

INTERTEMPORAL SUBSTITUTION AND SMOOTHNESS OF CONSUMPTION*

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I. Introduction.

Background

An undisputed fact about U.S. macroeconomic data is that consumption is much less volatile than income. The traditional explanation of this is a version of Milton Friedman's Permanent Income Hypothesis (PIH) which incorporates a Trend Stationary representation for income. In a recent paper Angus Deaton has argued that the Trend model is implausible, and that income is more sensibly modelled as a Difference Stationary process. He then shows that a version of the PIH that incorporates the Difference model has the false implication that consumption is more volatile than income. This result is referred to as "Deaton's Paradox" since it contradicts widespread opinion which holds that the PIH and the relative smoothness of consumption are virtually synonymous.

Deaton's Paradox has been interpreted by some to imply that economic theory cannot account for the observed smoothness of consumption relative to income. This conclusion is unwarranted. Very small modifications of the version of the PIH which incorporates the Difference model result in implications for the relative volatility of consumption that are empirically plausible. One such modification was pointed out by Deaton himself. He shows that the Trend and Difference models are so similar that even the most sophisticated statistical techniques cannot determine which fits the data better. For this reason, Deaton's criticism of the Trend model is an intuitive one (see Deaton, p.22.) and is not based on formal statistical

reasoning. Thus, one avenue of escape from the Deaton paradox is to return to the traditional version of the PIH which incorporates the Trend model of income. A second possibility is that other aspects of the PIH are misspecified. This is suggested by results in Christiano (1987), where I show that consumption is substantially less volatile than income in a version of Gary Hansen's (1985) real business cycle model (RBC) in which equilibrium income has a Difference representation. Since both the PIH and the RBC are representative agent growth models, their differences can be reduced to differences in their specification of preferences and technology.¹ Although there are several such differences, I argued that the key difference between the PIH and RBC lies in the nature of the technology shocks assumed to underly disturbances in income. Both assume that innovations to technology shocks have a permanent effect. However, the RBC assumes that a positive shock drives up the return on investment, thereby activating both the substitution and income effects on consumption. Because these effects offset each other, the response of consumption to an innovation in income in the RBC is weak relative to what is implied by the PIH. The PIH assumes technology shocks do no affect the marginal return on capital and so they only trigger an income effect on consumption. Because shocks have a permanent effect, the income effect is very strong, accounting for the implication emphasized by Deaton that consumption is more volatile than income.

Purpose of Paper.

This paper extends the results in Christiano (1987) in several ways.

¹Sargent (1986) and Hansen (1985) describe an equilibrium interpretation of the PIH, in which preferences are quadratic in consumption and production is linear in capital, with nonstochastic marginal product of capital.

First, those results are based on decision rules obtained by solving a log linear-quadratic approximation to the RBC. Little is known about the accuracy of that approximation, raising the possibility that the results in Christiano (1987) are distorted by approximation error. This possibility is ruled out in section II which shows that the log linear-quadratic approximate solution is highly accurate. I do this by comparing it with the exact solution, obtained by a numerical method. Second, I show that the RBC implies consumption is approximately a random walk. In this sense the RBC and the PIH are quite similar, since the latter implies consumption is exactly a random walk.

The random walk hypothesis for consumption is widely thought to be falsified by the empirical evidence of correlation between quarterly consumption changes and lagged income and other variables. However, this inference depends sensitively on the assumption that the time interval of the random walk coincides with the data sampling interval. If the random walk obtains over a time interval finer than the data sampling interval, then one expects consumption changes to be correlated with lagged variables. Accordingly, section III investigates the implications of the assumption that measured data are sampled averages of data generated from a version of the RBC with a timing interval finer than the data sampling interval. I show that this time aggregated RBC implies an empirically plausible amount of serial correlation for consumption and income. This is consistent with results in Christiano, Eichenbaum and Marshall (1987), who show there is surprisingly little evidence against a continuous time version of the PIH. Section IV argues that time aggregation effects improve the ability of the RBC to account for the Deaton Paradox. Section V concludes the paper.

