psi-1)\sin(\omega\Delta)]\, e^{-(\psi-1)\Delta}}{\omega^2+(\psi-1)^2}.$$

Using the solution (19) for B and replacing $\sin(\omega\Delta)/\omega$ using (17) gives

$$G(\Delta) = \frac{\omega^2+\psi^2}{\omega^2+(\psi-1)^2}\frac{1-\left(1-\frac{\psi-1}{\sqrt{\psi^2+\frac{\eta}{\sigma^2/2}}}\right)\cos(\omega\Delta)e^{-(\psi-1)\Delta}}{1-\left(1-\frac{\psi}{\sqrt{\psi^2+\frac{\eta}{\sigma^2/2}}}+\frac{\omega^2+\psi^2}{\left(\psi+\sqrt{\psi^2+\frac{\eta}{\sigma^2/2}}\right)\sqrt{\psi^2+\frac{\eta}{\sigma^2/2}}}\right)\cos(\omega\Delta)e^{-\psi\Delta}}. \quad (27)$$

If $\psi > 1$, both $e^{-(\psi-1)\Delta}$ and $e^{-\psi\Delta}$ converge to zero as $\Delta \to \infty$. Since $\cos(\omega\Delta) \to -1$ as $\Delta \to \infty$, this implies $G(\Delta) \to (\psi/(\psi-1))^2$. If $\psi = 1$, then the second factor in the expression for $G(\Delta)$ goes to 2 because $\cos(\omega\Delta) \to -1$ as $\Delta \to \infty$. Then $G(\Delta) \to \infty$ because the first factor goes to ∞ as $\omega \downarrow 0$ when $\Delta \to \infty$. Finally, if $\psi \in (-\infty, 1)$, then the first factor converges to $(\psi/(\psi-1))^2 \in (0,\infty)$ as $\Delta \to \infty$. But the second factor goes to ∞ as $\Delta \to \infty$ because $\cos(\omega\Delta) \to -1$ and $e^{-(\psi-1)\Delta} > 1$ dominates $\max\{1, e^{-\psi\Delta}\}$.

For the compact interval Ψ, a uniform lower bound on $G(\Delta)$ can be constructed as follows. The condition (17) for ω implies that $|\cos(\omega\Delta)|$ is bounded away from zero for all $\Delta \geq 1$. It follows that there are positive constants M and K so that

$$G(\Delta) \geq \frac{(\pi/2)^2+\Delta^2\psi^2}{\pi^2+\Delta^2(\psi-1)^2}\frac{1+Me^{(1-\psi)\Delta}}{1+Ke^{-\psi\Delta}} \geq \frac{(\pi/2)^2+\Delta^2\psi^2}{\pi^2+\Delta^2(1-\psi)^2}\frac{Me^{\min\{1,1-\psi\}\Delta}}{1+K}$$

for all $\psi \in \Psi$ and $\Delta \geq 1$. The function $e^x/(\pi^2+x^2)$ is increasing in x, and hence $\psi \geq 1$ implies $e^{(1-\psi)\Delta}/(\pi^2+\Delta^2(1-\psi)^2) \geq 1/\pi^2$. This implies that $G(\Delta) \geq (\Delta/\pi)^2 M/(1+K)$ for all $\psi \in [1/2, 1] \subset \Psi$. For $\psi \in \Psi \setminus [1/2, 1]$ a multiple of $e^{\Delta/2}$ can serve as a lower bound. Since $e^{\Delta/2} \geq \Delta^2$ for all $\Delta \geq 1$, this means that there is a constant multiple of Δ^2 that provides a uniform lower bound on $G(\Delta)$ for all $\psi \in \Psi$ and $\Delta \geq 1$.

D Proof of Proposition 3

The limiting stationary distribution has a density $f(u) = \psi^2 u e^{-\psi u}$. For any $\lambda < \psi$, this implies

$$\int_0^\infty e^{\lambda u} f(u)\mathrm{d}u = \left(\frac{\psi}{\psi-\lambda}\right)^2.$$

Applying this to $\lambda \in \{0, 1, -\xi\}$ gives

$$\begin{aligned}\int_0^\infty \rho U(a+u) f(u)\mathrm{d}u &= \frac{\xi}{1+\xi}\int_0^\infty \left(e^u - 1 - \frac{1-e^{-\xi u}}{\xi}\right)\psi^2 u e^{-\psi u}\mathrm{d}u \\ &= \frac{\xi}{1+\xi}\left(\frac{\psi}{\psi-1}\right)^2 + \frac{1}{1+\xi}\left(\frac{\psi}{\psi+\xi}\right)^2 - 1.\end{aligned}$$

