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DYNAMIC PROPERTIES OF TWO APPROXIMATE SOLUTIONS TO A PARTICULAR GROWTH MODEL

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ABSTRACT

This paper investigates two methods for approximating the optimal decision rules of a stochastic, representative agent model which exhibits growth in steady state and cannot be expressed in linear--quadratic form. Both methods are modifications on the linear quadratic approximation technique proposed by Kydland and Prescott. It is shown that one of the solution methods leads to bizarre dynamic behavior, even with shocks of empirically reasonable magnitude. The other solution technique does not exhibit such bizarre behavior.

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1. Introduction

Frequently, it is of interest to study the properties of dynamic, equilibrium models which cannot be transformed into a form in which the objective is quadratic and the constraints linear. In this case exact solutions are computationally burdensome, and this has prompted the search for reliable approximate solution methods.[1/] This paper reports on the dynamic properties of two versions of the linear quadratic (LQ) approximation method proposed by Kydland and Prescott (1982). The KP method involves replacing the non-LQ growth model by an LQ model which approximates it in a neighborhood of its steady state. Since this paper is concerned with a model which exhibits steady state growth in capital, consumption, and output, the KP method does not apply directly. However, the fact that ratios of these variables do converge to a steady state is exploited to transform the original model into an alternative, equivalent one to which the KP approximation method does apply. This alternative model is called the "star economy" below. Simple modifications of the approximate decision rules for the star economy produce the desired approximate decision rules for the original economy of interest.

The paper reports the results of experiments using two methods for approximating the optimal decision rules of the star economy. Under one method, the "linear method", the approximate decision rules in the star economy are linear in the state variables.[2/] Under the other method, the "log method" the decision rules are log linear in the state variables. The paper shows that, even for shocks[3/] of reasonable size, the linear approximation method performs poorly, exhibiting bizarre dynamic behavior that almost certainly does not characterize the true decision rules that it supposedly approximates. For example, for large shocks--larger than those observed in the data--a positive productivity shock induces negative capital investment in the period it occurs, followed by a more reasonable positive response in subsequent periods. Since all output is allocated either to consumption or investment, the national income identity implies a very powerful initial positive response of consumption followed by a substantial correction in the subsequent period. When the shock size is reduced to a more reasonable range, the reaction of capital investment strengthens, putting it in the positive--though small--range. The initial strong reaction of consumption with subsequent correction remains. Thus, for shocks of reasonable magnitude, the model implies that consumption is more volatile than output and that consumption growth is negatively autocorrelated. The log linear method, on the other hand, produces decision rules that exhibit reasonable dynamic behavior, even when the exogenous shocks are large.[4/]

To execute the experiments on which these conclusions are based, values had to be assigned to the parameters of the model. These were chosen so as to match the model's implications for variables like the consumption to output ratio and the capital stock to output ratio with the corresponding average values in the U.S. data.

Following is an outline of the paper. Section 2 describes the model. Section 3 describes the approximation techniques used. Section 4 analyzes and compares the shock responses of the two approximation methods used. The reasons for the poor performance of the linear approximation relative to the log linear approximation method are described there. Section 5 reports the second moment implications for the two approximation methods, which are quite different.

2. Model:

Economy-wide Resource Constraint

$$C_t + K_t - (1-\delta)K_{t-1} = (z_t H_t)^{(1-\theta)} K_{t-1}^{\theta}.$$

Here, C_t--total consumption, K_t--end of period stock of capital, H_t--total hours worked, z_t--productivity shock. In per capita terms, the resource constraint is:

(1) $$c_t + k_t - [(1-\delta)/n]k_{t-1} = n^{-\theta}(z_t h_t)^{(1-\theta)} k_{t-1}^{\theta},$$

where lower case denotes per capita terms, and n is the gross (constant) rate of population growth. Time is measured in quarters.

Technology Shock

I assume:

(2a) $$z_t = z_{t-1}\exp(x_t),$$

where

(2b) $$x_t = \mu + \rho x_{t-1} + \varepsilon_t - \psi\varepsilon_{t-1}, |\rho| < 1, |\psi| \leq 1,$$

$$\varepsilon_t = 0 \quad t \leq 0.$$

Then,

(3) $$(1-\rho L)(1-L)\log z_t = \mu + \varepsilon_t - \psi\varepsilon_{t-1}.$$

When $|\psi| < 1$, this is a difference stationary representation for $\log z_t$. When $\psi = 1$, $\log z_t$ is an AR(1) about a deterministic trend:

(4) $$\log z_t = [\mu/(1-\rho)]t + (1-\rho L)^{-1}\varepsilon_t + f(t) \qquad t > 0,$$

where f(t) is deterministic, and converges to a constant for large t. (When $\rho = 0$, $f(t) = \log z_0$, all t.) In addition, for large enough t, $(1-\rho L)^{-1}\varepsilon_t$ is approximately covariance stationary.

Preferences

A representative agent orders stochastic consumption and hours streams as follows:

(5) $$E_0 \sum_{t=0}^{\infty} \beta^t \{\ln c_t - \gamma h_t\}, \ \gamma > 0.$$

Gary Hansen (1985) has shown that the linearity of hours in the representative agent's utility function can be interpreted as arising from a nonconvexity in individual agents' leisure choice set. Under Hansen's interpretation, (5) does not require that individual agents' utility functions be linear in leisure.

3. Approximate Solution:

The stand-in agent chooses contingency plans for c_t, h_t, k_t to maximize (5) subject to (1) and (2). Approximations to these contingency plans may be obtained by first transforming the problem into one in which all decision variables have a steady state.

The Transformed ("Star") Economy

Let

(6) $$c_t^* = c_t/z_t, \ k_t^* = k_t/z_t.$$

Consider the following problem: Maximize over plans for c_t^*, h_t, k_t^* contingent on x_t, k_{t-1}^*, ε_t to maximize

$$(7) \qquad E_0 \sum_{t=0}^{\infty} \beta^t \{\ln c_t^* - \gamma h_t\}$$

subject to

$$(8) \qquad c_t^* + k_t^* - [(1-\delta)/n]\exp(-x_t)k_{t-1}^*$$
$$= n^{-\theta}\exp(-\theta x_t)(k_{t-1}^*)^{\theta} h_t^{(1-\theta)}$$

and (2b). Equation (8) is just (1) divided by z_t (>0) and (7) and (5) coincide (except for an additive constant). Consequently, plans for c_t^*, k_t^*, h_t solve (7) - (8) if, and only if, the implied plans for c_t, k_t, h_t maximize (5) subject to (1) and (2).

Note that a positive innovation to productivity in the original economy is a negative innovation to productivity and a positive innovation to capital depreciation in the star economy. Therefore, one expects the optimal plan for k_t^* to respond negatively to an innovation in x_t.

The transformed economy (the "star economy") is one in which all the decision variables converge to a steady state when $x_t = \mu/(1-\rho)$, a constant for all t. For this reason, this representation of the original economy is convenient from the point of view of applying versions of the Kydland-Prescott linear quadratic approximation. Below, I describe two such versions. In each case, however, it is convenient first to substitute c_t^* out of (7) using the resource constraint, (8). This yields:

$$(9) \qquad E_0 \sum_{t=0}^{\infty} \beta^t u(k_{t-1}^*, k_t^*, h_t, x_t).$$

Denote the steady state (i.e., $\varepsilon_t \equiv 0$, $t > 0$) values of k_t^*, x_t, and h_t by k^*, x, and h, respectively. Obviously, $x = \mu/(1-\rho)$.

Two versions of the linear quadratic approximation method for getting approximate decision rules for k_t^* and h_t are described next (decision rules for c_t^* and y_t^* then follow trivially from the resource constraint, (7), and the production function.) What I call the "level method" involves approximating u about the levels of k_{t-1}^*, k_t^*, x_t, h_t and what I call the "log method" involves approximating u about the logs of k_{t-1}^*, k_t^*, h_t, and the level of x_t.[5/] These two methods are now described in turn.

The Level Method

I approximated u in (9) by U, the second order Taylor series expansion of u about $k_{t-1}^* = k^*$, $k_t^* = k^*$, $x_t = x$, and $h_t = h$. The resulting problem is easy to solve using standard methods. Denote the solutions by:

(10) $$k_t^* - k^* = \lambda(k_{t-1}^* - k^*) + d(x_t - x) + q\varepsilon_t.$$

(11) $$h_t - h = h_1(k_{t-1}^* - k^*) + h_2(x_t - x) + h_3\varepsilon_t.$$

Here, λ, d, q, h_1, h_2, h_3 are functions of the structural parameters of the model: μ, ρ, ψ, β, γ, n, θ, δ. The approximate decision rule for c_t^* is obtained by substituting (10) and (11) into (8). Gross investment in the star economy, dk_t^*, is defined as follows:

(12) $$dk_t^* = k_t^* - [(1-\delta)/n]\exp(-x_t)k_{t-1}^*.$$

Evidently, given the decision rule for k_t^* in (10), a decision rule for dk_t^* follows trivially from (12). Finally, output in the star economy is given by

(13) $$y_t^* = c_t^* + dk_t^*.$$

Notice that the trivariate, linear system formed by $[k_t^*-k^*, h_t-h, x_t-x]$ has a unique steady state--[0,0,0]--and that steady state is globally stable.

The approximate decision rules for the original economy are the following:

(14) $$k_t = \{k^*+\lambda[k_{t-1}/z_{t-1}-k^*]+d(x_t-x)+q\varepsilon_t\}z_t$$

(15) $$h_t = h + h_1[(k_{t-1}/z_{t-1})-k^*] + h_2(x_t-x) + h_3\varepsilon_t.$$

The decision rules for c_t and dk_t are then derived in an obvious way. Finally, gross output, y_t, is just $y_t^* z_t$.

