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Equilibrium Selections*

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ABSTRACT

This paper studies the outcome of fully insured random selections among multiple competitive equilibria. This defines an iterative procedure of reallocation which is Pareto improving at each step. The process converges to a unique Pareto optimal allocation in finitely many steps. The key requirement is that random selections be continuous, which is a generic condition for smooth exchange economies with strictly concave utility functions.

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1. Introduction

The presence of multiple equilibria poses a tough problem for economic theory. Uniqueness of competitive equilibria can be guaranteed only under extremely restrictive assumptions about agents' preferences, such as either identical preferences or representability by Cobb-Douglas utility functions. In fact, recent work by Mercenier (1994) demonstrates that nonuniqueness of equilibrium prices also arises empirically in applied general equilibrium models.

In asking what can be done to alleviate the difficulties caused by this multiplicity, we wish to explore the consequences of imagining that competitive equilibria are somehow chosen according to a probability distribution concentrated on equilibrium prices. We consider the nicest possible case—the probabilistic selection of prices depends continuously on the data, and the random selection rule is itself common knowledge among all agents in the economy. Results due to Allen (1985a, 1985b) and Mas-Colell and Nachbar (1991) prove that "typical" smooth exchange economies permit continuous random selections from their equilibrium price correspondences. We therefore restrict ourselves to these generic economies having a finite or countable number of equilibria for all possible distributions of the total initial endowment vector. Yet this situation causes risk-averse traders to wish to insure themselves against the price uncertainty induced by the random selection among multiple equilibria. Our paper examines the ultimate outcomes that occur when this happens. We view the hypothesis of complete markets and the availability of actuarially fair insurance policies as desiderata for an idealized perfectly competitive economy.

One result of random equilibrium selection is that it generates nontrivial distributions over competitive equilibrium allocations. Due to risk aversion, these distributions fail to satisfy Pareto optimality. The First Welfare Theorem fails ex ante when equilibrium prices are randomly selected even though it holds ex post for any particular competitive price vector and associated equilibrium allocations. Introduction of the random selection alters the economy—not only does its equilibrium change, but more fundamentally, the entire specification switches to include some risks that were previously absent but now must be hedged. The net result of this process of realizing that randomization among competitive equilibria will occur and making fair insurance contracts is that a new and *unique* equilibrium is reached after finitely many steps whenever the original economy featured a continuous random selection from its equilibrium price correspondence. Our equilibrium outcome is Pareto optimal. We further show that any regular competitive equilibrium allocation of the original economy can be reached in two steps as the unique equilibrium of our process provided that a suitable continuous random selection (which always exists) is followed.

The uncertainty induced by randomization among equilibria has been discussed elsewhere. Cass and Shell (1983) provide examples of random selections as sunspot equilibria, and Chichilnisky, Dutta, and Heal (1992) study the market structure necessary to support insurance against the choice of equilibrium. We are concerned with a clear statement of the convergence of the implied sequence of reallocations.

Randomization among equilibria has close connections with extrinsic uncertainty or sunspots (Azariadis (1981), Balasko (1983), Cass and Shell (1983)). This introduces uncertainty which has no direct effect on endowments or preferences. The fact that our procedure needs more than one step is a direct consequence of their "irrelevance result" that if individuals share common subjective probabilities, extrinsic uncertainty cannot affect Pareto optima and therefore the competitive equilibria of complete market economies. Extrinsic uncertainty can matter if markets are incomplete (e.g., Azariadis (1981)) or if traders have diverse prior beliefs (Kurz (1993)). Our problem will typically not converge to an optimal allocation under these conditions.

The remainder of this paper is organized as follows: Section 2 introduces notation for our model, and Section 3 shows the genericity of our critical property that guarantees that there is a continuous random selection from the equilibrium price correspondence as we redistribute the total initial endowment vector among agents. Section 4 contains the statements and proofs of our main results. Finally, Section 5 concludes with further interpretations of our conclusions.

2. The Model

We study a basic model of a smooth pure-exchange economy having finite numbers of consumers and commodities. For technical reasons (see Section 3), C^∞ utilities and demands are required for our analysis. Moreover, we focus on economies that satisfy a generic property that excludes the presence of a continuum of competitive equilibria for any feasible redistribution of initial endowment vectors. (Again, see Section 3 below.)