II. The RBC Model.

This section describes the RBC model and reports some of its second moment implications. In particular, I show that the model implies an empirically plausible amount of volatility of consumption relative to income. Weaknesses of the model include its counterfactual implications that log consumption is approximately a random walk and that output growth is nearly serially uncorrelated. It is shown that it makes virtually no difference whether the model's second moment implications are deduced using the exact or the approximate solution method.

The Model

At date t a representative agent chooses contingency plans for $\{k_{t+j}, h_{t+j}, c_{t+j}, y_{t+j}, dk_{t+j} \geq 0; j=0,1,2,\dots\}$ to maximize

$$(1) \quad E_t \sum_{j=0}^{\infty} \beta^j \{ \ln(c_{t+j}) - \gamma h_{t+j} \}, \quad 0 < \beta < 1, \gamma > 0,$$

subject to $h_{t+j} \leq H$,

$$(2) \quad c_t + dk_t \leq y_t = n^{-\theta} (z_t h_t)^{(1-\theta)} k_{t-1}^{\theta}$$

a given level of k_{t-1} and the following law of motion for z_t :

$$(3) \quad z_t = z_{t-1} \exp(x_t).$$

Here y_t , c_t , k_t , h_t denote gross output, total consumption, end-of-period capital, and hours, respectively. Also, $dk_t \equiv k_t - [(1-\delta)/n]k_{t-1}$ denotes gross capital investment. All variables are in per capita terms. According to (2), gross output is related by a Cobb-Douglas function to hours, capital, and the exogenous productivity term, z_t . The parameter $\theta \in (0,1)$, is the share of capital in income, and $\delta \in (0,1)$ is the depreciation rate on capital. The parameter, n , is the gross rate of population growth. Also, x_t is an independently, identically distributed, discrete random variable with mean $\mu > 0$ and standard deviation σ_x . The distribution of x_t is discussed later.

The risk free rate of interest is defined as follows:

$$(4) \quad 1 + r_t = [d \log(c_t) / dC_t] / [\beta E_t d \log(c_{t+1}) / dC_{t+1}],$$

where $C_t \equiv c_t N_t$ is total consumption and N_t is the population in quarter t . Here, $N_t / N_{t-1} = n$.

Appendix A describes the exact and log linear-quadratic methods used to compute contingency plans for h_t , k_t , c_t , y_t , dk_t . There it is shown that the model has the balanced growth property that each of k_t , c_t , y_t and dk_t can be represented as the product of z_t and a covariance stationary stochastic process. As a result, these variables inherit z_t 's property of being a Difference Stationary process in logs. By contrast, h_t and r_t are predicted to be covariance stationary.

Parameter Values

To deduce the model's second moment implications, values must be assigned to its parameters. I chose the following: $\mu = .003589$, $\beta = .99$, $\delta = .018$, $n =$

1.00324, $\theta = .39$, $\gamma = .00263$, $\sigma_x = .019$.² The value of n is the average quarterly growth in the quality adjusted, working-age population in 1952-84. With this value, $\delta = .018$ is required if the gross investment series implied by $dk_t = k_t - [(1-\delta)/n]k_{t-1}$ using U.S. capital stock data is to resemble the gross investment series published by the U.S. Department of Commerce. The chosen values for θ , γ and μ have the effect that the model's implications for the average values of h_t , c_t/y_t and k_t/y_t roughly match their empirical counterparts in U.S. data for 1956,2-1984,1. The implied averages (and empirical values) for these variables are 320.7 (320.4), .72 (.72), and 11.21 (10.58), respectively. These parameter values imply an average value for r_t of .017 per quarter. Given the values assigned to θ and n , and using data on h_t , y_t , k_t , a time series on x_t can be computed using the production function. The chosen values of μ and σ_x are the sample mean and standard deviation, respectively, of these x_t 's.

