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THE MULTI-PERIOD CONTROL PROBLEM UNDER UNCERTAINTY

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The multi-period control problem analyzed assumes the data are generated by the simple regression model with an unknown slope coefficient. There is a tradeoff between stabilization and experimentation to learn more about the unknown coefficient. When parameter uncertainty is large, experimentation becomes an important consideration.

OPTIMAL CONTROL when there is uncertainty in the effect of a policy instrument is receiving the increased attention of the economics profession. Single period problems have been analyzed by [**1**, **4**, **8**, and **9**], but the multi-period problem has largely been ignored.[2] The latter is far more complicated because one must consider not only the effect of a decision upon the current period's expected loss but also its effect upon the resulting expected information. For example, as Drèze [**3**] points out, it may be optimal for a monopolist to experiment with price to learn more about the elasticity of demand. This action reduces expected profit in the current period, but possibly the loss can be recouped in subsequent periods using the improved information.

1. THE THEORY

The process assumed to generate the data is a simple regression model, namely,

$$y_t = \beta x_t + u_t \quad \text{for} \quad t = 1, 2, \ldots, T, \tag{1}$$

where y_t is the tth observation of the dependent variable, x_t is the tth value of a control variable, β is an unknown scalar parameter, and u_t is the tth unobserved random error term. The u_t's are normally and independently distributed, each with mean zero and common known variance, which without loss of generality will be taken to be one. Further, we assume our prior knowledge at the time x_1 is selected can be represented by a normal distribution with mean m_1 and precision (the reciprocal of the variance) h_1. It is readily verified that the distribution on the unknown parameter at the time of the tth decision will be normal with precision satisfying the difference equation

$$h_t = h_{t-1} + x_{t-1}^2, \tag{2}$$

and mean satisfying

$$m_t = (m_{t-1}h_{t-1} + x_{t-1}y_{t-1})/h_t. \tag{3}$$

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[2] There are numerous examples of problems of this type in the mathematical statistics, control engineering, and management science literature. The only economic applications to our knowledge are [**5** and **7**], where two period examples are considered. The latter has developed approximate solutions for two period simple regression models under the assumption of diffuse priors. See [**2**] for a summary of what has been done in the area and an extensive bibliography.

If one had a locally uniform initial prior on β and previous observations were available, m_t would be the least squares estimator and $1/h_t$ its variance. Subsequently $N(m, h)$ will denote a normal distribution function with mean m and precision h.

Given initial prior $N(m_1, h_1)$ on β, the control problem is to select the x_t sequentially so as to minimize the sum of the expected losses

$$(4) \qquad E\left[\sum_{t=1}^{T} q_t(y_t)\right]$$

where the q_t are the non-negative losses, E is the expectation operator, and $x_t^2 \leqslant Kh_t$. Besides assuming $E[q_t(y_t)]$ exists when β is normally distributed, we require its derivative with respect to m to exist and be continuous with respect to x_t for $x_t^2 \leqslant Kh_t$.

REMARK 1: This constraint set is invariant to the units in which x_t is measured, a result needed to prove Theorem 1, and is compact, which is needed to insure existence of optimal decisions. Alternatively we could have removed this constraint and imposed conditions on the loss functions to insure all optimal decisions were finite.

Let

$$(5) \qquad f_t(m_t, h_t) = \inf E\left[\sum_{i=t}^{T} q_i(y_i)|m_t, h_t\right]$$

for $t = 1, \ldots, T$. This is the infimum for the sum of the expected losses for periods t through T inclusive given prior $N(m_t, h_t)$ on β at time t. As the initial prior has been assumed normal, the prior at the time of the tth decision will necessarily be normal. By backward induction

$$(6) \qquad f_t(m_t, h_t) = \min E[q_t(y_t) + f_{t+1}(m_{t+1}, h_{t+1})|x_t, m_t, h_t], \qquad x_t^2 \leqslant Kh_t,$$

for $t = 1, \ldots, T$ with $f_{T+1} = 0$. Because x is constrained to a compact set, the infimum is obtained and the minimum operator may be used in (6). The first term in the expectation measures the effect of decision x_t upon the loss in the current period while the second the effect upon future losses given optimal future behavior. From (2), the larger x_t^2, the more precise will be the future knowledge of β as the precision (variance) of the posteriori will be larger (smaller). Current decisions affect future as well as current losses so there will be a trade off between stabilization and experimentation.

