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Three Contributions to the Theory of Decision Under Uncertainty

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Preface

This working paper consists of three studies, dealing with the relationship between expected utility theory and some of its alternatives. These alternative theories have begun to be used for modelling and estimation in areas such as macroeconomics and financial economics. (Examples include Attanasio and Weber 1989 and Epstein and Zin 1989.) We hope that our research will contribute to better understanding the issues regarding such models.

"Bayes Contingent Plans," by Green and Park, provides a new revealed-preference characterization of subjective expected-utility maximization.[1] This characterization applies to choice among arbitrary actions, rather than being restricted (as are previous characterizations) to choice among lotteries or actions that can be redefined as lotteries. We show that our axiomatization has only limited ability to discriminate between preferences based on expected utility versus maximin utility, but we show via an example that it can discriminate between expected utility and another alternative, known as "weighted utility." Weighted utility is fairly closely related to the specifications used in the econometric work cited above.

"A Revealed-Preference Implication of Weighted-Utility Decisions under Uncertainty," by Park, contains a strong result that complements the example just described. Roughly speaking, the result is that an objective-probability analogue of our axiomatic characterization succeeds in discriminating between expected-utility-maximizing and weighted-utility-maximizing patterns of choice except in a few degenerate cases. Moreover, this result is proved in a constructive way that has clear implications for how an econometric study of alternative preference specifications might be constructed.

"Reconciling Some Conflicting Evidence on Decision Making under Uncertainty," by Green, uses our revealed-preference characterization provide one possible explanation of a scientific puzzle.[2] Expected-utility models of economic agents solving naturally occurring problems seem to fit data closely and to impute a high degree of success to agents in acting rationally. In contrast, researchers who conduct laboratory experiments report that subjects' behavior is often inconsistent with expected-utility maximization in obvious and systematic ways. The proposed resolution of this paradox relies on features that our specific to our expected-utility representation theorem.

References

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Epstein, L., and S. Zin, "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: an Empirical Analysis," *Journal of Political Economy* **99**, 1991, 263–86.

[1] This paper is available in both TeX and postscript formats from the Economics Working Paper Archive at Washington University. The reference number of the paper is ewp-game/9307002. The archive can be accessed at econwpa.wustl.edu by email or ftp, or via the World Wide Web.

[2] This paper is also available from the Washington University archive. Its reference number is ewp-game/9509002.

Bayes Contingent Plans*

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Abstract

An intuitively natural consistency condition for contingent plans is necessary and sufficient for a contingent plan to be rationalized by maximization of conditional expected utility. One alternative theory of choice under uncertainty, the weighted-utility theory developed by Chew Soo Hong (1983) does not entail that contingent plans will generally satisfy this condition. Another alternative theory, the minimax theory as formulated by Savage (1972), does entail the consistency condition (at least for singleton-valued plans).

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1. Introduction

Social scientists must undertake the formidable task of modelling agents who are trying simultaneously to learn about their environment and to change it. For those social scientists who impose on themselves the intellectual discipline of articulating their ideas as explicit formal models, L. J. Savage's ([1954] 1972) Bayesian decision-theoretic characterization of coherent decision making and learning under uncertainty has become the benchmark representation of this process of simultaneous learning and action. In the context of this symposium, it is fitting to mention the field of industrial organization as an example. The study of strategic interaction among producers in an industry has been conducted primarily in the context of Bayesian game-theoretic models during the past two decades. Cyert and DeGroot (1970) introduced the Bayesian model to this field.

In this paper, we consider the specification of Bayesian decision theory as a "behavioral" theory—a term strongly emphasized by Savage in statistics as well as by Cyert, Simon, and others in economics. A theory of decision is to be considered behavioralistic if it has to do with predicting how agents make decisions or with advising agents about how to make decisions, rather than with agents' introspection about their decisions. Savage emphasizes the sharp contrast in this respect between his approach and the "verbalistic" approach taken by most of his contemporaries in statistics.

Savage recognizes (especially in the concluding paragraphs of Chapter 2.5 and in Chapter 5.5 of [1954] 1972) that the practical employment of his theory by a researcher would require the researcher's own understanding of which consequences are the salient ones. This burden on the researcher is exacerbated by the status of acts in Savage's theory. Savage does not take acts to be logically primitive entities. Rather, he defines them in terms of two other types of entity: consequences and states of nature. He takes the position that the identification of salient consequences "is an operation in which we all have much experience, and one in which there is in practice considerable agreement," but he acknowledges that "what are often thought of as consequences ... are in reality highly uncertain ... [and] can perhaps never be well approximated." ([1954 1972, pp. 16–17, 84.)

In this paper, we address Savage's problem by formulating a new version of Bayesian subjectivist decision theory . In this theory, acts and observations of evidence are the logically primitive terms. States of nature are defined to be infinite sequences of observations. The utility of an act is defined formally to be a state-dependent function of the act itself, so that reference to consequences is avoided completely. Our main object of study is a *contingent plan* which completely specifies an agent's plan about which act to perform if the choice of act were to be based on various possible sequences of observations.

After having set forth this theory, we investigate its strength. First we provide a necessary and sufficient condition for a contingent plan to be rationalizable in terms of maximization of conditional expected utility (that is, in terms of the Bayesian decision-theoretic criterion). Then we show that this condition is also satisfied by single-valued plans (that is, plans that recommend a unique act for each observation sequence) that can be rationalized by either a minimax-loss or minimax-regret criterion. In view of the heavy emphasis that Savage placed on the distinction between Bayesian and minimax decision making, this result shows a clear sense in which our behavioral formulation of the Bayesian theory has only weak observational implications. However, we also show that the observational implications of this formulation can fail to be satisfied by a contingent plan determined according to Chew's (1983) weighted-utility criterion. Thus the formulation is sufficiently strong to serve as a basis for a study of whether an actual decision maker would conform more closely to the Bayesian theory or to a prominent alternative theory.

We wish to emphasize that the equivalence result of this paper is proved in a finite-state, finite-action setting. It is an open question whether or not this result extends to a wider class of models. Also, we wish to make it clear that we do not view behavioralistic theories as being necessarily the only theories of value for social science. (See Chomsky (1957), for instance, for an argument that agents' judgments and intuitions constitute important non-behavioral evidence to be predicted and explained.)

2. Bayesian concept of rationality in decisions

The Bayesian decision-theoretic concept of rationality incorporates two principles. First, in any situation an agent ought to maximize expected utility. Second, the probability measures with respect to which expected utility is computed in successive situations ought to be related by conditionalization on whatever evidence the agent may have observed in their interim. This paper provides a representation theory for such a two-part Bayesian concept of rationality. The theory contemplates an agent who has available a set of feasible actions. The agent makes a sequence of observations. Before he begins to make these observations, and subsequently after each observation that he makes, the agent reports which action he would be most inclined to take on the basis of what he has observed to date. (If the agent considers several actions to be tied for the best, he will report all of them.) A *contingent plan* is a record of such a sequence of reports for every possible sequence of observations. The main contribution of this paper is to show that a necessary and sufficient condition (to be called *consistency*) exists for a contingent plan to be rationalizable in terms of conditional expected utility.

A simple example will illustrate the sort of contingent plan that would violate this necessary and sufficient condition. Consider someone who can choose either to go to a concert or to go to the opera or to stay at home (the feasible actions). Suppose that this person knows what is the program of the concert, but is uncertain which of two operas is being performed. He is initially inclined to go to the opera. However, regardless of which opera is announced in the newspaper (the possible outcomes of making an observation), upon reflection, he is inclined to stay at home. The pattern of decisions that is envisioned in this example is very specific. Prior to making an observation, the agent would choose a particular action if he had to make an immediate choice. After making the observation, the agent would choose a different action which is always the same, regardless of exactly what he observes. Call a contingent plan *consistent* if it does not stipulate such a pattern of decisions at any point.

The example violates Bayes rationality because the prior expectation of conditional expected utility must be equal to prior expected utility. Regardless of what he reads in the newspaper, the agent's subsequent inclination would imply that the conditional expected utility of staying at home must be greater than the conditional expected utility of going to the opera. If that were always so, then the prior expected utility of staying at home should have been greater than the prior expected utility of going to the opera. Thus, contrary to the example, the agent should initially have been inclined to stay at home rather than to go to the opera.

The setting of these results is closely related to that of the characterization by Green and Osband (1991) of expected-utility maximization. The main difference is that, while Green and Osband assume directly that an agent's decision is a function of a probability measure, here it is only assumed that the agent's decision is a function of what the agent has observed. It may be helpful also to mention some ways in which the theorem to be proved here differs from representation theorems for subjective expected utility such as that of Savage ([1954] 1972). As mentioned in Section 1, actions are taken here to be primitive entities of the theory. Savage interprets them to be functions from a specified set of states of the world to a set of consequences. The contingent plans studied here specify only the agent's optimal actions. Savage specifies the agent's preferences among suboptimal actions as well. Conditionalization of probabilities is represented explicitly here. Savage represents conditionalization implicitly in the context of specific assumptions about features of preferences (including "state-independence") and about the structure of the set of actions. The probability measure and utility function shown here to rationalize a contingent plan that satisfies the sufficient condition for Bayes rationality are not necessarily unique. Savage's assumptions imply their uniqueness.

3. Formalization of the problem

Consider an agent who observes at each date $t \in \mathbb{N}_+$ a piece of evidence $x_t \in \mathbf{X}$, where $\mathbf{X} = \{1, \ldots, m\}$ is a finite set. At some date $t \in \mathbb{N}$, and after having observed the values $x_1, \ldots, x_t$, the agent must take an action a from a finite set A.

The Bayesian concept of rationality interprets the agent as having beliefs represented by a probability space $(\Omega, \mathcal{B}, \Pr)$ and a state-contingent utility function $u: A \times \Omega \to \mathbb{R}$, and interprets the observations x_t as the values of random variables $X_t: \Omega \to \mathbf{X}$. A Bayesian-rational agent always chooses an action that maximizes the posterior expected utility

$$U(a, \omega, t) = \frac{\int_{B(\omega,t)} u(a, \theta)\, d\Pr(\theta)}{\Pr(B(\omega, t))}, \tag{1}$$

where

$$B(\omega, t) = \{\theta \in \Omega | \forall s \leq t \;\; X_s(\theta) = X_s(\omega)\}. \tag{2}$$

(Note that $B(\omega, 0) = \Omega$.) That is, if $\alpha(X_1(\omega), \ldots, X_t(\omega)) \subseteq A$ is the set of actions that the agent might decide to take after having observed $X_1(\omega), \ldots, X_t(\omega)$, then

$$\begin{aligned} &\forall \omega \in \Omega \;\; \forall t \in \mathbb{N} \;\; \forall a \in A \;\; \big[a \in \alpha(X_1(\omega), \ldots, X_t(\omega)) \\ &\qquad \iff \forall a' \in A \quad U(a', \omega, t) \leq U(a, \omega, t)\big]. \end{aligned} \tag{3}$$

Note that this formulation does not represent the agent as using the information that t, rather than another date, is when the action is to be taken. Implicitly it is assumed that this date is determined independently of the random variables X_t and independently of the sections $u(a,\cdot)$ of the utility function.

Now define the set of *observation sequences*, $\mathbf{X}^* = \bigcup_{t\in\mathbb{N}} \mathbf{X}^t$, and define $\lambda: \mathbf{X}^* \to \mathbb{N}$ by $\lambda(\vec{x}) = t \iff \vec{x} \in \mathbf{X}^t$. Given any measurable space $(\Omega, \mathcal{B})$ and infinite sequence of $\mathcal{B}$-measurable functions $X_t: \Omega \to \mathbf{X}$, define

$$B^*(\vec{x}) = \{\omega | \forall t \leq \lambda(\vec{x}) \;\; X_t(\omega) = \vec{x}_t\}. \tag{4}$$

Note that a measurable space and sequence of random variables can always be constructed from $\mathbf{X}$ by taking $\Omega = \mathbf{X}^{\mathbb{N}}$ and $X_t(\omega) = \omega_t$, and by taking $\mathcal{B}$ to be the smallest σ-algebra with which all of the projection functions X_t are measurable. If $\forall t \leq \lambda(\vec{x}) \;\; X_t(\omega) = \vec{x}_t$, then $B^*(\vec{x}) = B(\omega, \lambda(\vec{x}))$. Define $\xi: \Omega \times \mathbb{N} \to \mathbf{X}^*$ by

$$\xi(\omega, t) = \langle X_1(\omega), \ldots, X_t(\omega)\rangle. \tag{5}$$

Throughout this paper, attention will be confined to stochastic processes for which every observation sequence occurs with positive probability. That is, it is assumed that

$$\forall \vec{x} \in \mathbf{X}^* \;\; \Pr(B^*(\vec{x})) > 0. \tag{6}$$

Condition (6) makes it possible to define posterior expected utility analogously to (1), but in terms of $\mathbf{X}^*$, by

$$U^*(a, \vec{x}) = \frac{\int_{B^*(\vec{x})} u(a,\theta)\, d\Pr(\theta)}{\Pr(B^*(\vec{x}))}. \tag{7}$$

A *contingent plan* is a correspondence $\alpha: \mathbf{X}^* \to\!\!\to A$. A *Bayes* contingent plan is one for which there exists a representation of form (1)–(6). Say that a contingent plan α is *Bayes for* $\langle \Omega, \mathcal{B}, \Pr, \{X_t\}_{t\in\mathbb{N}_+}\rangle$ (or simply *Bayes for* $\{X_t\}_{t\in\mathbb{N}_+}$) if α satisfies

$$\forall \vec{x} \in \mathbf{X}^* \;\; \forall a \in A \;\; \left[a \in \alpha(\vec{x}) \iff U^*(a,\vec{x}) = \max_{a'\in A} U^*(a', \vec{x})\right] \tag{8}$$

with respect to some von Neumann-Morgenstern utility function and to that stochastic process.

4. A necessary condition

This section concerns the formulation of a condition on contingent plans, and the proof that this condition must necessarily be satisfied by a Bayes contingent plan. First some notation is introduced to formulate and analyze the condition.

Define the *immediate extensions* of $\vec{x} \in \mathbf{X}^*$ to be those observation sequences that consist of $\vec{x}$ followed by a single observation. Let $\vec{x} * y$ denote the immediate extension of $\vec{x}$ by y. The immediate extensions of all observation sequences can be represented by a correspondence $\epsilon: \mathbf{X}^* \to\!\!\to \mathbf{X}^*$ defined by $\vec{y} \in \epsilon(\vec{x}) \iff \exists y \in \mathbf{X} \;\; \vec{y} = \vec{x} * y$.