It remains only to specify the remaining features of the distribution of the x_t 's. As a check on the robustness of the calculations, four were used. In all cases, the distributions are restricted to be symmetric.³ In particular, denote the possible values of x_t by $x(i)$, with probability $\pi(i)$, $i=1, \dots, M$. Of the four distributions, three set $M = 3$, and the other, $M = 5$. The details for the five and three point distributions appear in Table 1 and Table 2, respectively. The three 3 point distributions are distinguished by the degree of kurtosis, κ , they imply. (The restrictions in Table 2 require α

²The approximate decision rules for k_t and h_t implied by these parameter values are $k_t = z_t \exp\{9.728593363 + .94889790842[\log(k_{t-1}/z_{t-1}) - 9.728593363 - (x_t - .003589)]\}$ and $h_t = \exp\{5.770449327 - .4531385034[\log(k_{t-1}/z_{t-1}) - 9.728593363 - (x_t - .003589)]\}$

³The skewness statistic based on the 111 computed x_t 's is -.355, which is not significantly different from zero.

$$= \kappa^{(1/2)}.)$$

Second Moment Results

Second moment results are reported in Tables 3 and 4, based on the four shock distributions, and the two methods for computing the contingency plans. A notable feature of the results in the Tables is that the second moment properties are virtually insensitive to the form of the x_t distribution, despite the substantial variety of distributions tried. Second, the approximate and exact results are very similar. In view of the extreme disparity in computational costs, this should stimulate interest in the approximate solution method. To be sure, there are some discrepancies between approximate and exact solutions, but one has to look carefully to find them: The approximate method seems consistently to overstate the sampling variation in σ_c/σ_y , σ_{dk}/σ_y , and to understate the sampling variation in $\rho_{r,\Delta c}(0)$. Despite these minor discrepancies, the approximate and exact solutions give essentially the same picture regarding the empirical performance of the model. In particular, the model predicts precisely the amount of relative consumption volatility observed in the data. (See the last column in Table 3 for the empirical results.) On other dimensions the model does less well. In particular, capital fluctuates a bit too much, and hours too little, relative to what is observed. Moreover, the relative volatility in real interest rates is about 23 standard deviations lower than the empirical measure, and the correlation between consumption growth and interest rates in the RBC is too high. It is an open question whether this reflects a failure of the model, or of the empirical measure of interest rates.

A particularly interesting feature of the RBC model is its implication

that the log of consumption is approximately a random walk. The tables report the RBC's implied correlation of consumption growth with the first lagged value of every other variable in the model. The mean value of these objects is always close to zero, and they display considerable sampling variation. The approximate random walk implication for consumption presumably reflects the very small amount of variation in interest rates implied by the model.

Another empirical shortcoming of the RBC is its implication that output growth is roughly a random walk. As shown in Christiano (1987), this reflects that, in this model, the dynamic properties of output growth closely mimic the dynamic properties of the growth of z_t , which is serially uncorrelated.

III. The Time Aggregated RBC Model.

A major empirical failing of the quarterly RBC is its implication that log consumption is approximately a random walk. On the other hand, it has been known at least since Working (1960) that if a process is a random walk over a time interval finer than the data sampling interval, then the time averaged and sampled process is not a random walk. This follows from the fact that in this case the change in measured consumption between quarter t and quarter $t+1$ is composed of the uncorrelated revisions to consumption effected by households from the beginning of quarter t to the end of quarter $t+1$. Similarly, the difference between measured consumption in quarter t and quarter $t-1$ is composed of revisions to consumption from the beginning of $t-1$ to the end of quarter t . Because of the overlap of revisions in quarter t , the change in measured consumption is correlated at lag one. A simple extension of this argument establishes that measured consumption changes are correlated with other information lagged 1 period, but uncorrelated with all information lagged 2 periods and more. Using several measures of consumption and income, and several sample periods, Christiano, Eichenbaum and Marshall (1987) test these implications and find them to be reasonably consistent with the data. For these reasons, it seems promising to investigate the empirical performance of a time aggregated version of the RBC.

I considered versions of the RBC with timing interval equal to $1/N$ quarters, for $N = 2, 4, \text{ and } 8$. The parameters of the RBC which are time dependent are β , σ_x , μ , γ , δ , and n . The values of these parameters in the time aggregated RBC are related to the $N = 1$ version of the model by:

$$(5) \quad \beta^{(N)} = .99^{1/N}, \quad \sigma_x^{(N)} = .019/N, \quad \mu^{(N)} = .003589/N, \quad \gamma^{(N)} = .00263 \times N,$$