THEOREM 1: *The functions f_t have the following property:*

$$(7) \qquad f_t(km_t, h_t/k^2) = f_t(m_t, h_t)$$

for $k \neq 0$.

PROOF: Consider the transformed problem

$$y_t = \beta^* x_t^* + u_t$$

where $\beta^* = k\beta$ and $x_t^* = x_t/k$. If the prior on β is $N(m_t, h_t)$, the prior on β^* is $N(km_t, h_t/k^2)$. In addition if $x_t^2 \leqslant Kh_t$, then $(x_t^*)^2 \leqslant Kh_t/k^2 = Kh_t^*$. The equivalence of these two problems imply (7).

Setting $k = h_t^{\frac{1}{2}}$ and then $k = -h_t^{\frac{1}{2}}$ in (7) yields

$$f_t(m_t h_t^{\frac{1}{2}}, 1) = f_t(m_t, h_t) = f_t(-m_t h_t^{\frac{1}{2}}, 1).$$

Thus, the minimal obtainable expected loss depends only upon the absolute value of the location parameter

$$s_t = m_t h_t^{\frac{1}{2}} \tag{8}$$

of the normal prior on β. For this decision problem it measures the degree of uncertainty in the unknown coefficients. Let

$$f_t(s_t) \equiv f_t(s_t, 1).$$

The system of functional equations become

$$f_t(s_t) = \min_{x_t^2 \leqslant K} E[q_t(y_t) + f_{t+1}(s_{t+1}) | s_t, x_t]. \tag{9}$$

From (2), (3), and (8), the distribution of s_{t+1} is $N(s_t\sqrt{1 + x_t^2}, x_t^{-2})$.

REMARK 2: This simplification is important for with a single state variable numerical solutions may be obtained in a fraction of a minute with the aid of a high speed computer. If there were two state variables, the time required would increase by a factor of 100. To see why computation costs increase so rapidly, consider the method of solution. One begins with $f_{T+1} = 0$ and uses (9) to compute f_T which in turn is used to compute f_{T-1}, etc., until f_1 is obtained. It does not appear that these functions can be characterized by a few parameters. Rather the functions must be tabulated for a grid of points. If there are 100 points per dimension and n state variables, the functions must be evaluated at 100^n points. For each evaluation a number of numerical integrations are required when searching for the optimal decision.

Let

$$h_t(x, s) = E[q_t(y_t) + f_{t+1}(s') | x, s], \tag{10}$$

the s' denoting the location parameter of the posteriori and let

$$B_t = \lim_{n \to \infty} \sup_{s^* \in N_n} |f_t(s) - f_t(s^*)|/|s - s^*|$$

where N_n is a neighborhood of s with radius $1/n$.

LEMMA A. *For any* s, B_t *is finite.*

PROOF: The theorem is trivial for $T + 1$ as $f_{T+1} = 0$. Let $g(x, s) = E[q_t(y_t)|s, x]$. By assumption $g_2(x, s)$ exists and is continuous with respect to x for $x^2 \leqslant K$. Therefore for ε_1 sufficiently small

$$|g(x, s) - g(x, s^*)| \leqslant [1 + \sup_{x^2 \leqslant K} |g_2(x, s)|]|s - s^*| \tag{11}$$

whenever $|s - s^*| \leqslant \varepsilon_1$ and $x^2 \leqslant K$.

There are $\delta > 0$ and $\varepsilon_2 > 0$ such that

$$|E[f_{t+1}(s')|x, s] - E[f_{t+1}(s')|x, s^*]| \leqslant |B_{t+1} + 1|\,|s - s^*| \tag{12}$$

whenever $x^2 \leqslant \delta$ and $|s - s^*| \leqslant \varepsilon_2$ given the assumption that the theorem is true for $t + 1$.[3]

Let $B = \sup f_{t+1}(s)$ and

$$B(s') = B \quad \text{if} \quad s' \leqslant \sqrt{1 + x^2}(s + s^*)/2,$$

$$B(s') = -B \quad \text{otherwise.}$$

Observe

$$|E[f(s')|x, s] - E[f(s')|x, s^*]| \leqslant |E[B(s')|x, s] - E[B(s')|x, s^*]|$$
$$= 2 \Pr [|s' - \sqrt{1 + x^2}(s + s^*)/2| \leqslant 2\sqrt{1 + x^2}|s - s^*|].$$