To fix notation, suppose that there are ℓ goods $(j = 1, \ldots, \ell)$ and N consumers $(h = 1, \ldots, N)$, each of whom has consumption set $\mathbb{R}^\ell_{++}$. Let $e_h \in \mathbb{R}^\ell_{++}$ denote the initial endowment of household h, and write u_h for the utility function of trader h, where for each $h = 1, \ldots, \ell$ the function $u_h \in \mathcal{U} = \{u : \mathbb{R}^\ell_{++} \to \mathbb{R} \mid u$ is C^∞, is strictly differentiably monotone $(Du(x) \gg 0$ for all $x \in \mathbb{R}^\ell_{++})$, is strictly differentiably concave $(D^2u(x)$ is negative definite for all $x \in \mathbb{R}^\ell_{++})$, and satisfies the boundary condition that the closure in $\mathbb{R}^\ell$ of the upper contour set $\{x \in \mathbb{R}^\ell_{++} \mid u(x) \geq u(\bar{x})\}$ through $\bar{x}$ is disjoint from the boundary of $\mathbb{R}^\ell_+$ for all $\bar{x} \in \mathbb{R}^\ell_{++}\}$. Let E denote the set of all feasible reallocations of initial endowments, so that $E = \{y = (y_1, \ldots, y_N) \in \mathbb{R}^{\ell N}_{++} \mid \hat{y} =$

$\sum_{h=1}^{N} y_h = \sum_{h=1}^{N} e_h = \hat{e}\}$. Where no confusion can result, we use the circumflex symbol to designate the operation of adding vectors over individuals. We also write vectors without subscripts to denote the entire "profile"— i.e., $e = (e_1, \ldots, e_N)$. We normalize prices to belong to the (open) unit simplex $\Delta = \{p \in \mathbb{R}^\ell_{++} \mid \sum_{j=1}^{\ell} p_j = 1\}$. Write $x_h : \Delta \to \mathbb{R}^\ell_{++}$ for individual h's demand function (which depends also on u_h and e_h), where for $p \in \Delta$, $x_h(p; u_h, e_h) = \arg\max \{u_h(x) \mid x \in \mathbb{R}^\ell_{++} \text{ and } p \cdot x \leq p \cdot e_h\}$; note that x_h is a C^∞ function of $p \in \Delta$ and $e_h \in \mathbb{R}^\ell_{++}$.

3. The Critical Property

In order to prove our main result, we must hypothesize continuity of the random equilibrium selection. However, this property need not be problematic as it holds for a generic subset of economies. To be specific, each pure exchange economy in a dense G_δ subset possesses such a continuous random selection from its equilibrium price correspondence as initial endowment vectors vary. This means that any economy satisfying the assumptions of our model can be approximated arbitrarily closely by economies with continuous random selections.

This claim is simply the continuous random selection result discovered by Allen (1985a, 1985b), although we appeal directly to the formulation provided by Mas-Colell and Nachbar (1991) since they perturb utility functions rather than aggregate excess demand functions in the definition of generic subsets of economies. The argument is that, for a countable intersection of open and dense subsets of economies, each redistribution of endowments corresponds to an at most countable number of competitive equilibrium prices. From this, one can show the existence of a continuous random selection from the equilibrium price correspondence.

More formally, let $\Psi : E \to \Delta$ denote the *equilibrium price correspondence*, where for $e = (e_1, \ldots, e_N) \in E \subset I\!R^{\ell N}_{++}$, $\Psi(e) = \{p \in \Delta \mid \sum_{h=1}^{N} x_h(p; u_h, e_h) = \sum_{h=1}^{N} e_h\}$. Write $\mathcal{M}(\Delta)$ for the set of probability measures defined on Borel subsets of Δ, and endow $\mathcal{M}(\Delta)$ with the topology of weak convergence of probability measures. [Alternatively, this is called the weak* topology and is defined by the condition that $\mu_n \to \mu \in \mathcal{M}(\Delta)$ if and only if $\int_\Delta f(p) d\mu_n(p) \to \int_\Delta f(p) d\mu(p)$ (pointwise) for all continuous and bounded real-valued functions f on Δ.] A *continuous random selection* from the equilibrium price correspondence is a (single-valued) function $\mu : E \to \mathcal{M}(\Delta)$ having the following two properties: (i) μ is continuous (when $\mathcal{M}(\Delta)$ is endowed with its weak* topology), and (ii) μ is a selection from Ψ (that is, for all $e \in E$, $\mu(\Psi(e)) = 1$).

