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Technical Appendix
to
"Optimal Fiscal Policy in a Business Cycle Model"

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1. The Model Economy

Households

Households maximize, by choice of $C(s^t)$, $\ell(s^t)$, $t \geq 0$,

$$(1) \qquad \max \sum_{t=0}^{\infty} \sum_{s^t} \tilde{\beta}^t \pi(s^t) U[C(s^t), \ell(s^t)]$$

subject to

$$(2) \qquad C(s^t) + K(s^t) + B(s^t) = (1 - \tau(s^t))W(s^t)\ell(s^t) + R_b(s^t)B(s^{t-1}) + R_k(s^t)K(s^{t-1}) + T(s^t),$$

subject to $K(s^{-1}) = K_0$, $R_b(s^0)B(s^{-1}) = a_0$, where K_0 and a_0 are given parameters. Here, $s^t = \{s_0, s_1, \ldots, s_t\}$ denotes the history of the exogenous shocks and $C(s^t)$, $K(s^t)$, $B(s^t)$ denote period t consumption and end-of-period t capital stocks and loans to the government, respectively. Also, $R_b(s^t)$ and $R_k(s^t)$ denote the gross returns on $B(s^{t-1})$ and $K(s^{t-1})$, and $T(s^t)$ denotes period t transfers from the government. Here,

$$(3) \qquad R_k(s^t) = 1 + (1 - \theta^e(s^{t-1}))(R(s^t) - \tilde{\delta}),$$

where $\theta^e(s^{t-1})$ denotes the effective period t tax rate on capital income, which is a function of s^{t-1}, $\tilde{\delta}$ denotes the rate of depreciation on a unit of capital, and R is the rental rate on capital. Finally, $\tau(s^t)$ denotes the period t labor tax rate, $W(s^t)$ denotes the wage rate and $\ell(s^t)$ denotes period t hours worked. The first order necessary conditions for the household's problem are:

$$(4) \qquad U_\ell(s^t) + U_c(s^t)(1 - \tau(s^t))W(s^t) = 0$$

$$(5) \qquad U_c(s^t) = \tilde{\beta} \sum_{s^{t+1}} \pi(s^{t+1} | s^t) U_c(s^{t+1}) R_k(s^{t+1})$$

$$(6) \qquad U_c(s^t) = \tilde{\beta} \sum_{s^{t+1}} \pi(s^{t+1} | s^t) U_c(s^{t+1}) R_b(s^{t+1}).$$

We assume the following parametric form for the utility function:

$$(7) \qquad U(C,\ell) = [C^{(1-\gamma)}(1-\ell)^{\gamma}]^{\psi}/\psi, \quad \text{when } \psi \neq 0$$

$$= (1-\gamma)\log C + \gamma \log(1-\ell), \quad \text{when } \psi = 0.$$

Firms

Firms rent capital and labor and combine these with technology, $x(s^t)$, to produce gross output $F(s^t)$ as follows:

$$(8) \qquad F(s^t) = [K(s^{t-1})]^{\alpha}[\exp(\mu t + x(s^t))\ell(s^t)]^{(1-\alpha)}.$$

Since firms are assumed to operate in competitive markets, their actions imply:

$$(9) \qquad R(s^t) = F_{k,t}$$

$$(10) \qquad W(s^t) = F_{\ell,t},$$

where $F_{k,t}$ and $F_{\ell,t}$ denote the partial derivatives of F with respect to $K(s^{t-1})$ and $\ell(s^t)$, respectively.

Equilibrium

The resource constraint for our economy is

$$(11) \qquad C(s^t) + G(s^t) + K(s^t) - (1-\tilde{\delta})K(s^{t-1}) = F(s^t).$$

It is convenient to scale the variables as follows:

$$(12) \qquad c(s^t) = \exp(-\mu t)C(s^t), \quad k(s^t) = \exp(-\mu t)K(s^t), \quad b(s^t) = \exp(-\mu t)B(s^t),$$

$$g(s^t) = \exp(-\mu t)G(s^t),$$

$$f(s^t) = \exp(-\alpha\mu)[k(s^{t-1})]^{\alpha}\,[\exp(x(s^t))\ell(s^t)]^{(1-\alpha)} = \exp(-\mu t)F(s^t),$$

$$u(s^t) = U(c(s^t), \ell(s^t))$$

$$\beta = \tilde{\beta} \exp[\mu(1-\gamma)\psi],$$

$$\delta = 1 - (1-\tilde{\delta}) \exp(-\mu),$$

$$r_k(s^t) = \exp(-\mu) + (1 - \theta^e(s^{t-1}))(f_k(s^t) + 1 - \exp(-\mu) - \delta),$$

$$r_b(s^t) = \exp(-\mu)R_b(s^t),$$

where f_k denotes the partial derivative of f with respect to k. Given our functional forms:

$$u_c(s^t) = U_c(s^t) \exp\{-\mu t[(1-\gamma)\psi - 1]\}$$

$$u_\ell(s^t) = U_\ell(s^t) \exp\{-\mu t(1-\gamma)\psi\}$$

$$f_\ell(s^t) = F_\ell(s^t) \exp(-\mu t)$$

$$f_k(s^t) = \exp(-\mu)F_k(s^t).$$

Here, u_c, u_ℓ denote the partial derivatives of u with respect to c and ℓ, and f_ℓ denotes the partial derivatives of f with respect to ℓ. Using this scaling of variables our first order conditions and resource constraint are:

$$(13) \qquad 1 + \frac{u_c(s^t)}{u_\ell(s^t)} (1 - \tau(s^t))f_\ell(s^t) = 0$$

$$(14) \qquad \sum_{s^{t+1}} \pi(s^{t+1} | s^t) \left\{ \frac{\beta u_c(s^{t+1})}{u_c(s^t)} r_k(s^{t+1}) - 1 \right\} = 0$$

$$(15) \qquad c(s^t) + g(s^t) + k(s^t) - (1-\delta)k(s^{t-1}) = f(s^t).$$

These are the equilibrium conditions for a nongrowing version of our economy in which the discount rate is β, the depreciation rate is δ, the utility function has the form given in (7), the after-tax return on capital is $r_k(s^t)$ and the production function is $\exp(-\alpha\mu)[k(s^{t-1})]^{\alpha}[\exp(x(s^t))\ell(s^t)]^{(1-\alpha)}$.