Thus,

$$|E[f(s')|x, s] - E[f(s')|x, s^*]| \leqslant 2B\sqrt{1 + x^2}|s - s^*|/x.$$

For $x^2 \geqslant \delta$ and $x^2 \leqslant K$, this implies

$$|E[f(s')|x, s] - E[f(s')|x, s^*]| \leqslant 2B\sqrt{1 + K^2}\delta^{-\frac{1}{2}}|s - s^*|. \tag{13}$$

Results (11), (12), and (13) insure

$$|f_t(s) - f_t(s^*)| \leqslant B_t|s - s^*|$$

provided $|s - s^*| \leqslant \min(\varepsilon_1, \varepsilon_2)$ and $x^2 \leqslant K$ where B_t is the maximum of the three bounds. By backward induction all the B_t are finite given B_{T+1} is finite.

THEOREM 2: *The functions $f_t(s)$ are continuous everywhere and differentiable almost everywhere.*

PROOF: The existence of B_t for a given s insures the Lipschitz condition will be satisfied in an interval containing s. This implies the function will be absolutely continuous in this interval and therefore differentiable almost everywhere within it. Given this is true for all s, the theorem follows.

Previously the f_t have been considered functions of a real variable but now will be functionals on the space of distribution functions P. The symbol s now denotes the distribution function $N(s, 1)$.

[3] Note the f_{t+1} are bounded, as the decision rule $x_s = 0$ for all s results in finite expected loss.

LEMMA B: *The functions $f_t(p)$, $p \in P$, are convex and*

(14) $$Ef_t(p) \leqslant f_t(Ep).$$

PROOF: For $p_1, p_2 \in P$, and $0 \leqslant \theta \leqslant 1$, $p_\theta = \theta p_1 + (1-\theta)p_2$ will be a distribution function. Then

$$\theta f_t(p_1) + (1-\theta)f_t(p_2) = \theta \min_{x^2 \leqslant K} E[q_t(y_t) + f_{t+1}(p')|p_1, x]$$
$$+ (1-\theta)\min_{x^2 \leqslant K} E[q_t(y_t) + f_{t+1}(p')|p_2, x] \leqslant \min_{x^2 \leqslant K} E[q_t(y_t)$$
$$+ f_{t+1}(p')|p_\theta, x] = f_t(p_\theta),$$

the p' denoting the posteriori distribution. Convexity and Jensen's inequality imply (14).

THEOREM 3: *The functions f_t are non decreasing in $|s|$ and bounded.*

PROOF: Selecting $x_t = 0$ for all t results in expected loss

$$\sum_{t=1}^{T} E[q_t(u_t)]$$

where u_t are $N(0, 1)$. This is a bound for f_t. Since the f_t are symmetric, only positive s need be considered. First

$$E_s N(s, 1) = N[m, (1+\sigma^2)^{-1}]$$

if s has distribution $N(m, \sigma^{-2})$. By Lemma B, above, and Theorem 1,

(15) $$E_s f_t(s) \leqslant f_t[E_s N(s, 1)] = f_t[N(m, (1+\sigma^2)^{-1})] = f_t(m/\sqrt{1+\sigma^2}).$$

Suppose f_t has a relative minimum at $m^* > 0$ and is increasing at that point. Consider sequence (σ_i, m_i) such that σ_i decreases to 0 while $m_i/\sqrt{1+\sigma_i^2} = m^*$. For i sufficiently large

(16) $$Ef_t(s) > f_t(m^*) = f_t(m_i/\sqrt{1+\sigma_i^2})$$

for $s \sim N(m_i, \sigma_i^{-2})$, since by Theorem 2 the function is absolutely continuous. This contradiction to (15) establishes that f_t has no relative minimum except possibly at $m = 0$. If it did, however,

(17) $$Ef_t(s) > f_t(0)$$

if $s \sim N(0, \sigma^2)$ for σ sufficiently small given f_t is symmetric. This contradicts (15) establishing the result that $f_t(s)$ has no minimum. Thus the functions are non-increasing in $|m|$, proving the Theorem.

We were surprised at the difficulty encountered in proving this obvious result that more precise is better than less precise information.

