Elections and Strategic Voting: Condorcet and Borda

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- voting rule (social choice function)

  method for choosing social alternative (candidate) on basis of voters' preferences (rankings, utility functions)

- prominent examples
  - Plurality Rule (MPs in Britain, members of Congress in U.S.)

    choose alternative ranked first by more voters than any other
  - Majority Rule (Condorcet Method)

    choose alternative preferred by majority to each other alternative

- Run-off Voting (presidential elections in France)
  - choose alternative ranked first by more voters than any other, unless number of first-place rankings

    less than majority

    among top 2 alternatives, choose alternative preferred by majority
- Rank-Order Voting (Borda Count)
  - alternative assigned 1 point every time some voter ranks it first, 2 points every time ranked second, etc.
  - choose alternative with lowest point total
- Utilitarian Principle
  - choose alternative that maximizes sum of voters' utilities

- Which voting rule to adopt?
- Answer depends on what one wants in voting rule
  - can specify *criteria* (axioms) voting rule should satisfy
  - see which rules best satisfy them
- One important criterion: *nonmanipulability*
  - voters shouldn't have incentive to misrepresent preferences, i.e., vote *strategically*
  - otherwise

    not implementing intended voting rule

    decision problem for voters may be hard

- But basic negative result

  Gibbard-Satterthwaite (GS) theorem

  - if 3 or more alternatives, *no* voting rule is always nonmanipulable

    (except for dictatorial rules - - where one voter has all the power)
- Still, GS overly pessimistic
  - requires that voting rule *never* be manipulable
  - but some circumstances where manipulation can occur may be unlikely
- In any case, natural question:

  Which (reasonable) voting rule(s) nonmanipulable *most often*?
- Paper tries to answer question

- X = finite set of social alternatives
- society consists of a continuum of voters [0,1]
  - typical voter $i \in [0,1]$
  - reason for continuum clear soon
- utility function for voter i $U_i : X \to \mathbb{R}$
  - restrict attention to *strict* utility functions

    if $x \neq y$, then $U_i(x) \neq U_i(y)$

    $\mathscr{U}_X$ = set of strict utility functions
- profile $U_{\cdot}$ - - specification of each individual's utility function

- voting rule (generalized social choice function) F

$$\text{for all profiles } U_{\cdot} \text{ and all } Y \subseteq X, \quad F(U_{\cdot}, Y) \in Y$$

  - $F(U_{\cdot}, Y) =$ optimal alternative in Y if profile is $U_{\cdot}$

- definition isn’t quite right - - ignores ties
  - with plurality rule, might be two alternatives that are both ranked first the most
  - with rank-order voting, might be two alternatives that each get lowest number of points
- But exact ties unlikely with many voters
  - with continuum, ties are *nongeneric*
- so, correct definition:

$$\text{for } \textit{generic} \text{ profile } U_{\cdot} \text{ and all } Y \subseteq X \quad F(U_{\cdot}, Y) \in Y$$

plurality rule:

$$f^{P}\left(U_{\cdot},Y\right)=\left\{a\middle|\mu\left\{i\middle|U_i\left(a\right)\geq U_i\left(b\right)\text{ for all }b\right\}\right.$$
$$\left.\geq\mu\left\{i\middle|U_i\left(a'\right)\geq U_i\left(b\right)\text{ for all }b\right\}\text{ for all }a'\right\}$$

majority rule:

$$f^{C}\left(U_{\cdot},Y\right)=\left\{a\middle|\mu\left\{i\middle|U_i\left(a\right)\geq U_i\left(b\right)\right\}\geq\tfrac{1}{2}\text{ for all }b\right\}$$

rank-order voting:

$$f^{B}\left(U_{\cdot},Y\right)=\left\{a\middle|\int r_{U_i}\left(a\right)d\mu\left(i\right)\leq\int r_{U_i}\left(b\right)d\mu\left(i\right)\text{ for all }b\right\},$$

where $r_{U_i}\left(a\right)=\#\left\{b\middle|U_i\left(b\right)\geq U_i\left(a\right)\right\}$

utilitarian principle:

$$f^{U}\left(U_{\cdot},Y\right)=\left\{a\middle|\int U_i\left(a\right)d\mu\left(i\right)\geq\int U_i\left(b\right)d\mu\left(i\right)\text{ for all }b\right\}$$

What properties should reasonable voting rule satisfy?