The exogenous shocks are given by x, g, θ^e, τ, T. They are considered uncontrollable by households and firms. Let

$$(16) \qquad s_t = \begin{bmatrix} x_t - x \\ \log(g_t/g) \\ \theta_t^e - \theta^e \\ \tau_t - \tau \\ \log(T_t/T) \end{bmatrix}$$

Here, a variable without a time subscript denotes its value in nonstochastic steady-state.

We suppose that the exogenous variables can take on only one of a finite set of N_s values:

$$(17) \qquad s_t \in S \equiv \{s(1),\ldots,s(N_s)\},$$

where s(i) is a 5×1 vector of numbers, $i = 1, \ldots, N_s$. Let

$$(18) \qquad \pi = \begin{bmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1N_s} \\ \pi_{21} & \pi_{22} & \cdots & \pi_{2N_s} \\ \vdots & & & \\ \pi_{N_s1} & \pi_{N_s2} & \cdots & \pi_{N_sN_s} \end{bmatrix},$$

where

$$(19) \qquad \pi_{ij} = \mathrm{Prob}[s_{t+1} = s(j) \,|\, s_t = s(i)].$$

We define an equilibrium for the nongrowing version of our economy as a set of functions

$k(s^t) = g(k(s^{t-1}),s_t)$, $c(s^t) = h(k(s^{t-1}),s_t)$, and $\ell(s^t) = n(k(s^{t-1}),s_t)$ such that (13)-(15) are satisfied and the transversality condition, $\beta^t\pi(s^t)u_c(s^t)k(s^t) \rightarrow 0$ as $t \rightarrow \infty$, is satisfied. In addition, we require that the government budget constraint be satisfied (given (15), this is equivalent to requiring (2)) and that $\beta^t\pi(s^t)u_c(s^t)b(s^t) \rightarrow 0$ as $t \rightarrow \infty$. An equilibrium for our growing economy is obtained by scaling in the obvious way. We consider equilibria which are "path independent" in the sense that all histories s^t which give rise to the same $(k(s^{t-1}),s_t)$ produce the same decisions, $k(s^t)$, $c(s^t)$, $\ell(s^t)$.

In practice we approximate g and n by log-linear functions $\hat{g}$ and $\hat{n}$, as described in Appendix A. The function h is approximated by $\hat{h}$, which is obtained by combining (15) with $\hat{g}$ and $\hat{n}$. In Section 3 we discuss our strategy for verifying that the government budget constraint is satisfied.

2. The Value Function and Equilibrium Government Debt

In this section we discuss our strategy for approximating the value function and the equilibrium debt function. We show how to use the equilibrium debt function to verify that the government budget constraint is satisfied. Each function satisfies a particular functional equation on a continuous state space defined by all (s,k) such that $s \in S$, $k \geq 0$. In each case we adopt a computational strategy similar to Coleman, Judd, or Marcet by restricting the state space to a finite number of points (s,k) such that $s \in S$ and $k \in K$, where K is an M-element set of values for the capital stock. This converts our functional equation into a system of MN_s equations. Our approximate value and debt functions are MN_s parameter functions which solve these equations exactly.

2.1 Equilibrium Government Debt

Evaluate (2) at $t + 1$, multiply by $\tilde{\beta}\pi(s^{t+1}|s^t)$, sum over all $s_{t+1} \in S$ and take (4)-(6) into account, to get

$$(20)\quad B(s^t) = \frac{1}{U_c(s^t)}\,\tilde{\beta}\sum_{s_{t+1}} \pi(s^{t+1}|s^t)\{U_c(s^{t+1})[c(s^{t+1}) + K(s^{t+1}) + B(s^{t+1})] + U_\ell(s^{t+1})\ell(s^{t+1})\}$$

$$- K(s^t).$$

Multiply both sides of the above equation by $\exp(-\mu t)$ to get

$$(21)\quad b(s^t) = \frac{1}{u_c(s^t)}\,\beta\sum_{s_{t+1}} \pi(s^{t+1}|s^t)\{u_c(s^{t+1})[c(s^{t+1}) + k(s^{t+1}) + b(s^{t+1})] + u_\ell(s^{t+1})\ell(s^{t+1})\}$$

$$- k(s^t),$$

using the scaled notation introduced above. Let b and k denote the beginning-of-period stock of debt and capital, and let ℓ denote hours worked in the current period. Let primes denote next period's value for these variables. Then, in equilibrium, $k' = g(k,s)$, $c = h(k,s)$, $\ell = n(k,s)$. Similarly, the current period marginal utilities of consumption and labor are functions of k and s only, and we denote these by $u_c(k,s)$ and $u_\ell(k,s)$. Then, when evaluated in equilibrium, (21) defines a mapping from the space of debt functions, $b' = \varphi(k,s)$, *into itself. The equilibrium debt rule is the fixed* point of this mapping. To make this more precise, define

$$(22)\quad a(k,s;\varphi,g,h,n) = \frac{1}{u_c(k,s)}\,\beta\sum_{s'} \pi(s'|s)\{u_c(g(k,s),s')[h(g(k,s),s') + g(g(k,s),s')$$

$$+ \varphi(g(k,s),s')] + u_\ell(g(k,s),s')n(g(k,s),s')\} - g(k,s),$$

$$(23)\quad \mathscr{E}(k,s;\varphi) = \varphi(k,s) - a(k,s;\varphi,g,n,h)$$

for all $k \geq 0$ and $s \in S$. Then, the equilibrium debt rule has the property that

$$(24)\quad \mathscr{E}(k,s;\varphi,g,n,h) = 0 \quad \text{for all } k \geq 0,\ s \in S.$$