It is also useful to have the risk free rate in this economy. Denote the price of a unit of capital, K_{t-1}, at the beginning of date t by P_t. If P_t is denominated in utility terms, then,

(16) $$P_t = \frac{\partial v(k_{t-1},z_{t-1},x_t,\varepsilon_t)}{\partial k_{t-1}} \frac{1}{N_{t-1}}$$

where v is the value of the optimal plan when the state is k_{t-1}, z_{t-1}, x_t, ε_t, and where $N_{t-1}(=[\partial k_{t-1}/\partial K_{t-1}]^{-1})$ is the population at date t-1. In particular:

(17) $$v(k_{t-1},z_{t-1}x_t,\varepsilon_t) = \max_{k_t,h_t}\{u(k_{t-1}/z_{t-1},k_t/z_t,h_t,x_t) + \beta E_t v(k_t,z_t,x_{t+1},\varepsilon_{t+1})\}.$$

Then, the risk free rate of interest in this economy is

(18) $$1 + r_t = P_t/(E_t\beta P_{t+1}).$$

In steady state, this is $n[\exp(x)/\beta]$, where n is the gross rate of population growth, x is the rate of growth in per capita consumption, and β is the discount factor. By the envelope theorem,

$$(19) \qquad \partial v(k_{t-1},z_{t-1},x_t,\varepsilon_t)/\partial k_{t-1} = \partial u(k_{t-1}/z_{t-1},k_t/z_t,h_t,x_t)/\partial k_{t-1}$$

$$= [\partial u(k^*_{t-1},k^*_t,h_t,x_t)/\partial k^*_{t-1}]\partial k^*_{t-1}/\partial k_{t-1}$$

$$= [\partial u(k^*_{t-1},k^*_t,h_t,x_t)/\partial k^*_{t-1}]/z_{t-1},$$

with the second and third arguments of u evaluated at the optimal plan. (The function u is implicitly defined in [9].) I approximated $\partial u/\partial k^*_{t-1}$ by $\partial U/\partial k^*_{t-1}$. This completes the discussion of solving the model using the level method.

The Log Method

Define:

$$(20) \qquad lk^*_t = \log k^*_t,\ lh_t = \log h_t,$$

$$lk^* = \log k^*,\ lh = \log h,$$

where log denotes the natural logarithm. Using this notation, write,

$$(21) \qquad u(k^*_{t-1},k^*_t,h_t,x_t) = u(\exp(lk^*_{t-1}),\exp(lk^*_t),\exp(lh^*_t),x_t),$$

$$\equiv lu(lk^*_{t-1},lk^*_t,lh^*_t,x_t).$$

Define the following optimization problem. Maximize over plans for lk^*_t, and lh^*_t, contingent on lk^*_{t-1}, x_t, and ε_t, the criterion:

$$(22) \qquad E_0 \sum_{t=0}^{\infty} \beta^t lu(lk^*_{t-1},lk^*_t,lh^*_t,x_t).$$

It is easy to see that the exponential of plans that solve (22) solve (9). Also, the log of plans that solve (9) solve (22). Thus, (9) and (22) are equivalent representations of the star economy.

The method of logs involves replacing lu in (22) by LU, the second order Taylor series expansion of lu about $lk^*_{t-1} = lk^*$, $lk^*_t = lk^*$, $x_t = x$, and $lh_t = lh$. The resulting optimization problem is easy to solve since it is quadratic. Denote its solution by:

(23) $$lk^*_t - lk^* = l\lambda\left(lk^*_{t-1} - lk^*\right) + ld(x_t - x) + lq\varepsilon_t$$

(24) $$lh_t - lh = lh_1\left(lk^*_{t-1} - lk^*\right) + lh_2(x_t - x) + lh_3\varepsilon_t.$$

The parameters, $l\lambda$, ld, lq, lh_1, lh_2, lh_3, are functions of the structural parameters of the model. The approximate decision rule for c^*_t is obtained by exponentiating lk^*_t and lh_t and substituting the result into (8). The gross investment rule, dk^*_t, is derived using (12) after obtaining k^*_t and k^*_{t-1} by exponentiating lk^*_t and lk^*_{t-1}, respectively.

The approximate decision rules for the original economy are the following:

(25) $$k_t = z_t \exp\{lk^* + l\lambda[\log(k_{t-1}/z_{t-1}) - lk^*] + ld(x_t - x) + lq\varepsilon_t\}$$

(26) $$h_t = \exp\{lh + lh_1[\log(k_{t-1}/z_{t-1}) - lk^*] + lh_2(x_t - x) + lh_3\varepsilon_t\}.$$

Note that these decision rules express $\log k_t$ and $\log h_t$ as linear functions of $\log k_{t-1}$, $\log z_t$, x_t and ε_t. Consequently, the rules for k_t and h_t in (25) and (26) are monotone in k_{t-1}, z_t, x_t, and ε_t. This is a property probably shared by the true decision rules, but not by the approximate ones generated by the level method. On this dimension, then, the log method probably produces a superior approximation to the correct decision rule.

Next, we discuss the computation of the risk free return in this economy. The discussion in (17) - (19) remains valid. However, in the present context, it is convenient to make the following change to (19) using (21):

$$(27) \quad \partial v(k_{t-1},z_{t-1},x_t,\varepsilon_t)/\partial k_{t-1} = \partial u(k_{t-1}/z_{t-1},k_t/z_t,h_t,x_t)/\partial k_{t-1}$$

$$= \partial lu[\log(k_{t-1}/z_{t-1}),\log(k_t/z_t),\log(h_t),x_t]/\partial k_{t-1}$$

$$= [\partial lu(lk^*_{t-1},lk^*_t,lh_t,x_t)/\partial lk^*_{t-1}]\partial lk^*_{t-1}/\partial k_{t-1}$$

$$= [\partial lu(lk^*_{t-1},lk^*_t,x_t)/\partial lk^*_{t-1}]/k_{t-1},$$

where the second and third arguments of lu are evaluated at the optimal plan. The function $\partial lu/\partial lk^*_{t-1}$ was approximated by $\partial LU/\partial lk^*_{t-1}$. The risk free rate was then obtained by substituting the approximation to $\partial v/\partial k_{t-1}$ from (27) into (16) and (18).

4. Shock Response of the Approximate Decision Rules

This section describes the shock responses of the approximate decision rules obtained by the level method and by the log method. First, I present the parameter values used for the shock response experiments, and the resulting steady state properties. These are deduced without the use of approximations. Then I report the decision rules implied by these parameter values. Finally, the shock response functions themselves are reported. It is shown that the level method produces shock responses that are perverse, even for shocks of reasonable magnitude. In contrast, the log method produces "reasonable" shock responses, even for large shocks.

Parameter Values and Mean Properties of the Model

I chose the following parameter values for the model:

$$\rho = 0,\ \mu = .003589,\ \gamma = .00263,\ n = 1.00324 \tag{28}$$

$$\beta = .99,\ \delta = .018,\ \theta = .39,\ \psi = 0.$$

These parameter values imply the average values indicated in Table 1.[6]

The Decision Rules

The parameters values in (28) imply the following decision rules. The approximate decision rules yielded by the level method are:

$$k_t = z_t\{16790.92 + .949[(k_{t-1}/z_{t-1}) - 16790.92] - 15932.87(x_t - .003589)\} \tag{29}$$

$$h_t = 320.68 - .0087[(k_{t-1}/z_{t-1}) - 16790.32] + 145.31(x_t - .003589). \tag{30}$$

The approximate decision rules yielded by the log method are:

$$k_t = z_t\exp\{9.729 + .949[\log(k_{t-1}/z_{t-1}) - 9.729] - .949(x_t - .003589)\} \tag{31}$$

$$h_t = \exp\{10.18 - .453[\log(k_{t-1}/z_{t-1}) - 9.729] + .453(x_t - .003589)\}. \tag{32}$$

The reduced form parameters themselves are:

$$\lambda = .949,\ k^* = 16790.92,\ d = -15932.87,\ q = 0 \tag{33}$$

$$h = 320.68,\ h_1 = -.0087,\ h_2 = 145.31,\ h_3 = 0,\ x = .003589,$$

$$l\lambda = .949,\ lk^* = 9.729,\ ld = -.949,\ lq = 0,$$

$$lh = 10.18,\ lh_1 = -.453,\ lh_2 = .453,\ lh_3 = 0.$$

Note from the sign of d and ld, that capital in the star economy responds negatively to an innovation in technology in the original economy. This is because, as noted above, a positive innovation in technology in the original economy implies a drop in productivity and a rise in the rate of capital depreciation in the star economy. Hours respond positively to a positive innovation in technology (see h_2 and lh_2), presumably reflecting the dominant effect of the jump in capital depreciation. (Steady state hours is an increasing function of δ, at least in a neighborhood of the parameter values in [28].)

Shock Response of Approximate Decision Rules Obtained by Level Method

Following is a discussion of the shock response behavior of (29) and (30). These are the approximate decision rules obtained using the level method. Casual inspection of (29) reveals that decision rule's capacity to generate perverse behavior. The reason lies in the fact that it is the product of a linear function (k_t^*) of the technology innovation and an exponential function of the same innovation (z_t), i.e., $k_t = k_t^* z_t$. The linear form of k_t^* and the signs of its parameters have the effect that the further k_{t-1}^* is below k^*, and/or the more positive the shock ε_t, the greater, in percentage terms, the negative effect of ε_t on k_t^*.[7/] On the other hand, the effect on z_t of a shock is a constant, in percent terms. Therefore, for an initial capital stock that is far below a steady state growth path, or a sufficiently large positive shock, the response of capital in the approximate decision rule will perversely be negative. Since the response in (29) is positive for small shocks, it follows in addition that the approximate decision rule, (29), is not monotone in ε_t. Since the true decision rule for capital probably is monotone in ε_t, the non-monotone behavior of (29) must reflect approximation error.