Now the necessary condition for a contingent plan to be Bayes can be stated in terms of three subsidiary conditions.

Contingent plan α satisfies *existence of an optimum at* $\vec{x}$ if

$$\alpha(\vec{x}) \neq \emptyset.$$

Contingent plan α satisfies *dominance-inclusiveness at* $\vec{x}$ if

$$\forall a \ \Big\{ [\forall y \ \ a \in \alpha(\vec{x} * y)] \Longrightarrow a \in \alpha(\vec{x}) \Big\}.$$

Contingent plan α satisfies *dominance-restrictiveness at* $\vec{x}$ if

$$\forall a \ \forall b \ \Big\{ [\forall y \ \ a \in \alpha(\vec{x} * y) \quad \text{and} \quad \exists y \ \ b \notin \alpha(\vec{x} * y)] \Longrightarrow b \notin \alpha(\vec{x}) \Big\}.$$

Now we come to the main definition. Define a contingent plan α to be *consistent at* $\vec{x}$ if it satisfies at $\vec{x}$ the three subsidiary conditions just presented: existence of optimum, dominance-inclusiveness, and dominance-restrictiveness. It is easily seen that the conjunction of these subsidiary conditions is equivalent to the set-theoretic condition that

$$\alpha(\vec{x}) \neq \emptyset \text{ and } \left[\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) = \emptyset \quad \text{or} \quad \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) = \alpha(\vec{x}) \right]. \tag{9}$$

Theorem 1. *A Bayes contingent plan must be consistent at every observation sequence.*

Proof: First, a Bayes contingent plan must be nonempty valued (that is, must satisfy $\alpha(\vec{x}) \neq \emptyset$ at every $\vec{x}$) because expected utility attains a maximum on a finite set of alternative actions. Suppose that α is Bayes for a stochastic process $\{X_t\}_{t \in \mathbb{N}_+}$ and that $\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) \neq \emptyset$. Specifically, suppose that $a \in \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$. Consider any $a' \in A$. Since $a \in \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$, (3) implies that $\forall \vec{y} \in \epsilon(\vec{x}) \ \ U^*(a', \vec{y}) \leq U^*(a, \vec{y})$.

By the law of iterated expectation,

$$U^*(a, \vec{x}) = \frac{\sum_{\vec{y} \in \epsilon(\vec{x})} \Pr(B^*(\vec{y})) U^*(a, \vec{y})}{\Pr(B^*(\vec{x}))} \text{ and } U^*(a', \vec{x}) = \frac{\sum_{\vec{y} \in \epsilon(\vec{x})} \Pr(B^*(\vec{y})) U^*(a', \vec{y})}{\Pr(B^*(\vec{x}))}. \tag{10}$$

Therefore $U^*(a', \vec{x}) \leq U^*(a, \vec{x})$. Since this inequality holds for all $a' \in A$, (3) implies that $a \in \alpha(\vec{x})$. That is, either $\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) = \emptyset$ or else $\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) \subseteq \alpha(\vec{x})$.

Now suppose instead that $a \in \alpha(\vec{x}) \setminus \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$. By (3) and (10),

$$\forall a' \in A \ \ \sum_{\vec{y} \in \epsilon(\vec{x})} \Pr(B^*(\vec{y})) U^*(a', \vec{y}) \leq \sum_{\vec{y} \in \epsilon(\vec{x})} \Pr(B^*(\vec{y})) U^*(a, \vec{y}). \tag{11}$$

Since $a \notin \bigcap_{\vec{y}\in\epsilon(\vec{x})} \alpha(\vec{y})$, suppose that $\vec{z} \in \epsilon(\vec{x})$ and $a \notin \alpha(\vec{z})$. Since $\alpha(\vec{z})$ is nonempty, suppose that $a' \in \alpha(\vec{z})$. Then (3) implies that $U^*(a,\vec{z}) < U^*(a',\vec{z})$. Therefore, in order for (11) to hold, there must be some other $\vec{w} \in \epsilon(\vec{x})$ such that $U^*(a',\vec{w}) < U^*(a,\vec{w})$. By (3), $a' \notin \alpha(\vec{w})$ and hence $a' \notin \bigcap_{\vec{y}\in\epsilon(\vec{x})} \alpha(\vec{y})$. Since a' is arbitrary except that $a' \neq a$, this argument shows that $\bigcap_{\vec{y}\in\epsilon(\vec{x})} \alpha(\vec{y}) \subseteq \{a\}$. However, $a \notin \bigcap_{\vec{y}\in\epsilon(\vec{x})} \alpha(\vec{y})$ by assumption. Thus $\bigcap_{\vec{y}\in\epsilon(\vec{x})} \alpha(\vec{y}) = \emptyset$.

What has just been established is that either $\bigcap_{\vec{y}\in\epsilon(\vec{x})} \alpha(\vec{y}) = \emptyset$ or else $\alpha(\vec{x}) \subseteq \bigcap_{\vec{y}\in\epsilon(\vec{x})} \alpha(\vec{y})$. The set inclusion must in fact be an equality, since it has earlier been established that either $\bigcap_{\vec{y}\in\epsilon(\vec{x})} \alpha(\vec{y}) = \emptyset$ or else $\bigcap_{\vec{y}\in\epsilon(\vec{x})} \alpha(\vec{y}) \subseteq \alpha(\vec{x})$. ∎

5. Consistent plans are Bayes

The converse of Theorem 1 is also true. Theorem 2 states that, if one requires only that a state-contingent expected-utility function should be bounded above, then each consistent contingent plan is Bayes for every stochastic process that satisfies (6). Theorem 3 states that, if one requires that a state-contingent expected-utility function should be bounded below as well as above, then a consistent plan is Bayes for some stochastic process (that may depend on the plan) that satisfies (6).

Theorem 2. *Suppose that α is consistent, and let $\langle \Omega, \mathcal{B}, \Pr, \{X_t\}_{t\in\mathbb{N}_+}\rangle$ be any stochastic process that satisfies (6). Then there is a state-contingent expected-utility function $u: A \times \Omega \to \mathbb{R}$ that makes α Bayes for $\{X_t\}_{t\in\mathbb{N}_+}$, and such that u is bounded above.*

Proof: For each $a \in A$, define a martingale $\{V_t^a: \Omega \to \mathbb{R}\}_{t\in\mathbb{N}_+}$ recursively as follows. Let $V_0^a(\omega) = 0$ if $a \in \alpha(\emptyset)$ and $V_0^a(\omega) = -1$ if $a \notin \alpha(\emptyset)$. Suppose that V_t^a has been defined. Then begin to define V_{t+1}^a by setting

$$V_{t+1}^a(\omega) = \begin{cases} \max\{V_t^{a'}(\omega) | a' \in A\}, & \text{if } a \in \alpha(\xi(\omega,t+1)) \text{ and } \bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))} \alpha(\vec{y}) \neq \emptyset \quad (12) \\ \max\{V_t^{a'}(\omega) \\ \quad +2^{-t} | a' \in A\}, & \text{if } a \in \alpha(\xi(\omega,t+1)) \text{ and } \bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))} \alpha(\vec{y}) = \emptyset \quad (13) \\ V_t^a(\omega), & \text{if } a \notin \bigcup_{\vec{y}\in\epsilon(\xi(\omega,t))} \alpha(\vec{y}) \,. \quad (14) \end{cases}$$

Condition (12) defines V_{t+1}^a except at those $\omega \in \Omega$ such that $\exists \vec{y} \in \epsilon(\xi(\omega,t))$ $[a \in \alpha(\vec{y})$ but $a \notin \alpha(\xi(\omega,t+1))]$. At these states, V_{t+1}^a will be defined by a martingale condition. To accomplish this, first define

$$G(a,\omega,t) = \{\theta | a \in \alpha(\xi(\theta,t+1))\} \cap B(\omega,t). \tag{15}$$

The condition that $\exists \vec{y} \in \epsilon(\xi(\omega,t))$ $[a \in \alpha(\vec{y})$ but $a \notin \alpha(\xi(\omega,t+1))]$ is equivalent to $\omega \notin G(a,\omega,t)$ and $G(a,\omega,t) \neq \emptyset$. In this case, define $V_{t+1}^a(\omega)$ by imposing the martingale condition

$$V_t^a(\omega)\Pr(B(\omega,t)) = \int_{G(a,\omega,t)} V_{t+1}^a(\theta)\, d\Pr(\theta) + V_{t+1}^a(\omega)\Pr(B(\omega,t)\backslash G(a,\omega,t)). \tag{16}$$

It is now proved by induction that each $\{V_t^a\}_{t\in\mathbb{N}}$ is a martingale and that for each date τ,

$$\forall a\ \forall\omega\ \left[a \in \alpha(\xi(\omega,\tau)) \iff V_\tau^a(\omega) = \max\{V_\tau^{a'}(\omega)|a' \in A\}\right]. \tag{17}$$

The martingale condition on $V_0^a,\ldots,V_t^a$ is vacuous for $t=0$, and V_0^a has been defined explicitly to assure (17). For the induction step, suppose the hypothesis that $V_0^a,\ldots,V_t^a$ is a martingale for each a and that (17) holds. These conditions must be shown to hold also for the random variables $\{V_{t+1}^a|a \in A\}$.

Begin by verifying condition (17). Consider an action a and a state ω. First suppose that the condition $a \in \alpha(\xi(\omega,t+1))$ and $\bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))}\alpha(\vec{y}) \neq \emptyset$ of (12) is satisfied. Since $\bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))}\alpha(\vec{y}) \neq \emptyset$, condition (13) is not satisfied at ω for any action a^*. Therefore $V_{t+1}^{a^*}(\omega)$ is determined for every action a^* by either (12) or (14) or (16). In all of these three cases, $V_{t+1}^{a^*}(\omega) \leq V_{t+1}^a(\omega) = \max\{V_t^{a'}(\omega)|a' \in A\}$. Therefore the condition that $a \in \alpha(\xi(\omega,t+1))$ and $\bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))}\alpha(\vec{y}) \neq \emptyset$ implies that $V_{t+1}^a(\omega) = \max\{V_{t+1}^{a'}(\omega)|a' \in A\}$.

Second suppose that the condition $a \in \alpha(\xi(\omega,t+1))$ and $\bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))}\alpha(\vec{y}) = \emptyset$ of condition (13) is satisfied. Then $V_{t+1}^a(\omega)$ is the highest number that can be assigned to any $V_{t+1}^{a'}(\omega)$ by any of conditions (12)–(14), so again $V_{t+1}^a(\omega) = \max\{V_{t+1}^{a'}(\omega)|a' \in A\}$.

Third suppose that the condition $a \notin \bigcup_{\vec{y}\in\epsilon(\xi(\omega,t))}\alpha(\vec{y})$ of condition (14) is satisfied. Here there are two subcases. The first is that, if the condition $\bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))}\alpha(\vec{y}) \neq \emptyset$ of (12) is satisfied, then this condition and the defining condition (9) of Bayes consistency imply that $a \notin \alpha(\xi(\omega,t))$ and that therefore $V_t^a(\omega) < \max\{V_t^{a'}(\omega)|a' \in A\}$. Some a^* is an element of $\bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))}\alpha(\vec{y})$, and $V_{t+1}^a(\omega) < V_{t+1}^{a^*}(\omega) = \max\{V_{t+1}^{a'}(\omega)|a' \in A\}$. The second subcase is that the condition $\bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))}\alpha(\vec{y}) = \emptyset$ of (13) is satisfied. Then (13) specifies a higher value of $V_{t+1}^{a^*}(\omega)$ for any $a^* \in \alpha(\xi(\omega,t+1))$ (of which (9) requires that there should be at least one) than the value of $V_{t+1}^a(\omega)$.

Finally consider the case that $\omega \notin G(a,\omega,t)$ and $G(a,\omega,t) \neq \emptyset$. There are two subcases, depending on whether or not $a \in \alpha(\xi(\omega,t))$. If $a \in \alpha(\xi(\omega,t))$ and $\omega \notin G(a,\omega,t)$, then $\bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))}\alpha(\vec{y}) = \emptyset$ by the definition (9) of Bayes consistency and the definition (15) of $G(a,\omega,t)$. Therefore $V_{t+1}^a(\theta) = V_t^a(\theta) + 2^{-t}$ for $\theta \in G(a,\omega,t)$. The condition that $G(a,\omega,t) \neq \emptyset$ implies that $\Pr(G(a,\omega,t)) > 0$ by (6), so the martingale condition (15) entails that $V_{t+1}^a(\omega) < V_t^a(\omega)$. Also by (9), there is some action $a^* \in \alpha(\xi(\omega,t+1))$, and $V_t^a(\omega) < V_{t+1}^{a^*}(\omega)$ by (13). Together these inequalities establish that $V_{t+1}^a(\omega) < V_{t+1}^{a^*}(\omega)$. In the alternative subcase, that $a \notin \alpha(\xi(\omega,t))$, continue to let a^* denote an element of $\alpha(\xi(\omega,t+1))$, and also let a' denote an element of $\alpha(\xi(\omega,t))$ which is nonempty as well by the Bayes consistency condition (9). By the induction hypothesis, $V_t^a(\omega) < V_t^{a'}(\omega)$. The value of $V_{t+1}^{a^*}(\omega)$ must be determined by either (12) or (13), and in either case $V_t^{a'}(\omega) \leq V_{t+1}^{a^*}(\omega)$. Also $V_{t+1}^a(\omega) \leq V_t^a(\omega)$, either by (14) or else by an argument from (16) parallel to the one just given. Again in this subcase, then, $V_{t+1}^a(\omega) < V_{t+1}^{a^*}(\omega)$.

The foregoing argument shows that (17) holds at date $\tau = t+1$. Now consider the martingale condition that, for every $a \in A$ and every event B measurable with respect to $X_1,\ldots,X_t$,

$$\int_B V_t^a(\omega)\,d\Pr(\omega) = \int_B V_{t+1}^a(\omega)\,d\Pr(\omega). \tag{18}$$

It is sufficient to verify this condition on the sets $B(\xi(\omega,t))$. If $a \in \bigcap_{\vec{y}\in\epsilon(\xi(\omega,t))} \alpha(\vec{y})$, then (18) is guaranteed by (12). If $a \notin \bigcup_{\vec{y}\in\epsilon(\xi(\omega,t))} \alpha(\vec{y})$, then (18) is guaranteed by (14). Otherwise (18) is guaranteed by (16).