In practice, we do not compute φ exactly. Instead, we approximate $\varphi(k,s)$ by a function, $\hat{\varphi}(k,s;\underset{\sim}{b})$, which is piecewise linear in k for each fixed $s \in S$. The vector $\underset{\sim}{b}$ gives the values of $\hat{\varphi}$ for (s,k) such that $s \in S$ and $k \in K = \{k_1,\ldots,k_M\}$, where $M \times N_s$ is the number of elements in $\underset{\sim}{b}$. By analogy with (24), our method for approximating φ involves selecting $\underset{\sim}{b}$ so that

$$(25) \qquad \mathscr{E}(k,s;\underset{\sim}{b},\hat{g},\hat{n},\hat{h}) = 0$$

for all (s,k) such that $s \in S$ and $k \in K$. It turns out that finding $\underset{\sim}{b}$ in (25) requires solving a system of linear equations. To see this, suppose we begin with some set of parameters $\underset{\sim}{b}$. Then, $a(k,s;\underset{\sim}{b},\hat{g},\hat{n},\hat{h})$ for $k \in K$ and $s \in S$ defines a new vector, $\underset{\sim}{b}'$. This mapping from $\underset{\sim}{b}$ to $\underset{\sim}{b}'$ is linear and can be written $\underset{\sim}{b}' = T(\underset{\sim}{b})$, where $T(\underset{\sim}{b}) = Z + B\underset{\sim}{b}$. Then, according to (25) we seek a $\underset{\sim}{b}$ such that $\underset{\sim}{b} = T(\underset{\sim}{b})$. But this simply requires solving the linear system of equations $\underset{\sim}{b} = Z + B\underset{\sim}{b}$ for $\underset{\sim}{b}$. The $MN_s \times 1$ vector Z and the $MN_s \times MN_s$ matrix B are described in detail in Appendix B.

Given our (approximate) equilibrium decision rules and government debt function, we can verify that the government budget constraint is satisfied. Since the resource constraint is satisfied by our construction of the equilibrium consumption decision rule, we need only verify that the household's budget constraint, (2), is satisfied. But, for t = 1, 2, 3, ..., we can in effect force it to be satisfied by appropriate choice of $R_b(s^t)$. Thus, we need only verify that (2) is satisfied at $t = 0$. Solving (2) for $R_b(s^0)B(s^{-1})$ and imposing the equilibrium decision rules, we get:

$$(26) \qquad R_b(s^0)B(s^{-1}) = \hat{h}(k_0,s_0) + \hat{g}(k_0,s_0) + \hat{\varphi}(k_0,s_0) - (1 - \tau(s_0))f_\ell(s_0)n(k_0,s_0)$$
$$- r_k(s_0)k_0 - T(s_0),$$

where $k_0 \equiv \exp(-\mu)K_0$. The government budget constraint is satisfied if $R_b(s^0)B(s^{-1}) = a_0$.

2.2 The Value Function

Next, we require the discounted utility, (1), associated with our approximations to the equilibrium decision rules, $\hat{g}$, $\hat{h}$, $\hat{n}$. Let this be given by the function $\nu(k,s)$. Then,

$$(27) \qquad \nu(k,s) = u(\hat{h}(k,s),\hat{n}(k,s)) + \beta E[\nu(k',s')|k,s].$$

The function ν, when evaluated at the initial values of k and s, corresponds (apart from an additive constant when $\psi = 0$) to (1). We approximate ν using a slight modification of the approach used to approximate φ. In particular, we approximate ν by $\hat{\nu}$, a function which is piecewise linear in k for fixed s. We pin down $\hat{\nu}$ by imposing the condition that it satisfies (27) for all s, k such that $s \in S$ and $k \in K$ fixed point condition. See Appendix B for the details.

3. Assigning Parameter Values

For parameter estimation purposes, it is convenient to display equations (13)–(15) in nonstochastic steady-state. In this section we adopt a notation which is slightly inconsistent with that in the previous section, by letting a variable without the s^t argument denote its value in nonstochastic steady state. Then, in nonstochastic steady state, (13)–(15) become

$$(28) \qquad 1 - \frac{1-\gamma}{\gamma}\,\frac{1-\ell}{\ell}\,(1-\tau)(1-\alpha)\,\frac{f}{c} = 0$$

$$(29) \qquad \tilde{\beta}\exp[(1-\gamma)\mu\psi][\exp(-\mu) + (1-\theta^e)(\alpha f/k + 1 - \exp(-\mu) - \delta)] - 1 = 0$$

$$(30) \qquad \frac{c}{f} = \frac{\exp[(1-\alpha)x - \alpha\mu]z^{\alpha} - \delta z}{\exp[(1-\alpha)x - \alpha\mu]z^{\alpha}} - \frac{g}{f}$$

$$(31) \qquad \frac{f}{k} = \exp[(1-\alpha)x - \alpha\mu]z^{(\alpha-1)},$$

where $z = k/\ell$.

The model parameters are γ, $\tilde{\beta}$, ψ, μ, τ, θ^e, $\tilde{\delta}$, g. We set $\mu = 0.016$, the average annual per capita growth rate in GNP in the post-war period. Also, we set $\tilde{\delta} = 0.083$, the rate of depreciation implicit in post-war U.S. per capita investment (public and private) and capital stock data. In addition, we set $g/f = 0.18$, $k/f = 2.7$, $(1-\ell)/\ell = 3.28$, their post-war sample averages. Our estimate for $(1-\ell)/\ell$ reflects the assumption that the representative household has a time endowment of 15 hours per day (i.e., 1,369 hours per quarter). In addition, it reflects an estimate that per capita hours worked has averaged 320 hours per quarter in the post-war period. Finally, we set $\alpha = 0.34$, the post-war sample average of the ratio of employee compensation plus proprietor's income to total gross output. For a further discussion of the data on which these calculations are based, see Christiano (1988).

We adopted the normalization, $x = 0$. Given these ratios and parameter values, (31) implies $z = 4.47$ and (30) implies $c/f = 0.56$. The latter coincides with the post-war sample average of the private consumption to gross output ratio. We still have six parameters to pin down: g, γ, τ, θ^e, ψ, and $\tilde{\beta}$. We set the ratio, τ/θ^e, equal to 0.87, the ratio of the postwar average marginal income tax rate to the average tax rate on capital. The average marginal income tax rate, 25 percent, is based on data in Barro and Sahasakal (1983) and the average effective tax rate on capital income, 28 percent, is based on Jorgenson and Sullivan (1981)'s data. We chose the remaining parameters to be consistent with $g/y = 0.18$, (28), (29), and the postwar average U.S. debt to gross output ratio, 0.51. We also took into account the model's implication for the Frisch labor supply elasticity. The debt to output ratio is the average, for the period 1950–1980, of the ratio to GNP of the market value of privately held federal, state, and local debt, plus the present volume of depreciation allowances. For the construction of their time series, see Chari, Christiano, and Kehoe (1991). The Frisch labor supply elasticity, $\mathscr{E}$, is the elasticity of hours worked with respect to the after-tax real wage, holding the marginal utility of wealth constant. Evaluated in nonstochastic steady state, this is:[1]

$$\epsilon = \frac{1 - (1-\gamma)\psi}{1 - \psi}\,\frac{1 - \ell}{\ell}.$$

Note that as $\psi \to -\infty$, ϵ converges to a lower bound of $(1-\gamma)(1-\ell)/\ell$.