Suppose before selecting decision t, the results of the experiment x (the distribution of y being $N(\beta x, 1)$) may be observed. The location parameter of the distribution on β at time t will be s' rather than s as a result of this additional observation. From (10)

$$E[f_t(s')|s, x] = E[\min_z h_t(z, s')|s, x] \leqslant \min_z E[h_t(z, s')|s, x] \tag{18}$$
$$= \min_z h_t(z, s) = f_t(s).$$

This proves any experiment x can be expected to reduce, or at least not increase, future expected losses.

Suppose two experiments x_1 and x_2 may be observed or alternatively one experiment x_3 with $x_3^2 = x_1^2 + x_2^2$, before selecting $x_t, \ldots, x_T$. Since both s_{12}, the location parameter with experiments x_1 and x_2, and s_3, the location parameter with experiment x_3, will be distributed $N(s\sqrt{1 + x_3^2}, x_3^{-2})$,

$$E[f_t(s_3)|x_3, s] = E[f_t(s_{12})|x_1, x_2, s].$$

Assuming x_1 is the first experiment

$$E[f_t(s_{12})|x_1, x_2, s] = E\{E[f_t(s_{12})|x_2, s_1]|x_1, s\} \leqslant E[f_t(s_1)|x_1, s],$$

by (18). Thus

$$E[f_t(s_3)|x_3, s] \leqslant E[f_t(s_1)|x_1, s]$$

where $x_3^2 = x_1^2 + x_2^2 \leqslant x_1^2$. This discussion can be summarized by the following theorem.

THEOREM 4: *The larger x_t^2, the more informative is the experiment; that is,*

$$E[f_{t+1}(s')|s, x_{t1}] \leqslant E[f_{t+1}(s')|s, x_{t2}]$$

if $x_{t1}^2 \geqslant x_{t2}^2$.

This result implies the optimal decision will be larger in absolute value than the one which minimizes expected loss in the current period, so the optimal policy is to sacrifice some stability in order to gain information.

The optimal decision for $h_t = 1$ is of the form

$$x_t^0 = x_t^0(s_t),$$

where x_t^0 minimizes the right side of (9). In general the optimal decision will be

$$x_t^0 = h_t^{\frac{1}{2}} x_t^0(s_t); \tag{19}$$

the optimal decision in each period is equal to the scale parameter of the prior at time t times a function of the location parameter.

2. NUMERICAL ANALYSIS FOR QUADRATIC LOSS

In this section the loss functions will be restricted to be quadratic:

$$q_t(y_t) = (y_t - d_t)^2,$$

the d_t being the desired or target levels for the performance variable. All the variables and targets have been divided by the standard deviation of the error term so the variances of the transformed disturbances are 1. The single period problem, $T = 1$, has been analyzed by Zellner and Geisel [**8**] though they did not assume the variance of the additive disturbance known. They explored many forms for the loss function finding the quadratic reasonably robust for the class of symmetric loss functions. Given this result, we thought the quadratic loss most interesting for purposes of the quantitative exploration of the importance of experimentation. The analysis could equally well have been performed using other loss functions without increasing computation costs.

To apply the theorem of the previous section, the x_t^2 must be constrained by Kh_t for some K. We let

$$K = \sum_{t=1}^{T} d_t^2 .$$

For larger x_t^2, the expected loss in the current period would exceed the sum of the expected losses in the remaining periods if $x_s = 0$ for $s \geqslant t$. This constraint will never be binding and any larger value for K would have yielded the same results.

2.1. *Two Alternative Decision Rules*

For this problem the conventional decision approach, which is called the certainty equivalence decision rule in [**9**], is to select

$$x_t = d_t/m_t . \tag{20}$$

This rule was obtained by determining the optimal decision rule if β were known and substituting the expected value for this unknown parameter. The expected loss in period t for rule (20) is

$$E[(y_t - d_t)^2 | x_t = d_t/m_t] = 1 + d_t^2 m_t^{-2} h_t^{-1} = 1 + (d_t/s_t)^2, \tag{21}$$

which is large when $|s_t|$ is small. Selecting $x_t = 0$ results in smaller expected loss than rule (20) if $|s_t| \leqslant 1$. Given this result, we define our *certainty equivalence rule* as follows:

$$x_t^{ce} = \begin{cases} d_t/m_t & \text{if} \quad |s_t| > 1, \\ 0 & \quad\quad\ \text{otherwise}. \end{cases}$$

A procedure such as the one above would probably be recommended by a non-Bayesian econometrician, for he would recommend (20) only if the estimate of β were statistically significant. The certainty equivalent decision rule depends upon the significance level and therefore is not unique.[4] Some will surely question whether in fact $x_t = 0$ is the appropriate passive policy. Possibly a better definition would be the policy of not changing the instrument. In fact if x_t and y_t are measured as deviations from their value in period 0, $x_t = 0$ corresponds to the alternative definition.