Thus it has been shown by induction that the processes $\{\{V_t^a\}_{t\in\mathbb{N}} | a \in A\}$ are martingales that satisfy (17). Each martingale $\{V_t^a\}_{t\in\mathbb{N}}$ is bounded above by 2, so the Martingale Convergence Theorem implies that there is an integrable random variable $u^a: \Omega \to \mathbb{R}$ bounded above by 2 and such that $u^a(\omega) = \lim_{t\to\infty} V_t^a(\omega)$ almost surely. Defining $u: A \times \Omega \to \mathbb{R}$ by $u(a,\omega) = u^a(\omega)$ completes the proof, since the Martingale Convergence Theorem implies that $V_t^a(\omega) = U(a,\omega,t)$ for all a, ω and t. ∎

Theorem 3. *If α is consistent, then there exists a stochastic process $\langle\Omega, \mathcal{B}, \Pr^*, \{X_t\}_{t\in\mathbb{N}_+}\rangle$ satisfying (6) and a bounded, state-contingent, expected-utility function $u^*: A \times \Omega \to \mathbb{R}$ that makes α Bayes for $\{X_t\}_{t\in\mathbb{N}_+}$.*

Proof: Begin with a stochastic process $\langle\Omega, \mathcal{B}, \Pr, \{\mathbf{X}_t\}_{t\in\mathbb{N}_+}\rangle$ satisfying (6) and with the utility function $u: A \times \Omega \to \mathbb{R}$ that was constructed in Theorem 2. Define a function $\delta: \Omega \to \mathbb{R}_+$ by

$$\delta(\omega) = -\min\{-1, \min\{u(a,\omega) | a \in A\}\}. \tag{19}$$

Now define

$$\Pr^*(B) = \frac{\int_B \delta(\omega)\, d\Pr(\omega)}{\int_\Omega \delta(\omega)\, d\Pr(\omega)} \quad \text{and} \quad u^*(a,\omega^*) = \frac{u(a,\omega^*)}{\delta(\omega^*)} \int_\Omega \delta(\omega)\, d\Pr(\omega)\,. \tag{20}$$

By (19) and (20), Pr is absolutely continuous with respect to $\Pr^*$ so $\Pr^*$ satisfies (6). By (20), $\int_B u^*(a,\omega)\, d\Pr^*(\omega) = \int_B u(a,\omega)\, d\Pr(\omega)$ for all $B \in \mathcal{B}$ and $a \in A$ so u^* makes α Bayes. By (19), u^* satisfies the bounds $\forall a\ \forall\omega\ \ -1 \le u^*(a,\omega) \le 2$. ∎

6. Comparing the Bayes and miminax criteria

A prominent alternative to maximization of expected utility in the history of decision theory has been minimization of maximum loss—that is, the minimax criterion. An important reason to have characterized Bayes contingent plans is to compare them with the contingent plans that alternative criteria such as minimax would recommend. In order to do so, a necessary condition to be a minimax contingent plan is now derived. This condition coincides almost exactly with the consistency condition (9). In particular, the condition is equivalent to consistency in the important case that a contingent plan is singleton valued.

The minimax criterion is formulated in terms of the loss $\ell(a,\omega)$ of an action in a state of nature. The loss may be conceived either as the negative of the state-contingent utility of an action,

$$\ell(a,\omega) = -u(a,\omega)\,, \tag{21}$$

or else as a state-contingent regret which is defined by

$$\ell(a,\omega) = \max_{a' \in A} u(a',\omega) - u(a,\omega)\,. \tag{22}$$

Action a is minimax, conditional on observation sequence $\vec{x}$, if

$$\sup_{\omega \in B^*(\vec{x})} \ell(a,\omega) = \min_{a' \in A} \sup_{\omega \in B^*(\vec{x})} \ell(a',\omega)\,. \tag{23}$$

A contingent plan α is *minimax* (for utility function $u\colon A \times \Omega \to \mathbb{R}$) if

$$\forall \vec{x} \in \mathbf{X}^* \;\; \alpha(\vec{x}) = \{a \mid \sup_{\omega \in B^*(\vec{x})} \ell(a,\omega) = \min_{a' \in A} \sup_{\omega \in B^*(\vec{x})} \ell(a',\omega)\}\,. \tag{24}$$

Theorem 4. *If a contingent plan $\alpha\colon \mathbf{X}^* \to A$ is minimax, then for every $\vec{x} \in \mathbf{X}^*$,*

$$\alpha(\vec{x}) \neq \emptyset \;\; \text{and} \;\; \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) \subseteq \alpha(\vec{x})\,, \tag{25}$$

and

$$\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) \neq \emptyset \implies \exists \vec{y} \in \epsilon(\vec{x}) \;\; \alpha(\vec{x}) \subseteq \alpha(\vec{y})\,. \tag{26}$$

A singleton-valued minimax contingent plan is consistent at every observation sequence $\vec{x}$ and hence is Bayes.

Proof: Suppose that α is minimax. Clearly α must be nonempty valued. Suppose that $a \notin \alpha(\vec{x})$. Then $\ell(a,\omega^*) > \sup_{\omega \in B^*(\vec{x})} \ell(a',\omega)$ for some $\omega^* \in B^*(\vec{x})$. Since $\omega^* \in B^*(\xi(\omega^*, \lambda(\vec{x})+1))$ and $\xi(\omega^*, \lambda(\vec{x})+1) \in \epsilon(\vec{x})$, $a \notin \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$. This shows that $\bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y}) \subseteq \alpha(\vec{x})$, establishing (25).

Now suppose that $a \in \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$ and that $a' \in \alpha(\vec{x}) \setminus \bigcap_{\vec{y} \in \epsilon(\vec{x})} \alpha(\vec{y})$. Let $\gamma = \sup_{\omega \in B^*(\vec{x})} \ell(a',\omega)$. Since $a' \in \alpha(\vec{x})$, for every $n \in \mathbb{N}_+$ there exists a state of nature $\omega_n \in B^*(\vec{x})$ such that $\ell(a,\omega_n) < \gamma + 1/n$. Because $\epsilon(\vec{x})$ is finite, $\{n \mid \omega_n \in B^*(\vec{y}^*)\}$ is infinite for some $\vec{y}^* \in \epsilon(\vec{x})$. Thus $\sup_{\omega \in B^*(\vec{y}^*)} \ell(a',\omega) \leq \gamma \leq \sup_{\omega \in B^*(\vec{y}^*)} \ell(a,\omega)$. Since $a \in \alpha(\vec{y}^*)$, these weak inequalities must actually be equalities and therefore $a' \in \alpha(\vec{y}^*)$. Since the choice of $\vec{y}^*$ in this argument depends only on γ which is the same for all $a' \in \alpha(x)$, the argument actually establishes assertion (26).

If α is singleton-valued (that is, if the correspondence α is actually a single-valued function), then condition (25) is equivalent to consistency of α (that is, to condition (9)). Hence α is Bayes by theorems 2 and 3. ∎

7. Comparing the Bayes and weighted-utility criteria

Another illuminating comparison is between Bayes contingent plans and those that would satisfy the analogous optimality definition with respect to the "weighted" utility functions introduced by Chew (1983). Chew replaced the "Independence" axiom of the expected utility theory with weaker axioms ("Betweenness" axiom and "Substitution-Independence" axiom), and identified a class of preferences among uncertain prospects that are representable by a weighted mean of the utility values of sure prospects (weighted utility function). These utility functions are intended to generalize expected utility as narrowly as possible while accommodating Allais' paradox.

Suppose, according to an expected utility function or minimax criterion, an action is optimal for probability measures $\Pr^1$ and $\Pr^2$ on Ω. Then, the action remains to be optimal for any convex combination of $\Pr^1$ and $\Pr^2$. However, if the underlying preference is a weighted utility function, this is not generally true any more. (Park (1993) characterizes the mapping, from the probability simplex on a finite set of decision-relevant events to a finite set of actions, that may be rationalized by weighted utility. In particular, the boundary of the set of probability measures at which one action is preferred to another is the set of roots of a quadratic function of the probabilities.) As a consequence, weighted utility functions may produce contingent plans that violate consistency as is illustrated by an example below. This is not a degenerate example because any weighted utility function "sufficiently close" to the one in the example, produces the same phenomenon.

Chew's (1983) original definition of weighted utility is stated in von Neumann and Morgenstern's framework involving preferences among lotteries, but it can be paraphrased to define conditional weighted utility in a way that is analogous to the definition of conditional expected utility in (1). In addition to a state-contingent utility function u, a "weighing function" $\gamma\colon \mathbb{R} \to \mathbb{R}_{++}$ is used to characterize weighted utility. This function γ takes the state-contingent utility of an action as its argument. (In Chew's formulation, its argument is a lottery payoff which would correspond to a consequence in Savage's formulation of the expected-utility theory.) Expected utility is the special case of weighted utility in which the weighing function is constant. Here is the definition of conditional weighted utility provided that appropriate measurability conditions are satisfied.

$$\tilde{U}(a,\omega,t) = \frac{\int_{B(\omega,t)} \gamma(u(a,\theta))u(a,\theta)\, d\Pr(\theta)}{\int_{B(\omega,t)} \gamma(u(a,\theta))\, d\Pr(\theta)} \tag{27}$$

Corresponding to the definition of a Bayes contingent plan, say that a contingent plan α is *weighted-rational for* $\langle \Omega, \mathcal{B}, \Pr, \{X_t\}_{t\in\mathbb{N}_+}\rangle$ (or simply *weighted-rational for* $\{X_t\}_{t\in\mathbb{N}_+}$) if α satisfies

$$\forall \vec{x} \in \mathbf{X}^* \ \ \forall a \in A \ \left[a \in \alpha(\vec{x}) \iff \tilde{U}^*(a,\vec{x}) = \max_{a'\in A} \tilde{U}^*(a',\vec{x})\right] \tag{28}$$

with respect to some state-contingent weighted utility function and weighting function and to that stochastic process. ($\tilde{U}^*$ is defined analogously to U^*.)

Now we construct an example of a weighted-rational contingent plan that fails to be consistent. Let $\Omega = \{\omega_1, \omega_2\}$, with $\Pr(\omega_1) = \Pr(\omega_2) = 1/2$. Define $\mathbf{X} = \{1, 2\}$, $X_1(\omega_1) = 1$ and $X_1(\omega_2) = 2$.

Let $A = \{a, b\}$. Define $u(a,\omega_1) = 1$, $u(a,\omega_2) = 3$, $u(b,\omega_1) = 2$, and $u(b,\omega_2) = 4$. Select a continuous (or, smooth) weighting function γ such that $\gamma(1) = \gamma(4) = 1$ and $\gamma(2) = \gamma(3) = 4$. For the resulting weighted-utility preference, a simple calculation shows that $\tilde{U}(a,\omega,0) = 13/5$ and $\tilde{U}(b,\omega,0) = 12/5$ so that $\alpha(\emptyset) = \{a\}$.

Next, find the optimal action after the realization of X_1. For $X_1(w) = 1$, or equivalently, for $\omega = \omega_1$,

$$\tilde{U}(a,\omega,1) = \frac{\gamma(u(a,\omega))u(a,\omega)}{\gamma(u(a,\omega))} = u(a,\omega_1) = 1,$$
$$\tilde{U}(b,\omega,1) = \frac{\gamma(u(b,\omega))u(b,\omega)}{\gamma(u(b,\omega))} = u(b,\omega_1) = 2.$$

Similarly, for $X_1(w) = 2$, or equivalently, for $\omega = \omega_2$,

$$\tilde{U}(a,\omega,1) = u(a,\omega_2) = 3,$$
$$\tilde{U}(b,\omega,1) = u(b,\omega_2) = 4.$$

Hence, $\alpha(\langle 1\rangle) = \alpha(\langle 2\rangle) = \{b\}$. This is a violation of consistency at $\emptyset$, namely,

$$\bigcap_{\vec{y}\in\epsilon(\emptyset)} \alpha(\vec{y}) = \{b\} \notin \{\emptyset, \alpha(\emptyset)\}.$$

This example illustrates the violation of consistency described in section 2 when actions a and b are going to the opera and staying at home, respectively, and different values of X_1 indicate different operas being announced to be performed.

8. Conclusion

An intuitively natural consistency condition for contingent plans has been shown to be necessary and sufficient for a contingent plan to be rationalized by maximization of conditional expected utility. One alternative theory of choice under uncertainty, the weighted-utility theory developed by Chew Soo Hong (1983) does not entail that contingent plans will generally satisfy this condition. Another alternative theory, the minimax theory as formulated by Savage ([1954] 1972), does entail the consistency condition at least for single-valued plans.

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A Revealed-Preference Implication of Weighted Utility Decisions under Uncertainty

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Abstract

Revealed preference of the weighted utility theory of Chew (1983) is investigated in the same set-up as Green and Osband (1991) for expected utility theory; the structure of the partition of the state simplex according to the chosen act is examined. It is shown that the boundary between two chosen acts generated by a weighted utility is the solution set to a quadratic equation. Moreover, except for a small class of pairs of acts, the actual shape of the observable boundary between chosen acts is non-affine within a generic subset of weighted utilities, so that it is distinguishable from boundaries generated by expected utilities which are affine according to Green and Osband.

Keywords: revealed preference, expected utility, weighted utility, behavior partition

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1. Introduction

The expected utility theory, which dates back to Bernoulli and was given axiomatic foundations by von Neumann and Morgenstern (1944) and by Savage (1954), has been the standard analytic framework of decision under uncertainty. However, Allais and others have suggested that people violate expected-utility maximization in systematic ways. Such reservations have led various authors to develop alternative theories that generalize the expected utility theory. These "non-expected utility theories" make sharply different predictions from the expected utility theory about the observed behavior of agents in some contexts (such as laboratory experiments) where the possibilities for observation are very rich. The relationship between the expected utility theory and its generalizations in terms of more restrictive observational settings (such as the typical situation of non-experimental economists) is less well understood.

Achieving such an understanding has been difficult because individuals' preferences are not directly observable. But these preferences can be inferred from people's behavior if certain preferences generate distinctive behavior patterns. So the expected utility theory and alternative theories can be evaluated systematically as descriptions of behavior, using a revealed preference approach that is well established in the study of choice under certainty. This paper investigates such a revealed preference approach to the weighted utility theory of Chew (1983) and compares with the corresponding characterization of expected utility theory by Green and Osband (1991). The main result is that these two theories induce systematically different characterizations, and under minor conditions the behavior rationalized by the weighted utility theory is distinguishable from that of the expected utility theory in a "generic" sense.