Definition. An Edgeworth box E for the economy given by utilities and endowments $\{(u_1, e_1), \ldots, (u_N, e_N)\}$ has the *critical property* if for all $e' \in E = \{(e'_1, \ldots, e'_N) \in I\!R^{\ell N}_{++} \mid \sum_{h=1}^{N} e'_h = \sum_{h=1}^{N} e_h\}$, the set $\Psi(e')$ is at most countable. In words, this says that for any feasible redistribution of initial endowments, the resulting set of competitive equilibrium price vectors is at most countable, even at critical economies.

Our interest in the critical property derives from the results that it is generic and that it suffices to ensure the existence of a continuous random selection from the equilibrium price correspondence.

Proposition. Given endowments $e_1, \ldots, e_N \in I\!R^{\ell N}_{++}$ and utilities $u_2, \ldots, u_N$ for all but one person in the economy (where for each $h = 2, \ldots, N$, $u_h \in \mathcal{U}$), for u_1 belonging to a countable intersection of open dense subsets (in the C^∞ compact open topology on utility functions $u_1 : I\!R^{\ell}_{++} \to I\!R$) of $\mathcal{U}$, the critical property is satisfied. Moreover, all economies in this generic set possess continuous random selections from their equilibrium price correspondences.

Proof. This follows from Mas-Colell and Nachbar (1991) combined with the observation that differentiable strict concavity is satisfied by a countable intersection of (nonempty) open subsets of utilities. ∎

Remark. The hypothesis that all utilities and demands are infinitely continuously differentiable is needed for the proofs given in Allen (1985a, 1985b) and in Mas-Colell and Nachbar (1991), which use very sophisticated tools from differential topology. Similarly, these methods do not permit one to strengthen the result to yield an open and dense set of well-behaved economies rather than a dense G_δ subset. Note also that, while Allen (1985a, 1985b) obtains the generic finiteness of the equilibrium price set for all parameters, the Mas-Colell and Nachbar (1991) framework gives only the conclusion that, for all parameter values, the set of equilibria is at most countable for a residual subset of utilities.

4. The Main Result

This section presents the main theorem and its proof. As indicated above, the critical property is needed to guarantee the existence of a continuous random selection from the equilibrium price correspondence for all possible redistributions of endowments.

Theorem. For a countable intersection of open and dense subsets of C^∞ utilities in $\mathcal{U}$, starting from any initial endowment vectors $e_h \in \mathbb{R}^\ell_{++}$, $h = 1, \ldots, N$, the sequence of redistributions defined inductively by, for $h = 1, \ldots, N$,

$$e_h^0 = e_h$$

and

$$e_h^{n+1} = \int_\Delta x_h(p; u_h, e_h^n) d\mu(e^n)(p)$$

converges to an allocation $\bar{e} = (\bar{e}_1, \ldots, \bar{e}_N) \in \mathbb{R}^{\ell N}_{++}$, where $\bar{e}_h = \lim_{n\to\infty} e_h^n$ for each $h = 1, \ldots, N$, such that $\bar{e}$ has a unique competitive equilibrium price vector (i.e.,

$\#\Psi(\bar{e}) = 1$). Moreover, the convergence occurs in finitely many steps—given $u_1, \ldots, u_N$ in the generic set and $e_1, \ldots, e_N$, there exists M such that $e^n = \bar{e}$ for all $n \geq M$.

Proof. Let $\bar{E}$ denote the closure of E so that $\bar{E} = \{(e'_1, \ldots, e'_N) \in I\!R_+^{\ell N} \mid \sum_{h=1}^{N} e'_h = \sum_{h=1}^{N} e_h\}$. Note that feasibility of total demand $\left(\sum_{h=1}^{N} x_h(p; u_h, e_h^n)\right)$ at any competitive equilibrium price $p \in \operatorname{supp} \mu(e^n) \subseteq \Psi(e^n)$ implies that the convex combinations e^{n+1} are also feasible, so that $e^{n+1} \in E$ for all n. Then $\bar{E}$ is compact and therefore the sequence $\{e^n\}_{n=1}^{\infty}$ has a convergent subsequence, call it $\{e^{n'}\}_{n'=1}^{\infty}$, such that $\lim_{n' \to \infty} e^{n'} = \bar{e} \in \bar{E}$. To show that $\bar{e} \in E$ notice that (iterated) individual rationality of the $x_h(p; u_h, e_h^n)$, which are competitive equilibrium allocations for the economy with utilities $u_1, \ldots, u_N$ and endowments $e_1^n, \ldots, e_N^n$, implies that for all $h = 1, \ldots, N$ and all $n = 1, 2, \ldots$, $u_h(e_h^{n+1}) \geq u_h(e_h^n) \geq u_h(e_h)$. This, combined with the boundary condition and the fact that $e_h \in I\!R_{++}^{\ell}$ for all h, implies that $\bar{e} \in E \subset I\!R_{++}^{\ell N}$.