The second approach considered is the myopic decision rule which minimizes expected loss in the current period without regard for the future. The expected loss in period t conditional on x_t and the prior's parameters is

$$E[(y_t - d_t)^2 | x_t, m_t, h_t] = (m_t x_t - d_t)^2 + x_t^2/h_t + 1. \tag{22}$$

It is easily shown *the myopic decision rule* is[5]

$$x_t^{my} = d_t m_t/(m_t^2 + 1/h_t), \tag{23}$$

and by substituting (23) into (22) the corresponding expected loss,

$$E[(y_t - d_t)^2 | x_t^{my}] = 1 + d_t^2/(1 + s_t^2). \tag{24}$$

In thc subsequent section the performance of these alternative decision rules will be compared with the optimal procedure.

2.2. *Some Results*

The optimal, certainty equivalence, and myopic decision rules and their corresponding expected losses were computed for a number of sets of targets $\{d_t\}$. Table I presents the minimal expected loss as a function of the location parameter s_1 for a number of examples with the same target in every period. The expected losses are decreasing functions of $|s_1|$, demonstrating more precise knowledge of the unknown parameter is preferable.

Table II gives the value of the decision rules at selected s_1 for the optimal and the alternative procedures. The table assumes the prior on β has precision 1, but it may be used to find the appropriate decisions in the more general situation by multiplying its entries by the scale parameter $h_t^{\frac{1}{2}}$. The table may also be used to obtain decision values for negative s_1 as the functions are asymmetric. With the optimal procedure different decision rules are used in each period. The optimal rule for period t is, of course, the optimal first period rule for the $T - t + 1$ problem. In other words, the optimal decision t depends only upon the prior at the time of the tth decision and the targets in the periods remaining in the process.

[4] The null hypothesis $\beta = 0$ seemed appropriate in this situation, for if $\beta = 0$, control is impossible. With diffuse priors s_t corresponds roughly to the t statistic so the required level of significance is lower than conventional ones. It was selected because it is the best level for the single period problem and, given the results of Section 1, has uniformly smaller expected loss for multi-period problems than any higher level.

[5] This rule has been called the sequential updating procedure by [7].

TABLE I

MINIMAL EXPECTED LOSSES: EQUAL TARGETS

Location Parameter	Targets = 1.0			Targets = 4.0			Targets = 16.0		
$\|s_1\|$	2 period	4 period	6 period	2 period	4 period	6 period	2 period	4 period	6 period
0.0	3.99	7.49	10.48	28.24	35.80	39.14	334.0	346.2	349.0
0.2	3.86	7.24	10.17	25.51	32.46	35.73	300.0	311.2	313.9
0.4	3.65	6.91	9.80	22.44	28.87	32.02	260.8	270.4	273.1
0.7	3.30	6.39	9.19	17.99	23.76	26.81	201.2	209.3	212.0
1.0	2.99	5.87	8.57	14.29	19.51	22.45	151.2	158.1	160.7
1.4	2.67	5.29	7.82	10.60	15.15	17.97	103.4	108.9	111.4
2.0	2.39	4.75	7.08	7.09	10.75	13.39	60.9	64.9	67.5
3.0	2.20	4.38	6.55	4.48	7.39	9.88	30.1	33.2	35.7
4.0	2.12	4.23	6.33	3.51	6.19	8.61	18.4	21.3	23.7
5.0	2.08	4.15	6.22	3.04	5.60	7.97	13.0	15.8	18.3