The decision making set-up is as follows: A decision maker has a finite set of acts which induce a *consequence* for each state of nature. Facing uncertainty represented by a probability measure over states, the decision maker chooses a best act in the sense that the random prospect (a probability measure over consequences) that this act induces is preferred to the ones induced by other acts according to his or her preference. Then, the probability simplex on states of nature is partitioned according to the chosen act. This partition is called a *behavior partition.*[1]

Green and Osband completely characterize behavior partitions induced by preferences conforming to the expected utility theory: A partition of the probability simplex on states is "rationalized" by expected utility if and only if it consists of convex polyhedral partition blocks satisfying a certain integrability condition. In particular, the boundary between two partition blocks (this boundary represents the set of probability assessments over states for which the acts chosen in either block are indifferent) is an affine surface.[2] Because of its simplicity, "affine boundary" restriction is practically a more useful empirical implication of expected utility theory than the integrability condition. However, it is only valid to draw the inference from the blocks of an agent's behavior partition being polyhedral to the agent being an expected-utility maximizer if, were the agent be-

[1] Another decision making situation explored in the study of the revealed preference of the expected utility theory is to observe the choice of acts by changing the set of feasible acts while fixing the probability distribution of the states. This is used in Border (1992), Fishburn (1975), Kim (1991) and Ledyard (1986) among others.

[2] In this paper, a "surface" means a smooth manifold of codimension 1 in the space it is embedded. A surface is "affine" ("quadratic", respectively) if it is part of solution set to a non-vacuous linear (quadratic, respectively) equation.

having according to some other criterion than expected utility, the partition blocks would not be polyhedral (that is, their boundaries would be curved).

This paper concerns the curvature of the boundaries between behavior-partition blocks for a particular class of preferences among random prospects: the "weighted-utility" preferences introduced by Chew (1983). This is a particularly interesting class to study because Chew designed it deliberately to be as parsimonious a generalization of expected utility as possible, subject to being able to accommodate the pattern of choices predicted by Allais' paradox. Specifically, Chew weakened von Neumann and Morgenstern's Independence axiom to an axiom (Betweenness) that still shares the implication that the agent's indifference curves among random prospects are affine. One might naturally think that affine indifference curves would induce a behavior partition with polyhedral blocks,[3] and in that case the Green-Osband criterion would fail to distinguish between expected-utility and weighted-utility preferences. (Indeed, the present research began as an attempt to prove that such a failure must occur.[4]) So weighted-utility preferences are a "hard case" for the Green-Osband characterization, relative to less parsimonious generalizations of expected utility (such as by Machina, 1982) for which one would expect ex ante that decision boundaries would display curvature.[5]

Surprisingly, the boundaries between behavior-partition blocks induced by weighted-utility preferences do generically display curvature, as well. That is the main result of this paper. Before turning to its formal statement and proof, consider the intuition for it. Note that, if we follow Savage in defining an act as a mapping from states of nature to a set of consequences, then a probability measure on states of nature and an act together induce a probability measure on consequences. In most treatments of choice among acts under uncertainty, the simplex of such induced probability measures is studied. This is the probability simplex to which the Betweenness axiom is applied. However, Green and Osband deal instead with the probability simplex on the underlying states of nature themselves. This is the simplex that is divided into blocks by the behavior partition. For a probability measure p on states of nature to be on the boundary between the blocks where acts a and b are chosen respectively, it is necessary that, starting from p, a and b induce probability measures on consequences such that the agent is indifferent between them. Each of the two acts induces an affine mapping from the state-probability simplex to the consequence-probability simplex, but the weighted-utility function is not affine, so the locus of probability measures p satisfying this induced-indifference condition is not necessarily affine. Specifically, it is shown in section 3 to be a quadratic surface.[6]

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[3] This may look more likely to be the case by the observation that, under quadratic utilities, both the indifference curves and the boundaries between partition blocks are quadratic surfaces. See footnote (4).

[4] Failure to distinguish expected utility and non-expected utility preferences has been observed in other context. See Dardanoni (1993).

[5] Besides being a parsimonious generalization of the Independence axiom, Betweenness is a substantively important generalization that has already had noteworthy application (particularly by Epstein and Zin, 1989) in empirical economics.

[6] Chew, Epstein and Segal (1994) have shown that weighted utility and quadratic utility are the only preferences that satisfy a "projective independence" axiom. Note that quadratic utility implies that indifference curves are quadratic in the simplex of induced probability measures on consequences, rather than directly imposing any condition concerning the underlying simplex of probability measures on states of nature. It is shown in section 3 that quadratic utilities also generate partition blocks with quadratic boundaries.

Since affine surfaces are a special form of quadratic surfaces, a weighted-utility preference may produce partition blocks that are not distinguishable from those produced by expected-utility preferences. However, it is also shown that, unless the observed acts are "mirror-images" or one of them is "constant" as specified in section 4, this phenomenon is only exceptional; the boundary between two observed acts produced by a weighted-utility preference is non-affine except for a small subset of weighted-utility preferences whose closure has measure zero in the space of weighted-utility preferences. Therefore, the revealed preference of a generic subset of weighted-utility preferences is distinguishable from that of expected-utility preferences. Moreover, if the utility of acts are state-dependent, the condition that acts are neither mirror-images nor constant is automatically satisfied.

The rest of the paper is organized as follows. Section 2 formulates the model and reviews the two decision theories discussed in this paper. Section 3 establishes a necessary condition for a behavior partition to be consistent with weighted utility theory. Section 4 identifies a resticted class of pairs of acts such that the boundary between the pair of acts in this class is necessarily affine under weighted utility. Section 5 shows the generic distinction of the two theories in revealed preference for acts that are not in the class discussed in section 4.

2. Model

Let $\mathbf{S} = \{s_1, \ldots, s_n\}$ and $\mathbf{Z} = \{1, \ldots, K\}$ be finite sets of states of nature and consequences, respectively. The simplex of $\mathbf{S}$,

$$\Delta(\mathbf{S}) \equiv \{p \in I\!R^n : p_j \geq 0 \text{ and } \sum_{j=1}^{n} p_j = 1\} \qquad (2.a)$$

represents the set of all probability assessments that the agent might have concerning the state of nature. The simplex of $\mathbf{Z}$, denoted by $\Delta(\mathbf{Z})$, is defined analogously and represents random propects over consequences in $\mathbf{Z}$. Typical elements of $\Delta(\mathbf{S})$ and $\Delta(\mathbf{Z})$ are denoted by row vectors p and σ, respectively. An act a is a function from $\mathbf{S}$ to $\mathbf{Z}$ which is naturally extended to a linear function from $\Delta(\mathbf{S})$ to $\Delta(\mathbf{Z})$ represented by a $|\mathbf{Z}| \times |\mathbf{S}|$ transformation matrix M_a: the i-th column of M_a is a unit vector with 1 in the coordinate for $a(s_i)$ and 0's elsewhere. Then, $M_a p^t \in \Delta(\mathbf{Z})$ denotes the random prospect that act a induces under assessment $p \in \Delta(\mathbf{S})$. Here and throughout the paper, superscript t denotes the transpose.

A decision maker's preference $\succeq$ is defined on the random prospects over $\mathbf{Z}$, that is, $\succeq \subset \Delta(\mathbf{Z}) \times \Delta(\mathbf{Z})$. For $\sigma, \sigma' \in \Delta(\mathbf{Z})$, we write $\sigma \succeq \sigma'$ to mean $(\sigma, \sigma') \in \succeq$. Given a preference $\succeq$ and a finite set of acts $\mathbf{A}$, define the *best-act mapping* $BA_{\{\succeq,\mathbf{A}\}} : \Delta(\mathbf{S}) \to\!\!\to \mathbf{A}$ as usual;

$$BA_{\{\succeq,\mathbf{A}\}}(p) \equiv \{a \in \mathbf{A} : M_a p^t \succeq M_b p^t \text{ for all } b \in \mathbf{A}\} \qquad (2.b)$$

Faced with an assessment $p \in \Delta(\mathbf{S})$, a decision maker chooses an act in $BA_{\{\succeq,\mathbf{A}\}}(p)$.

A finite partition Π of $\Delta(\mathbf{S})$ is a *behavior partition* if there exist a set $\mathbf{A}$ of acts from $\mathbf{S}$ to $\mathbf{Z}$ and a preference $\succeq$ on $\Delta(\mathbf{Z})$ satisfying

(i) there is a bijection $\beta : \mathbf{A} \to \Pi$ such that $a \in BA_{\{\succeq,\mathbf{A}\}}(p)$ for all $p \in \beta(a)$, and

(ii) each $a \in \mathbf{A}$ is the unique best act for some $p \in \Delta(\mathbf{S})$.

A behavior partition describes the pattern of rational choice of acts by an agent with preference $\succeq$ and available acts $\mathbf{A}$, where each partition element consists of all assessments over states of nature for which one particular act is chosen. Condition (ii) essentially says that the agent could not be assured to do as well if any act were removed from $\mathbf{A}$. Without (ii), any partition would be rationalized as a behavior partition by a trivial preference that ranks all random prospects indifferently.

Topologocally we view elements of Π as embedded in $\Delta(\mathbf{S})$ and below we define some terminology used later. The terminology is selected to capture "typical" situations even though the definitions are more complex than the terminology itself might suggest to comprise "non-typical" situations as explained in Remark 2.1.

We are interested in surfaces in $\Delta(\mathbf{S})$ that divide $\Delta(\mathbf{S})$ into nonempty subsets. Hereafter, we use a *surface in* $\Delta(\mathbf{S})$ to mean a smooth manifold of codimension 1 in $\Delta(\mathbf{S})$ whose intersection with the interior of $\Delta(\mathbf{S})$ is nonempty. (Notice that a surface may be entirely contained in the interior of $\Delta(\mathbf{S})$. A surface intersected with the interior of $\Delta(\mathbf{S})$ is obviously a surface.) A partition element of Π is called a *block* if it's interior is non-empty. If a behavior partition is rationalized by a continuous preference, condition (ii) above implies that every partition element is a block. The intersection of closures of two blocks is called a *boundary* if it is a finite union of surfaces in $\Delta(\mathbf{S})$. On the boundary of two blocks, both acts chosen in either block are best acts.

Remark 2.1 : *A block is not necessarily a connected set under preferences more general than expected utility. It will become apparent later that the behavior partition generated by a nontrivial class of weighted utilities may contain a block consisting of two or more disjoint components. Therefore, a boundary may consist of multiple surfaces. It is an unexpected finding that a boundary can be a union of two surfaces that intersect with each other in the interior of* $\Delta(\mathbf{S})$ *for a special case as explained in Remark 4.5.*

Different classes of preferences have different structural implications on behavior partition. One of most prominent (so that easy to check) feature of a behavior partition is the curvature of boundaries between blocks. The contrast of curvature as "linear versus non-linear" is easy to detect and therefore have a strong empirical implication. This paper separates the weighted utility theory in this sense from the expected utility theory. We conclude this section by reviewing the two theories.

Expected utility representation : A preference $\succeq$ on $\Delta(\mathbf{Z})$ is an *expected utility* if there exists a function $U : \mathbf{Z} \to I\!R$ (an expected-utility function) such that for all $\sigma, \sigma' \in \Delta(\mathbf{Z})$,

$$\sigma \succeq \sigma' \iff \sum_{z \in \mathbf{Z}} U(z)\sigma(z) \geq \sum_{z \in \mathbf{Z}} U(z)\sigma'(z) \qquad (2.c)$$

The axiomatization of expected utility representation is well known; a preference $\succeq$ on $\Delta(\mathbf{Z})$ is represented by an expected utility if and only if the strict preference $\succ$ is a weak order, and satisfies the Archimedean (or continuity) axiom and the Independence (or substitution) axiom. Chew (1983) replaced the Independence axiom with a weaker axiom of Betweenness which says that a convex combination of two lotteries is strictly between them in preference ordering. Requiring in addition the axiom of Substitution-independence, Chew provided

Weighted utility representation : A preference $\succeq$ on $\Delta(\mathbf{Z})$ is a *weighted utility* if there exist functions $u : \mathbf{Z} \to \mathbb{R}$ and $\alpha : \mathbf{Z} \to \mathbb{R}_{++}$ (a utility function and a weighting function) such that

i) $[u(z) = u(z')] \implies [\alpha(z) = \alpha(z')]$,[7] and
ii) for all $\sigma, \sigma' \in \Delta(\mathbf{Z})$,

$$\sigma \succeq \sigma' \iff \frac{\sum_{z \in \mathbf{Z}} u(z)\alpha(z)\sigma(z)}{\sum_{z \in \mathbf{Z}} \alpha(z)\sigma(z)} \geq \frac{\sum_{z \in \mathbf{Z}} u(z)\alpha(z)\sigma'(z)}{\sum_{z \in \mathbf{Z}} \alpha(z)\sigma'(z)} \qquad (2.d)$$

We refer precise formulation of above mentioned axioms to known references on choice theory (e.g, Kreps, 1988) and Chew (1983). Here, we discuss implication of these axioms on indifference surfaces in $\Delta(\mathbf{Z})$. Independence axiom generates indifference surfaces which are affine and parallel to one another. Betweenness implies affine indifference surfaces, and Substitution-independence implies that indifference surfaces have a "common axis" outside of $\Delta(\mathbf{Z})$ (so-called "fan-shaped" indifference surfaces). Typical indifference surfaces for both theories are presented in Figure 1 for three consequences case.

[**Figure 1** here]

3. Weighted Utilities Generate Quadratic Boundaries

Given $\mathbf{S}$ and $\mathbf{Z}$ as before and a preference $\succeq$ on $\Delta(\mathbf{Z})$, the *border* of acts a and b is

$$I_{ab}(\succeq) \equiv \{p \in \Delta(\mathbf{S}) : \ M_a p^t \sim M_b p^t\} \qquad (3.a)$$

When the underlying preference is obvious, we write I_{ab} instead of $I_{ab}(\succeq)$. If $\succeq$ is represented by a functional $v : \Delta(\mathbf{Z}) \to \mathbb{R}$, then I_{ab} is the solution set to

$$v(M_a p^t) - v(M_b p^t) = 0 \qquad (3.b)$$

As long as the left hand side of (3.b) is a smooth function in p for which 0 is a regular value, which is generically the case for preferences considered in this paper, the solution set is a smooth manifold of codimension 1 in the relevant domain. (See the Preimage Theorem, Guillemin and Pollack (1974), p 21.) The boundary between two parition blocks for which a and b are chosen respectively, is by definition a full-dimensional subset of I_{ab}. In this section we investigate the curvature of borders hence that of boundaries of behavior partition.