Let $\bar{e} \in E$ be a limit of a convergent subsequence $\{e^{n'}\}_{n'=1}^{\infty}$. Since $\lim_{n' \to \infty} e^{n'} = \bar{e}$ and since $\mu : E \to \mathcal{M}(\Delta)$ is weakly continuous while the x_h are continuous (in fact, C^∞) in prices and endowments, $\bar{e}_h = \int_\Delta x_h(p; u_h, \bar{e}_h) d\mu(\bar{e})(p)$. If $\mu(\bar{e})$ is a Dirac measure, then necessarily $x_h(p; u_h, \bar{e}_h) = \bar{e}_h$ for $p \in \Psi(\bar{e})$ and, because $\bar{e} \in E$ implies $\sum_{h=1}^{N} x_h(p; u_h, \bar{e}_h) = \sum_{h=1}^{N} \bar{e}_h = \sum_{h=1}^{N} e_h$, the First Welfare Theorem says that $\bar{e}$ is Pareto optimal. If $\mu(\bar{e})$ is not a Dirac measure, strict concavity of u_h and individual rationality of competitive equilibrium allocations implies that $u_h(\bar{e}_h) = u_h\left(\int_\Delta x_h(p; u_h, \bar{e}_h) d\mu(\bar{e})(p)\right) > \int_\Delta u_h\big(x_h(p; u_h, \bar{e}_h)\big) d\mu(\bar{e})(p) \geq \int_\Delta u_h(\bar{e}_h) d\mu(\bar{e})(p) = u_h(\bar{e}_h)$, which is a contradiction. Therefore, $\bar{e}$ is Pareto optimal.

To show that the entire sequence—and not just a subsequence—converges to $\bar{e}$, suppose not. Then $\{e^n\}_{n=1}^{\infty}$ must have subsequences with at least two distinct limits. As before, let $\{e^{n'}\}_{n'=1}^{\infty}$ denote a convergent subsequence (the existence of which was demonstrated above), and let $\{e^{m'}\}_{m'=1}^{\infty}$ be a subsequence of $\{e^n\}_{n=1}^{\infty}$ having a different limit, say $\bar{\bar{e}} \in E$, so that $\bar{e} = \lim_{n' \to \infty} e^{n'} \neq \lim_{m' \to \infty} e^{m'} = \bar{\bar{e}}$, where both $\bar{e}$ and $\bar{\bar{e}}$ are Pareto optimal. Recall that Pareto optimal initial allocations lead to unique competitive equilibrium prices and unique competitive equilibrium allocations when preferences are smooth and strictly convex.

Notice that, for all h, the sequences $\{e_h^n\}_{n=1}^{\infty}$ of redistributions of endowments are strictly improving in the sense that $u_h(e_h^n) \leq u_h(e_h^{n+1})$ for all n, with strict inequality unless $\mu(e^n)$ is a Dirac measure (which is surely the case whenever $\Psi(e^n)$ is a singleton). To see this, observe that because $x_h(p; u_h, e_h^n)$ is consumer h's demand at price vector $p \in \Delta$ when he has initial endowment vector $e_h^n \in I\!R_{++}^{\ell}$, necessarily $u_h\big(x_h(p; u_h, e_h^n)\big) \geq u_h(e_h^n)$. Strict concavity of utilities then implies that $u_h\big(\int_\Delta x_h(p; u_h, e_h^n) d\mu(e^n)(p)\big) > \int_\Delta u_h\big(x_h(p; u_h, e_h^n)\big) d\mu(e^n)(p) \geq u_h(e_h^n)$ whenever $\mu(e^n)$ is not a Dirac measure. In this case, the Pareto optimality of both $\bar{e}$ and $\bar{\bar{e}}$ means that there are consumers h' and h'' for which $u_{h'}(\bar{e}_{h'}) < u_{h'}(\bar{\bar{e}}_{h'})$ and $u_{h''}(\bar{\bar{e}}_{h''}) < u_{h''}(\bar{e}_{h''})$. This contradicts monotonicity of $u_h(e_h^n)$ in n for all $h = 1, \ldots, N$ whenever $\bar{e} \neq \bar{\bar{e}}$. If $\Psi(e^n)$ is a singleton—or, more generally, if $\mu(e^n)$ is a Dirac measure—then $e_h^{n+1} = x_h\big(\Psi(e^n); u_h, e_h^n\big)$ for all h and hence $e^{n+1} = e^{n+2} = \ldots$, so that $u_h(e_h^n) \leq u_h(e_h^{n+1}) = u_h(e_h^{n+2}) = \ldots$ and $M \leq n+1$. In this case, we must have $\bar{e} = \bar{\bar{e}}$.