By themselves the preceding observations are perhaps not surprising, since we know that for sufficiently large shocks, any approximation must eventually break down. The real issue is whether the considerations raised above are significant relative to the magnitude of variation in the data. Below I supply evidence that they are significant. Fortunately the approximate decision rules obtained by the log method do not share the deficiencies of (29) and (30).

I begin by formalizing the preceding argument. I focus the discussion on the decision rule for capital investment, rather than the rule for the capital stock. In the star economy, the decision rule for capital investment is given by

(34) $$dk_t^* = k^* + \lambda(k_{t-1}^* - k^*) + d(x_t - x) + q\varepsilon_t - [(1-\delta)/n]k_{t-1}^*$$

$$= k^* + \lambda(k_{t-1}^* - k^*) + d\rho(x_{t-1} - x) + (d+q)\varepsilon_t$$

$$- [(1-\delta)/n]\exp[-x - \rho(x_{t-1} - x) - \varepsilon_t]k_{t-1}^*,$$

where $x_t - x = \rho(x_{t-1} - x) + \varepsilon_t$ has been used (for now, $\psi = 0$). Also,

(35) $$z_t = z_{t-1}\exp[x + \rho(x_{t-1} - x) + \varepsilon_t].$$

From this, we get

$$(36) \qquad \partial dk_t/\partial\varepsilon_t = z_t \partial dk_t^*/\partial\varepsilon_t + dk_t^* \partial z_t/\partial\varepsilon_t$$

$$= z_t[k^* + d + \lambda(k_{t-1}^* - k^*) + d\rho(x_{t-1} - x) + d\varepsilon_t].$$

This expression shows that if the capital stock is far below a steady state path (eg., $k_{t-1}^* - k^*$ very negative), or if ε_t is sufficiently big and positive, the approximate decision rule could imply a negative reaction of investment to a favorable productivity shock. To get an idea of the numerical importance of this, several shock response functions were graphed. These appear in Figures 1 - 6.

First, I discuss Figures 1 - 3. Each figure reports results based on two simulations of length 112 ("period 1" through "period 112"). In the first simulation, paths for c_t, dk_t, h_t, y_t were calculated assuming the economy started on a steady state growth path and $\varepsilon_t = 0$. In particular, I chose an initial ("period 0") z such that the resulting k/z was 16790.2 and k was equal to the U.S. capital stock in 1956,1 (= 52010.09, in 1982 dollars). The second sample path was identical to the first for periods 1 and 2, but in period 3 ε_t is non-zero, and takes on the value indicated in the figures. The paths reported in Figures 1 - 3 represent 100 times the log deviation of the shocked path from the unshocked one. The measured standard deviation of ε_t using U.S. data (1956,2 - 1984,1) and the parameter values in (28) is about .019. This is a useful benchmark to keep in mind.

Figure 3a shows the response of the approximate decision rules to a very small shock, .0019. This response seems "reasonable", with capital investment responding strongly, output jumping very quickly to its new, higher steady state growth path due to an assist from greater work effort. Note the damped response of consumption to the shock. The motive to smooth consumption (eg., the income effect) which derives from the diminishing marginal utility in the period utility function is offset somewhat by the motive to invest when the returns are high ("the substitution effect"), so that consumption does not jump immediately to its new, higher steady state growth path. It is noteworthy that the accompanying interest rate move is very tiny. (This is not reported in the figure.) In particular, the interest rate jumps by an incredibly small .006 basis points in period 3 and slowly returns to its steady state value of 1.017 (= 7 percent AR). Figure 3b displays the response of the model to a shock of -.0019 in period 3. The response is the mirror image of the one in Figure 3a, reflecting the approximate linearity of the model for such small shocks.

Figure 2a shows the response to a shock of .019. For the most part, Figure 2a is just Figure 3a scaled up by 10. An exception is given by the first period, in which the surge in capital investment is not quite so strong, resulting in an extra boost to consumption (this shows up as a small "hook" in consumption's impulse response).

Equation (36) leads us to expect that with a bigger shock, the negative impact on k_t^* begins to play a larger role in determining the effect on k_t. This is confirmed by Figure 1a, which shows the effect of a very strong shock in period 3. There the "hook" in consumption's response has grown so large that consumption jumps more than output in the period of the shock. Of course, it follows that capital investment--perversely--must fall in this period.

Figures 2b and 1b show, respectively, the response of the approximate decision rules to a moderate and strong negative technology shock. Those figures suggest that the first period response of capital to this shock is probably too large, indeed with a shock = -.19 the negative effect on investment is so strong that consumption perversely rises initially.

Thus, examination of Figures 1 - 3 suggests, not surprisingly, that the level method of approximation is inaccurate for large shocks. On the other hand, there is not much evidence in these figures of a breakdown in accuracy for shocks of "reasonable" magnitude, i.e., ± .019. A curious feature of all the impulse responses is that when things do seem to go wrong, it is principally in the first period of the response. In addition, "problems", when they do occur, seem to be in the response of consumption and capital investment. The response of hours always looks reasonable, even for large shocks. A consequence of this is that output also behaves reasonably, since in addition to hours, its only other argument is the stock of capital, which cannot move much on short notice.

Unfortunately, the fact that the shock response function is apparently well behaved for shocks of reasonable magnitude when the system is on a steady state growth path does not rule out perverse behavior. As equation (36) suggests, shocks of reasonable magnitude can induce a perverse reaction if they "hit" when the system is far enough below a steady state growth path. Figures 4a - 5b illustrate this possibility. In these figures the baseline path is above or below a steady state growth path, as indicated. The deviations from the growth path considered are small in that deviations even greater than this were observed when the system was hit by a sequence of 112 independent, normal shocks with mean zero and standard deviation .019. Like before, in each case, it is the initial response to the shock that seems "perverse", and the shock response function one period after the shock and later seems "reasonable". Note, interestingly, that in most of the figures, the initial reaction of consumption to the shock is greater than is the reaction of income. This suggests that consumption fluctuates more in a system like this than does income. In addition, since consumption seems to "correct" itself after the initial response, one anticipates some negative serial correlation in the growth rate of consumption, or at least some high frequency movements.

Figures 6a - 6d illustrate the response of the system to small shocks, when away from the steady state. (The magnitude of the deviation from steady state was chosen after observing the response of the system to a sequence of 112 shocks and noticing that the upper bound of the deviation from steady state was about 1.8 percent). For the most part, the shock response functions in these figures look "reasonable", suggesting that--for the indicated shock magnitude--the approximate solutions mimick closely the exact solutions. (Of course, as the shocks become smaller, we can claim with confidence that the approximate solutions become arbitrarily accurate.)

Shock Response of Approximate Decision Rules Obtained by Log Method

In the brief discussion after equation (26), it was noted that the decision rules obtained by the log method are monotone in their arguments. The reason, of course, is that an innovation to technology in this case has a constant percent effect on k_t^*, in addition to z_t. Thus, this method seems to avoid the major deficiency of the level method. To confirm this, I plotted in Figures 7 and 8 the shock response of these decision rules in circumstances that produced the most perverse behavior by the decision rules computed using the level method. The dramatically suspicious responses we observed in the latter decision rules do not occur here. For example, Figures 7a and 7b show the response to a large shock when the system starts in a steady state growth path. The pattern there looks quite similar, except for scale, to the response of the decision rules derived by the level method to small shocks, as in Figures 3a and 3b. In addition Figures 8a and 8b depict the response of the decision rules to a shock of reasonable magnitude, starting from a capital stock below the steady state growth path. Those response patterns also closely mimick the response in Figures 3a and 3b (except for scale), and seem "reasonable". They should be compared with the bizarre response patterns plotted in Figures 4a and 5a. I conclude that there is no evidence of suspicious behavior in the approximate decision rules derived using the log method, even for large shocks. This stands in striking contrast with the results on the decision rules derived using the level method.

5. Second Moment Properties of the Model

This section presents some second moment properties of the variables in the approximate solutions to the model. The variables I look at are consumption, capital investment, hours, output, and the risk free rate. Since some of the data generated by the model are not stationary, transformations have to be made to induce (at least approximate) covariance stationarity. Without this, sample moments are not meaningful. Two transformations were used. The first is the one proposed by Hodrick and Prescott ( ), and I label this the "HP data transformation". This involves first logging all variables except the interest rate, and then computing their deviations from a trend line.[8/] The other data transformation--called the "growth transformation"--used exploits the fact that c_t, dk_t, y_t are the product of a covariance stationary process and z_t. Thus, the log first difference of these three variables is covariate stationary. Under the growth transformation the log first difference of c_t, dk_t, and y_t were computed, and h_t, r_t where left untransformed. The latter two are predicted by the model to be covariance stationary.

Results are presented both for the level and log methods of approximation. The results are consistent with what we saw in the shock response functions. In particular, Table 2a shows that for shocks of reasonable size (eg., with standard deviation .019) consumption is more variable than output when the growth transformation is used. In addition, in this case the autocorrelation of consumption growth is negative. This is what one would have anticipated given the sharp "hook" evident in the shock response functions. For small and "tiny" (eg., .00019) shocks the variance of consumption is about half that of output, which is also to be expected given the shock response functions for small shocks. The relative volatility statistics for the HP transform seem slightly less sensitive the bizarre shape of the shock response functions for the level approximation method. Nevertheless, for shocks of reasonable magnitude, the volatility of consumption is still almost that of output, but falls substantially with a fall in σ_ε. Tables 3a and 3b report the results for the log method of approximation. The results there also mirror what we saw in the shock response functions. In particular, the relative volatilities for reasonably sized shocks are roughly the same as what they are for small and tiny shocks. I conjecture, based on the reasoning in the preceding section, that that reflects the greater accuracy of that approximation.