In the benchmark case that $\succeq$ is represented by an expected-utility function U, the functional v in (3.b) is a linear operation of premultiplying by a row vector $\mathbf{u} \equiv (U(z))_{z=1}^{K}$. So, I_{ab}

[7] This condition says that two consequences of the same utility level are indistinguished as incorporated in random prospects. In Chew's original representation, this is automatic because consequences are identified by their utility levels. In this paper, condition *i*) is required mainly to avoid unnecessary technical complication.

solves a linear equation $\mathbf{u}(M_a - M_b)p^t = 0$ and, therefore, is affine. This is part of the complete characterization of behavior partition by Green and Osband (1991) under expected utility.

Proposition 3.1 (Green and Osband) **:** *Under an expected utility $\succeq$, the border I_{ab} is affine.*

Let us for a moment consider I_{ab} when $\succeq$ is a quadratic utility of Chew, Epstein and Segal (1991); v is quadratic in probabilities of *consequences*. Then, (3.b) becomes

$$(M_a p^t)^t R(M_a p^t) - (M_b p^t)^t R(M_b p^t) = p(M_a^t R M_a - M_b^t R M_b)p^t = 0$$

where R is a $|\mathbf{Z}| \times |\mathbf{Z}|$ matrix. This equation is quadratic in p and so I_{ab} is a quadratic surface.

Notice the coincidence of curvature between the indifference surfaces in $\Delta(\mathbf{Z})$ and the borders (equivalently, boundaries) in $\Delta(\mathbf{S})$: they are all affine under expected utility while quadratic under quadratic utility. This relationship, however, does not extend to under weighted utility; indifference surfaces are affine but boundaries are quadratic.

To see this, let $\mathbf{S} = \{s_1, \ldots, s_n\}$ and $\mathbf{Z} = \{1, \ldots, K\}$ be finite sets of states and consequences, respectively. Represent a weighted utility $\succeq$ by a pair of vectors $\mathbf{u} = (u_1, \ldots, u_K)$ and $\alpha = (\alpha_1, \ldots, \alpha_K)$ which are, respectively, utility levels and weights of pure consequences. Let D be the $K \times K$ diagonal matrix with α_k as k-th diagonal element and $\mathbf{e}$ be the K-vector $(1, \ldots, 1)$. Then, the weighted utility level of the random prospect $M_a p^t$ induced by an act a under p, is

$$\frac{\mathbf{u} D M_a p^t}{\mathbf{e} D M_a p^t}$$

Since the same argument applies to another act b, I_{ab} is the solution to

$$\frac{\mathbf{u} D M_a p^t}{\mathbf{e} D M_a p^t} - \frac{\mathbf{u} D M_b p^t}{\mathbf{e} D M_b p^t} = \frac{p(M_a^t D^t \mathbf{u}^t \mathbf{e} D M_b - M_b^t D^t \mathbf{u}^t \mathbf{e} D M_a)p^t}{p M_a^t D^t D M_b p^t} = 0 \qquad (3.c)$$

Since the denominators in (3.c) are strictly positive as weighted sums of probabilties and their product, equation (3.c) reduces to a quadratic equation. Therefore, we established

Proposition 3.2 : *Given acts a and b from $\mathbf{S}$ to $\mathbf{Z}$ and a weighted utility $\succeq$ on $\Delta(\mathbf{Z})$,*

$$I_{ab}(\succeq) = \{p \in \Delta(\mathbf{S}) : \; pQp^t = 0\} \qquad (3.d)$$

for some $|\mathbf{S}| \times |\mathbf{S}|$ matrix Q.

The simplex $\Delta(\mathbf{S})$ is often identified with the projection on the first $(n-1)$ coordinates, $\tilde{\Delta}(\mathbf{S}) = \{p \in I\!R_+^{n-1} : \sum_{i=1}^{n-1} p_i \leq 1\}$. Let $\tilde{I}_{ab}$ denote the projected image of I_{ab} on $\tilde{\Delta}(\mathbf{S})$. Then, substituting p_n with $(1 - p_1 - \cdots - p_{n-1})$ in I_{ab}, we have the following expression for $\tilde{I}_{ab}$ in $\tilde{\Delta}(\mathbf{S})$:

$$\tilde{I}_{ab} = \{p \in \tilde{\Delta}(\mathbf{S}) : \; p\tilde{Q}p^t + p\tilde{q} + c = 0\} \qquad (3.e)$$

for some $(n-1) \times (n-1)$ matrix $\tilde{Q}$, some $(n-1)$-vector $\tilde{q}$ and some real number c. The coefficients of the quadratic equation in (3.e) (i.e, elements of $\tilde{Q}$ and $\tilde{q}$) are expressed in terms of utility levels and weights of consequences.

Expected utilities are a special class of weighted utilities where all consequences have the same weight, say $\alpha > 0$. Then, $D = \alpha I_K$ and equation (3.c) becomes $\mathbf{u}(M_a - M_b)p^t = 0$ which determines an affine surface in $\Delta(\mathbf{S})$. Moreover, since an affine surface is a special form of a quadratic surface, a weighted utility which is not an expected utility may generate boundaries which are affine. We say that a surface is *genuinely* quadratic if it is not affine. The next section characterizes pairs of acts that always have affine borders under weighted utility. Section 5 shows that other pairs of acts "almost always" have genuinely quadratic borders under weighted utility.

4. When Weighted Utilities Generate Affine Boundaries

If weighted utilities that may produce affine boundaries are "negligible," given a behavior partition with affine boundaries, we can "safely" (in the sense of probability 1) reject the hypothesis that it has been generated by a weighted utility. In this respect, we are concerned about the set of weighted utilities that may produce affine borders.

In equation (3.c), M_a and M_b are determined by the pair of acts under consideration. Generally, given M_a and M_b, (3.c) may become linear for some weighted utilities, i.e., for some specification of D and $\mathbf{u}$. For some restricted pairs of acts, however, M_a and M_b are structured in such a way that equation (3.c) becomes linear for all D and $\mathbf{u}$. This section identifies these pairs of acts. To maintain the focus of the paper, the discussion is restricted to weighted utilities even though the results in this section can be proved for all preferences satisfying the Betweenness axiom.

Firstly, if an act is *constant*, that is, if it leads to the same consequence regardless of states, then the weighted utility level from this act is constant regardless of p. So, one of the terms in the left hand side of (3.c) is a constant and the equation becomes linear.

Lemma 4.1 : *Let a and b be acts one of which is constant. Then, the border I_{ab} is affine under weighted utility.*

The other case is when there are only two states s_1 and s_2 so that the simplex $\Delta(\mathbf{S})$ is 1-dimensional, that is, $\Delta(\mathbf{S})$ is a unit interval. Then, solutions to a quadratic equation consist of at most two points of the unit interval $\Delta(\mathbf{S})$, which trivially constitute affine boundaries of a behavior partition of $\Delta(\mathbf{S})$. Note, however, that the number of consequences induced by either of the acts from the two states are generally more than two, so that the simplex $\Delta(\mathbf{Z})$ is of higher dimensional and weighted utilities are a richer class of preferences than expected utilities. Conceptually the same argument applies to three states for the specific pairs of acts such that *i*) both acts lead to the same consequence from the third state and *ii*) the consequences from the other two states are switched between a and b. In this case, the third state is "irrelevant" in determining I_{ab}. Predictably, when these states are duplicated, the borders remains to be affine while gaining additional dimensions corresponding to the duplicated states. The discussions in this paragraph are formalized below.

Definition 4.2 : *Let a and b be acts from* $\mathbf{S}$ *to* $\mathbf{Z}$. *States s and s' are equivalent for a and b if $a(s) = a(s')$ and $b(s) = b(s')$. Denote the equivalence class of s by $[s]$. A pair of acts a and b are mirror-images (of each other) if*

a) *there are only two equivalence classes, or*

b) *there are exactly three equivalence classes, say $[s_1]$, $[s_2]$ and $[s_3]$, such that $a(s_1) = b(s_2)$, $a(s_2) = b(s_1)$ and $a(s_3) = b(s_3)$.*

Lemma 4.3 : *Given a weighted utility $\succeq$ and a pair of acts a and b that are mirror-images, $I_{ab}(\succeq)$ is a finite union of affine surfaces.*

Proof : First, consider case a) in Definition 4.2. Let $[s_1]$ and $[s_2]$ be the two equivalence classes and let q_j, $j = 1, 2$, denote the probability of $[s_j]$:

$$q_j \equiv \sum_{\{i:[s_i]=[s_j]\}} p_i \qquad (4.a)$$

where $p \in \Delta(\mathbf{S})$. Clearly, $q_1 + q_2 = 1$. So (3.e) becomes a quadratic equation in a single variable q_1. It is well known that this equation has at most two solutions and each of them is also the solution to an obvious linear equation in q_1. Substituting (4.a) for q_1, we have at most two linear equations in p whose solutions constitute I_{ab}. This completes the proof for case a).

Next, consider case b) of Definition 4.2 and define q_j, $j = 1, 2, 3$, as above. Represent equation (3.e) in two variables q_1 and q_2. Suppose this equation is not vacuous because otherwise the proof is trivial. If $q_1 = q_2$, acts a and b lead to the same lottery, i,e., $M_a p^t = M_b p^t$. Hence, $\{p \in \Delta(\mathbf{S}) : q_1 = q_2\} \subset I_{ab}$. This implies that the left hand side of equation (3.e) is factored into a product of $(q_1 - q_2)$ and another linear factor and, therefore, the solution set to (3.e) is represented by two linear equations. Substituting (4.a) for q_1 and q_2, we have two linear equations in p whose solutions constitute I_{ab}. This completes the proof. ∎

Remark 4.4 : *For pairs of acts that are mirror-images in sense b) of Definition 4.2, the requirement that the consequences from states s_1 and s_2 are switched between a and b are necessary for Lemma 4.3. In particular, "$a(s_1) \succ a(s_2)$ while $b(s_2) \succ b(s_1)$" is not sufficient. What leads to the conclusion is that when $[s_1]$ and $[s_2]$ have the same probability, the random prospects that a and b induce are identical (not merely indifferent), which determines a part of I_{ab} which is affine. Notice that this conclusion comes from the acts alone without reference to $\succeq$. Even though $a(s_3) = b(s_3)$, if the above mentioned requirement fails this argument also fails because I_{ab} is governed by the preference $\succeq$.*

Remark 4.5 : *The behavior partition for a pair of acts that are mirror-images in sense b) can take the following form which may not be naturally predicted: The 2-dimensional simplex $\Delta(\{s_1, s_2, s_3\})$ is divided into four blocks by two straight lines crossing at one point inside the simplex. One of the straight lines connects the vertex for s_3 and the middle point of the side $\overline{s_1 s_2}$, and the other straight line is parallel to $\overline{s_1 s_2}$. One of the acts is chosen in one pair of blocks with no common boundary and the other act is chosen in the other pair of blocks. It is easily verified that the behavior partition of $\Delta(\{s_1, s_2, s_3\})$ always takes this form if there is one indifference curve in $\Delta(\mathbf{Z})$ that is parallel to the side of $\Delta(\mathbf{Z})$ connecting the two consequences induced from s_1 and s_2. (Notice that $\Delta(\mathbf{Z})$ is also 2-dimensional.) This behavior partition has affine boundaries but does not satisfy the integrability condition of Green and Osband.*

5. Weighted Utilities Generate Genuinely Quadratic Boundaries

Pairs of acts shown in the last section to have affine borders under weighted utility, are very restricted; either they are mirror-images or one of them is constant. Essentially, "mirror-images" picks up situations where $\Delta(\mathbf{S})$ is 1-dimensional so that quadratic equations are inherently undistinguished from linear equations in the structure of their solution components. In particular, if the utility of an act is defined to be a state-dependent function of the act itself, then by definition no act is constant and no pair of acts are mirror-images. In this section we establish, for acts that are neither constant nor mirror-images, that the boundaries are genuinely quadratic under weighted utility by verifying non-affine curvature of borders in $\Delta(\mathbf{S})$.

It is not to be hoped even for these acts that boundaries are genuinely quadratic under all weighted-utility preferences. Expected utilities are a subset of weighted utilities and, as will be clarified shortly, there are weighted utilities which are not expected utilities but still produce affine boundaries. The result shown is for "generic" weighted utilities in the sense specified below.

Given $\mathbf{Z} = \{1, \ldots, K\}$, a weighted utility on $\Delta(\mathbf{Z})$ is represented by a pair of vectors $\mathbf{u} = (u_1, \ldots, u_K) \in \mathbb{R}^K$ and $\alpha = (\alpha_1, \ldots, \alpha_K) \in \mathbb{R}^K_{++}$ where u_z and α_z are the utility level and the weight of consequence $z = 1, \ldots, K$. So, we define a space of weighted utilities as

$$\mathcal{W} \equiv \mathbb{R}^K \times \mathbb{R}^K_{++}$$

A subset of $\mathcal{W}$ is *negligible* if it is contained in a closed set with measure 0. A subset of $\mathcal{W}$ is *generic* if it is the complement of a negligible set.[8]

Proposition 5.1 : *Suppose a and b are acts from $\mathbf{S}$ to $\mathbf{Z}$ that are neither constant nor mirror-images. Then, the set of weighted utilities under which I_{ab} contains an affine surface in $\Delta(\mathbf{S})$, is a negligible subset of $\mathcal{W}$.*

Proof of Proposition 5.1 is basically to find the utility levels and weights of weighted utility for which the quadratic equation in (3.e) admits a representation by linear equations, and verify that these weighted utilities form a negligible subset. Apparently, if the coefficient matrix $\tilde{Q}$ in (3.e) is 0, the equation is linear. Less obviously, for certain coefficients, a quadratic equation can be represented by a union of linear equations. (For example, $p_1^2 - p_2^2 + 2p_1 + 1 = 0$ is equivalent to $p_2 = p_1 + 1$ or $p_2 = -p_1 - 1$.)

If the coefficients of the quadratic equation are independently determined (as in the case of state-dependent utilities of acts), either of the above mentioned reduction to linearity occurs only if the parameter values of weighted utility solve a certain non-vacuous polynomial equation and so the conclusion of the Proposition follows. (See Lemma A in the Appendix.) However, since different combinations of state and act may lead to identical consequences, coefficients of the quadratic equation can be interrelated. To deal with this complication, the proof consists of case-by-case arguments as provided in the Appendix. Here, we sketch the main structure of the proof.