$$\sigma^{(N)} = 1.00324^{1/N}, \delta^{(N)} = 1 - (1 - .018)^{1/N}$$

where the superscript (N) signifies that the associated parameter belongs to the RBC with timing interval N. Contingency plans for the time aggregated RBC were computed using the approximation method. They were used to simulate 1000 artificial data sets, each of length $112 \times N$, on consumption, income, capital investment, hours worked, output, and the one period real interest rate, measured at a quarterly rate. An N period moving sum of each data set other than the real rate of interest was then computed, and every N-th observation of the resulting series was sampled and stored for subsequent use. The real interest rate was simple sampled point-in-time. This produced 1000 data sets on "measured" consumption, income, capital investment, etc., each of length 112.⁴ I then computed the first and second moments of the same 13 statistics studied in Tables 3 and 4. The simulations were executed by drawing from the distribution of x_t 's appearing in Table 1. The results for N = 2, 4, 8 are reported in Table 5. Results for N = 1 and based on the U.S. data are also reproduced in the Table for convenience. These were taken from the first and last columns of Table 3, respectively.

Several things stand out in Table 5. First, the ratios of standard deviations are not much affected by time aggregation. In particular, the model continues to imply an empirically plausible amount of consumption volatility. On the other hand, the approximate random walk behavior of consumption that appears in the +ROR column disappears as N increases. Moreover, the correlation between consumption growth and variables lagged one period is empirically plausible in most cases. In addition, the first order

⁴An implicit assumption is that real returns are measured point-in-time. This is in fact not the case, since point-in-time nominal returns are adjusted using a price index that is best thought of as an average of prices over the sampling interval.

serial correlation properties of output growth is also close to its corresponding empirical value. In fact, time aggregation has improved the empirical performance of the model on almost all dimensions. An exception is the correlation between consumption growth and real returns, which is further from the corresponding empirical measure. This has to be interpreted with caution, however, since the model's interest rate is a one period real return (measured at a quarterly rate), which is not a one quarter return when N exceeds 1.

IV. The Time Aggregated RBC and Deaton's Paradox.

[to be added...]

V. Conclusion.

[to be added...]

Table 1: Five Point Distribution of x_t 's.

$$\begin{aligned}x(1) &= \mu - \alpha\sigma_x, \quad x(2) = \mu - \sigma_x, \quad x(3) = \mu, \quad x(4) = \mu + \sigma_x, \quad x(5) = \mu + \alpha\sigma_x \\ \sum_i \pi(i)(x(i)-\mu)^2 &= \sigma_x^2, \quad \sum_i x(i)\pi(i) = \mu, \quad [\sum_i \pi(i)(x(i)-\mu)^4] / \sigma_x^4 = \kappa \\ \pi(1) &= \pi(5), \quad \pi(2) = \pi(4), \quad \alpha = 10, \quad \kappa = 6.\end{aligned}$$

Table 2: Three Point Distributions for x_t

$$\begin{aligned}x(1) &= \mu - \alpha\sigma_x, \quad x(2) = \mu, \quad x(3) = \mu + \alpha\sigma_x, \quad \pi(1) = \pi(3) \\ \sum_i \pi(i)(x(i)-\mu)^2 &= \sigma_x^2, \quad [\sum_i \pi(i)(x(i)-\mu)^4] / \sigma_x^4 = \kappa \\ \kappa &= 3, 6, 20.\end{aligned}$$