Finally, to show that the convergence occurs in finitely many steps, let $e^n \to \bar{e}$ which is Pareto optimal. Then, for every $\epsilon > 0$, there is $M(\epsilon)$ such that $||e^n - \bar{e}|| < \epsilon$ for every $n \geq M(\epsilon)$. By Theorem 4.5.3 and Corollary 4.5.4 of Balasko (1988, pp. 104-105), every Pareto optimal allocation $\bar{e} \in E$ has the property that there is $\epsilon(\bar{e}) > 0$ such that every $e \in E$ with $||e - \bar{e}|| < \epsilon(\bar{e})$ is regular and has a unique competitive equilibrium price vector. Setting $M - 1 = M\big(\epsilon(\bar{e})\big)$ guarantees that $\Psi(e^{M-1})$ is a singleton, so that $\mu(e^{M-1})$ must be a Dirac measure. Then e^M is Pareto optimal, and therefore $e^M = e^{M+1} = \ldots = \bar{e}$. ∎

Remark. Any regular competitive equilibrium can be achieved in two steps by using a continuous random selection that assigns probability one to the competitive equilibrium price vector associated with the desired Walrasian allocation.

Corollary. The convergent sequence e^n identified in the theorem features Pareto improvement at each step, and furthermore, its limit $\bar{e}$ is a Pareto optimal allocation.

Proof. By the individual rationality of competitive equilibrium allocations and by concavity of each u_h, we have $u_h(e_h^{n+1}) = u_h\big(\int_\Delta x_h(p; u_h, e_h^n)d\mu(e^n)(p)\big) \geq \int_\Delta u_h\big(x_h(p; u_h; e_h^n)\big)d\mu(e^n)(p) \geq u_h(e_h^n)$, with strict inequality whenever $\#\Psi(e^n) > 1$. This shows that strict Pareto improvement occurs at each step, except possibly if $\#\Psi(e^n) = 1$. In that case, the proof of the theorem shows that the last step moves directly to the contract curve—i.e., in symbols, if $M-1$ is defined to be the first integer for which $\#\Psi(e^{M-1}) = 1$, then $\mu(e^{M-1})$ is precisely the Dirac measure concentrated at the point $\Psi(e^{M-1})$ in Δ and (by strict concavity) each $x_h(p; u_h, e_h^{M-1})$ is a singleton when evaluated at $p = \Psi(e^{M-1})$. Hence, for all h, $e_h^M = x_h\big(\Psi(e^{M-1}); u_h, e_h^{M-1}\big)$, which is Pareto optimal by the First Welfare Theorem. ∎

Remark 1. The statement of the theorem is reminiscent of the concept of quasi-stability analyzed in Hahn and Negishi (1962), in that we show that some set—in our case, a subset of the set of Pareto optimal allocations—serves as a sink for a certain discrete-time process that involves trades conducted at "false" prices, where no consumption occurs before the limit of the process has been reached. However, we do not claim stability or its analogue for our setting, nor do we consider questions of how one can find competitive prices. Instead, our process features perfectly competitive price vectors at each stage, albeit traders myopically act as if the randomization and our resulting further redistributions will not occur. Note also that we hit the limit after a finite number of steps, so that the common objection that nontâtonnement processes let agents starve before arriving at the limit is alleviated here.

Remark 2. The MDP planning procedure for public goods (proposed by Drèze and de la Vallée Poussin (1971) and Malinvaud (1972a)) also has the property that reallocations are Pareto improving at each step.