Table 1: Averages

Variable	Model	U.S. Economy (1956,3 - 1984,1)
c_t/y_t	.72	.72
k_t/y_t	11.21	10.58
h_t	320.7	320.4
$(c_t-c_{t-1})/c_{t-1}$	.0036	.0039
$(y_t-y_{t-1})/y_{t-1}$	.0036	.0041
$(h_t-h_{t-1})/h_{t-1}$	.0	.00051
$(k_t-k_{t-1})/k_{t-1}$	.0036	.0046
r_t	1.017	1.010

Table 2a: Results for Growth Transformation Based on
Decision Rule Calculated Using Level Method*

σ_ε	$\frac{\sigma_c}{\sigma_y}$	$\frac{\sigma_{dk}}{\sigma_y}$	$\frac{\sigma_r}{\sigma_y}$	$\frac{\sigma_h}{\sigma_y}$	$\rho_c(1)$
.019	1.407	4.363	.094	411.6	-.417
	(.173)	(2.16)	(.0006)	(10742.60)	(.102)
.0019	.508	2.38	.088	411.76	.041
	(.003)	(.024)	(.0005)	(10760.03)	(.146)
.00019	.491	2.35	.088	411.76	.063
	(.0001)	(.0004)	(.0005)	(10759.84)	(.106)

*Based on 1000 simulations, each of length 112. Initial condition for each simulation: steady state k/z and initial k = 52010.09. Column 1: standard error of ε_t in (2b). Column 2: average across simulations of ratio of standard deviation of detrended consumption to standard deviation of detrended income (σ_y). Column 3: average across simulations of ratio of detrended capital investment to σ_y. Column 4: average across simulations of ratio of standard across simulations of ratio of standard deviation of hours to σ_y. Column 6: average across simulations of first order autocorrelation of detrended consumption. Numbers in parentheses: standard error across simulations.

Table 2b: Results for HP Transformation Based on Decision Rule Calculated Using Level Method*

σ_ε	$\frac{\sigma_c}{\sigma_y}$	$\frac{\sigma_{dk}}{\sigma_y}$	$\frac{\sigma_r}{\sigma_y}$	$\frac{\sigma_h}{\sigma_y}$	$\rho_c(1)$
.019	.879	3.13	.044	.516	.207
	(.044)	(.524)	(.00005)	(.00004)	(.158)
.0019	.511	2.34	.035	.515	.709
	(.0003)	(.0037)	(.0000008)	(.00004)	(.080)
.00019	.505	2.34	.035	.515	.723
	(.0001)	(.0004)	(.0000002)	(.00004)	(.067)

*See note to Table 2a for all except columns 4 and 5. Column 4: ratio of standard deviation of detrended interest rate to σ_y. Column 5: ratio of standard deviation of detrend hours to σ_y.

Table 3a: Results for Growth Transformation Based on Decision Rule Calculated Using Log Method*

σ_ε	$\frac{\sigma_c}{\sigma_y}$	$\frac{\sigma_{dk}}{\sigma_y}$	$\frac{\sigma_r}{\sigma_y}$	$\frac{\sigma_h}{\sigma_y}$	$\rho_c(1)$
.019	.493	2.40	.088	412.04	.063
	(.0010)	(.028)	(.0005)	(10870.48)	(.107)
.0019	.491	2.35	.088	411.77	.064
	(.0011)	(.00043)	(.0005)	(10764.94)	(.106)
.0019	.491	2.35	.088	411.76	.064
	(.00010)	(.0002)	(.0005)	(10760.25)	(.106)

*See note to Table 2a.

Table 3b: Results for HP Transformation Based on Decision Rule Calculated Using Log Method*

σ_ε	$\frac{\sigma_c}{\sigma_y}$	$\frac{\sigma_{dk}}{\sigma_y}$	$\frac{\sigma_r}{\sigma_y}$	$\frac{\sigma_h}{\sigma_y}$	$\rho_c(1)$
.019	.507	2.38	.035	.515	.723
	(.0011)	(.031)	(.0000006)	(.00004)	(.067)
.0019	.506	2.34	.035	.515	.723
	(.00012)	(.0006)	(.0000002)	(.00004)	(.067)
.00019	.506	2.33	.035	.515	.723
	(.00011)	(.0004)	(.0000002)	(.00004)	(.067)

*See note to Table 2b.

Table 5: Results for Growth (HP) Transformation
Based on Actual U.S. Data, 1956 Q2 - 1984 Q4*

$\frac{\sigma_c}{\sigma_y}$	$\frac{\sigma_{dk}}{\sigma_y}$	$\frac{\sigma_r}{\sigma_y}$	$\frac{\sigma_h}{\sigma_y}$	$\rho_c(1)$
.487	1.920	2.243	699.59	.271
(.407)	(2.188)	(.788)	(.840)	(.820)

*Note: First row corresponds to results based on growth transformation. Second row corresponds to results based on HP transformation. For columns in row one, see note to Table 2a. For columns in row two, see note to Table 2b.

Footnotes

1/For recent work that pursues alternative approximate solution strategies to the one described here, see Gagnon and Taylor (1986), Labadie (1986), and Sims (1984).

2/For papers which use this approximation method, see Christiano (1986) and G. Hansen (1986).

3/A "productivity shock" is defined relative to the model of this paper, in which output is produced using a Cobb-Douglas function of capital and labor, and a multiplicative productivity shock. Once parameters are assigned to the Cobb-Douglas production function--as I must to carry out the experiments in the paper--then the productivity shock can be computed directly from the data as in Solow (1956) and Prescott (1986). A "large" productivity shock is an innovation to productivity that is ten times the standard deviation of the measured innovation to productivity. A "small" productivity shock is one tenth the standard deviation observed in the data.

4/Since I do not know the properties of the true decision rules, it is hard to be certain about what constitutes "plausible" and "implausible" behavior on the part of the approximate decision rules. Statements of this kind are based on conjecture about the true decision rules. I hope to make this conjecture rigorous in subsequent drafts. For example, it may be possible to show that the true decision rules are monotone functions of their arguments. In this case, my finding that in the linear approximation, investment responds positively to small productivity shocks, and negatively to larger ones would be shown to be an artifact of approximation error.

5/Actually, what I call the "level method" is really a combination of level method and the log method, to be described subsequently in the text. It is a combination because, in calculating the approximation around x_t, the level method in fact is approximating about the log of z_t/z_{t-1}. The fact that the "level method" involves a mixture of approximating about logs and levels is at the root of its problems, which will be documented below.

6/The formulas used to calculate the results in Table 1 were derived from the first order necessary conditions for an optimum in steady state. A value of $\rho = 0$ was chosen to simplify the exposition. In fact the parameter values in (28) together with U.S. data suggest a value of ρ in the neighborhood of -.1.

7/In particular, the greater $|\partial \log k_t^* / \partial \varepsilon_t|$, where $|\ |$ denotes the absolute value.

8/On each simulation, the number of observations on each variable was 112. A trend line was computed for a given vector of 112 observations by multiplying it by a particular 112 by 112 matrix. For details about the construction of this matrix, see Hodrick and Prescott ( ).