The steady-state debt to output ratio in our model is obtained by evaluating (21) in steady state:

$$\text{(32)}\qquad \frac{b}{y} = \left\{\beta\left[\frac{c}{y} + \frac{k}{y} - \frac{1-\gamma}{\gamma}\,\frac{c}{y}\,\frac{\ell}{1-\ell}\right] - \frac{k}{y}\right\}/(1-\beta).$$

In our benchmark model, we set $\psi = 0$, a value which is standard in the real business cycle literature (see, e.g., ...). Then, with $g = 0.07$, $\tilde{\beta} = 1.016$, $\gamma = 0.748$, $\tau = 0.237$, $\theta^e = 0.271$, (28)-(29) $q/g = 0.18$, $b/y = 0.51$ are satisfied with this parameterization, $\mathscr{E} = 3.27$. The 1.6 percent discount rate is lower than the 3 percent figure used in Christiano and Eichenbaum (1992), despite their application of the same estimation strategy used here. The difference reflects their assumption that $\theta^e = 0$. From (29), it is clear that for a given capital-output ratio, the estimated discount rate decreases with θ^e. In effect, one must assume lower impatience in order to rationalize a given capital-labor ratio when the return to capital is reduced by a higher tax rate.

We also considered a smaller value of ψ. One reason for this is based on estimates of ϵ found in the empirical labor literature. In their review of that literature, Rotemberg and Woodford (forthcoming) report estimates of ϵ for males near zero and for females in the range 0.5–1.5. As noted above, there is no ψ which can produce an ϵ less than $(1-\gamma) = 0.8$. The range 0.8–1.5 for ϵ corresponds to a range $(-\infty, -2.69)$ for ψ, given our estimates of $(1-\ell)/\ell$ and γ. These considerations lead us to consider $\psi = -8$ as an alternative to $\psi = 0$. Under this alternative, $\epsilon = 1.1$, and $\tilde{\beta} = 1.016$. Rather than work with a value of $\tilde{\beta}$ greater than 1, in our alternative (*high curvature*) parameterization, we hold $\tilde{\beta}$ to its value in the benchmark parameterization.

Footnote

[1]The Frisch elasticity is defined as follows. Let $\lambda = u_c$. Write the intratemporal first order condition as $u_\ell + \lambda w = 0$, when w is the after tax real wage. Totally differentiating these two equations with respect to c, ℓ, and w, holding λ fixed, the Frisch elasticity is defined as $\epsilon \equiv d \log \ell / d \log w$.

Appendix A:

Approximating the Equilibrium Decision Rules

In this appendix, we describe our strategy for obtaining an approximate solution to the model. The strategy involves first replacing the first order conditions, (13)–(14), by a log-linear approximation about steady state. We then solve the resulting equations by an undetermined coefficients method.

1. Log-Linearizing the Euler Equations

In this appendix we work with a version of (13) and (14) in which $c(s^t)$ has been substituted out using (15). Also, to simplify notation, we set $k_{t+1} = k(s^t)$, $\ell_t = \ell(s^t)$, $\theta_t^e = \theta^e(s^t)$, $\tau_t = \tau(s^t)$, $f_t = f(s^t)$, $g_t = g(s^t)$, $x_t = x(s^t)$, $T_t = T(s^t)$. Write

$$\text{(A.1)}\quad q(k_{t+1},k_t,\ell_t,g_t,x_t,\tau_t) = 1 + \frac{u_{c,t}}{u_{\ell,t}}(1-\tau_t)f_{\ell,t},$$

$$h(k_{t+2},k_{t+1},k_t,\ell_{t+1},\ell_t,x_{t+1},x_t,g_{t+1},g_t,\theta_t^e)$$

$$= \frac{u_{c,t+1}}{u_{c,t}}\left[\exp(-\mu) + (1-\theta_t^e)(f_{k,t+1} + 1 - \exp(-\mu) - \delta)\right] - 1.$$

Then, the first order conditions are:

$$\text{(A.2)}\quad q(k_{t+1},k_t,\ell_t,g_t,x_t,\tau_t) = 0$$

$$\text{(A.3)}\quad E[h(k_{t+2},k_{t+1},k_t,\ell_{t+1},\ell_t,x_{t+1},x_t,g_{t+1},g_t,\theta_t^e)\,|\,\tau_t,x_t,\theta_t^e,T_t,g_t] = 0.$$

Since we seek a log-linear expansion of q and h, it is useful to introduce the following change of variables:

$$(A.4)\quad \tilde{k}_t = \log k_t,$$

$$\tilde{\ell}_t = \log \ell_t$$

$$\tilde{g}_t = \log g_t.$$

Then,

$$(A.5)\quad q(k_{t+1},k_t,\ell_t,g_t,x_t,\tau_t) = q[\exp(\tilde{k}_{t+1}), \exp(\tilde{k}_t), \exp(\tilde{\ell}_t), \exp(\tilde{g}_t), x_t, \tau_t]$$

$$\equiv \tilde{q}(\tilde{k}_{t+1},\tilde{k}_t,\tilde{\ell}_t,\tilde{g}_t,x_t,\tau_t).$$

Similarly define

$$(A.6)\quad \tilde{h}(\tilde{k}_{t+2},\tilde{k}_{t+1},\tilde{k}_t,\tilde{\ell}_{t+1},\tilde{\ell}_t,x_{t+1},x_t,\tilde{g}_{t+1},\tilde{g}_t,\theta^e_t).$$

Let Q and H denote the first order linear expansion of $\tilde{q}$ and $\tilde{h}$ about nonstochastic steady-state. Then, the first order conditions in the log-linearly expanded system are:

$$(A.7)\quad Q(\tilde{k}_{t+1},\tilde{k}_t,\tilde{\ell}_t,\tilde{g}_t,x_t,\tau_t) = 0$$

$$(A.8)\quad E[H(\tilde{k}_{t+2},\tilde{k}_{t+1},\tilde{k}_t,\tilde{\ell}_{t+1},\tilde{\ell}_t,x_{t+1},x_t,\tilde{g}_{t+1},\tilde{g}_t,\theta^e_t)\,|\,\tilde{T}_t,\tilde{g}_t,x_t,\tau_t,\theta^e_t] = 0.$$