Table 3: Selected Second-Moment Properties.^a

Statistic ^b	5 point, $\kappa = 6$		3 point, $\kappa = 6$		U.S. ^d Estimates
	Approximation Mean (stdv) ^c	Exact Mean (stdv)	Approximation Mean (stdv)	Exact Mean (stdv)	
σ_c/σ_y	.49 (.031)	.50 (.012)	.49 (.034)	.49 (.012)	.49
σ_{dk}/σ_y	2.40 (.171)	2.38 (.091)	2.41 (.183)	2.38 (.095)	1.91
σ_h/σ_y	414.1 (99.8)	413.7 (99.3)	412.3 (97.4)	412.3 (97.3)	669.6
σ_r/σ_y	.088 (.021)	.104 (.017)	.088 (.021)	.105 (.018)	.561
σ_y	.017 (.002)	.017 (.002)	.017 (.002)	.017 (.002)	.012
Er_t	.017 (.001)	.017 (.001)	.017 (.001)	.017 (.001)	.002
$\rho_{r,\Delta c}(0)$.493 (.034)	.331 (.086)	.492 (.035)	.351 (.077)	.085
$\rho_{\Delta c,\Delta c}(1)$.066 (.106)	.053 (.105)	.060 (.106)	.052 (.103)	.271
$\rho_{\Delta c,\Delta y}(1)$.053 (.099)	.050 (.099)	.047 (.099)	.048 (.098)	.204
$\rho_{\Delta c,h}(1)$.086 (.126)	.084 (.124)	.085 (.124)	.083 (.120)	-.057
$\rho_{\Delta c,\Delta dk}(1)$.044 (.095)	.047 (.096)	.039 (.095)	.045 (.096)	.161
$\rho_{\Delta c,r}(1)$.087 (.126)	.153 (.104)	.086 (.124)	.147 (.110)	.104
$\rho_{\Delta y,\Delta y}(1)$	-.001 (.095)	-.002 (.095)	-.006 (.095)	-.008 (.095)	.361
CPU time ^e	0	432	0	257	

^aSecond moments computed by simulating 1000 sample paths, each of length 112 observations, using either the exact or approximate the contingency plan, as indicated, with the x_t 's drawn randomly from the indicated 5 or 3 point discrete distribution.

^b σ , E are the variability and mean, respectively, of the indicated variable; $\rho_{u,v}(\tau)$ is the correlation between $u(t)$ and $v(t-\tau)$, $\tau=0,1$; and $\Delta u(t)$ denotes $\log u(t) - \log u(t-1)$.

^cMean and stdv denote the mean and standard deviation of the indicated statistic, across 1000 realizations.

^dBased on 112 quarters, 1956,2-1984,1.

^eTotal CPU time, in minutes, needed to compute contingency plans on IBM 3033 mainframe, where 0 means less than .00 minutes.

Table 4: Selected Second-Moment Properties.^a

Model Simulations (3 point shock distribution)

Statistic	$\kappa = 3$		$\kappa = 20$	
	Approximation Mean (stdv)	Exact Mean (stdv)	Approximation Mean (stdv)	Exact Mean (stdv)
σ_c/σ_y	.49 (.031)	.50 (.011)	.49 (.043)	.50 (.012)
σ_{dk}/σ_y	2.40 (.162)	2.38 (.085)	2.42 (.243)	2.38 (.122)
σ_h/σ_y	410.1 (98.9)	410.3 (98.9)	415.0 (90.4)	415.5 (89.9)
σ_r/σ_y	.088 (.021)	.102 (.018)	.088 (.019)	.107 (.019)
σ_y	.017 (.001)	.017 (.001)	.016 (.004)	.016 (.004)
Er_t	.017 (.001)	.017 (.001)	.017 (.001)	.017 (.001)
$\rho_{r,\Delta c}(0)$.492 (.033)	.346 (.081)	.493 (.040)	.314 (.091)
$\rho_{\Delta c,\Delta c}(1)$.058 (.104)	.049 (.104)	.064 (.096)	.047 (.093)
$\rho_{\Delta c,\Delta y}(1)$.045 (.096)	.045 (.096)	.050 (.088)	.043 (.088)
$\rho_{\Delta c,h}(1)$.081 (.126)	.079 (.124)	.090 (.121)	.086 (.116)
$\rho_{\Delta c,\Delta dk}(1)$.038 (.091)	.042 (.093)	.041 (.083)	.040 (.086)
$\rho_{\Delta c,r}(1)$.082 (.126)	.139 (.109)	.091 (.121)	.158 (.107)
$\rho_{\Delta y,\Delta y}(1)$	-.007 (.091)	-.008 (.091)	-.004 (.084)	-.002 (.085)
CPU time (minutes)	0	251	0	245

^aSee notes to Table 3.