Recall (25). A laborious derivative calculation yields

$$\frac{\partial}{\partial\psi}\int_0^\infty \rho U(a+u) f(u)\mathrm{d}u = \frac{2\psi\xi}{1+\xi}\left(\frac{1}{(\psi+\xi)^3} - \frac{1}{(\psi-1)^3}\right) + \frac{(\xi+3\psi-2)\psi^2}{(\psi-1)^2(\psi+\xi)^3} \times \frac{\partial\xi}{\partial\psi}.$$

Since $\xi > 0$, $\psi > 1$, and $\partial\xi/\partial\psi < 0$, this is negative.

We also need the slope of

$$\int_0^\infty e^{a+u} f(u)\mathrm{d}u = e^a\left(\frac{\psi}{\psi-1}\right)^2.$$

By (26), e^a is increasing in ψ. But the mean of e^u is not. Another laborious calculation shows

$$\begin{aligned}&\frac{\partial}{\partial\psi}\frac{-\psi+\sqrt{\psi^2+\delta}}{1-\psi+\sqrt{\psi^2+\delta}}\frac{\delta+2\psi-1}{\delta}\left(\frac{\psi}{\psi-1}\right)^2 = \\ &-\frac{-\psi+\sqrt{\psi^2+\delta}}{\delta\psi(\psi-1)}\frac{(1+\psi)\psi+2\delta+2(\psi-1)\sqrt{\psi^2+\delta}}{\sqrt{\psi^2+\delta}}\left(\frac{\psi}{\psi-1}\right)^2\end{aligned}$$

Since $\psi > 1$, this is negative. Also, the mean of e^{a+u} converges to 1 as $\psi \to \infty$ because part (ii) of Lemma 1 says that $e^a \to 1$ as $\psi \to \infty$. On the other hand, $\psi \downarrow 1 > (1-\delta)/2$ implies that e^a has a positive limit as $\psi \downarrow 1$, and so the mean of e^{a+u} diverges.

E Proof of Proposition 4

Proposition 3 already says that finite-Δ equilibria satisfy $\psi_\Delta > 1$ for all large enough Δ. For any $\psi \in [1, \infty)$ and $\Delta \in [1, \infty)$, let $\omega\Delta \in (\pi/2, \pi)$ and $f(\cdot)$ be as defined in (16)-(19). The equilibrium condition for the Δ economy can be restated as $s_\Delta(\psi) = d_\Delta(\psi)$, where

$$\begin{aligned}d_\Delta(\psi) &= \frac{1}{1+(\varepsilon-1)e^a\int_0^\infty e^u f(u)\mathrm{d}u} \\ s_\Delta(\psi) &= \frac{\phi}{\eta+\frac{1}{2}\sigma^2\left(\omega^2+\psi^2\right)}\frac{\mathcal{E}(q/w)}{\mathcal{L}(q/w)}, \qquad \frac{q}{w} = \frac{1}{\rho}\frac{\xi}{1+\xi}\int_0^\Delta\left(e^u - 1 - \frac{1-e^{-\xi u}}{\xi}\right)f(u)\mathrm{d}u.\end{aligned}$$

Since $\psi \geq 1$ implies $\zeta_+ > 1$, these functions are well defined, positive, and real-valued for any $\psi \in [1, \infty)$ and $\Delta \in [1, \infty)$. Write $s_\infty(\psi)$ and $d_\infty(\psi)$ for the pointwise large-Δ limits of these functions. These limiting functions satisfy $d_\infty(1) = 0 < s_\infty(1)$ and $\lim_{\psi \to \infty} s_\infty(\psi) = 0 < 1/\varepsilon = \lim_{\psi \to \infty} d_\infty(\psi)$. Proposition 3 implies they are monotone.