Writing out Q explicitly,

$$(A.9)\quad Q(\tilde{k}_{t+1},\tilde{k}_t,\tilde{\ell}_t,\tilde{g}_t,x_t,\tau_t) = Q_1(\tilde{k}_{t+1}-\tilde{k}) + Q_2(\tilde{k}_t-\tilde{k}) + Q_3(\tilde{\ell}_t-\tilde{\ell}) + Q_4(\tilde{g}_t-\tilde{g})$$

$$+ Q_5(x_t-x) + Q_6(\tau_t-\tau),$$

where

$$(A.10)\quad Q_1 = -\left[\frac{u_c}{u_\ell}\right][\eta_c-\epsilon_c](1-\tau)f_\ell(k/c)$$

$$Q_2 = -Q_1[f_k+1-\gamma] + \frac{u_c}{u_\ell} (1-\tau)f_{k\ell}k$$

$$Q_3 = -Q_1 f_\ell \frac{\ell}{k} + \left[\frac{u_c}{u_\ell}\right] (1-\tau)[f_{\ell\ell} + (\eta_\ell - \epsilon_\ell)f_\ell]$$

$$Q_4 = Q_1(g/k)$$

$$Q_5 = -Q_1 f_x/k + \frac{u_c}{u_\ell} (1-\tau)f_{\ell x}$$

$$Q_6 = -\frac{u_c}{u_\ell} f_\ell .$$

Here,

(A.11) $$\eta_c = \frac{d \log u_c}{d \log c}, \quad \epsilon_c = \frac{d \log u_\ell}{d \log c}, \quad \eta_\ell = \frac{d \log u_c}{d \log \ell}, \quad \epsilon_\ell = \frac{d \log u_\ell}{d \log \ell},$$

evaluated in steady-state.

Also, for our utility function,

(A.12) $$\eta_c = (1-\gamma)\psi - 1, \quad \eta_\ell = \gamma\psi \frac{\ell}{(1-\ell)},$$

$$\epsilon_c = -(1-\gamma)\psi, \quad \epsilon_\ell = -(\gamma\psi-1) \frac{\ell}{(1-\ell)},$$

$$\frac{u_c}{u_\ell} = -\left[\frac{1-\gamma}{\gamma}\right] \frac{1-\ell}{c}.$$

The function H is

$$\text{(A.13)}\quad H(\tilde{k}_{t+2},\tilde{k}_{t+1},\tilde{k}_t,\tilde{\ell}_{t+1},\tilde{\ell}_t,x_{t+1},x_t,\tilde{g}_{t+1},\tilde{g}_t,\theta^e_t)$$

$$= H_1(\tilde{k}_{t+2}-\tilde{k}) + H_2(\tilde{k}_{t+1}-\tilde{k}) + H_3(\tilde{k}_t-\tilde{k}) + H_4(\tilde{\ell}_{t+1}-\tilde{\ell}) + H_5(\tilde{\ell}_t-\tilde{\ell})$$

$$+ H_6(x_{t+1}-x) + H_7(x_t-x) + H_8(\tilde{g}_{t+1}-\tilde{g}) + H_9(\tilde{g}_t-\tilde{g}) + H_{10}(\theta^e_t-\theta^e).$$

Here,

$$\text{(A.14)}\quad H_1 = -\eta_c k/c,\quad H_2 = -H_1(f_k+1-\delta+1) + \beta(1-\theta^e)f_{kk}k,\quad H_3 = H_1(f_k+1-\delta),$$

$$H_4 = -H_1 f_\ell\, \ell/k + \eta_\ell + \beta(1-\theta^e)f_{k\ell}\ell,\quad H_5 = H_1 f_\ell\, \ell/k - \eta_\ell,$$

$$H_6 = -H_1 f_x/k + \beta(1-\theta^e)f_{kx},\quad H_7 = H_1 f_x/k,\quad H_8 = -H_1(f_k+1-\delta+1)g/k$$

$$H_9 = -H_1 g/k,\quad H_{10} = -\beta(f_k + 1 - \exp(-\mu) - \delta).$$

It is convenient to substitute out for hours worked using the Q first order condition:

$$\text{(A.15)}\quad \tilde{\ell}_t - \tilde{\ell} = -\frac{Q_1}{Q_3}(\tilde{k}_{t+1}-\tilde{k}) - \frac{Q_2}{Q_3}(\tilde{k}_t-\tilde{k}) - \frac{Q_4}{Q_3}(\tilde{g}_t-\tilde{g}) - \frac{Q_5}{Q_3}(x_t-x) - \frac{Q_6}{Q_3}(\tau_t-\tau)$$

for all t. Substitute this into the function H and collect terms:

$$
\begin{aligned}
&\left[H_1 - \frac{Q_1}{Q_3} H_4\right](\tilde{k}_{t+2}-\tilde{k}) + \left[H_2 - \frac{Q_2}{Q_3} H_4 - \frac{Q_1}{Q_3} H_5\right](\tilde{k}_{t+1}-\tilde{k}) \\
&\quad + \left[H_3 - \frac{Q_2}{Q_3} H_5\right](\tilde{k}_t-\tilde{k}) + \left[H_6 - \frac{Q_5}{Q_3} H_4\right](x_{t+1}-x) \\
\text{(A.16)}\qquad &\quad + \left[H_7 - \frac{Q_5}{Q_3} H_5\right](x_t-x) + \left[H_8 - \frac{Q_4}{Q_3} H_4\right](\tilde{g}_{t+1}-\tilde{g}) \\
&\quad + \left[H_9 - \frac{Q_4}{Q_3} H_5\right](\tilde{g}_t-\tilde{g}) + H_{10}(\theta^e_t-\theta^e) - \frac{Q_6}{Q_3} H_4(\tau_{t+1}-\tau) \\
&= \bar{H}_1(\tilde{k}_{t+2}-\tilde{k}) + \bar{H}_2(\tilde{k}_{t+1}-\tilde{k}) + \bar{H}_3(\tilde{k}_t-\tilde{k}) + \bar{H}_4(x_{t+1}-x) \\
&\quad + \bar{H}_5(x_t-x) + \bar{H}_6(\tilde{g}_{t+1}-\tilde{g}) + \bar{H}_7(\tilde{g}_t-\tilde{g}) + \bar{H}_8(\theta^e_t-\theta) \\
&\quad + \bar{H}_9(\tau_{t+1}-\tau) + \bar{H}_{10}(\tau_t-\tau)
\end{aligned}
$$