TABLE II

VALUES OF THE DECISION VARIABLE FOR THE ALTERNATIVE RULES: EQUAL TARGETS

Location Parameter	Certainty Equivalence Rule	Myopic Rule	First Period Optimum Decision $T = 2$	$T = 4$	$T = 6$
		Targets = 1.0			
0.0	0.0	0.0	.13	.56	.71
0.2	0.0	.19	.40	.69	.80
0.4	0.0	.34	.50	.72	.85
0.7	0.0	.47	.53	.68	.81
1.0	0.0	.50	.52	.58	.67
2.4	.71	.47	.48	.50	.53
2.0	.50	.40	.41	.41	.42
3.0	.33	.30	.30	.30	.31
4.0	.25	.24	.24	.24	.24
5.0	.20	.19	.19	.19	.19
		Targets = 4.0			
0.0	0.0	0.0	1.58	2.00	2.08
0.2	0.0	.77	1.93	2.30	2.34
0.4	0.0	1.38	2.21	2.55	2.68
0.7	0.0	1.88	2.40	2.63	2.70
1.0	0.0	2.00	2.30	2.54	2.55
1.4	2.86	1.89	2.09	2.20	2.22
2.0	2.00	1.60	1.68	1.76	1.76
3.0	1.33	1.20	1.23	1.25	1.26
4.0	1.00	.94	.96	.97	.97
5.0	.80	.77	.78	.78	.79
		Targets = 16.0			
0.0	0.0	0.0	5.00	5.00	5.01
0.2	0.0	3.08	6.05	6.26	6.27
0.4	0.0	5.52	7.43	7.66	7.67
0.7	0.0	7.52	8.40	8.40	8.40
1.0	0.0	8.00	8.53	8.53	8.53
1.4	11.43	7.57	7.88	7.88	7.88
2.0	8.00	6.40	6.53	6.55	6.55
3.0	5.33	4.80	4.84	4.84	4.84
4.0	4.00	3.76	3.78	3.79	3.79

As can be seen from Table II, the optimal x_1^0 are larger in absolute value than the myopic x_1^{my}, the difference reflecting the degree of experimentation. The longer the planning horizon and the greater the degree of uncertainty, the more experimentation is optimal. For values of the location parameter larger than 2.0, the first period optimum decision is almost the same for $T = 2$, $T = 4$, and $T = 6$ and about the same as that for the myopic rule but not the certainty equivalence rule.

A comparison of the performance of the alternative with the optimal rules is found in Table III. The certainty equivalence rule performs well only when uncertainty in β is small; say, $|s_t| \geqslant 2$ if the targets are 1 or $|s_t| \geqslant 4$ if the $d_t = 16$.

TABLE III

EXPECTED LOSSES AS PER CENT OF MINIMUM FOR ALTERNATIVE DECISION RULES: EQUAL TARGETS

Location Parameter $\lvert s_1 \rvert$	Certainty Equivalence Rule $T = 2$	$T = 4$	$T = 6$	Myopic Rule $T = 2$	$T = 4$	$T = 6$
			Targets = 1.0			
0.0	100	107	115	100	106	112
0.2	104	111	118	101	106	111
0.4	110	116	122	101	104	108
0.7	121	125	131	100	102	105
1.0	134	136	140	100	101	102
1.4	111	110	110	100	100	101
2.0	104	104	104	100	100	100
3.0	101	101	101	100	100	100
4.0	100	100	100	100	100	100
5.0	100	100	100	100	100	100
			Targets = 4.0			
0.0	120	190	261	120	165	181
0.2	133	209	285	112	131	138
0.4	151	236	319	106	117	122
0.7	189	286	380	103	111	114
1.0	238	349	454	102	109	112
1.4	125	130	142	101	106	108
2.0	113	107	124	101	103	104
3.0	104	103	103	100	101	101
4.0	102	101	100	100	100	100
5.0	101	100	100	100	100	100
			Targets = 16.0			
0.0	153	297	442	153	193	196
0.2	171	330	491	108	113	114
0.4	197	380	565	102	106	107
0.7	255	491	727	101	104	105
1.0	340	650	959	100	103	104
1.4	140	141	145	100	102	102
2.0	121	131	143	100	101	101
3.0	109	111	113	100	100	100
4.0	104	104	103	100	100	100
5.0	103	102	102	100	100	100

The myopic rule is superior to the certainty equivalence rule with near optimal performance over a wider range that probably includes most cases encountered in economics. Its performance is worse the greater the degree of uncertainty and longer the planning horizon, namely when one would expect experimentation to be important.