The truncated and proper mean of e^u are related via

$$\int_0^\Delta e^u f(u) \mathrm{d}u = G(\Delta) \leq G(\Delta) + \frac{[1 - F(\Delta)] e^\Delta}{\zeta_+ - 1} = \int_0^\infty e^u f(u) \mathrm{d}u. \tag{28}$$

Upper and lower bounds for the value of entry are

$$\frac{1}{\rho} \max\left\{0, \frac{\xi G(\Delta)}{1 + \xi} - 1\right\} \leq \frac{q}{w} = \frac{1}{\rho} \frac{\xi}{1 + \xi} \int_0^\Delta \left(e^u - 1 - \frac{1 - e^{-\xi u}}{\xi}\right) f(u) \mathrm{d}u \leq \frac{1}{\rho} \frac{\xi G(\Delta)}{1 + \xi}, \tag{29}$$

for all $\psi \in [1, \infty)$ and $\Delta \in [1, \infty)$. The contribution of the right tail to the mean of e^u has an upper bound given by

$$[1 - F(\Delta)]\, e^\Delta \leq \frac{1}{\psi + \sqrt{\psi^2 + \frac{\eta}{\sigma^2/2}}} \frac{(\pi/\Delta)^2 + \psi^2}{\sqrt{\psi^2 + \frac{\eta}{\sigma^2/2}}} \times e^{-(\psi - 1)\Delta}. \tag{30}$$

The expression (27) for $G(\Delta)$, together with $\omega\Delta \in (\pi/2, \pi)$, implies that there are positive constants M and K so that for all $\psi \in [1, \infty)$ and $\Delta \in [1, \infty)$,

$$\frac{(\pi/2)^2 + \Delta^2 \psi^2}{\pi^2 + \Delta^2 (\psi - 1)^2} \frac{1}{1 + M e^{-\psi \Delta}} \leq G(\Delta) \leq \frac{\pi^2 + \Delta^2 \psi^2}{(\pi/2)^2 + \Delta^2 (\psi - 1)^2} \times \left(1 + K e^{-(\psi - 1)\Delta}\right). \tag{31}$$

Consider $\psi \in [1, \psi_*]$ for some $\psi_* > 1$ close to 1. For all $\psi \in [1, \psi_*]$ the lower bound on $G(\Delta)$ given in (31) is itself bounded below by $1/(((\pi/\Delta)^2 + (\psi_* - 1)^2)(1 + M))$. This bound does not depend on ψ and converges to $1/((\psi_* - 1)^2 (1 + M))$ as Δ becomes large. This can be made arbitrarily large by taking $\psi_* - 1 > 0$ small. In light of (28) and (29), this means that it is possible to take $\psi_* - 1 > 0$ close enough to zero to ensure that $\sup_{\psi \in [1, \psi_*]} d_\Delta(\psi) < \inf_{\psi \in [1, \psi_*]} s_\Delta(\psi)$ for all large enough Δ. So there can be no solution to $d_\Delta(\psi) = s_\Delta(\psi)$ on $[1, \psi_*]$ for any Δ large enough.

Alternatively, consider $\psi \in [\psi^*, \infty)$ for some large ψ^*. On this domain, the upper bound on $G(\Delta)$ given in (31) converges to $(\psi/(\psi - 1))^2 \leq (\psi^*/(\psi^* - 1))^2$ as Δ becomes large. Since $\xi \downarrow 0$ as $\psi \to \infty$, (29) then implies that there is a large enough ψ^* that will ensure that $\sup_{\psi \geq \psi^*} s_\Delta(\psi)$ is close to zero for all large enough Δ. At the same time, the right-hand side (30) can also be made small throughout $[\psi^*, \infty)$ by taking Δ large enough. Since $e^a \uparrow 1$ as $\psi \to \infty$, this means that there is a ψ^* large enough to ensure that

$\inf_{\psi \in [\psi^*, \infty)} d_\Delta(\psi)$ is close to $d_\infty(\infty) = 1/\varepsilon > 0$ for all large Δ. So there can be no solution to $d_\Delta(\psi) = s_\Delta(\psi)$ on $[\psi^*, \infty)$ for any Δ large enough.

These observation imply that the solutions to $d_\Delta(\psi) = s_\Delta(\psi)$ must be in some compact interval $[\psi_*, \psi^*] \subset (1, \infty)$ for all Δ large enough. On this interval, the bounds (30)-(31) together with similar bounds for the truncated mean of $e^{-\xi u} \in (0, 1)$ can be used to prove the uniform convergence of $d_\Delta(\psi) \to d_\infty(\psi)$ and of $s_\Delta(\psi) \to s_\infty(\psi)$ as $\Delta \to \infty$. This implies that the solutions to $d_\Delta(\psi) = s_\Delta(\psi)$ have to converge to the unique solution to $d_\infty(\psi) = s_\infty(\psi)$.

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