Table 5: Selected Second-Moment Properties.^a

Statistic ^b	Time Aggregated RBC				U.S. ^d Estimates
	N = 1	N = 2	N = 4	N = 8	
	Mean (stdv) ^c	Mean (stdv)	Mean (stdv)	Mean (stdv)	
σ_c/σ_y	.49 (.031)	.50 (.023)	.49 (.019)	.49 (.016)	.49
σ_{dk}/σ_y	2.40 (.171)	2.38 (.106)	2.38 (.076)	2.38 (.056)	1.91
σ_h/σ_y	414.1 (99.8)	473.3 (114.)	494.0 (116.)	498.3 (112.)	669.6
σ_r/σ_y	.088 (.021)	.100 (.023)	.103 (.023)	.103 (.022)	.561
σ_y	.017 (.002)	.010 (.001)	.007 (.001)	.005 (.0004)	.012
Er_t	.017 (.001)	.017 (.001)	.017 (.0004)	.017 (.0003)	.002
$\rho_{r,\Delta c}(0)$.493 (.034)	.557 (.033)	.576 (.033)	.579 (.033)	.085
$\rho_{\Delta c,\Delta c}(1)$.066 (.106)	.234 (.101)	.287 (.099)	.305 (.098)	.271
$\rho_{\Delta c,\Delta y}(1)$.053 (.099)	.221 (.093)	.276 (.091)	.295 (.092)	.204
$\rho_{\Delta c,h}(1)$.086 (.126)	.160 (.124)	.180 (.124)	.185 (.118)	-.057
$\rho_{\Delta c,\Delta dk}(1)$.044 (.095)	.209 (.089)	.264 (.086)	.285 (.089)	.161
$\rho_{\Delta c,r}(1)$.087 (.126)	.221 (.108)	.290 (.097)	.324 (.085)	.104
$\rho_{\Delta y,\Delta y}(1)$	-.001 (.095)	.159 (.090)	.212 (.088)	.231 (.091)	.361

^aSecond moments computed by simulating 1000 sample paths, each of length 112 observations on sampled, summed data, using either the exact or approximate the contingency plan, as indicated, with the x_t 's drawn randomly from the 5 point discrete distribution. See section III for further explanation.

^b σ , E are the variability and mean, respectively, of the indicated variable; $\rho_{u,v}(\tau)$ is the correlation between $u(t)$ and $v(t-\tau)$, $\tau=0,1$; and $\Delta u(t)$ denotes $\log u(t) - \log u(t-1)$.

^cMean and stdv denote the mean and standard deviation of the indicated statistic, across 1000 realizations.

^dBased on 112 quarters, 1956,2-1984,1.

References

- Bertsekas, Dimitri P., 1976. Dynamic Programming and Stochastic Control. Academic Press.
- Christiano, Lawrence J., 1986. "Why Does Inventory Investment Fluctuate So Much?" paper presented at Catholic University of Lisbon/University of Rochester Conference on Real Business Cycles, Lisbon, Portugal.
- Christiano, Lawrence J., 1987. "Is Consumption Excessively Insensitive to Innovations in Income?", American Economic Review Papers and Proceedings.
- Christiano, Lawrence J., Martin Eichenbaum and David Marshall, 1987. "The Permanent Income Hypothesis Revisited", Federal Reserve Bank of Minneapolis, October.
- Deaton, Angus, 1986. "Life-Cycle Models of Consumption: Is the Evidence Consistent With the Theory?", NBER Working Paper No. 1910, April.
- Hansen, Gary D., 1985. "Indivisible Labor and the Business Cycle." Journal of Monetary Economics, November, 16, 309-27.
- Hansen, Lars Peter, 1985. "Econometric Modeling of Asset Prices Under Rational Expectations," paper presented to the Fifth World Congress of the Econometric Society.
- Sargent, Thomas J., 1986. "Equilibrium Investment Under Uncertainty: Measurement Errors and the Investment Accelerator," Hoover Institution, Stanford University.
- Working, H., 1960. "Note on the Correlation of First Differences of Averages in a Random Chain." Econometrica, 28: 916-918.

Appendix A: Solving the RBC Model.

According to (3), $\log(z_t)$ is a random walk with positive drift, and so z_t is expected to grow without bound. As a consequence, optimal y_t , k_t and c_t are also Difference Stationary processes in logs, expected to grow without bound. These facts make it difficult to solve the model directly using the techniques applied in this paper. Instead, it is convenient to first transform the problem into an alternative, equivalent, form in which the optimal values of the decision variables do not exhibit growth. The transformation exploits the "balanced growth" property of this model: although the levels of c_t , y_t , k_t grow without bound, their ratio to z_t is stationary.