- *Pareto Property* (P): if $U_i(x) > U_i(y)$ for all i and $x \in Y$, then $y \neq F(U_{\cdot}, Y)$

  – if everybody prefers x to y, y should not be chosen

- *Anonymity* (A): suppose $\pi : [0,1] \rightarrow [0,1]$ measure-preserving permutation. If $U_i^{\pi} = U_{\pi(i)}$ for all i, then

$$F\left(U_{\cdot}^{\pi}, Y\right) = F\left(U_{\cdot}, Y\right) \text{ for all } Y$$

  – alternative chosen depends only on voters' *preferences* and not *who* has those preferences
  – voters treated symmetrically

- *Neutrality* (N): Suppose $\rho : Y \to Y$ permutation.

  If $U_i^{\rho,Y}\left(\rho\left(x\right)\right) > U_i^{\rho,Y}\left(\rho\left(y\right)\right) \Leftrightarrow U_i\left(x\right) > U_i\left(y\right)$ for all x, y, i,

  then

  $$F\left(U_{\cdot}^{\rho,Y}, Y\right) = \rho\left(F\left(U_{\cdot}, Y\right)\right).$$

  – alternatives treated symmetrically

- All four voting rules – plurality, majority, rank-order, utilitarian – satisfy P, A, N

- Next axiom most controversial

  still

  - has quite compelling justification
  - invoked by both Arrow (1951) and Nash (1950)

- *Independence of Irrelevant Alternatives* (I):

$$\text{if } x = F\left(U_{\cdot}, Y\right) \text{ and } x \in Y' \subseteq Y$$

then

$$x = F\left(U_{\cdot}, Y'\right)$$

  - if *x* chosen and some non-chosen alternatives removed, *x* still chosen

  - Nash formulation (rather than Arrow)

  - no “spoilers” (e.g. Nader in 2000 U.S. presidential election, Le Pen in 2002 French presidential election)

- Majority rule and utilitarianism satisfy I, but others don't:
  - plurality rule

.35	.33	.32
x	y	z
y	z	y
z	x	x

$$f^{P}\left(U_{\square},\{x,y,z\}\right)=x$$

$$f^{P}\left(U_{\square},\{x,y\}\right)=y$$

  - rank-order voting

.55	.45
x	y
y	z
z	x

$$f^{B}\left(U_{\square},\{x,y,z\}\right)=y$$

$$f^{B}\left(U_{\square},\{x,y\}\right)=x$$

Final Axiom:

- *Nonmanipulability* (NM):

if $x = F\left(U_{\square}, Y\right)$ and $x' = F\left(U'_{\square}, Y\right)$,

where $U'_j = U_j$ for all $j \notin C \subseteq [0,1]$

then

$U_i\left(x\right) > U_i\left(x'\right)$ for some $i \in C$

- – the members of coalition C can't all gain from misrepresenting utility functions as U'_i

- NM implies voting rule must be *ordinal* (no cardinal information used)
- F is *ordinal* if whenever, for profiles $U_{\cdot}$ and $U'_{\cdot}$,

$$U_i(x) > U_i(y) \Leftrightarrow U'_i(x) > U'_i(y) \text{ for all } i, x, y$$

(*) $F(U_{\cdot}, Y) = F(U'_{\cdot}, Y)$ for all Y

- *Lemma*: If F satisfies NM and I, F ordinal
  - suppose $x = F(U_{\cdot}, Y)$ $\quad y = F(U'_{\cdot}, Y)$, where $U_{\cdot}$ and $U'_{\cdot}$ same ordinally
  - then $x = F(U_{\cdot}, \{x, y\})$ $\quad y = F(U'_{\cdot}, \{x, y\})$, from I
  - suppose $\begin{array}{c} C \\ \hline y \\ x \end{array} \quad \begin{array}{c} -C \\ \hline x \\ y \end{array}$
  - if $F(U'_C, U_{-C}, \{x, y\}) = y$, then C will manipulate
  - if $F(U'_C, U_{-C}, \{x, y\}) = x$, then $-C$ will manipulate
- NM rules out utilitarianism