For a and b to have an affine boundary, there should be one state for which a is preferred to b and another state for which the preference is reversed. Since a and b are not mirror-images, there is a third state not equivalent to either of the two above mentioned states. Now we focus on the 2-dimensional face of $\Delta(\mathbf{S})$ convex-spanned by these three states and look into the possibility that I_{ab} intersected with this face contains straight line segments. The equation in (3.e) confined to this

[8] The space of weighted utilities can be reduced by taking the quotient space of $\mathcal{W}$. However, generic sets are preserved between these two spaces because the quotient map is homeomorphic. Earlier version of this paper presents the results using the reduced space.

face is easier to handle because it has only two variables. By examining all possible consequences that may be induced by a and b from these three states,[9] we conclude that a pair of acts that are neither constant nor mirror-images are sufficiently differentiated so that it is not possible for the coefficients of the equation to be interrelated in such a way that the quadratic equation breaks down to linear ones.

We close this section with an implication of Proposition 5.1 on the size of the set of weighted utilities that may produce behavior partitions with affine boundaries for three or more acts. The proof of this corollary is immediate from Proposition 5.1.

Corollary 5.2 : *Suppose* $\mathbf{A}$ *is a finite set of acts from* $\mathbf{S}$ *to* $\mathbf{Z}$ *such that none of the acts is constant and at least one act, say* a*, is not a mirror-image of any other act in* $\mathbf{A}$*. Then, the set of weighted utilities that may produce a behavior partition consisting of two or more polyhedral blocks such that one of the blocks is for* a*, is a negligible subset of* $\mathcal{W}$.

It needs to be mentioned that if one of the acts is constant, the conclusion of Corollary 5.2 is not generally true because a non-affine border may be hidden in the polyhedral block for the constant act. The following example illustrates this possibility.

Example 5.3 : Let $\mathbf{S} = \{s_1, s_2, s_3, s_4\}$ and $\mathbf{Z} = \{1, 2, 3\}$ be the sets of states and consequences and let $\{a, b, c\}$ be the following acts;

$$a : \begin{cases} s_1 \to 3 \\ s_2 \to 1 \\ s_3 \to 1 \\ s_4 \to 1 \end{cases} \qquad b : \begin{cases} s_1 \to 2 \\ s_2 \to 2 \\ s_2 \to 2 \\ s_4 \to 2 \end{cases} \qquad c : \begin{cases} s_1 \to 1 \\ s_2 \to 1 \\ s_3 \to 2 \\ s_4 \to 3 \end{cases}$$

Notice that neither $\{a, c\}$ nor $\{b, c\}$ are mirror-images. First, consider the expected utility according to which $z_3 \succ z_2 \succ z_1$ and z_2 is indifferent to the lottery that z_1 realizes with probability 2/3 and z_3 realizes with probability 1/3. Using the diagram in Figure 2, a is chosen in the top tetrahedron, c is chosen in the bottom tetrahedron and b is chosen in between. I_{ac} is the shaded triangle. Notice that I_{ac} is entirely contained in the interior of the block for b. The borders between acts change continuously as the preference changes continuously. Hence, in the space of weighted utilities, there is a neighborhood of the above considered expected utility such that, if we pick a preference in that neighborhood the corresponding I_{ac} (which is not affine in general) is hidden in the polyhedral block for b. So, the weighted utilities in this neighborhood produce behavior partitions that are rationalized by expected utility.

[**Figure 2** here]

[9] The number of these consequences is in general more than 3 and so, the dimension of the probability simplex over these consequences is higher than 2.

Appendix: Proof of Proposition 5.1

Let $\mathbf{S} = \{s_1, \ldots, s_n\}$ be states and let $\mathbf{Z} = \{1, \ldots, K\}$ be consequences. We simplify the notation: a_i and b_i represent $a(s_i)$ and $b(s_i)$, respectively. In this appendix, we restrict our attention to the following generic subset of weighted utilities; no two pure consequences are indifferent. We verify the conclusion of Proposition 5.1 for a generic subset of this generic set of weighted utilities.

Suppose the border I_{ab} contains an affine surface B_{ab} in $\Delta(\mathbf{S})$. Then, by Substitution-independence, I_{ab} contains the affine hull of B_{ab} for which we use the same notation B_{ab}. It is not the case that all vertices of $\Delta(\mathbf{S})$ are contained in one of the closed half space of B_{ab}: If they were, B_{ab} would not intersect the interior of $\Delta(\mathbf{S})$ contrary to supposition. (Recall that a surface is assumed to contain interior points of $\Delta(\mathbf{S})$.) So, there is one state in each open half space of B_{ab}. That is, there are two states, say s_1 and s_2 such that

$$a_1 \succ b_1 \qquad \text{and} \qquad b_2 \succ a_2 \qquad (A.a)$$

Since a and b are not mirror-images, there is a third state s_3 not equivalent to s_1 or s_2, that is,

$$(a_3, b_3) \neq (a_i, b_i), \;\; i = 1, 2 \qquad (A.b)$$

Denote the simplex of $\{s_1, s_2, s_3\}$ in terms of p_2 and p_3:

$$\tilde{\Delta} = \{(p_2, p_3) \in \mathbb{R}^2_+ : \; p_2 + p_3 \leq 1\} \qquad (A.c)$$

¿From (3.e), $B_{ab} \cap \tilde{\Delta}$ is the solution to the quadratic equation

$$P(p_2, p_3) \equiv \gamma_{22} p_2^2 + \gamma_{33} p_3^2 + \gamma_{23} p_2 p_3 + \delta_2 p_2 + \delta_3 p_3 + c = 0 \qquad (A.d)$$

where the coefficients are as follows ($j = 2, 3$):

$$\begin{aligned}
\gamma_{jj} =& (\alpha_{a_j} u_{a_j} - \alpha_{a_1} u_{a_1})(\alpha_{b_j} - \alpha_{b_1}) \\
& - (\alpha_{b_j} u_{b_j} - \alpha_{b_1} u_{b_1})(\alpha_{a_j} - \alpha_{a_1}), \\
\gamma_{23} =& (\alpha_{a_2} u_{a_2} - \alpha_{a_1} u_{a_1})(\alpha_{b_3} - \alpha_{b_1}) \\
& - (\alpha_{b_2} u_{b_2} - \alpha_{b_1} u_{b_1})(\alpha_{a_3} - \alpha_{a_1}) \\
& + (\alpha_{a_3} u_{a_3} - \alpha_{a_1} u_{a_1})(\alpha_{b_2} - \alpha_{b_1}) \\
& - (\alpha_{b_3} u_{b_3} - \alpha_{b_1} u_{b_1})(\alpha_{a_2} - \alpha_{a_1}), \\
\delta_j =& (\alpha_{a_j} u_{a_j} - \alpha_{a_1} u_{a_1})\alpha_{b_1} + \alpha_{a_1} u_{a_1}(\alpha_{b_j} - \alpha_{b_1}) \\
& - (\alpha_{b_j} u_{b_j} - \alpha_{b_1} u_{b_1})\alpha_{a_1} - \alpha_{b_1} u_{b_1}(\alpha_{a_j} - \alpha_{a_1}), \\
c =& \alpha_{a_1} \alpha_{b_1}(u_{a_1} - u_{b_1}) \neq 0
\end{aligned}$$

We check necessary conditions for the solution set to ($A.d$) to be affine:

If $\gamma_{22} = \gamma_{33} = 0$, necessary conditions are

(c.1) $\gamma_{23} = 0$ or

(c.2) $\gamma_{23} c = \delta_2 \delta_3$.

If $\gamma_{22} \neq 0$, $(A.d)$ implies

$$p_2 = \frac{-(\gamma_{23}p_3 + \delta_2) \pm [(\gamma_{23}p_3 + \delta_2)^2 - 4\gamma_{22}(\gamma_{33}p_3^2 + \delta_3 p_3 + c)]^{\frac{1}{2}}}{2\gamma_{22}}$$

This equation is linear in p_3 only if the square root term resolves as linear, that is,

$$\text{(c.3)} \quad (\delta_2^2 - 4\gamma_{22}c)(\gamma_{23}{}^2 - 4\gamma_{22}\gamma_{33}) - (\gamma_{23}\delta_2 - 2\gamma_{22}\delta_3)^2 = 0.$$

Finally, if $\gamma_{33} \neq 0$, analogous argument leads to the necessary condition

$$\text{(c.4)} \quad (\delta_3^2 - 4\gamma_{33}c)(\gamma_{23}{}^2 - 4\gamma_{22}\gamma_{33}) - (\gamma_{23}\delta_3 - 2\gamma_{33}\delta_2)^2 = 0.$$

What we will show is that given a pair of acts a and b as specified in Proposition 5.1, those weighted utilities $(\mathbf{u}, \alpha)$ that satisfy one of the necessary conditions $(c.1)$ to $(c.4)$ form a negligible set. By virtue of the next Lemma, we only need to show that each condition is a polynomial equation in $(\mathbf{u}, \alpha)$ with at least one non-vanishing term (monomial). (We say a monomial *vanishes* if its coefficient is 0.)

Lemma A : *The solution set to a polynomial equation in $\mathbb{R}^N$ is a negligible subset of $\mathbb{R}^N$ if the polynomial has at least one non-vanishing monomial.*

Proof : This is an immediate corollary of Theorems 1 and 2 of Whitney (1957). ∎

The analysis depends on the consequences that each act induces. If each act-state pair induces a distinct consequence (i.e., the case of state-dependent utilities of acts), obviously every condition (c.1) to (c.4) is a non-vacuous polynomial. On the other hand, if some act-state pairs induce identical consequences, the analysis involves simple but laborious calculation. For each possible case, we find a monomial which does not vanish. The result is summarized below.

Since acts are not mirror-images, we can choose s_1 and s_3 such that $(a_1, b_1) \neq (b_3, a_3)$. For notational convenience, denote $a_1 \equiv s$ and $b_1 \equiv r$ where $s \succ r$.

<u>Case 1.</u> $a_3 = s \prec b_3 \equiv t$.

In this case γ_{33} is a vacuous polynomial (i.e, identically 0). Because a is not constant, there is a state, say s_2, such that $a_2 \neq s$.

[Subcase 1-1] $\gamma_{22} = 0$.

¿From the expression of γ_{22} in terms of $(\mathbf{u}, \alpha)$, this subcase happens only if either $a_2 = r$ and $b_2 = s$, or $b_1 = b_2 = r$.

In this subcase, condition (c.1) is

$$(\alpha_{a_2}u_{a_2} - \alpha_s u_s)(\alpha_t - \alpha_r) - (\alpha_t u_t - \alpha_r u_r)(\alpha_{a_2} - \alpha_s) = 0.$$

So, the term $\alpha_s \alpha_t u_s$ does not vanish.

Next, condition (c.2) is expressed in terms of $(\mathbf{u}, \alpha)$ as

$$\begin{aligned}
&[(\alpha_{a_2}u_{a_2} - \alpha_s u_s)(\alpha_t - \alpha_r) - (\alpha_t u_t - \alpha_r u_r)(\alpha_{a_2} - \alpha_s)] \cdot [\alpha_s \alpha_r (u_s - u_r)] \\
&- [(\alpha_{a_2}u_{a_2} - \alpha_s u_s)\alpha_r + \alpha_s u_s(\alpha_{b_2} - \alpha_r) - (\alpha_{b_2}u_{b_2} - \alpha_r u_r)\alpha_s + \alpha_r u_r(\alpha_{a_2} - \alpha_s)] \cdot \\
&\quad [\alpha_s u_s(\alpha_t - \alpha_r) - (\alpha_t u_t - \alpha_r u_r)\alpha_s] \\
&= 0
\end{aligned}$$

Consider the first possibility of $\gamma_{22}=0$: $a_2=r$ and $b_2=s$. Then, the term with u_s^2 is $\alpha_s^2\alpha_r(\alpha_t-\alpha_r)u_s^2$ which does not vanish. Consider the second possibility: $b_1=b_2=r$. If $a_2=t$, the term $\alpha_t^2\alpha_r\alpha_s u_t^2$ does not vanish. If $a_2\neq t$, the term $\alpha_t\alpha_s\alpha_r\alpha_{a_2}u_t u_s$ does not vanish.

[Subcase 1-2] $\gamma_{22}\neq 0$.

Since $\gamma_{33}=0$, condition (c.3) is $\gamma_{22}(\gamma_{23}\delta_2\delta_3-\gamma_{22}\delta_3^2-\gamma_{23}^2 c)=0$. Being too long, the expression in terms of $(\mathbf{u},\alpha)$ is omitted. Similar investigation to Subcase 1-1 can be carried out whose results are summarized below.

1) If $b_2=s$, the term $\alpha_s^3\alpha_t^2\alpha_{a_2}u_s^3$ does not vanish.
2) If $b_2=t=a_2$, the term $\alpha_t^2\alpha_s^2\alpha_r^2u_t^2u_s$ does not vanish.
3) If $b_2=t$ but $a_2\neq t$, the term $\alpha_t^3\alpha_s^3u_t^2u_s$ does not vanish.
4) If $b_2\neq s$, $b_2\neq t$ and $a_2=t$, the term $\alpha_t^2\alpha_s^2\alpha_r^2u_t^2u_s$ does not vanish.
5) If $b_2\neq s$, $b_2\neq t$ and $a_2\neq t$, the term $\alpha_t^2\alpha_s^2\alpha_{a_2}\alpha_{b_2}u_t^2u_s$ does not vanish.

<u>Case 2.</u> $a_3\equiv d\prec b_3=r$.

By exchanging the roles of a and b and those of s_1 and s_3, this is the same situation as Case 1.

<u>Case 3.</u> $a_3\equiv f$ and $b_3\equiv g$ where $f\prec g$, $f\neq s$, $g\neq r$ and $(g,f)\neq(s,r)$.

Denote $a_2\equiv x$ and $b_2\equiv y$ and recall that $(x,y)\neq(s,r)$ and $(x,y)\neq(f,g)$. From these conditions, it is easily checked that γ_{33} is a non-vanishing polynomial. So, we consider condition (c.4). Again, we omit the detailed process and summarize results for various cases: we divide cases depending on whether g, x and y are identical with s or not, and in each case we consider further variations.