The procedure analyzed in our theorem can be viewed as the iteration of a map. The properties of this map, which we call F, are studied in the next proposition.

Proposition. Define $F : \tilde{E} \to \tilde{E}$ by

$$F\big((y_1, \ldots, y_N)\big) = \Big(\int_\Delta x_1(p; u_1, y_1) d\mu(y)(p), \ldots, \int_\Delta x_N(p; u_N, y_N) d\mu(y)(p) \Big),$$

where $\tilde{E} = \{y = (y_1, \ldots, y_N) \in I\!R^{\ell N}_{++} \mid \sum_{h=1}^{N} y_h = \sum_{h=1}^{N} e_h$ and $u_h(y_h) \geq u_h(e_h)$ for all $h = 1, \ldots, N\}$ is the set of all individually rational feasible allocations and $\mu : \tilde{E} \to \mathcal{M}(\Delta)$ is a continuous random selection. The map F has the following properties:

(i) F is continuous and has a fixed point.

(ii) For each $y \in \tilde{E}$, either $F(y) = y$ or $F(y)$ Pareto dominates y—i.e., $y \neq y' \equiv F(y)$ implies that $u_h(y'_h) \geq u_h(y_h)$ for all $h = 1, \ldots, N$ with $u_h(y'_h) > u_h(y_h)$ for some $h = 1, \ldots, N$.

(iii) The point $y \in \tilde{E}$ is a fixed point of F if and only if y is a Pareto optimum.

Proof. To see (i), note that the continuity of F follows from the fact that the random selection μ is continuous combined with continuity of the demand functions $x_h(\cdot; u_h, \cdot)$ in $p \in \Delta$ and $y_h \in I\!R^{\ell}_{++}$ for any $u_h \in \mathcal{U}$. By Brouwer's Fixed Point Theorem, F has a fixed point on the compact and convex subset $\tilde{E}$ of $I\!R^{\ell N}_{++}$.

For (ii), use individual rationality to observe that $u_h\big(x_h(p; u_h, y_h)\big) \geq u_h(y_h)$.

The inequality is strict for some consumer $h = 1, \ldots, N$ whenever $x_h(p; u_h, y_h) \neq y_h$; this follows from the strict concavity of utilities $u_h \in \mathcal{U}$. Strict concavity also implies that $u_h\big(\int_\Delta x_h(p; u_h, y_h) d\mu(y)(p)\big) > \int_\Delta u_h\big(x_h(p; u_h, y_h)\big) d\mu(y)(p)$ for any nondegenerate distribution $\mu(y)$ on Δ. This proves that $u_h(y_h') > u_h(y_h)$ whenever $y_h' \neq y_h$, as desired.

Finally, one direction of (iii) follows from (ii). For the equivalence, note that every Pareto optimal allocation has a unique supporting competitive price vector and, at these prices, agents' demands are such that they do not wish to trade away from the optimal allocations. ∎

Observation 1. In general, whenever there are multiple equilibria for the original economy that receive strictly positive probability under the continuous random selection, one cannot achieve the limit allocation in a single step except for extraordinarily special configurations. Depending on the probability weights assigned to various equilibria, the first step gives a reallocation that lies in the convex hull of the competitive equilibrium allocations for the original economy; the first reallocation generally is not Pareto optimal and hence requires additional redistributions. The basic problem is that an overall equilibrium need not be achieved in a single step, so that the process must be repeated.

Observation 2. The iterative process of redistribution which we construct has three elements: (C) finding the set of competitive equilibria for a given endowment distribution, (D) assigning a probability distribution to these prices, and (I) allocating the average (according to the distribution) of the equilibrium allocations to the traders in the economy. Steps D and I can be understood as the outcome of complete insurance against randomization.