But majority rule also violates NM

- F^C not even always *defined*

.35	.33	.32
x	y	z
y	z	x
z	x	y

$$F^C\left(U_{\square},\{x,y,z\}\right)=\varnothing$$

  - example of *Condorcet cycle*
  - F^C must be extended to Condorcet cycles
  - one possibility

$$F^{C/B}\left(U_{\square},Y\right)=\begin{cases}F^C\left(U_{\square},Y\right), & \text{if nonempty}\\ F^B\left(U_{\square},Y\right), & \text{otherwise}\end{cases}$$

(Black's method)

  - extensions make F^C vulnerable to manipulation
  -

.35	.33	.32
x	y	z
y	z	x
z	x	y

$$F^{C/B}\left(U_{\square},\{x,y,z\}\right)=x$$

z
y
x

$$F^{C/B}\left(U'_{\square},\{x,y,z\}\right)=z$$

Theorem: There exists no voting rule satisfying P,A,N,I and NM

Proof: similar to that of GS

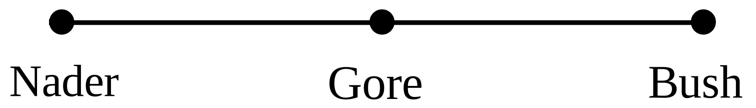
overly pessimistic - - many cases in which some rankings unlikely

Lemma: Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles

When can we rule out Condorcet cycles?

- preferences single-peaked

2000 US election



unlikely that many had ranking

Bush		Nader
	or	
Nader		Bush
Gore		Gore

- strongly-felt candidate
  - in 2002 French election, 3 main candidates: Chirac, Jospin, Le Pen
  - voters didn’t feel strongly about Chirac and Jospin
  - felt strongly about Le Pen (ranked him first or last)

- Voting rule F *works well* on domain $\mathscr{U}$ if satisfies P,A,N,I,NM when utility functions restricted to $\mathscr{U}$

  - e.g., F^C works well when preferences single-peaked

- *Theorem 1*: Suppose F works well on domain $\mathscr{U}$, then F^C works well on $\mathscr{U}$ too.
- Conversely, suppose that F^C works well on $\mathscr{U}^C$.

Then if there exisits profile $U_{\square}^{\circ}$ on $\mathscr{U}^C$ such that

$$F\left(U_{\square}^{\circ}, Y\right) \neq F^C\left(U_{\square}^{\circ}, Y\right) \text{ for some } Y,$$

there exists domain $\mathscr{U}'$ on which F^C works well but F does not

Proof: From NM and I, if F works well on $\mathscr{U}$, F must be ordinal

- Hence result follows from

  Dasgupta-Maskin (2008), *JEEA*

  - shows that Theorem 1 holds when NM replaced by ordinality

To show this D-M uses

Lemma: F^C works well on $\mathscr{U}$ if and only if $\mathscr{U}$ has no Condorcet cycles

- Suppose F works well on $\mathscr{U}$
- If F^C doesn't work well on $\mathscr{U}$, Lemma implies $\mathscr{U}$ must contain

  Condorcet cycle

$$\begin{matrix} x & y & z \\ y & z & x \\ z & x & y \end{matrix}$$

- Consider

$$U_{\square}^{1} = \begin{array}{ccc} \underline{1} & \underline{2}\ldots & \underline{n} \\ x & z & z \\ z & x & x \end{array}$$