A number of three period problems having targets varying among periods were evaluated and are summarized in Table IV. The larger future targets and the smaller the current target, the more important is experimentation. This is easily explained

TABLE IV

EXPECTED LOSS AS PER CENT OF MINIMUM FOR MYOPIC DECISION RULE: UNEQUAL TARGETS $\{d_i\}$ AND $T = 3$

Location Parameter	Value of Target: $d_1 = d_2 = 4$ and		
s_1	$d_3 = 1$	$d_3 = 4$	$d_3 = 16$
0.0	121	147	418
0.2	112	123	258
0.4	106	113	205
0.7	103	107	177
1.0	102	105	164
1.4	101	103	150
2.0	100	102	129
3.0	100	100	110
4.0	100	100	105
5.0	100	100	103
	Value of Target: $d_2 = d_3 = 4$ and		
	$d_1 = 1$	$d_1 = 4$	$d_1 = 16$
0.0	186	147	106
0.2	171	123	100
0.4	151	113	100
0.7	130	107	100
1.0	115	105	100
1.4	104	103	100
2.0	102	102	100
3.0	100	100	100
4.0	100	100	100
5.0	100	100	100

as increasing future targets increases the payoff for experimentation while reducing the current target reduces experimentation costs.

2.3. *The Moving Horizon Rule*

Three considerations lead us to examine the performance of a first order moving horizon scheme. With this rule, the decision maker looks one period ahead, selecting the decision which would be optimal if the next period were the last. First, Theil [**6**, pp. 154–6] suggested a moving horizon approach as an approximate solution to an infinite period planning problem. Second, in the previous section we found the amount of experimentation for the two and the six period problems differed by only a small amount, suggesting that little is gained by looking further into the future. Finally, it is easily computed and one is not constrained by computation considerations to a formulation involving but a single state variable.

With this rule for $t < T$, the x_t selected minimizes

$$E\{(y_t - d_t)^2 + \min_{x_{t-1}} E[(y_{t+1} - d_{t+1})^2 | s_{t+1}] | m_t, h_t, x_t\}, \tag{25}$$

TABLE V

EXPECTED LOSS AS PER CENT OF MINIMUM FOR MOVING HORIZON RULE: EQUAL TARGETS

Location Parameter s_1	Targets = 1.0 $T = 4$	Targets = 1.0 $T = 6$	Targets = 4.0 $T = 4$	Targets = 4.0 $T = 6$	Targets = 16.0 $T = 4$	Targets = 16.0 $T = 6$
0.0	103	106	101	102	100	100
0.2	102	104	101	102	100	100
0.4	101	103	101	102	100	100
0.7	101	102	101	101	100	100
1.0	100	101	101	101	100	100
1.4	100	101	100	101	100	100
2.0	100	100	100	100	100	100
3.0	100	100	100	100	100	100
4.0	100	100	100	100	100	100
5.0	100	100	100	100	100	100

or, using (24),

$$(26) \qquad 1 + (m_t x_t - d_t)^2 + x_t^2/h_t + E[1 + d_{t+1}^2/(1 + s_{t+1}^2)|m_t, h_t, x_t].$$

For $t = T$, the myopic rule is used.

For our problem this procedure is an excellent approximation, particularly when the targets are large. From Table V, its expected loss is less than a half a per cent greater than the minimal obtainable value when the targets are 16 in every period and within 2 per cent when they are 4. When $|s_t| < 2$, which generally is not the case in economic applications, this rule performs better than the myopic procedure.

2.4. *Misspecification of Error Variance*

As part of the problem statement, the variance of the error term σ_u^2 was assumed known. In most applications, however, this will not be the case, and σ_u^2 must first be estimated. This is not a serious drawback to our analysis as the optimal decision rule is surprisingly insensitive to errors in specifying σ_u^2, at least, in the range of uncertainty where experimentation is important. If $|s_t|$ is not small and experimentation unimportant, the myopic rule can be used. Since the myopic rule is easily computed (see [7]) even if σ_u^2 is treated as a second unknown parameter, there is no need to estimate σ_u^2 when using this rule.