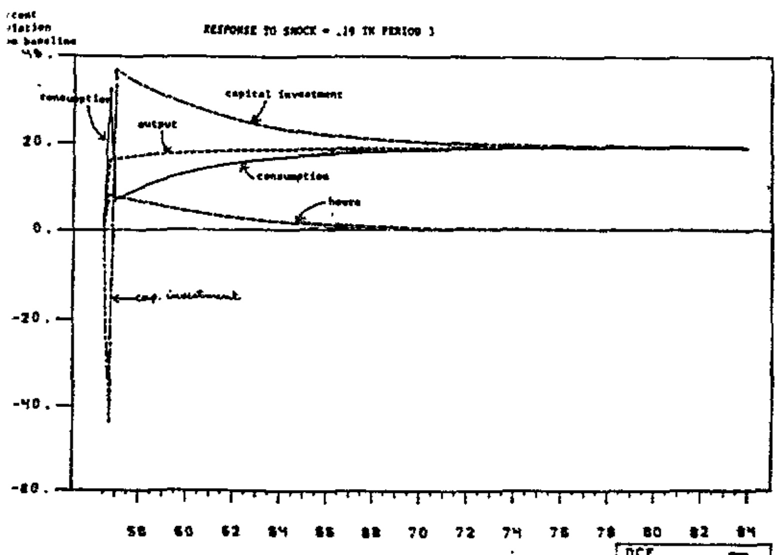


Fig. 1a

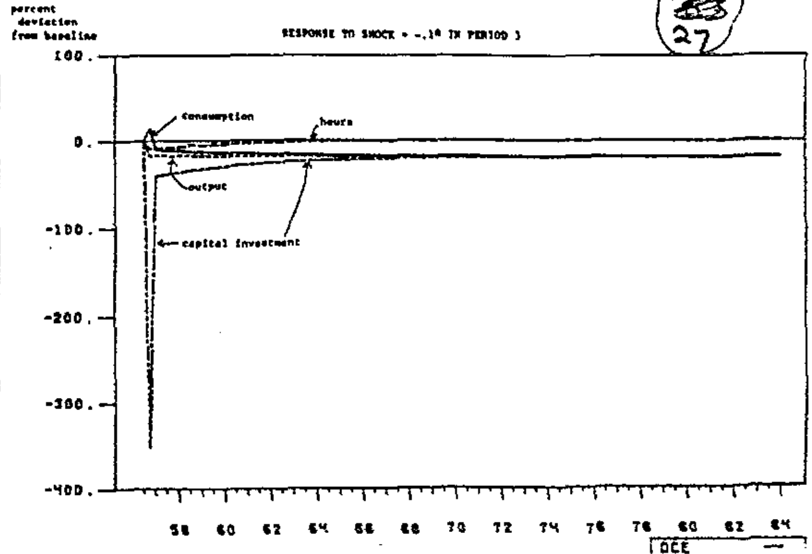


Fig. 1b

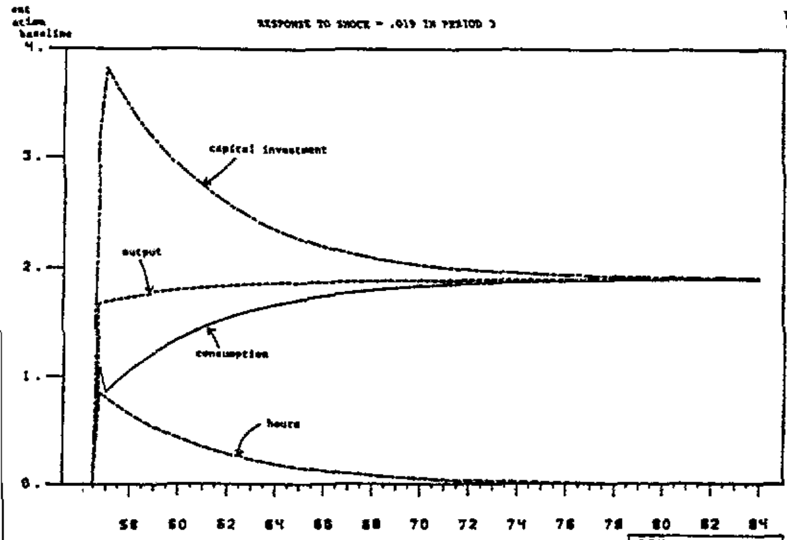


Fig. 2a

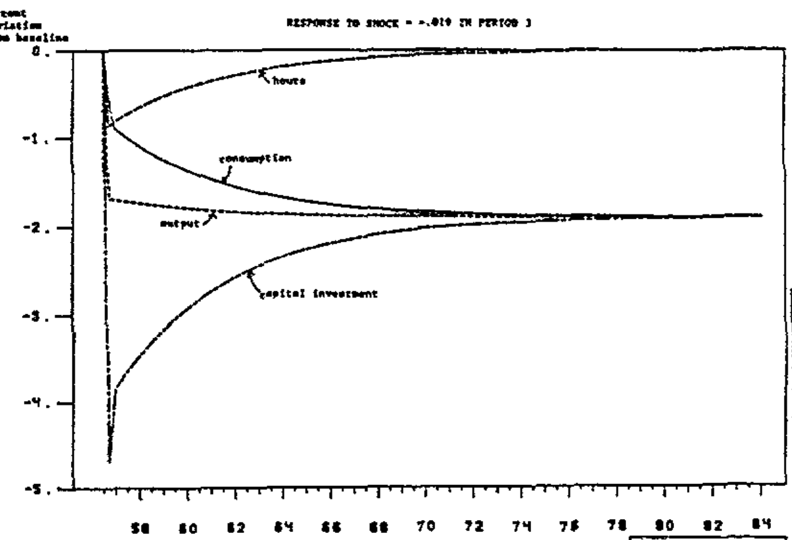


Fig. 2b

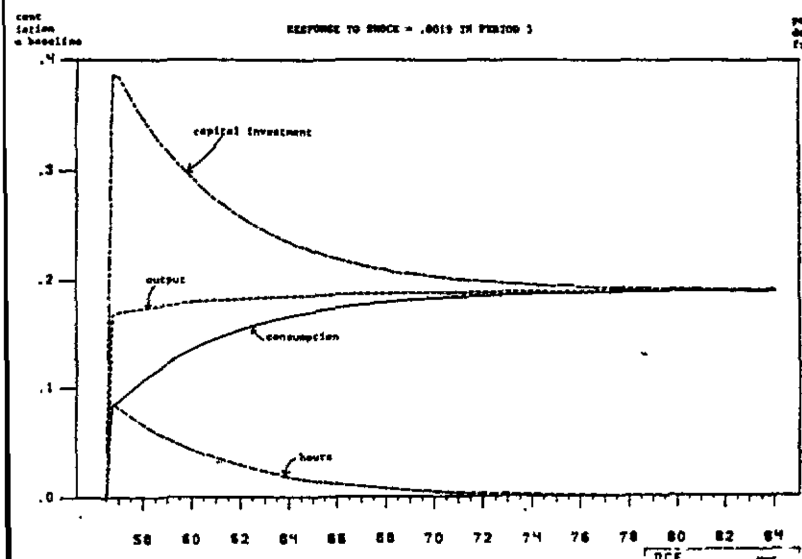


Fig. 3a

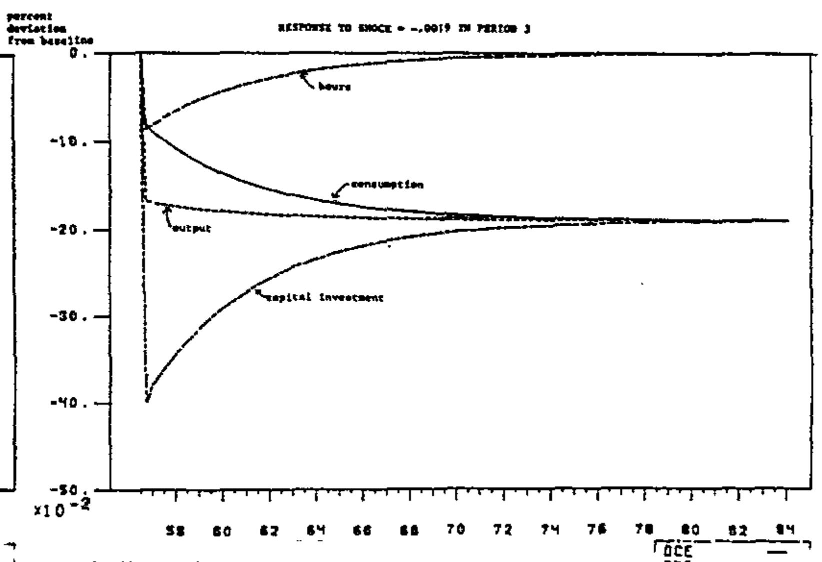
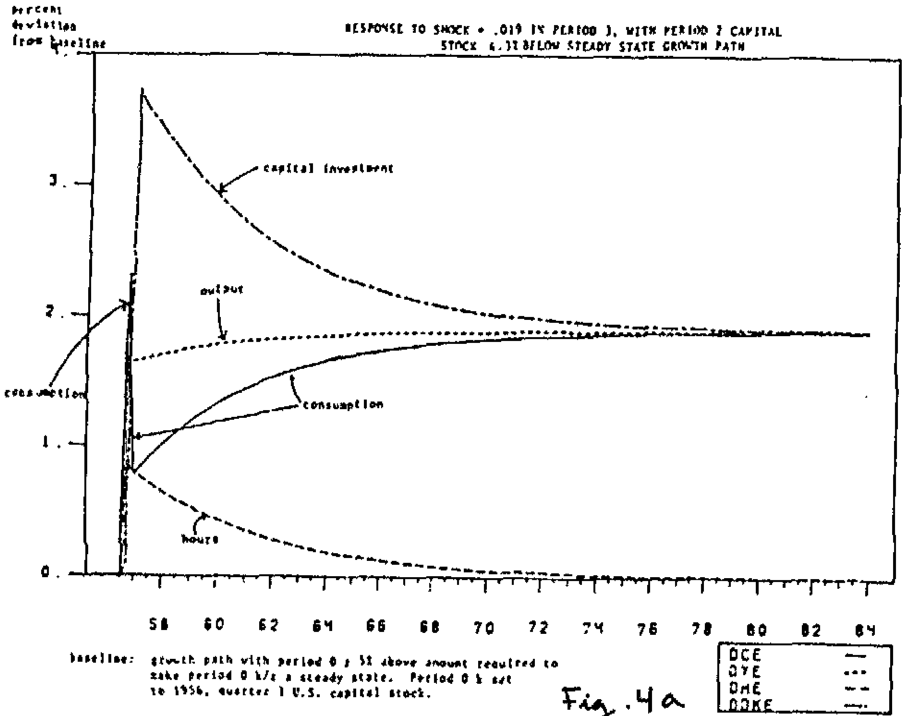
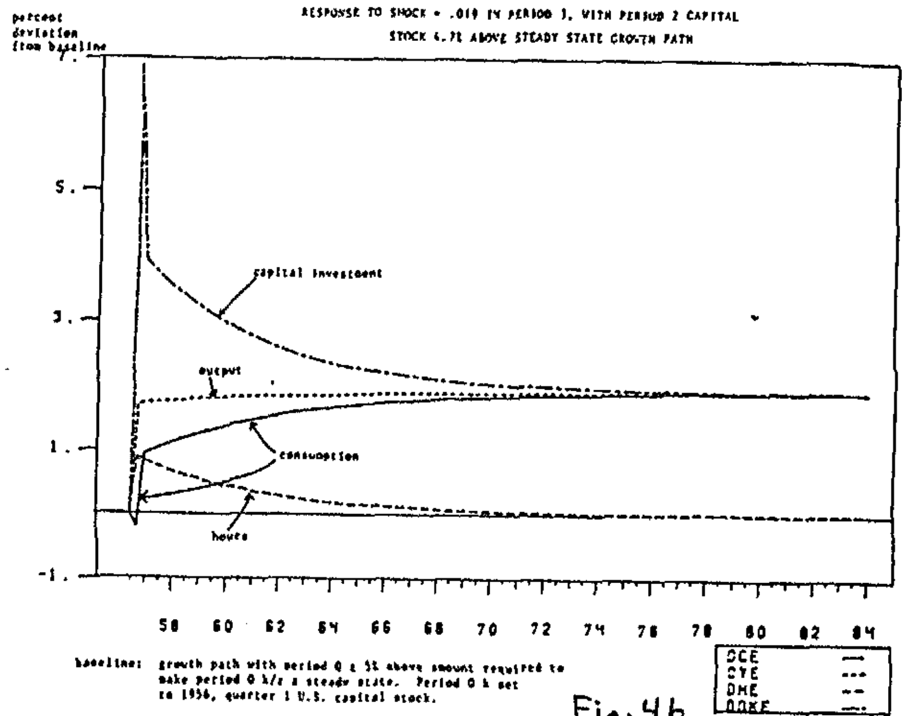