or,

$$
\begin{aligned}
\text{(A.17)}\quad & E[\bar{H}_1\bar{k}_{t+2}+\bar{H}_2\bar{k}_{t+1}+\bar{H}_3\bar{k}_t+\bar{H}_4\bar{x}_{t+1}+\bar{H}_5\bar{x}_t+\bar{H}_6\bar{g}_{t+1}+\bar{H}_7\bar{g}_t+\bar{H}_8\bar{\theta}^e_t \\
&+\bar{H}_9\bar{\tau}_{t+1}+\bar{H}_{10}\bar{\tau}_t \,|\, \theta^e_t,\tilde{T}_t,\tilde{g}_t,\tau_t,x_t,\tilde{k}_t] = 0,
\end{aligned}
$$

where a bar over a variable indicates deviation from nonstochastic steady state. Divide (26) by $\bar{H}_1$:

$$
\text{(A.18)}\quad E[\bar{k}_{t+2}-\phi\bar{k}_{t+1}+\nu\bar{k}_t+\gamma s_t+\psi s_{t+1} \,|\, s_t,\tilde{k}_t] = 0,
$$

where,

$$
\gamma = \left[\frac{\bar{H}_5}{\bar{H}_1}\ \frac{\bar{H}_7}{\bar{H}_1}\ \frac{\bar{H}_8}{\bar{H}_1}\ \frac{\bar{H}_{10}}{\bar{H}_1}\ 0\right]
$$

$$\psi = \left[\frac{\bar{H}_4}{\bar{H}_1} \ \frac{\bar{H}_6}{\bar{H}_1} \ 0 \ \frac{\bar{H}_9}{\bar{H}_1} \ 0 \right].$$

The system has now been reduced to a single first order condition in capital.

2. Solving the Log-Linearized System

Consider the following decision rule:

(A.19) $\bar{k}' = \lambda\bar{k} + d(s)$,

where a prime on a variable denotes next period's value. Also,

(A.20) $d(s) = d_i$, for $s = s(i)$, $i = 1, \ldots, N_s$.

The coefficients to be determined are

$$\lambda, d_1, \ldots, d_{N_s}.$$

Note,

$$\bar{k}'' = \lambda\bar{k}' + d(s') = \lambda^2\bar{k} + \lambda d(s) + d(s').$$

Substitute this into the capital first order condition, to get

$$E[\lambda^2\bar{k} + \lambda d(s) + d(s') - \phi\lambda\bar{k} - \phi d(s) + \nu\bar{k} + \gamma s + \psi s' | \bar{k}, s] = 0,$$

or,

(A.21) $E[(\lambda^2 - \phi\lambda + \nu)\bar{k} + (\lambda - \phi)d(s) + \gamma s + d(s') + \psi s' | \bar{k}, s] = 0.$

We use the condition that (30) be satisfied for all possible s and for all $\bar{k}$ to determine the unknown coefficients in (28). Thus, select λ so that

(A.22) $\lambda^2 - \phi\lambda + \nu = 0.$

Also, pick $d_1, \ldots, d_N$ so that

$$(\lambda-\phi)d_i + \tilde{\gamma}_i + \sum_{j=1}^{N_s} \pi_{ij}(d_j+\tilde{\psi}_j) = 0,$$

for $i = 1, \ldots, N_s$. Here,

$$\tilde{\gamma}_j = \gamma s(j), \quad \tilde{\psi}_j = \psi s(j), \quad j = 1, \ldots, N_s.$$

Let,

$$\tilde{\gamma} = \begin{bmatrix} \tilde{\gamma}_1 \\ \vdots \\ \tilde{\gamma}_{N_s} \end{bmatrix}, \quad \tilde{\psi} = \begin{bmatrix} \tilde{\psi}_1 \\ \vdots \\ \tilde{\psi}_{N_s} \end{bmatrix}, \quad d = \begin{bmatrix} d_1 \\ \vdots \\ d_{N_s} \end{bmatrix}.$$

Then,

(A.23) $(\lambda-\phi)d + \tilde{\gamma} + \pi(d+\tilde{\psi}) = 0,$

so that,

(A.24) $d = -[(\lambda-\phi)\ I_{N_s} + \pi]^{-1}[\tilde{\gamma}+\pi\tilde{\psi}].$

The decision rule for capital is:

(A.25) $k_{t+1} = k[k_t/k]^{\lambda} \exp(d(s(t))).$

The decision rule for labor is obtained by substituting the decision rule for capital into the linearized first order condition for labor:

$$(A.26)\quad \bar{\ell}_t = -\frac{Q_1}{Q_3}\ \bar{k}_{t+1} - \frac{Q_2}{Q_3}\ \bar{k}_t - \frac{Q_4}{Q_3}\ \bar{g}_t - \frac{Q_5}{Q_3}\ \bar{x}_t - \frac{Q_6}{Q_3}\ \bar{\tau}_t$$

$$= -\frac{Q_1}{Q_3}\ \lambda\bar{k}_t - \frac{Q_1}{Q_3}\ d(s(t)) - \frac{Q_2}{Q_3}\ \bar{k}_t - \frac{Q_4}{Q_3}\ \bar{g}_t - \frac{Q_5}{Q_3}\ \bar{x}_t - \frac{Q_6}{Q_3}\ \bar{\tau}_t$$

$$= e_k\bar{k}_t + e_d d(s(t)) + e_g\bar{g}_t + e_x\bar{x}_t + e_\tau\bar{\tau}_t,$$

where,

$$e_k = -\frac{1}{Q_3}\ [\lambda Q_1 + Q_2]$$

$$e_d = -\frac{Q_1}{Q_3}$$

$$e_g = -\frac{Q_4}{Q_3}$$

$$e_x = -\frac{Q_5}{Q_3}$$

$$e_\tau = -\frac{Q_6}{Q_3}$$

or,

$$\log(\ell_t/\ell) = e_k \log(k_t/k) + e_d d(s(t)) + e_g \log(g_t/g) + e_x(x_t - x) + e_\tau(\tau_t - \tau)$$

or,

$$(A.27)\quad \ell_t = \ell(k_t/k)^{e_k}(g_t/g)^{e_g} \exp[e_d d(s(t)) + e_x(x_t - x) + e_\tau(\tau_t - \tau)].$$