Suppose observations are available beginning with period $t = N < 0$, a locally uniform prior describes the decision maker's initial knowledge of β, and

$$\sigma_u^2 = 1 + e, \qquad |e| < 1,$$

rather than the assumed value of 1. The precision of the normal prior at time of decision t will be

$$h_t' = \sum_{s=-N}^{t-1} x_s^2$$

and mean

$$m_t = \sum_{s=-N}^{t-1} x_s y_s / h_t'.$$

But, the variance of u_t is not 1, so there is an error in specifying h_t (none in m_t). The correct precision is

$$h_t = h_t'/(1 + e).$$

We found this error of incorrectly specifying the precision of the prior has little effect upon the performance of the optimal decision rule when $|s_t| \leqslant 1$. On the other hand, when $|s_t|$ is reasonably large, the myopic procedure is nearly optimal. This suggests the following decision rule:

$$x_t^e = \begin{cases} \sqrt{1+e}\, x_t^0 (\sqrt{1+e}\, s_t), & |s_t| \leqslant 1 \text{ and } t < T, \\ x_t^{my}(s_t) & \text{otherwise.} \end{cases}$$

Note, when $|s_t| \leqslant 1$ and $t < T$, x_t^e is the decision value obtained by substituting the incorrect precision h_t' for h_t in the optimal decision rule (19).

For all the examples considered, the increase in expected losses resulting from the use of x_t^e is surprisingly small. When the error in σ_u^2 is 25 per cent, Table VI reveals a maximum increase of 3 per cent. Even with a 50 per cent error, the increase is at most 8 per cent and is generally much less. The percentage increases were almost the same for all planning horizons between 2 and 6, the only cases considered in this study. Similar results held when there were unequal targets. Given these results, it would be surprising if a scheme treating σ_u^2 as a second unknown parameter would perform significantly better than one where it is estimated.

2.5. *Summary*

A multi-period control problem was analyzed using numerical methods. The principle conclusions are these:

(i) The certainty equivalent approach is a reasonable procedure only when uncertainty in the unknown parameter is small, say when the ratio of the prior's mean to its standard deviation is at least 4 in absolute value.

(ii) The myopic procedure performance is nearly optimal over a wider range. But, when $|s_t| \leqslant 2$, experimentation becomes a relevant consideration; e.g., it pays to select a decision larger in absolute value than the one which minimizes current expected loss in order to obtain improved information about the unknown parameter.

(iii) The more periods remaining in the planning horizon, the more important is experimentation.

(iv) The first order moving horizon scheme is an excellent approximation to the optimal solution and is easily computed even for more complex problems.

TABLE VI

PERCENTAGE INCREASES IN EXPECTED LOSS RESULTING FROM ERRORS IN SPECIFYING σ_u^2: EQUAL TARGETS AND $T = 4$

Location Parameter $\lvert s_t \rvert$	Error = 25%	Error = −25%	Error = 50%	Error = −50%
		Targets = 1.0		
0	0	0	1	1
0.2	0	0	1	1
0.4	0	0	1	1
0.7	0	0	1	1
1.0	0	0	0	1
1.4	0	0	0	0
2.0	0	0	0	0
3.0	0	0	0	0
4.0	0	0	0	0
5.0	0	0	0	0
		Targets = 4.0		
0	0	1	2	4
0.2	1	1	3	5
0.4	1	1	4	6
0.7	1	2	4	6
1.0	2	1	3	6
1.4	3	3	3	4
2.0	1	1	2	2
3.0	1	1	1	1
4.0	0	0	0	0
5.0	0	0	0	0
		Targets = 16.0		
0	0	1	1	2
0.2	1	1	3	3
0.4	1	1	4	4
0.7	1	1	4	6
1.0	1	2	4	8
1.4	1	1	1	1
2.0	0	0	0	0
3.0	0	0	0	0
4.0	0	0	0	0
5.0	0	0	0	0

(v) The analysis is surprisingly insensitive to errors in specifying the error of the additive error term. A 25 per cent error increases expected losses by at most a few per cent for the examples considered.

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APPENDIX

Notes on the Computations

Piecewise linear functions were used to approximate the expected loss functions for the alternative decision rules. Letting $h_t(s_t|p)$ be the expected loss for periods t through T with policy p, backward induction implies

$$h_t(s_t|p) = E[(y_t - d_t)^2 + h_{t+1}(s_{t+1})|s_t, h_t = 1, x_t = p_t(s_t)]$$

with $h_{T+1} = 0$. For a set of points S, the above functions were evaluated and linear interpolations used for points in between. For $|s_t| > 500$, the functions h_t were approximated by $T - t + 1$, the expected loss if the decision maker knew the true value of β and used any of the decision rules considered. The points in S were in steps of 0.1 between 0 and 1, in steps of 0.2 between 1 and 2, in steps of 0.5 between 2 and 3, in steps of 2 between 6 and 20, in steps of 5 between 20 and 50, and 500. In addition the negatives of these points were included in S.

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