Fig. 3b



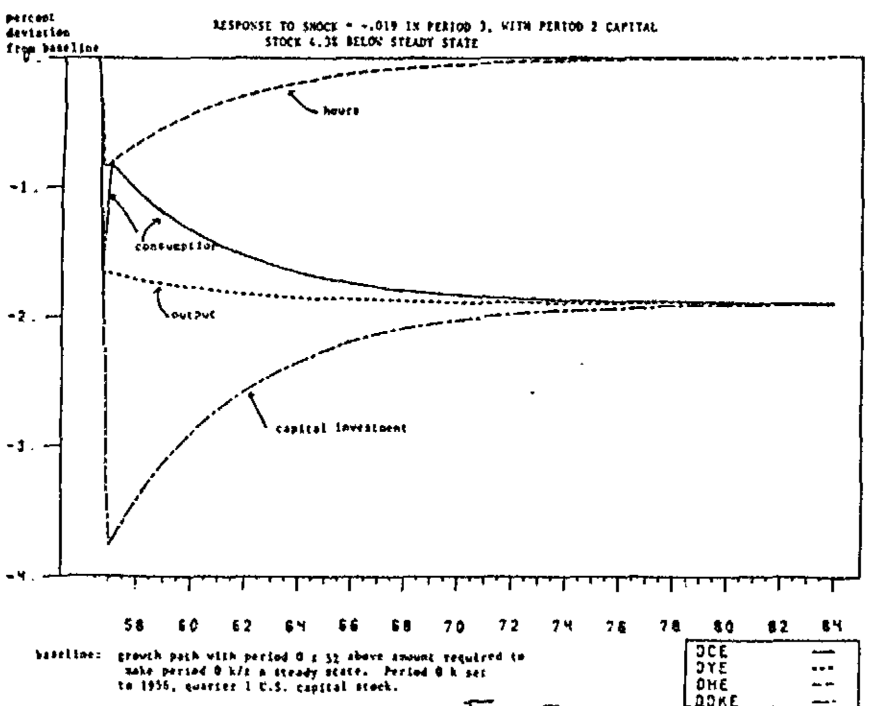
baseline: growth path with period 0 k 5% above amount required to make period 0 k/z a steady state. Period 0 k set to 1956, quarter 1 U.S. capital stock.

Fig. 4a



baseline: growth path with period 0 k 5% above amount required to make period 0 k/z a steady state. Period 0 k set to 1956, quarter 1 U.S. capital stock.

Fig. 4b



baseline: growth path with period 0 k 5% above amount required to make period 0 k/z a steady state. Period 0 k set to 1956, quarter 1 U.S. capital stock.