1) $g=s$, $x=s$, $y=s$: In this case, $f\neq r$ because $(g,f)\neq(s,r)$, and the term $\alpha_s^4\alpha_r^3\alpha_f u_s^4$ does not vanish.
2) $g=s$, $x=s$, $y\neq s$: In this case, $f\neq r$ because $(g,f)\neq(s,r)$, and $y\neq r$ because $(x,y)\neq(s,r)$. So, the term $\alpha_s^4\alpha_r^3\alpha_y u_s^4$ does not vanish.
3) $g=s$, $x\neq s$, $y=s$: In this case, $f\neq r$ because $(g,f)\neq(s,r)$, and $f\neq x$ because $(x,y)\neq(f,g)$. So, the term $\alpha_s^4\alpha_f^3\alpha_r u_s^4$ does not vanish.
4) $g\neq s$, $x=s$, $y=s$: Recall $f\neq g$ and $g\neq r$. In this case, the term $\alpha_s^4\alpha_g^2\alpha_r\alpha_f u_s^4$ does not vanish.
5) $g=s$, $x\neq s$, $y\neq s$: In this case, $f\neq r$ because $(g,f)\neq(s,r)$.
   If $x=f$ and $y\neq f$, the term $\alpha_s^4\alpha_f^3\alpha_r u_s^4$ does not vanish.
   In all other situations, the term $\alpha_s^4\alpha_f^4u_s^4$ does not vanish.
6) $g\neq s$, $x=s$, $y\neq s$: In this case, $y\neq r$ because $(x,y)\neq(s,r)$.
   If $y=g$, the term $\alpha_s^3\alpha_g^3\alpha_f\alpha_r u_s^3u_g$ does not vanish.
   If $y\neq g$, the term $\alpha_s^4\alpha_g^2\alpha_y^2u_s^3u_g$ does not vanish.
7) $g\neq s$, $x\neq s$, $y=s$: Recall $f\neq g$ and $g\neq r$.
   If $x=g$, the term $\alpha_s^4\alpha_g^4u_s^4$ does not vanish.
   If $x\neq g$, the term $\alpha_s^4\alpha_g^3\alpha_x u_s^4$ does not vanish.
8) $g\neq s$, $x\neq s$, $y\neq s$:
   If $x=y=g$, the term $\alpha_s^3\alpha_g^3\alpha_r^2u_s^3u_g$ does not vanish.

If $x = g$ but $y \neq g$, the term $\alpha_s^3\alpha_g^4\alpha_y u_s^3 u_g$ does not vanish.
If $x \neq g$ but $y = g$, the term $\alpha_s^3\alpha_g^4\alpha_x u_s^3 u_g$ does not vanish.
If $x \neq g$ and $y \neq g$, the term $\alpha_s^3\alpha_g^3\alpha_x\alpha_y u_s^3 u_g$ does not vanish.

Until now, we showed that, given a pair of non-constant acts a and b that are not mirror-images, those weighted utilities $(\mathbf{u}, \alpha)$ that satisfy one of the conditions (c.1) to (c.4) form a negligible set. Since one of these conditions is necessary for the border I_{ab} to contain an affine surface in $\Delta(\mathbf{S})$, the proof of Proposition 5.1 is complete.

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Figures

Figure 1

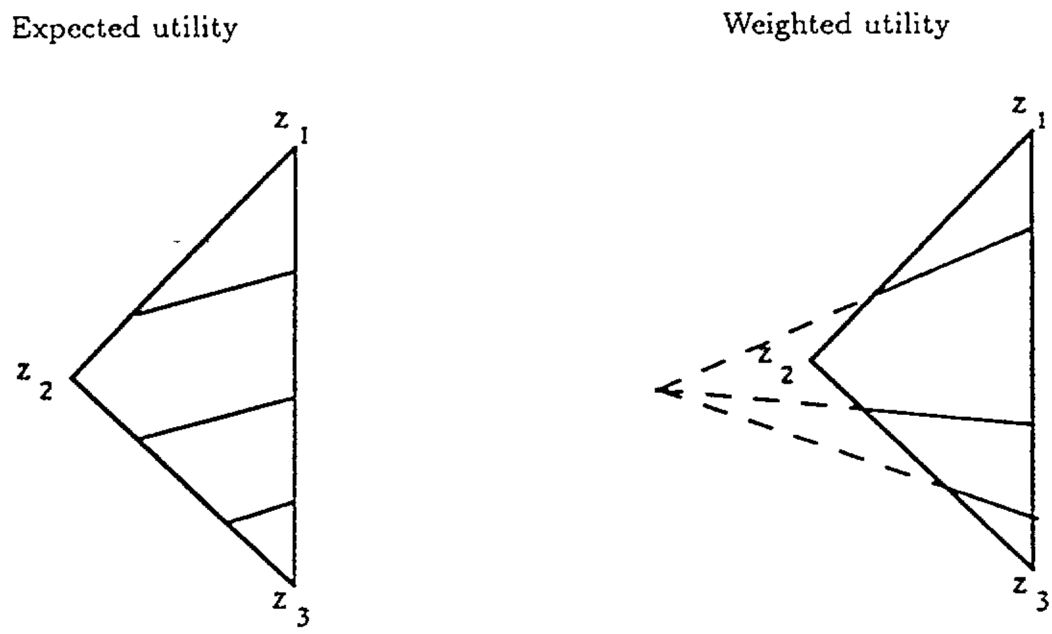
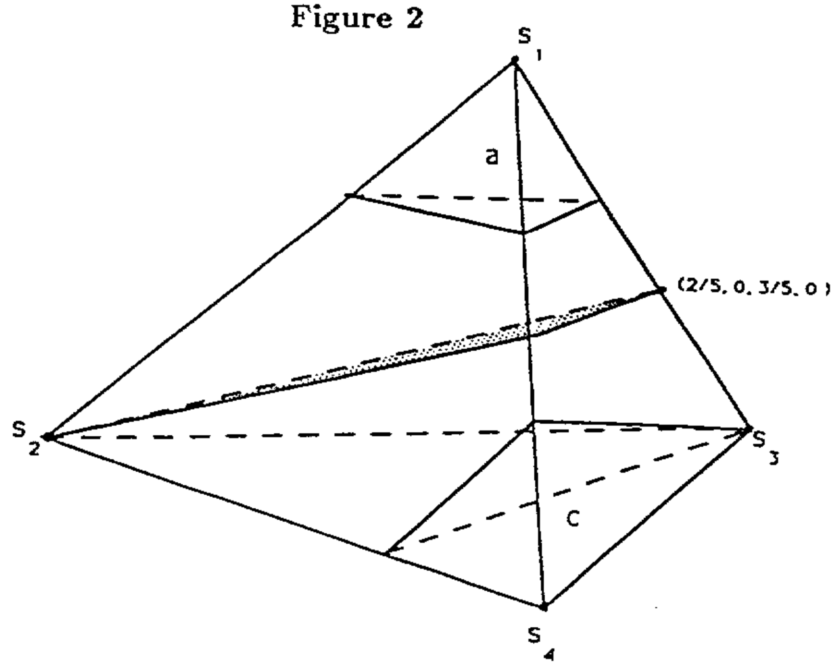


Figure 2



Reconciling Some Conflicting Evidence on Decision Making under Uncertainty*

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Abstract: Laboratory experiments concerning decision under uncertainty tend to uncover systematic violations of Bayesian rationality. When models that posit Bayesian rationality are compared to non-experimental data, though, they fit the data well. One possible explanation is that an agent's global pattern of choices may not be rationalizable, but that the pattern may satisfy weak conditions sufficent to rationalize the limited range of choices required by any particular decision protocol. Examples of such patterns are constructed here. An agent who adopts a protocol acts rationally, but an experimenter induces irrationality by imposing distinct protocols in various phases of the experiment.

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1. Introduction

When people make decisions under uncertainty, are their actions in accord with subjective-expected-utility (SEU) theory? That is, does a person tend to behave as though (*a*) his initial beliefs were encoded by a probability measure on possible events, (*b*) he considers evidence by conditioning this probability measure according to Bayes' rule, and (*c*) he chooses an action that would maximize the expectation of some utility function (which is always the same across the various decision problems that he faces) with respect to this conditional probability measure?

This question has been studied intensively. Researchers who use non-experimental methods tend to conclude that SEU theory accounts well for their data. In sharp contrast, experimental researchers tend to report that behavior inconsistent with SEU theory is pervasive, robust, and easy to elicit from subjects. This systematic disparity between experimental and non-experimental findings is a paradox that needs to be resolved.[1] In this paper I propose one resolution among the many complementary resolutions that can probably account for aspects of this puzzle.

The resolution to be proposed here has to do with how a decision maker often works: by obtaining answers to a sequence of questions, and then eventually choosing an action on the basis of these answers. Such questions typically have answers that are of a yes-or-no variety, or that are numerical quantities, or that otherwise partition the situation into possibilities that are jointly exhaustive and mutually exclusive. Each successive question refines the partition further. It is even common for a decision maker to work according to a protocol that raises these questions in a fixed order. Sometimes an order is determined by practical considerations (as when a detective goes quickly to the scene of the crime, where weather and traffic are likely to obliterate evidence such as footprints), but often a determinate order is followed purely as a matter of professional discipline (as when a doctor obtains a patient's history, if it is possible to do so, at the beginning of treating an illness—even though the nature of the illness is often obvious from signs and symptoms that the patient presents). This determinateness of order has the implication that there is a fixed sequence of increasingly fine partitions of events that uniformly characterize the decision maker's knowledge at successive stages of a family of related decision problems such as criminal investigations or medical cases.

Part of a decision maker's job is to decide when to stop asking questions and to do something. However, even before this point is reached, a decision maker will make judgments regarding what would be the best thing to do on the basis of current evidence. Sherlock Holmes might be confident that he will solve a case conclusively but might nevertheless judge that, if there were an imminent prospect of the criminal committing another heineous crime before further evidence could be unearthed, then the police should try to prevent it by arresting Moriarity. A surgeon might know that, if the hospital had run out of a dye needed to make a patient's blood vessels show clearly in an X-ray, then a particular operation should be performed without waiting to take such an X-ray

[1] This paradox has been discussed by the contributors to Hogarth and Reder (1987). In addition to the types of study discussed in that volume, a new kind of non-experimental study has subsequently become available: inference about decision making based on explicit modelling and estimation of data regarding decision making by an expert subject in the course of his actual work. A leading example of such a study is Rust (1987).

because prompt repair would be imperative if the patient were bleeding internally. I will assume that a researcher can observe such "premature" contingent decisions, either because exceptional circumstances such as those that I have just described do arise occasionally, or else because decision makers are able to make reliable reports about what they would do counterfactually. In this respect, I am making a generous assumption about how much data a non-experimental researcher can obtain.

To summarize, in non-experimental settings a decision maker will typically consider a fixed sequence of information partitions, which is dictated either by feasibility restrictions or by the decision maker's protocol. This fixed sequence of partitions is all that a non-experimental researcher can observe. In contrast, an experimental researcher can force a subject to consider information partitions that the researcher—not the subject—selects. In particular, a researcher can force a subject to use incomparable partitions to make decisions about cases that are otherwise similar. For example, a doctor might be required to choose a hypothetical treatment on the basis of information about whether or not the patient has "an elevated temperature" in one case, but on the basis of information about whether or not the patient has "a high fever" in another case. Those two information partitions are intuitively close to one another, but they are not identical. In medicine and in many other fields, authoritative protocols for decision making specify recommended information partitions exactly and discourage decision makers from substituting approximately equivalent partitions. Thus subjects, and especially experts who are well-trained professionals, can be forced in experimental settings to make decisions that they would systematically avoid in their actual work. I am going to argue that comparisons among such forced decisions in experimental settings greatly expand the opportunity to observe violations of the SEU theory. Some implications of this argument will be examined in the concluding section of the paper.

2. A formal description of decision-making data

I will describe the data from experimental and non-experimental studies of decision making in terms that closely resemble a standard version of SEU theory.[2] There is a finite Boolean algebra $\mathcal{E}$, the elements of which are to be interpreted as *events.*[3] I will assume informally that, except for the null event (denoted by $0 \in \mathcal{E}$), these are events to which the decision maker would consider seriously as possibly occurring.[4] Upper-case parameters and variables will denote events in $\mathcal{E}$.

There is also a finite set $\mathcal{A}$ of *acts* that the decision maker can choose. Lower-case parameters and variables will denote acts in $\mathcal{A}$.

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[2] A significant limitation of this description will be discussed in the conclusion.

[3] Finiteness is assumed here for convenience. It is not a hypothesis of the results of Green and Park (1993) that are adapted to this setting below.

[4] If there are nonzero events that the decision maker would totally discount, then they should constitute a lattice ideal. The Boolean algebra formed by factoring by this ideal will then satisfy my informal assumption. This is the technical reason why I am considering an abstract Boolean algebra of events, rather than a field of sets of "states of nature."

A *decision rule* is a correspondence

$$\delta: \mathcal{E} \to \mathcal{A}. \tag{1}$$

That is, δ is a function that assigns a subset of $\mathcal{A}$ to each event. Since $0 \in \mathcal{E}$ is to be interpreted as the impossible event, stipulate that

$$\delta(0) = \mathcal{A}. \tag{2}$$

This stipulation is innocuous, and it will simplify some definitions below.

I will assume that a researcher can observe a subject's entire decision rule in an experiment. All that can be non-experimentally observed, though, is a decision maker's *contingent plan.* A contingent plan consists of a sequence $\Pi = (\Pi_0, \ldots, \Pi_n)$ of partitions[5] of 1, together with a correspondence

$$\pi: \bigcup_{m \leq n} \Pi_m \to A. \tag{3}$$

Assume that the initial partition Π_0 is trivial, and the partitions Π_m are successive refinements of one another. That is,

$$\Pi_0 = \{1\}, \quad \text{and} \quad \forall m < n \;\; \Pi_{m+1} \text{ is a refinement of } \Pi_m. \tag{4}$$

Given a function π that is defined on the union of a sequence Π of partitions satisfying (4), a sequence Π' satisfying (4) can be constructed from the domain of π. The new Π can differ from the original Π only with respect to the timing of the resolution of uncertainty. (That is, a partition element of some Π_m may be partitioned further at a different time in Π' than in Π, but it must be partitioned in the same way in both sequences.) Since conditional expected utility is insensitive to the time at which uncertainty is resolved, the function π contains all of the information about the sequence Π that will be of interest here. Therefore the term 'contingent plan' will sometimes be applied to π itself, without specifying Π explicitly.

Finally, say that a decision rule δ and a sequence Π of partitions satisfying (4) *induce* the contingent plan π that is obtained by restricting δ to the set of events $\bigcup_{m \leq n} \Pi_m$.

---

[5] A partition of $C \in \mathcal{E}$ is a set $\mathcal{P}$ of non-null events (that is, $A \neq 0$) that are pairwise disjoint (that is, $A \wedge B = 0$ if $A \neq B$) and such that $\bigvee_{A \in \mathcal{P}} A = C$. Partition $\mathcal{P}'$ is a refinement of partition $\mathcal{P}$ if, for each element $A \in \mathcal{P}$, a subset $\mathcal{R} \subseteq \mathcal{P}'$ is a partition of A.