To see this, let $e^n = (e_1^n, \ldots, e_N^n) \in \mathbb{R}_{++}^{\ell}$ be the nth-step initial allocation of endowments, and suppose that there are multiple competitive equilibria in the exchange economy with endowments e^n. Each equilibrium price $p^{n,i} \in \Psi(e^n) \subset \Delta$ is associated with allocations (or demands) $x^{n,i} \in E \subset I\!R^{\ell N}_{++}$, where $x^{n,i} = (x^{n,i}_1, \ldots, x^{n,i}_N)$ and, for all $h = 1, \ldots, N$, $x^{n,i}_h = x_h(p^{n,i}; u_h, e^n_h)$. Let $\mu^n \in \{\mu^n \in \mathcal{M}(\Delta) \mid \mu^n(\Psi(e^n)) = 1\} = \{(\mu^{n,i})_i \mid \mu^{n,i} \geq 0, \sum_{i=1}^{\#\Psi(e^n)} \mu^{n,i} = 1\}$ be a probability distribution on $\Psi(e^n)$. By strict concavity of utilities, demands are single valued, and we can think of μ^n as a distribution on the corresponding $x^{n,i}$ vectors to simplify the notation. By the First Welfare Theorem, each allocation $x^{n,i}$ is Pareto optimal. However, the distribution μ^n is not. To see this, observe that strict concavity implies that every individual h prefers (strictly) the allocation $\bar{x}^n_h = \sum_i \mu^{n,i} x^{n,i}_h$ and $\bar{x}^n = (\bar{x}^n_1, \ldots, \bar{x}^n_N)$ is feasible. This is exactly the argument of Cass and Shell (1983) and Balasko (1983), which establishes that with strictly concave von Neumann-Morgenstern utility functions, extrinsic uncertainty cannot affect optimal allocations.

Consider an economy in which individual endowments are uncertain but there is no aggregate risk. The uncertainty can be described by a probability distribution ν on E. Malinvaud (1972b) shows that the competitive equilibria of this economy necessarily coincide with the equilibria of a fully insured economy in which endowments are given by $\tilde{e} = (\tilde{e}_1, \ldots, \tilde{e}_N)$, where $\tilde{e} = \int_E e d\nu(e)$. If we interpret the random selection as assigning the associated competitive equilibrium allocations to individuals, then Step I achieves full insurance and the following Step C determines the competitive equilibria of the insured economy. Such insurance could be achieved either by trade in insurance contracts (Malinvaud (1972b)), by trade in price-contingent derivative securities (Chichilnisky, Dutta and Heal (1992)), or by trade in Arrow (1953) securities which pay off in each state $i = 1, \ldots, \#\Psi(e^n)$, where the states are the price vectors selected from $\Psi(e^n)$. As

Observation 1 points out, this reallocation is typically not optimal.

Observation 3. One may wonder whether it is possible to characterize the limit points of our process completely. We have been unable to do this in general, due to the potentially complex nature of the set of Pareto optima in high dimensions and the fact that, by choosing appropriate continuous random selections, we can generate at least the convex hull of the set of regular competitive equilibrium allocations for the original economy in our first step. However, for the 2×2 Edgeworth box (i.e., $\ell = 2$ and $N = 2$), we can give a characterization. As the continuous random selection changes, we can generate a closed subset of the contract curve for a given initial endowment distribution in the original economy. The set of all limit points of our process contains only individually rational and Pareto optimal allocations, but not every such allocation can be reached because, for instance, the indifference curve through an original initial allocation may perhaps not contain the original economy's unique competitive equilibrium allocation. Thus, our redistribution process implicitly proposes a refinement of the core, and this refinement of a cooperative solution concept always gives a nonempty closed set of feasible allocations. On the other hand, with an atomless continuum of each type of agent in a 2×2 Edgeworth box for which the original initial endowment vector led to exactly three distinct competitive equilibria, relatively open subsets of the contract curve are generally included in our set of limit points of the process. We conclude that for large economies, our process does not necessarily yield the core or, equivalently, the competitive equilibrium allocations of the original economy.

Observation 4. A more abstract way to derive the existence of a limit set (for all random selections considered together) satisfying Pareto optimality and uniqueness of competitive equilibria could be based on Zorn's Lemma. Define the binary relation $\prec$

on $\tilde{E}$ by $x \prec y$ if and only if either y belongs to the relative interior of the convex hull of the competitive equilibrium allocations with endowments x and there are at least two such equilibria or y is a limit of such a chain beginning at x. The relation $\prec$ is asymmetric and acyclic; it is a subrelation of the strict Pareto dominance relation. Then the partial ordering $\prec$ has a maximal linearly ordered chain which has an accumulation point (because the nonempty set $\tilde{E}$ of feasible and individually rational allocations is compact). Conclude that a maximal chain has a maximal element, which is an allocation having a unique competitive equilibrium. However, this alternative method does not yield convergence in finitely many steps, nor does it guarantee the equality of all possible limits for a given (continuous) random selection.