Fig. 5a

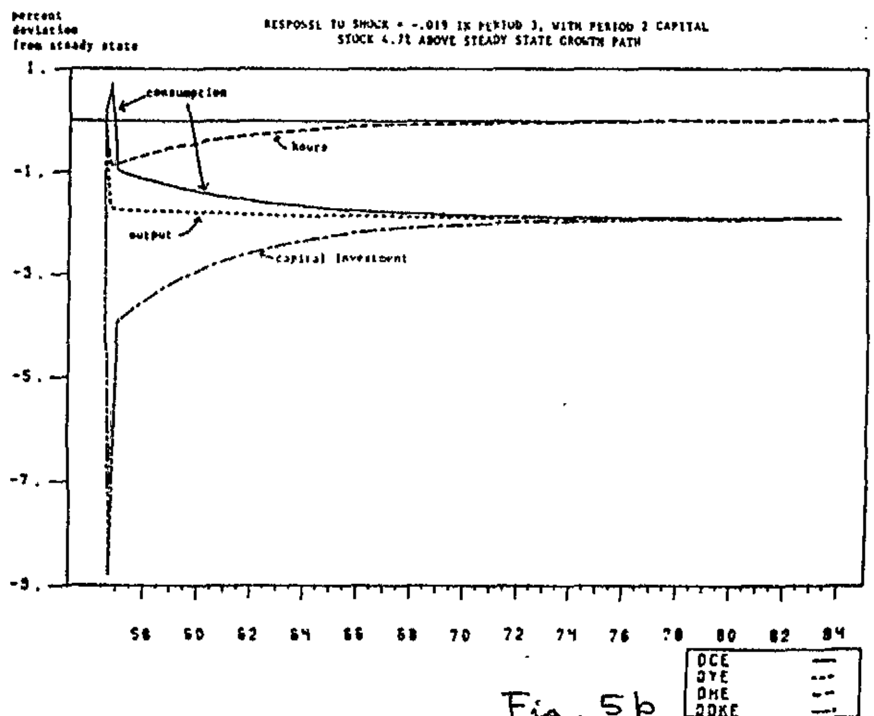


Fig. 5b

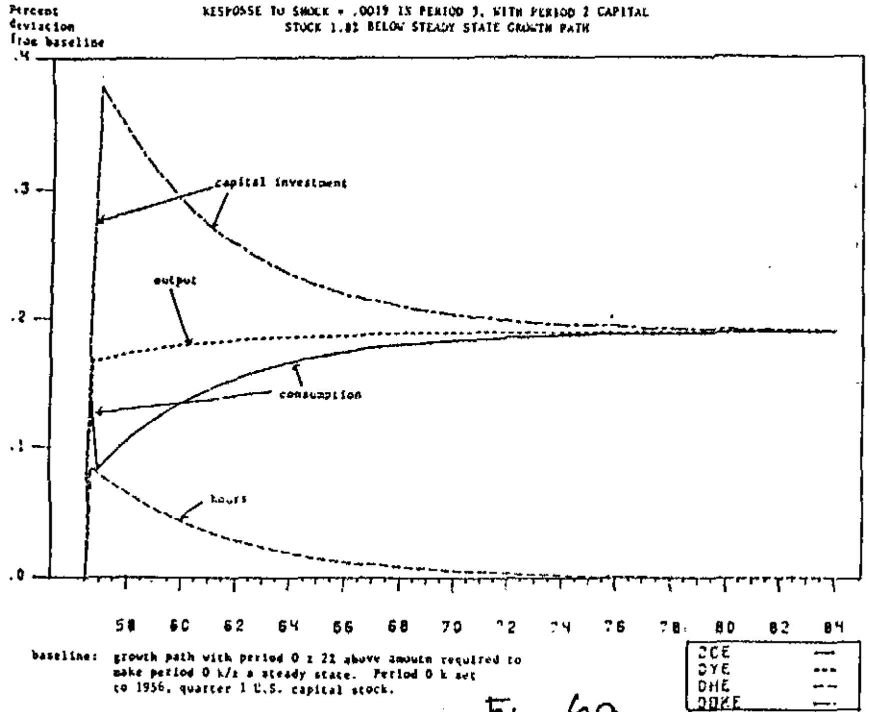


Fig. 6a

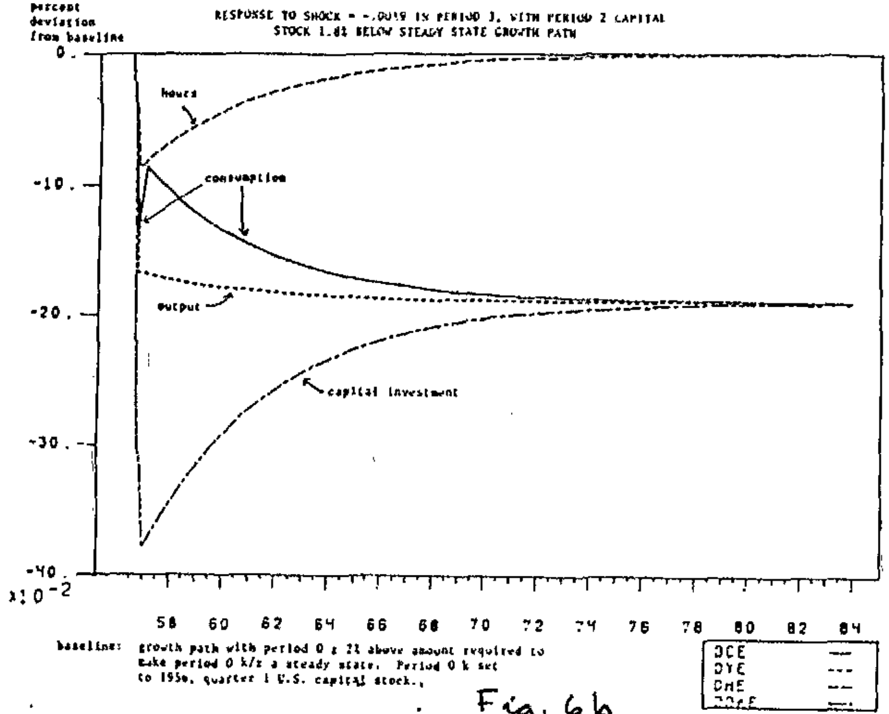


Fig. 6b

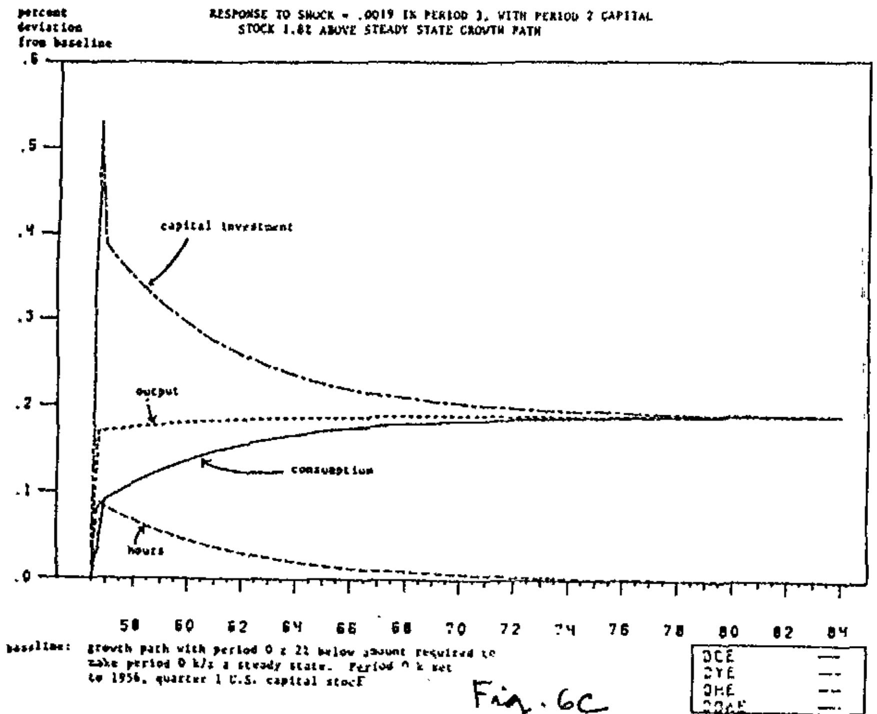


Fig. 6c

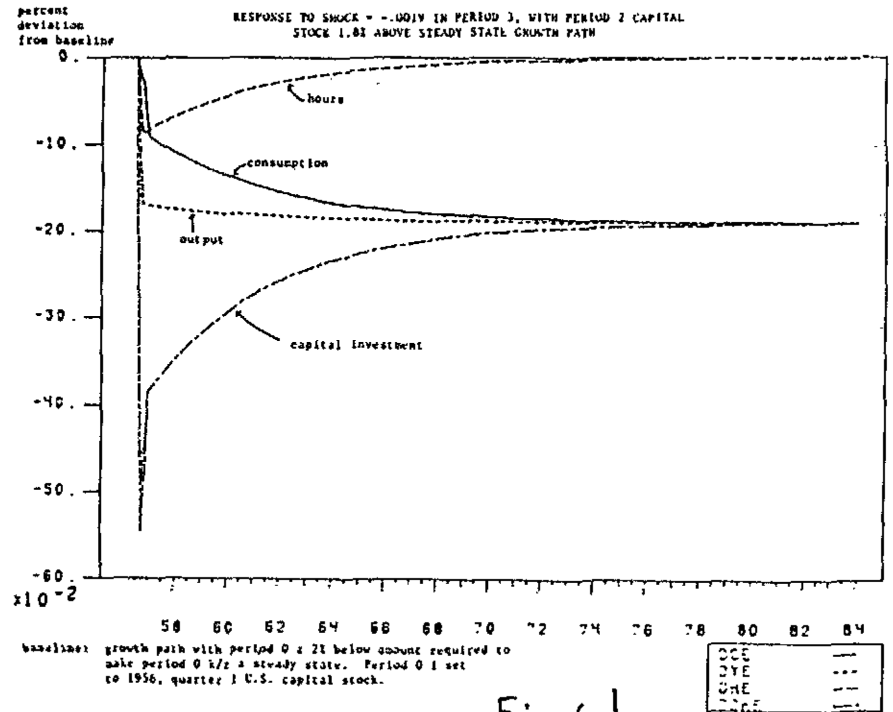


Fig. 6d

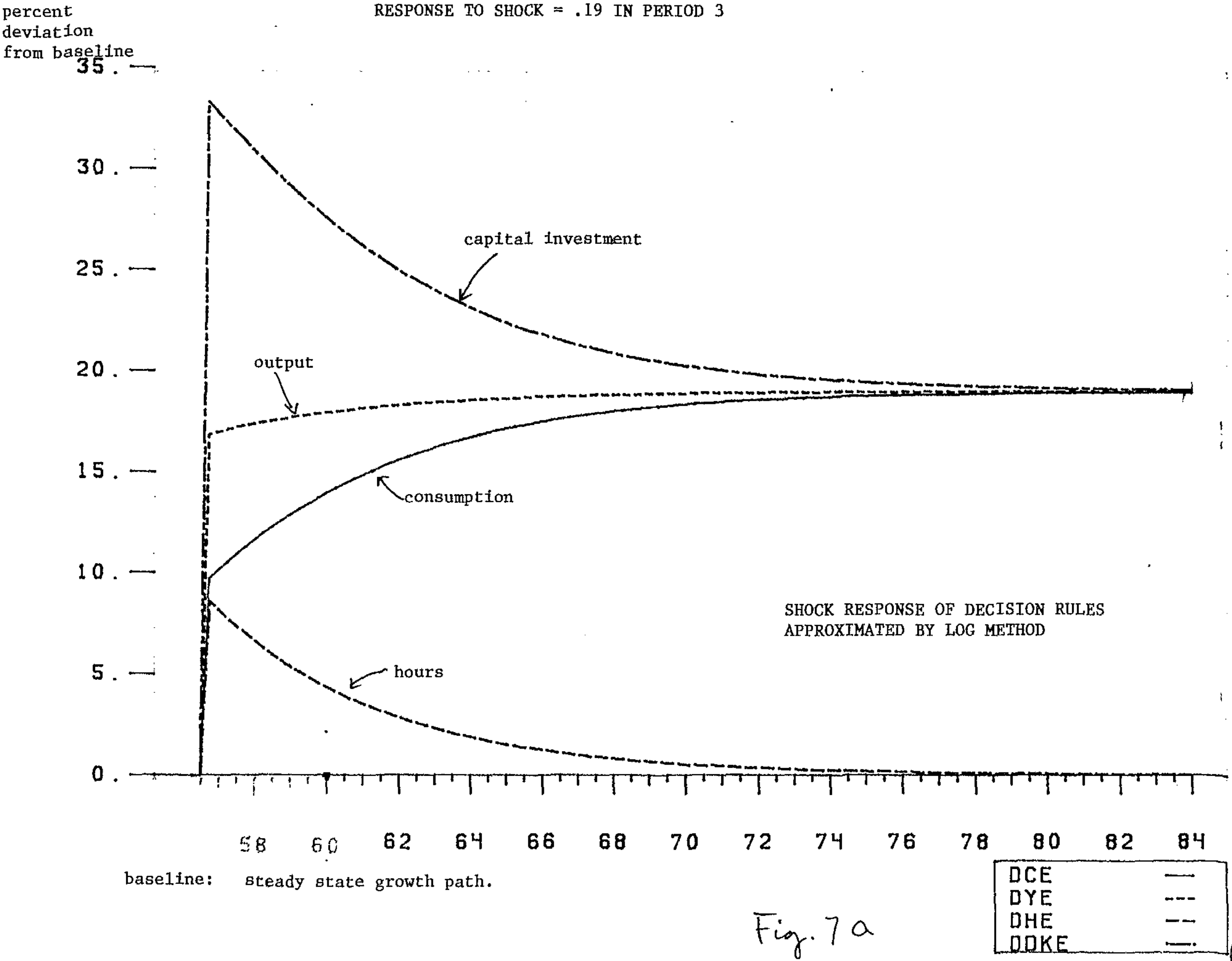
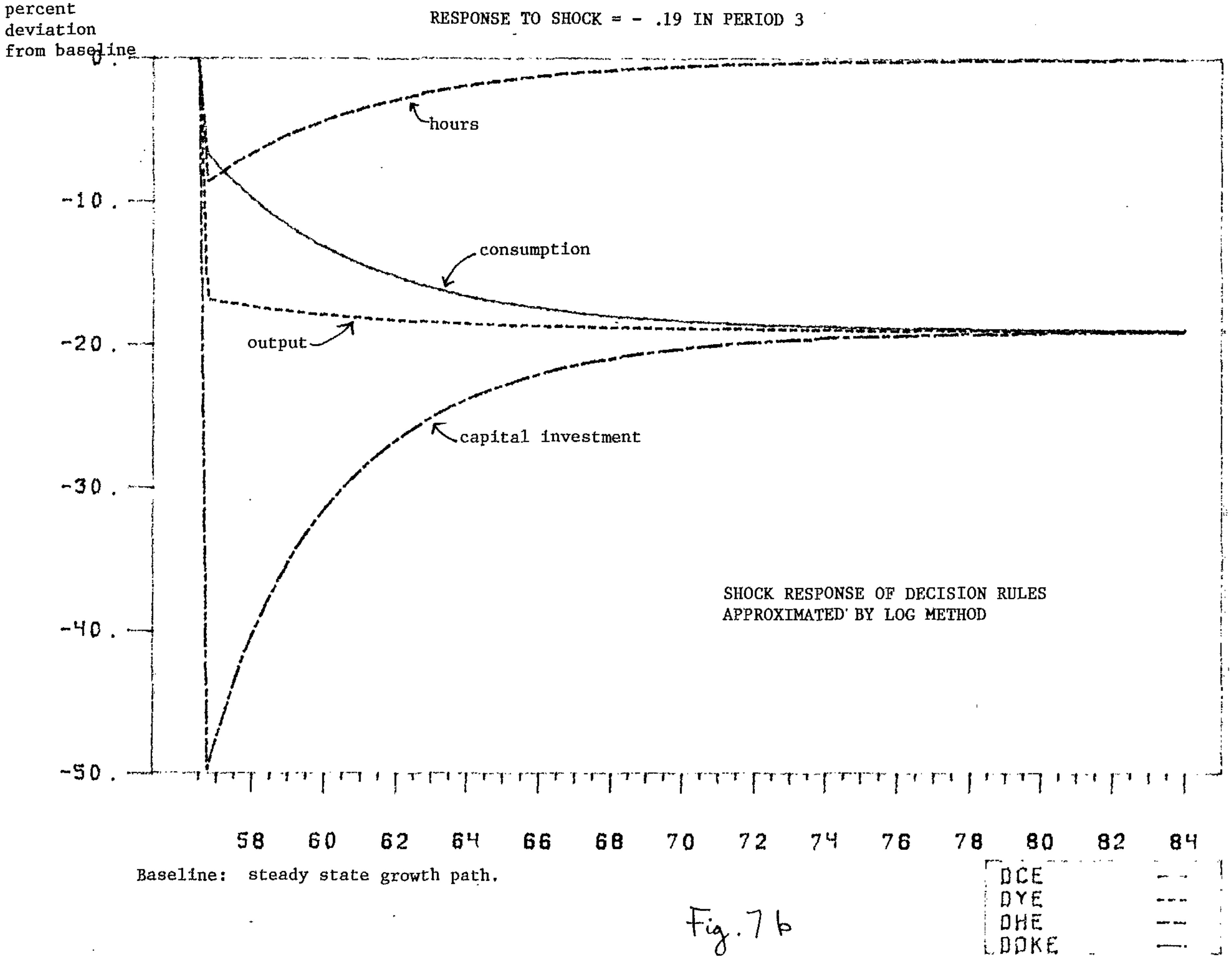