Because the return function is unbounded above, there is no loss of generality in replacing the weak inequality in (2) by an equality. Equation (2) can then be used to eliminate consumption as an independent decision variable. Accordingly, substitute out for c_t in (1) from (2) to get the following objective problem:

$$E_t \sum_{j=0}^{\infty} \beta^j \{ \ln[n^{-\theta} (z_{t+j} h_{t+j})^{(1-\theta)} k_{t-1+j}^{\theta} + \frac{1-\delta}{n} k_{t-1+j} - k_{t+j}] - \gamma h_{t+j} \}.$$

Next, factor z_{t+j} from the expression in square brackets to yield the following alternative representation of the objective function:

$$(A.1) \quad E_t \sum_{j=0}^{\infty} \beta^j \{ u(h_{t+j}, \tilde{k}_{t+j}, \tilde{k}_{t+j-1}, x_{t+j}) + \ln(z_{t+j}) \},$$

where,

$$(A.2) \quad u(h_t, \tilde{k}_t, \tilde{k}_{t-1}, x_t) = \ln[n^{-\theta} h_t^{(1-\theta)} \exp(-\theta x_t) (\tilde{k}_{t-1})^\theta] \\ + [(1-\delta)/n] \exp(-x_t) \tilde{k}_{t-1} - \tilde{k}_t - \gamma h_t$$

and

$$(A.3) \quad \tilde{k}_t \equiv k_t / z_t.$$

The original optimization problem can be posed in terms of equations (A.1)-(A.3) as follows: Maximize (A.1) over contingency plans for $\{\tilde{k}_{t+j}, h_{t+j} \geq 0; j=0,1,2,\dots\}$ subject to the law of motion for x_t , and the nonnegativity constraint on c_t . The optimal decision rule for k_t is obtained by multiplying the decision rule for \tilde{k}_t by z_t .

Solution by Numerical Dynamic Programming

The representation of the problem in (A.1)-(A.3) is well suited to solution by numerical dynamic programming methods since \tilde{k}_t and h_t are stationary processes which, under certain circumstances, can be restricted without loss of generality to fall in a closed, bounded set. Denote this set by $G = [0, H] \times [\underline{k}, \bar{k}]$, where $0 \leq \underline{k} < \bar{k} < \infty$.⁵ Our numerical procedures required

⁵A sufficient condition that the restriction $(\tilde{k}_t, h_t) \in G$ be nonbinding is that the smallest possible realization of x_t exceed $\log[(1-\delta)/n]$. In this case there is a $0 < \tilde{k}^* < \infty$ such that for any initial $0 < \tilde{k}_{t-1} \leq \tilde{k}^*$, all possible

further restricting G to the intersection of $[0, H] \times [\underline{k}, \bar{k}]$ and a finite set of grid points. The dynamic programming formulation of the problem is the following:

$$(A.4) v(\tilde{k}_{t-1}, x_t) = \max_{(\tilde{k}_t, h_t) \in G \cap A(\tilde{k}_{t-1}, x_t)} \{u(h_t, \tilde{k}_t, \tilde{k}_{t-1}, x_t) + \beta E_{x_{t+1}} v(\tilde{k}_t, x_{t+1})\},$$

where A constrains $c_t \geq 0$:

$$(A.5) A(\tilde{k}_{t-1}, x_t) = \{(\tilde{k}_t, h_t) : \tilde{k}_t \leq n^{-\theta} \exp(-\theta x_t) \tilde{k}_{t-1} h_t^{\theta(1-\theta)} + \frac{1-\delta}{n} \exp(-x_t) \tilde{k}_{t-1}\}.$$

A solution to the problem is a set of functions $\tilde{k}_t = k(\tilde{k}_{t-1}, x_t)$, $h_t = h(\tilde{k}_{t-1}, x_t)$ that solve (A.4). These were obtained by first computing v using the method of successive approximations described in Bertsekas (1976, p.237).⁶