A useful check on the decision rule formulas is to substitute them into the intertemporal condition for capital, (A.8), and determine whether $E[H|s,\bar{k}] = 0$ for all s and $\bar{k}$. Doing so, we get

$$
\begin{aligned}
&E\{[\lambda^2 H_1+\lambda H_2+H_3+H_4\lambda e_k+H_5 e_k]\bar{k} + [\lambda H_1+H_2+H_4 e_k+H_5 e_d]d(s) \\
&\quad + [H_1+H_4 e_d]d(s') + [H_6+H_4 e_x]\bar{x}' + [H_7+H_5 e_x]\bar{x} + [H_8+H_4 e_g]\bar{g}' \\
&\quad + [H_9+H_5 e_g]\bar{g} + H_{10}\bar{\theta}^e + H_4 e_\tau \bar{\tau}' + H_5 e_\tau \bar{\tau}|s,\bar{k}\}.
\end{aligned}
$$

Thus, we require

$$\lambda^2 H_1 + \lambda H_2 + H_3 + H_4\lambda e_k + H_5 e_k = 0.$$

To obtain the other restriction, first define

$$
\begin{aligned}
z(s) &= [H_1+H_4 e_d]d(s) + [H_6+H_4 e_x]\bar{x} + [H_8+H_4 e_g]\bar{g} + H_4 e_\tau \bar{\tau} \\
a(s) &= [\lambda H_1+H_2+H_4 e_k+H_5 e_d]d(s) + [H_7+H_5 e_x]\bar{x} + [H_9+H_5 e_g]\bar{g} + H_{10}\bar{\theta}^e + H_5 e_\tau 5.
\end{aligned}
$$

Then, we require

$$a(s) + \sum_{j=1}^{N_s} \pi_{sj} z(s_j) = 0, \quad s = s_1, \ldots, \quad s_{N_s},$$

or,

$$a + \pi z = 0,$$

where

$$a = \begin{bmatrix} a(s_1) \\ \vdots \\ a(s_{N_s}) \end{bmatrix}, \quad z = \begin{bmatrix} z(s_1) \\ \vdots \\ z(s_{N_s}) \end{bmatrix}.$$

Appendix B:

Computation of Equilibrium Debt and Utility

In this appendix we present computational details of our strategy for approximating the equilibrium government debt and value functions. For a general overview, see Section 4.

B.1 The Government Debt Function

Let $K = \{k_1,\ldots,k_M\}$ denote a grid of values for the capital stock, with $k_{j+1} \geq k_j$, $j = 1, \ldots, M - 1$. The interval, $[k_1,k_M]$, should contain an ergodic set for k_t in its interior. Let $i = 1, \ldots, N_sM$ be an enumeration of all possible combinations of k, s, for $k \in K$ and $s \in \{s(1),\ldots,s(N_s)\}$. In particular, we set

$$
\begin{aligned}
&i = 1 \rightarrow k = k_1, \quad s = s(1) \\
&\vdots \\
&i = N_s \rightarrow k = k_1, \quad s = s(N_s) \\
&i = N_s + 1 \rightarrow k = k_2, \quad s = s(1),
\end{aligned}
$$

and so on.

Fix some value of $i \in \{1,\ldots,N_sM\}$. This determines some k, s combination. Let $\psi^{(i)}$ denote the $1 \times N_s$ vector of all possible values of $\beta\pi_{ss'}u_c'/u_c$, where s' is next period's value of s, u_c' is next period's marginal utility of consumption, $\pi_{ss'}$ is the probability of passing from s to s' and u_c is the current period's equilibrium marginal utility of consumption. Let $D_1^{(i)}$ denote the $N_s \times 1$ vector of values of r_k', next period's gross, after tax return on capital. Similarly, let $D_2^{(i)}$ denote the $N_s \times 1$ vector of r_b', next period's returns on government debt.

Then, according to (14) and the scaled version of (6),

$$
\text{(B.1)} \quad 1 = \psi^{(i)}D_1^{(i)}
$$

(B.2) $1 = \psi^{(i)}D_2^{(i)}$.

Let $k^{(i)}$ and $b^{(i)}$ denote the equilibrium values of k' and b' given state i. Let $\tilde{b}^{(i)}$ denote the $N_s \times 1$ vector of next period's equilibrium debt. The j^{th} element of $\tilde{b}^{(i)}$ is next period's debt when $s' = s(j)$. Let $d_1^{(i)}$ denote the $N_s \times 1$ vector of values of

$$c' + k'' - (1-\tau')f_\ell'\ell' - T',$$

where c' is next period's consumption, k'' is next period's capital decision, f_ℓ' is next period's marginal product of labor, and so on. Again, the j^{th} element of $d_1^{(i)}$ corresponds to $s' = s(j)$. Then, next period's budget constraint, (2), implies

(B.3) $d_1^{(i)} + \tilde{b}^{(i)} = D_1^{(i)}k^{(i)} + D_2^{(i)}b^{(i)}$.

Premultiplying (B.3) by $\psi^{(i)}$ and taking (B.1)-(B.2) into account, we get,

(B.4) $b^{(i)} = \psi^{(i)}d_1^{(i)} + \psi^{(i)}\tilde{b}^{(i)} - k^{(i)}$,

which is a matrix representation of $a(k,s;\underset{\sim}{b},\hat{g},\hat{h},\hat{n})$ in (22).

Now, let G_i denote the $1 \times M$ vector containing the interpolation weights relating the capital decision, k', to the capital grid, K. That is,

$$k^{(i)} = G_iK,$$

where K is the $M \times 1$ vector of $[k_1,k_2,\ldots,k_M]'$. (To simplify notation, we use K to denote both the set $\{k_1,k_2,\ldots,k_M\}$ and the column vector $[k_1,k_2,\ldots,k_M]'$.) For example, if $k^{(i)} = w_ik_j + (1-w_i)k_{j+1}$, then the j and $j + 1$ elements of G_i contain w_i and $(1-w_i)$ and all other terms in G_i are zero.