(*) Suppose $F\left(U_{\square}^{1}, \{x, z\}\right) = z$

- $$U_{\square}^{2} = \begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & & \underline{n} \\ x & y & z & & z \\ y & z & x & & x \\ z & x & y & & y \end{array}$$

$F\left(U_{\square}^{2}, \{x, y, z\}\right) = x \quad \Rightarrow$ (from I) $F\left(U_{\square}^{2}, \{x, z\}\right) = x$, contradicts (*)

$F\left(U_{\square}^{2}, \{x, y, z\}\right) = y \quad \Rightarrow$ (from I) $F\left(U_{\square}^{2}, \{x, y\}\right) = y$, contradicts (*) (A,N)

so

$F\left(U_{\square}^{2}, \{x, y, z\}\right) = z$

- so $F\left(U_{\square}^{2}, \{y, z\}\right) = z$ (I)
- so for

$$U_{\square}^{3} = \begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & \ldots & \underline{n} \\ x & x & z & & z \\ z & z & x & & x \end{array}$$

$F\left(U_{\square}^{3}, \{x, z\}\right) = z$ (N)

- Continuing in the same way, let $U_{\square}^{4} = \begin{array}{ccc} \underline{1} & \ldots\underline{n-1} & \underline{n} \\ x & x & z \\ z & z & x \end{array}$

$F\left(U_{\square}^{4}, \{x, z\}\right) = z$, contradicts (*)

- So F can't work well on $\mathscr{U}$ with Condorcet cycle

- Conversely, suppose that F^C works well on $\mathscr{U}^C$ and

$$F\left(U^{\circ}_{\square}, Y\right) \neq F^C\left(U^{\circ}_{\square}, Y\right) \text{ for some } U^{\circ}_{\square} \text{ and } Y$$

- Then there exist α with $1-\alpha > \alpha$ and

$$U^{\circ}_{\square} = \begin{array}{cc} 1-\alpha & \alpha \\ \hline x & y \\ y & x \end{array}$$

such that

$$x = F^C\left(U^{\circ}_{\square}, \{x, y\}\right) \text{ and } y = F\left(U^{\circ}_{\square}, \{x, y\}\right)$$

- But not hard to show that F^C unique voting rule satisfying P,A,N, and NM when $|X| = 2$ - - contradiction

- Let’s drop I
  - most controversial
- *no* voting rule satisfies P,A,N,NM on $\mathscr{U}_X$
  - GS again
- *F works nicely* on $\mathscr{U}$ if satisfies P,A,N,NM on $\mathscr{U}$

Theorem 2: $|X| = 3$

- Suppose F works nicely on $\mathscr{U}$, then F^C or F^B works nicely on $\mathscr{U}$ too.

- Conversely suppose F^* works nicely on $\mathscr{U}^*$, where $F^* = F^C$ or F^B.

  Then, if there exisits profile $U_{\square}^{\circ\circ}$ on $\mathscr{U}^*$ such that

  $$F\left(U_{\square}^{\circ\circ}, Y\right) \neq F^*\left(U_{\square}^{\circ\circ}, Y\right) \text{ for some } Y,$$

  there exists domain $\mathscr{U}'$ on which F^* works nicely but F does not

Proof:

- F^C works nicely on any Condorcet-cycle-free domain

- F^B works nicely only when $\mathscr{U}$ is subset of Condorcet cycle

- so F^C and F^B complement each other
  - if F works nicely on $\mathscr{U}$ and $\mathscr{U}$ doesn't contain Condorcet cycle, F^C works nicely too
  - if F works nicely on $\mathscr{U}$ and $\mathscr{U}$ contains Condorcet cycle, then $\mathscr{U}$ can't contain any other ranking (otherwise *no* voting rule works nicely)
  - so F^B works nicely on $\mathscr{U}$.

Striking that the 2 longest-studied voting rules (Condorcet and Borda) are also

- *only two* that work nicely on maximal domains