The k and h functions are the argmax of the maximization problem to the right of the equality in (A.4). The solution to the original problem is then

feasible subsequent \tilde{k}_t 's also belong to $[0, \tilde{k}^*]$. (As a result, $[0, \tilde{k}^*]$ forms an ergodic set for \tilde{k}_t .) This may be verified by studying equation (2) expressed in terms of $\tilde{c}_t \equiv c_t/z_t$, h_t , \tilde{k}_t , x_t , taking into account the nonnegativity constraint on \tilde{c}_t , and the restriction $h_t \in [0, H]$. Therefore, there is no loss of generality in limiting \tilde{k}_t to $[\underline{k}, \bar{k}]$ for $\underline{k} = 0$ and $\bar{k} \geq \tilde{k}^*$. The other variable, $h_t \in [0, H]$ by construction.

⁶The successive approximations were carried out by starting with an initial v function, say v_0 , and then carrying out the calculations indicated on the right of the equality in (A.4) and calling the result v_1 . This was repeated, this time beginning with v_1 , and ending with a new value function, v_2 . Proceeding in this way, we obtained a sequence, v_j , $j = 1, 2, 3, \dots$. The function to which this sequence converged (in the sup norm sense) was taken as the solution to the functional equation, (A.4).

$$(A.6) \quad k_t = z_t k[(k_{t-1}/z_{t-1}), x_t], \quad h_t = h[(k_{t-1}/z_{t-1}), x_t].$$

Decision rules for y_t , c_t , and gross investment, $dk_t \equiv k_t - [(1-\delta)/n]k_{t-1}$, were obtained by substituting (A.6) into (2). Given initial values for k_{t-1} and z_{t-1} , these decision rules can be used to compute second moment properties of the endogenous variables of the model. Monte Carlo simulation methods were used for this.

In order that our computed second moments correspond to those of the version of the model in which the endogenous variables can take on a continuum of values, we repeated the above procedure for larger and larger intervals $[k, \bar{k}]$ and finer grids. We stopped when we found virtually no further change in the implied second moment properties.

Solution by Log Linear-Quadratic Approximation.

Following is the log linear-quadratic method for approximating k and h . To allow application of tools for solving linear-quadratic control problems, nonnegativity constraints are ignored throughout. First, replace the stochastic problem defined by (A.1)-(A.2) with the nonstochastic version in which the variance of x_t is zero. Then, use the first order necessary conditions for an interior optimum of (A.1)-(A.2) to compute steady state values of \tilde{k}_t and h_t . Denote these by \tilde{k}_s and h_s , respectively. Write

$$(A.7) \quad k^* = \log(\tilde{k}_s), \quad h^* = \log(h_s), \quad k_t^* = \log(\tilde{k}_t), \quad h_t^* = \log(h_t).$$

Use (A.7) to rewrite the return function, (A.2), as follows:

$$(A.8) \quad u^*(h_t^*, k_t^*, k_{t-1}^*, x_t) \equiv u[\exp(h_t^*), \exp(k_t^*), \exp(k_{t-1}^*), x_t]$$

Next, let U^* denote the second order Taylor series expansion of u^* about $h_t^* = h^*$, $k_t^* = k^*$, $k_{t-1}^* = k^*$, and $x_t = \mu$. We now have a log linear-quadratic version of (A.4):

$$(A.9) \quad V^*(k_{t-1}^*, x_t) = \max_{(k_t^*, h_t^*) \in \mathbb{R}^2} \{U^*(h_t^*, k_t^*, k_{t-1}^*, x_t) + \beta E_{x_{t+1}} V^*(k_t^*, x_{t+1})\}.$$

The solution to this problem, which is trivial to compute, is a set of decision rules: $k_t^* = K^*(k_{t-1}^*, x_t)$ and $h_t^* = H^*(k_{t-1}^*, x_t)$. These are used to obtain decision rules for the variables of interest, k_t and h_t , as follows:

$$(A.10) \quad k_t = z_t \exp\{K^*[\log(k_{t-1}/z_{t-1}), x_t]\}, \quad h_t = \exp\{H^*[\log(k_{t-1}/z_{t-1}), x_t]\}$$

In conjunction with (2), (A.10) can be used to compute implied decision rules for c_t , y_t and dk_t . Like (A.6), (A.10) can be used to compute second moment properties of the variables of the model once initial values for k_{t-1} and z_{t-1} are specified.