5. Interpretation

Multiplicity is sometimes offered as a "critique" of rational expectations (for instance, see Hahn (1991)). The objection begins with the statement that *even if* individuals know the entire structure of the economy, they may rationally expect some particular equilibrium to be realized. Rational expectations should require that such predictions be correct, yet economists have no theory of which equilibrium will actually arise. To formalize the argument, we should start with probability distributions. Once this is done—when the distributions are incorporated into the model—the original equilibria fail to remain equilibrium outcomes in the modified economy. The "no sunspot" result of Cass and Shell (1983) and Balasko (1983) assures us that a nondegenerate distribution cannot be a competitive equilibrium.

We could draw a negative conclusion from this, in that it says that competitive equilibrium price distributions do not exist or, alternatively, that equilibrium requires unmodelled coordination of beliefs. Instead, we believe that what we have here is a slightly different story. We show that even though one admits multiplicity in the fundamental model, the modifications needed to generate price distributions, when successively applied, necessarily lead to a unique allocation which is indeed Pareto optimal. The only requirement for this convergence—which occurs in a finite number of steps—is that the underlying economy be generic and that agents all take the same well-behaved random selection from the basic equilibrium price correspondence.

The process leads to a unique outcome as the limit of reallocations compatible with competitive equilibrium under complete markets. At each step, individuals are able to insure themselves against randomness in the equilibrium outcome. At the nth step, the maximum number of elementary securities (or basic insurance contracts) which must be traded is $\#\Psi(e^n)$, the number of competitive equilibria associated with the endowment redistribution e^n. By assumption, the number of these equilibria is at most countable at every step. With finitely many steps, the total number of securities required by the entire process is at most countable, provided that the utilities of our basic economy lie in the generic set for which our process is defined and converges. We view the completeness of markets as one aspect of an idealized perfectly competitive environment. Completeness at every stage is needed to insure against multiple equilibria and to guarantee that our process does not stop before reaching a unique competitive equilibrium (and hence Pareto optimal) allocation. Notice that the introduction of insurance alters the impact of randomization and, in fact, changes the equilibrium.

Our arguments require that the randomized selection from the competitive price correspondence be continuous. Such selections exist generically but definitely not for all economies. Whenever there are continuous random selections, typically there are uncountably many of them (unless, of course, the equilibrium price correspondence is single valued, in which case the equilibrium price function is the only selection, continuous or otherwise). We hypothesize that all agents use the same selection and that it is common knowledge. This requirement can be weakened somewhat so that it applies only to the finite number of redistribution points that arise during our process, but we must impose the restriction that all of the averages (with respect to the randomization) defining our redistribution process must use the same probabilities for all traders. (Otherwise, our redistributions need not constitute feasible allocations.) We further use the common probabilities feature to define the concept of fair insurance. Lack of agreement in evaluating probabilities usually leads to outcomes with less than full insurance, even in the standard competitive framework.

In global terms, given utilities in our generic set satisfying the critical property, the entire process continuously maps the open set E of feasible allocations onto its Pareto optimal subset. Each possible initial endowment distribution is mapped to a point on the contract curve. In fact, the fixed points of our entire process (i.e., its limit points) are precisely the contract curve.

The process could also be interpreted as an iterated planning procedure, where traders submit entire excess demand functions at each stage and the planner responds with a redistribution of endowments that equals the average demand at equilibrium prices according to a continuous random selection. Under this interpretation, no agent needs to know the probabilities associated with the selection. Moreover, we could envision the planner asking agents to send their excess demand functions for all prices and all individual endowment vectors. From this data, the planner could proceed to perform the "thought experiment" behind our process and then directly reallocate endowments to equal the limit point of our process. Clearly the planning procedure could stop after one round if the planner chooses any one of the competitive equilibria of the original economy and assigns it probability one. However, the interest in our process is that, by choosing a nondegenerate random selection, allocations other than competitive equilibria of the original economy can be achieved.

Our process suggests a comparison with a version of the social choice problem—namely, to aggregate data from the economic environment so as to obtain a unique individually rational and Pareto optimal outcome. For generic utilities, we do this in a way that exhibits continuous dependence on the original initial endowments and that satisfies uniqueness for a particular continuous random selection. Continuity with respect to utilities must remain a conjecture, since new mathematical machinery appears needed to show that there are random selections that depend continuously on preferences. We have not investigated the minimal information requirements or the manipulation possibilities for our process.

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