3. SEU rationality

Conformity of a decision rule or of a contingent plan to subjective-expected-utility theory can be defined in terms of signed measures. A signed measure on $\mathcal{E}$ is a function $\mu: \mathcal{E} \to \Re$ that satisfies

$$\mu(0) = 0 \quad \text{and} \quad \forall A \ \forall B \ \left[A \wedge B = 0 \Longrightarrow \mu(A \vee B) = \mu(A) + \mu(B)\right]. \tag{5}$$

Note that in a probability space, a signed measure is defined on the field of events (that is, measurable subsets of the sample space) by considering the integral of a measurable function (from the sample space to $\Re$) as a function of the domain of integration. In particular, suppose that $(\Omega, \mathcal{E}, \Pr)$ is a probability space and that $v: \mathcal{A} \times \Omega$ is a state-contingent utility function. Then, for each $a \in \mathcal{A}$, a signed measure μ_a is defined by $\mu_a(B) = \int_B v(a, \omega)\, d\Pr(\omega)$. The conditional expected utility of act a in event B is $\mu(B)/\Pr(B)$. Thus a' has higher conditional expected utility than does a in event B if and only if $\mu_{a'}(B) > \mu_a(B)$.

This observation motivates a definition of SEU rationality. To state this definition, let $\mathcal{M}$ denote the set of signed measures. If $u: \mathcal{A} \to \mathcal{M}$, then I will write $u_a(B)$ instead of $[u(a)](B)$ in order to avoid cumbersome notation.

Let $\mathcal{C} \subseteq \mathcal{E}$, and let $\gamma: \mathcal{C} \to \mathcal{A}$. Then γ is *SEU rational* if there is a mapping $u: \mathcal{A} \to \mathcal{M}$ such that

$$\forall C \in \mathcal{C} \quad \gamma(C) = \left\{a \middle| \forall a' \ \ u_{a'}(C) \leq u_a(C)\right\}. \tag{6}$$

4. A decision rule that is not SEU rational

The following is an example of a decision rule that is not SEU rational. Let $\mathcal{E}$ be the field of subsets of a set of four elements. Specifically, let

$$1 = \{a, b, c, d\}; \ \ \mathcal{E} = \{B | B \subseteq S\}. \tag{7}$$

For an event $B \in \mathcal{E}$, define $\#B$ to be the cardinality of B. Also let

$$\mathcal{A} = 1 \cup \{f, g\}. \tag{8}$$

Define $\delta: \mathcal{E} \to \mathcal{A}$ by

$$\delta(B) = \begin{cases} \mathcal{A}, & \text{if } B = \emptyset; \\ B, & \text{if } B = \{x\}; \\ \{f\}, & \text{if } \#B = 2 \text{ or } B = 1; \\ \{g\}, & \text{if } \#B = 3. \end{cases} \tag{9}$$

That δ is not SEU rational is easily demonstrated by contradiction. Let $A = \{a\}$, $J = \{b, c\}$, and $K = \{b, c, d\}$,. Suppose that the mapping $u: \mathcal{A} \to \mathcal{M}$ were to satisfy the SEU-rationality condition (6). By (6) and (9), $u_f(J) > u_g(J)$ and $u_f(A \vee J) < u_g(A \vee J)$. By the additivity property (5) of a signed measure, therefore, $u_f(A) < u_g(A)$. However, analogous reasoning involving substitution of K for J and transposition of f and g, leads to the contradictory conclusion that $u_g(A) < u_f(A)$.

It is well known that there exist decision rules that are not SEU rational. What makes this example noteworthy is the additional fact, to be proved below, that every contingent plan induced by δ (and an arbitrary sequence of partitions satisfying (4)) does conform to SEU theory. To provide some intuition for that result, let me explain now why the demonstration that δ is not SEU rational cannot be applied to any contingent plan induced by δ. Recall that the domain of a contingent plan π is the union of a sequence of partitions ordered by refinement. It is easily seen that, if A and B are any two elements of such a union, then either $A \wedge B = A$ or $A \wedge B = B$ or $A \wedge B = \emptyset$. (This can be proved by induction on the number of partitions in the sequence, and it is also intuitively obvious from drawing a Venn diagram of such a sequence.) In the demonstration that δ is not SEU rational, information about the images of δ at both $A \vee J$ and K are used. Observe that $(A \vee J) \wedge K = J$, and that $J \neq A \vee J$ and $J \neq K$, so that $A \vee J$ and K cannot both occur in a sequence of partitions ordered by refinement. In any contingent plan, then, the decision maker's choice at one of these events or the other will be unobserved.

The kind of example presented here is pervasive. To see this, extend the cardinality operator $\#$ to events in an arbitrary finite Boolean algebra by specifying that $\#B$ is the cardinality (in the usual sense) of the finest partition of B. (Thus $\#1$ denotes the "cardinality" of the unit element of $\mathcal{E}$. If $\mathcal{E}$ is the field of all subsets of a set S, then $1 = S$, so $\#1$ is the cardinality of S.) Also, if $A \in \mathcal{E}$, then define $\mathcal{E}_A$ to be the Boolean algebra of elements $A \wedge B$ for all $B \in \mathcal{E}$. (In particular, $\mathcal{E}_1 = \mathcal{E}$.) The following lemma, which is proved in the same manner as the analysis that has just been made of the example, shows how to construct a decision rule that violates SEU rationality at many places.

Lemma 1: Let $\mathcal{E}$ be a finite Boolean algebra. Define $\mathcal{C} = \{C \in \mathcal{E} | \#C \text{ is odd and } \#C \leq (\#1)/2\}$. Let $\mathcal{A} = \mathcal{C} \cup \{f, g\}$, where $f \neq g$ and $\mathcal{C} \cap \{f, g\} = \emptyset$. Define $\delta: \mathcal{E} \to \mathcal{A}$ by

$$\delta(B) = \begin{cases} \mathcal{A}, & \text{if } B = \emptyset; \\ \{B\}, & \text{if } B \in \mathcal{C}; \\ \{f\}, & \text{if } \#B \text{ is even}; \\ \{g\}, & \text{if } \#B \text{ is odd and } \#B > (\#1)/2. \end{cases} \tag{10}$$

Then, for any $A \in \mathcal{E}$ such that $\#A \geq (\#1)/2 + 3$, $\mathcal{E}_A$ is not SEU rational.

5. SEU-rational contingent plans

Although the decision rules constructed in the preceding section are not SEU rational, they always induce SEU-rational contingent plans. This fact will be proved by means of the following theorem, which adapts to the present context a result of Green and Park (1993). The theorem is proved in the appendix.

Theorem 1: A contingent plan π with domain $\bigcup_{m \leq n} \Pi_m$ is SEU rational if it satisfies three conditions for all events A, B, and C in its domain.

$$\text{If } A = B \vee C \text{ and } B \wedge C = 0 \text{ and } a \in \pi(B) \text{ and } a \in \pi(C), \text{ then } a \in \pi(A). \tag{11}$$

$$\text{If } A = B \vee C \text{ and } B \wedge C = 0 \text{ and } a \in \pi(B) \text{ and } a \in \pi(C) \text{ and } b \notin \pi(B), \text{ then } b \notin \pi(A). \tag{12}$$

$$\pi(A) \neq \emptyset. \tag{13}$$

Condition (11) states that, if event A is partitioned into events B abd C, and if the decision maker would be willing to choose act a if he were certain of either B or C, then he must be willing to chose a if he is certain of event A. Condition (12) states that if the decision maker has at least one such "sure-thing" act, and if there is one of the two specific events comprised by A in which he would be unwilling to choose another act b, then he must not be willing to choose b in event A. Condition (13), which states that in any event there must be at least one act that the decision maker would be willing to choose, is motivated by the idea that the decision maker's conditional choices reflect optimization with respect to consistent preferences on the finite set $\mathcal{A}$.

Green and Park (1993) use this theorem provide some idea of the stringency of conditions (11)—(13). They show that a single-valued minimax-loss or minimax-regret decision rule always (that is, for any partition sequence) induces a contingent plan that is SEU-rational. That result does not address the specific question being studied in this paper, though, because it is consistent with the possibility that a minimax-loss or minimax-regret decision rule may always be SEU rational on its full domain if it is single-valued. However, Green and Park do address the present question by constructing an example in which a decision rule induces a contingent plan that is not SEU rational. The decision rule in the example is obtained by maximizing an instance of Chew's (1983) "weighted-utility" preference. It is noteworthy that Chew introduced that class of preferences in order to provide a relatively parsimonious generalization of expected-utility theory that would accomodate experimental evidence regarding Allais' paradox. Thus the example would support the view that behavior that is inconsistent with SEU rationality should presumably be observable outside the laboratory, if it can be elicited from experimental subjects inside the laboratory. Now I turn to the analysis of the examples constructed in the proceding section, and I show that such a view does not necessarily have to be taken.

6. SEU rationality of contingent plans induced by SEU-irrational decision rules

The decision rules defined at the beginning of section 4 and in lemma 1 can be shown always to induce SEU-rational contingent plans, by showing that it satisfies conditions (11)—(13) of theorem 1.[6] Specifically, consider the decision rule δ defined by condition (10) in lemma 1. Clearly δ satisfies condition (13) of being nonempty valued. Moreover, since δ is single valued (except at 0), condition (11) implies condition (12). To see this, suppose that the antecedent of (12) is true. Then, by (11), $a \in \delta(A)$. Since $b \notin \pi(B)$, $B \neq 0$ and therefore $A \neq 0$. Since δ is single valued at A, $b \notin \delta(A)$. That is the conclusion of (12), so the implication (12) is true.

Thus only condition (11) has to be checked. Suppose, therefore, that $A = B \vee C$ and that $B \wedge C = 0$. To consider the case that $B = 0$ or $C = 0$, assume without loss of generality that $B = 0$. Then $A = C$, so (11) is tautological. If neither $B = 0$ nor $C = 0$, then there are three possibilities. Either both $\#B$ and $\#C$ are even, or else both $\#B$ and $\#C$ are odd and at least one of them is no larger than $(\#1)/2$ (since $B \wedge C = 0$), or else one of $\#B$ and $\#C$ is even and the other is odd. In the first case, $\#A$ is also even, so $\delta(B) = \delta(C) = \delta(A) = \{f\}$ by definition (10), so (11) is true. In both of the remaining cases, (10) implies that there are x and y such that $\delta(B) = \{x\}$ and $\delta(C) = \{y\}$ and $x \neq y$, so the antecedent of implication (11) must false because there can be no a such that $a \in \delta(B)$ and $a \in \delta(C)$. Therefore (11) is true. This proves the following result.

Theorem 2: If π is induced by the decision rule δ defined by (10), together with any sequence of partitions satisfying (4), then π is SEU rational.

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[6] Lemma 1 does not show that the decision rule in the example given at the beginning of section 4 is not SEU rational, because in that example $\#1 < (\#1)/2 + 3$. The bound $(\#1)/2 + 3$ can actually be improved to $(\#1)/2 + 2$, though, except in the case that $\#1$ is divisible by 2 but not by 4. That improvement subsumes the example.

Appendix: Proof of Theorem 1

Conditions (11) and (12) imply that

$$\text{If } \Theta \text{ is a partition of } \mathcal{A}, \text{ then } \bigcap_{B\in\Theta} \pi(B) = \pi(A) \text{ or } \bigcap_{B\in\Theta} \pi(B) = \emptyset. \tag{14}$$

Signed measures u_a for $a \in \mathcal{A}$ will be defined from a function $V: \mathcal{A} \times \bigcup_{m\leq n} \Pi_m \to \Re$. The function V is defined recursively on the partitions Π_m, as follows. The basis step is that, if $B \in \Pi_0$, then $B = 1$ and

$$V(a,1) = \begin{cases} 1, & \text{if } a \in \pi(1); \\ 0, & \text{if } a \notin \pi(1); \end{cases} \tag{15}$$

For the recursion step, let $\phi: \bigcup_{m\leq n} \Pi_m \to \mathcal{A}$ be a selection from π. (That is, $\forall A \in \bigcup_{m\leq n} \Pi_m \ \phi(A) \in \pi(A)$.) Suppose that $m < n$ and that V has been defined on $\mathcal{A} \times \Pi_m$. The next partition Π_{m+1} can be expressed uniquely as a union $\bigcup_{C\in\Pi_m} \Theta_C$, where each Θ_C is a partition of C. For each $a \in \mathcal{A}$ and for each $C \in \Pi_m$, define Θ_{aC} to be the set of $B \in \Theta_C$ such that either $a \in \pi(B)$ or else $\forall D \in \Theta_C \ a \notin \pi(D)$. Now, for each $a \in \mathcal{A}$, extend the definition of V to $\{a\} \times \Theta_{aC}$ by specifying that

$$V(a,B) = \begin{cases} V(\phi(C),C), & \text{if } a \in \pi(B) \text{ and } \bigcap_{B\in\Theta_C} \pi(B) \neq \emptyset; \\ V(\phi(C),C)+1, & \text{if } a \in \pi(B) \text{ and } \bigcap_{B\in\Theta_C} \pi(B) = \emptyset; \\ V(a,A), & \text{if } \forall D \in \Theta_C \ a \notin \pi(D); \end{cases} \tag{16}$$

Complete the extension of V to $\mathcal{A} \times \Theta_C$ by specifying that

$$\text{If } B \notin \Theta_{aC}, \text{ then } V(a,B) = \rho < V(a,C), \tag{17}$$

where ρ solves the equation

$$\sum_{D\in\Theta_{aC}} [V(a,D) \cdot \#D] + \rho[\#C - \sum_{D\in\Theta_{aC}} \#D] = V(a,C) \cdot \#C. \tag{18}$$

It must be proved that the equation (18) always possesses a solution, and that this solution satisfies the inequality in (17). Both of these facts are guaranteed by (14), as is shown by Green and Park (1993).

Conditions (17) and (18) are essentially a martingale condition, with respect to counting measure on the atoms of $\mathcal{E}$ (that is, the elements $D \in \mathcal{E}$ such that $\#D = 1$), for the conditional expected utility $V(a,B)$ of taking act a. By the martingale convergence theorem, for each a there is a signed measure $u_a \in \mathcal{M}$ that satisfies

$$\forall B \in \bigcup_{m\leq n} \Pi_m \quad u_a(B) = V(a,B). \tag{19}$$

By induction on m, using conditions (16) and (17),

$$\forall B \in \bigcup_{m\leq n} \Pi_m \quad \pi(B) = \{a | \forall a' \in \mathcal{A} \ V(a',B) \leq V(a,B). \tag{20}$$

Conditions (19) and (20) together imply condition (6) which defines SEU rationality. ∎

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