RESPONSE TO SHOCK = .19 IN PERIOD 3
percent deviation from baseline
35.
30.
25.
20.
15.
10.
5.
0.
capital investment
output
consumption
hours
SHOCK RESPONSE OF DECISION RULES
APPROXIMATED BY LOG METHOD
58
60
62
64
66
68
70
72
74
76
78
80
82
84
baseline: steady state growth path.
DCE
DYE
DHE
DDKE

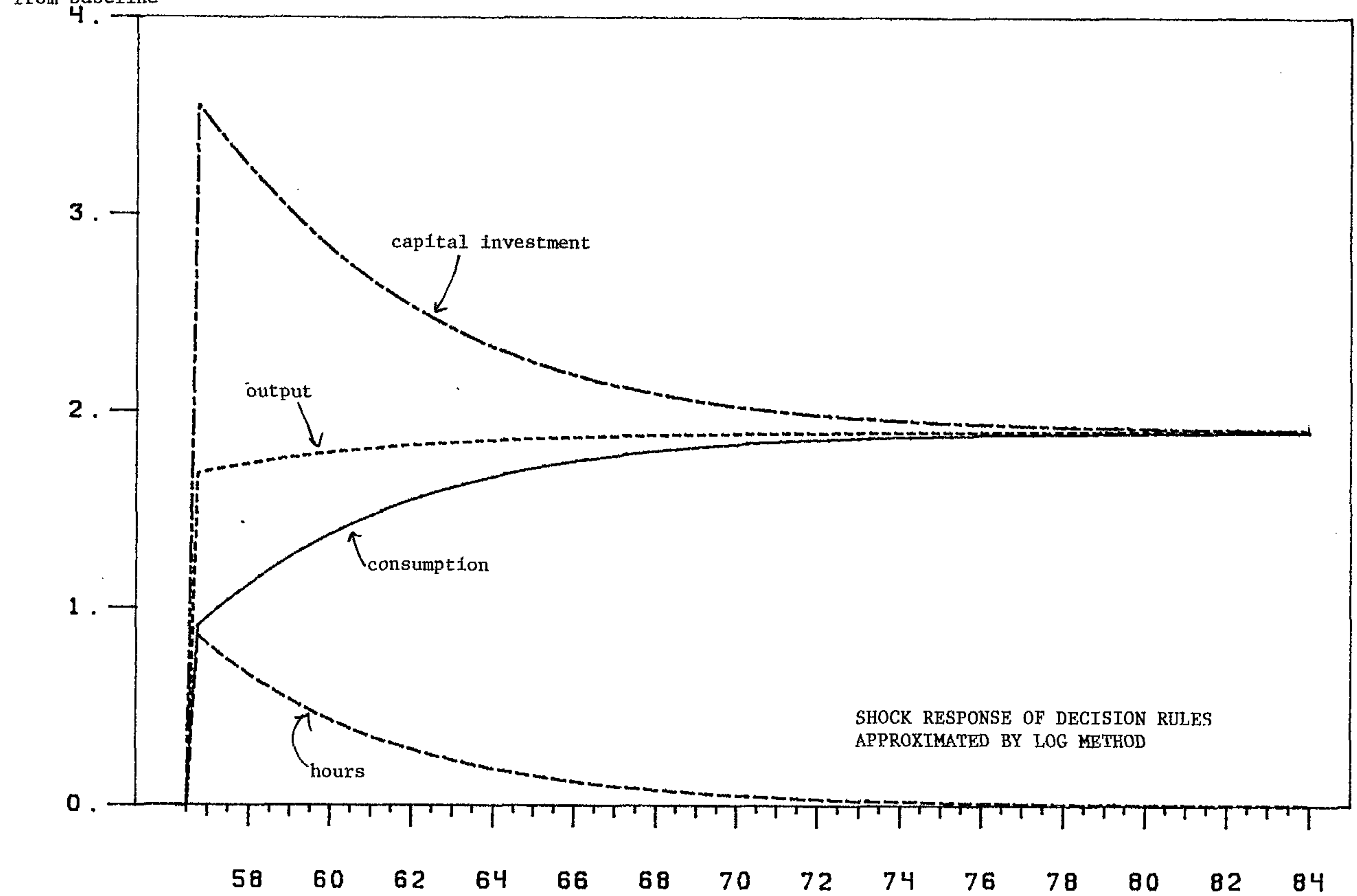
Fig. 7a



Baseline: steady state growth path.

Fig. 7b

RESPONSE TO SHOCK = .019 IN PERIOD 3, WITH PERIOD 2 CAPITAL STOCK 4.4% BELOW STEADY STATE GROWTH PATH

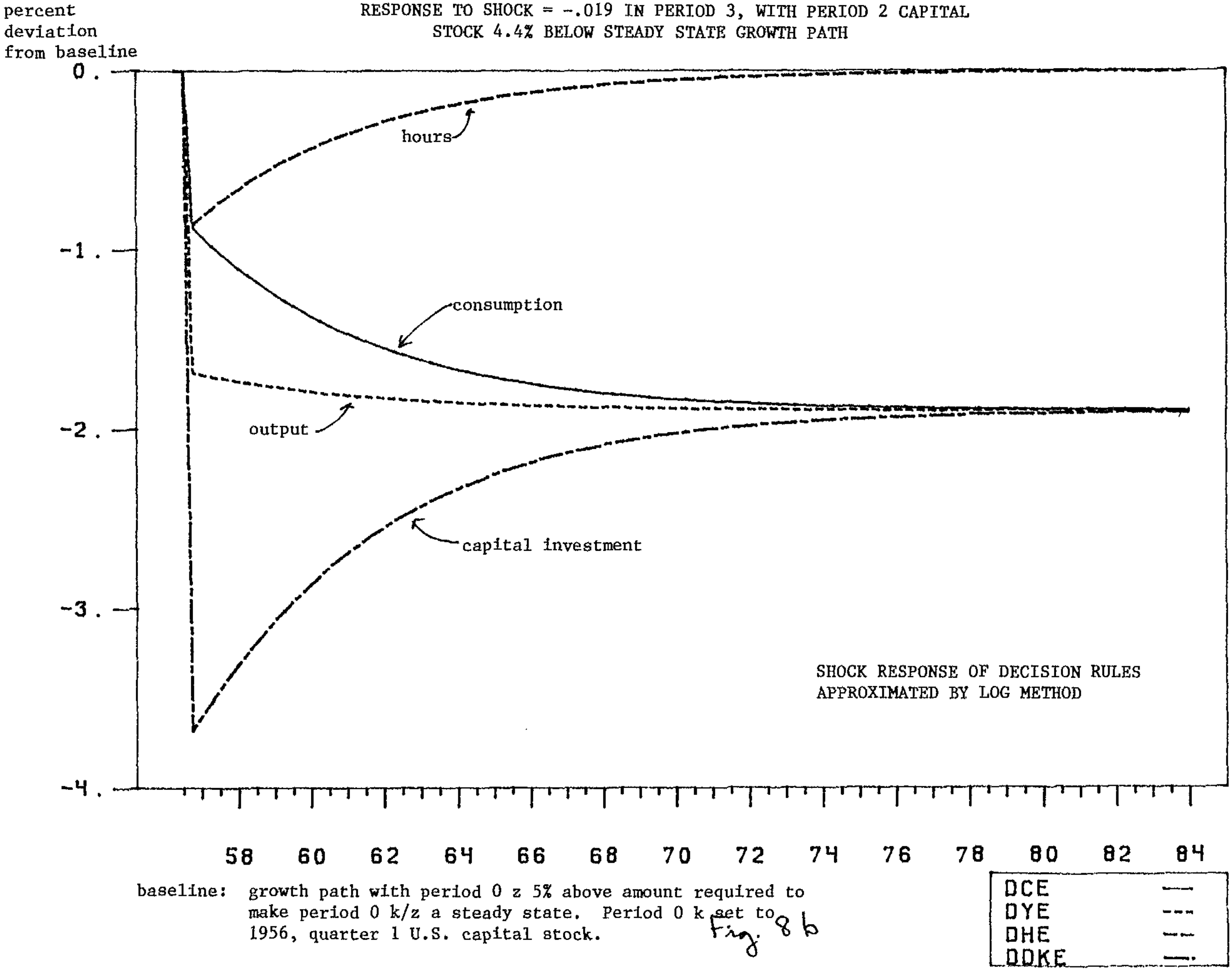


baseline: growth path with period 0 z 5% above amount required to make period 0 k/z a steady state. Period 0 k set to 1956, quarter 1 U.S. capital stock

DCE	——
DYE	----
DHE	-- --
DDKE	—-.

Fig. 8a

RESPONSE TO SHOCK = -.019 IN PERIOD 3, WITH PERIOD 2 CAPITAL STOCK 4.4% BELOW STEADY STATE GROWTH PATH



baseline: growth path with period 0 z 5% above amount required to make period 0 k/z a steady state. Period 0 k set to 1956, quarter 1 U.S. capital stock.

Fig. 8b

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