Let $\underset{\sim}{b}$ denote an $N_sM \times 1$ vector of parameters defining a debt rule, $\hat{\psi}$. Write

$$\text{(B.5)}\quad \underset{\sim}{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_M \end{bmatrix},$$

where b_j is an $N_s \times 1$ vector for $j = 1, \ldots, M$. The i^{th} element of b_j defines the value of $\hat{\varphi}(k_j,s_i;\underset{\sim}{b})$, $i = 1, \ldots, N_s$. Values of $\hat{\varphi}$ for values of k not on the grid, K, are defined by linear interpolation. Thus, if $k = wk_j + (1-w)k_{j+1}$, then $\hat{\varphi}(k,s;\underset{\sim}{b}) \equiv w\hat{\varphi}(k_j,s;\underset{\sim}{b}) + (1-w)\hat{\varphi}(k_{j+1},s;\underset{\sim}{b})$. It follows that,

$$\text{(B.6)}\quad \tilde{b}^{(i)} = [G_i \otimes I_{N_s}]\underset{\sim}{b}, \equiv \begin{bmatrix} \hat{\varphi}(k^{(i)},s_1;\underset{\sim}{b}) \\ \hat{\varphi}(k^{(i)},s_2;\underset{\sim}{b}) \\ \vdots \\ \hat{\varphi}(k^{(i)},s_{N_s};\underset{\sim}{b}) \end{bmatrix}.$$

In (B.6), $\otimes$ denotes the Kronecker product. Substitute (B.6) into (B.4):

$$\text{(B.7)}\quad b^{(i)} = \psi^{(i)}d_1^{(i)} - k^{(i)} + \psi^{(i)}(G_i \otimes I_{N_s})\underset{\sim}{b}.$$

Stacking (B.7),

$$\begin{bmatrix} b^{(1)} \\ b^{(2)} \\ \vdots \\ b^{(N_sM)} \end{bmatrix} = \begin{bmatrix} \psi^{(1)}d_1^{(1)} - k^{(1)} \\ \psi^{(2)}d_1^{(2)} - k^{(2)} \\ \vdots \\ \psi^{(N_sM)}d_1^{(N_sM)} - k^{(N_sM)} \end{bmatrix} + \begin{bmatrix} \psi^{(1)}(G_1 \otimes I_{N_s}) \\ \psi^{(2)}(G_2 \otimes I_{N_s}) \\ \vdots \\ \psi^{(N_sM)}(G_{N_sM} \otimes I_{N_s}) \end{bmatrix} \underset{\sim}{b},$$

or,

(B.8) $\underset{\sim}{b}' = Z + B\underset{\sim}{b}$,

in obvious notation. Here, Z is a $N_sM \times 1$ vector and B is an $N_sM \times N_sM$ matrix. Equation (B.8) is a map from the space of parameters of $\hat{\varphi}$ into itself. The fixed point is found by solving the following system of N_sM linear equations:

(B.9) $[I-B]\underset{\sim}{b} = Z$.

B.2 Value Function

Let $\nu^{(i)}$ denote the value of state $i \in \{1,\ldots,N_sM\}$. Let $\hat{\psi}^{(i)}$ denote the $1 \times N_s$ vector of all possible values of $\beta\pi_{s_is'}$, where s_i is the value of s in state i and s′ is the value of s in the next period. Let $\tilde{\nu}^{(i)}$ denote the $N_s \times 1$ vector of values associated with the N_s possible states next period. Then, we can express (27) in matrix notation as follows:

(B.10) $\nu^{(i)} = u^{(i)} + \hat{\psi}^{(i)}\tilde{\nu}^{(i)}$,

where $u^{(i)} = u(c_i,\ell_i)$ and c_i, ℓ_i are the levels of consumption and work effort implied by the equilibrium decision rules and the resource constraint. Write

(B.11) $\underset{\sim}{\nu} = \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_M \end{bmatrix}$,

where ν_j is an $N_s \times 1$ vector for $j = 1, \ldots, M$. The i^{th} element of ν_j defines the values of $\hat{\nu}(k_j,s_i;\underset{\sim}{\nu})$. Values of $\hat{\nu}$ for k not on the grid K are defined by linear interpolation. In particular, if $k = wk_j + (1-w)k_{j+1}$, then $\hat{\nu}(k,s;\underset{\sim}{\nu}) \equiv w\hat{\nu}(k_j,s;\underset{\sim}{\nu}) + (1-w)\hat{\nu}(k_{j+1},s;\underset{\sim}{\nu})$. Thus,

(B.12) $\tilde{\nu}^{(i)} = [G_i \otimes I_{N_s}]\underset{\sim}{\nu}$, where G_i was discussed above. Substituting (B.12) into (B.10),

(B.13) $\nu^{(i)} = u^{(i)} + \tilde{\psi}^{(i)}(G_i \otimes I_{N_s})\underset{\sim}{\nu}$.

Stacking (B.13),

$$\begin{bmatrix} \nu^{(1)} \\ \nu^{(2)} \\ \vdots \\ \nu^{(N_sM)} \end{bmatrix} = \begin{bmatrix} u^{(1)} \\ u^{(2)} \\ \vdots \\ u^{(N_sM)} \end{bmatrix} + \begin{bmatrix} \tilde{\psi}^{(1)}(G_1 \otimes I_{N_s}) \\ \tilde{\psi}^{(2)}(G_2 \otimes I_{N_s}) \\ \vdots \\ \tilde{\psi}^{(N_sM)}(G_{N_s}M \otimes I_{N_s}) \end{bmatrix} \underset{\sim}{\nu},$$

or,

(B.14) $\underset{\sim}{\nu}' = u + B_\nu \underset{\sim}{\nu}$.

Here, B_ν is an $N_sM \times N_sM$ matrix similar to B in (B.8) and u is the $N_sM \times 1$ vector $u = [u^{(1)}, u^{(2)}, \ldots, u^{(N_sM)}]'$. The fixed point we seek has the property $\underset{\sim}{\nu}' = \underset{\sim}{\nu}$, and so it solves the following system of linear equations,

(B.15) $[I - B_\nu]\underset{\sim}{\nu} = u$.