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Macro Theory III
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Lecture 4
Labor Indivisibility and the Workweek

This lecture follows closely the material in Section 3 of Hornstein and Prescott. The full reference is

"The Plant and the Firm in General Equilibrium Theory," with A. Hornstein, in Robert Becker, Michele Boldrin, Ronald Jones, and William Thomson, eds., *General Equilibrium, Growth, and Trade II: The Legacy of Lionel McKenzie,* Academic Press, 393–410, 1993.

This paper started from lecture notes that were not fully rigorous and complete.

For a history of the development of business cycle theory see

"Business Cycle Theory: Methods and Problems," August 1998. To appear in the International Institute for Economic Research XI Workshop Proceedings, *Cycle, Growth and Technological Change*.

Like the plant and firm paper, this paper was prepared for teaching purposes. Both papers can be downloaded from the web site:

http://research.mpls.frb.fed.us/~prescott/macro_theory/index.html

Environment:

People:

There are measure one people whose preferences are ordered by the expected value of $U(c,h)$.

Here $c \geq 0$ is consumption and $0 \leq h \leq 1$ is workweek length. Each individual owns $\bar{k} > 0$ of capital.

Technology:

The output of a plant that is operated h units of time and uses k units of capital and n workers is

$$h\,f(k,n),$$

where $f(k,0) = f(0,h) = 0$. Note: all the n workers at (h, k, n) plant work h units of time.

Question: Is this a convex economy? Are there increasing returns to scale?

The answer to these questions is immediate once the environment is represented as a Debreu economy.

Representing the Environment as a Debreu economy

Commodity Space:

The big problem is picking an appropriate commodity space. The space should be such that the preferences are convex and continuous and the aggregate production possibility set Y is convex and has an interior point. To simplify the analysis, I will assume that the sets of possible c, h and k are finite. Further H contains the point 0, which corresponds to not working. Let $s = (c,h,k) \in C \times H \times K = S$. The commodity space is $L = \Re^{\#S}$, where $\#S$ means the cardinality of set S. Since the set S is finite, the cardinality of S is just the number of points in S.

A commodity vector is $x=(x_s)_{s\in S}$. From the prospective of an individual, commodity $x_{(c,k,h)}$ is the probability of receiving c units of the consumption good, being required to work h, and being required to supply k units of capital services. From the perspective of the firm, it is the measure of people who get to receive c, work h, and provide k units of labor services.

Notation convention: Summations are over all indices unless otherwise indicated.

Consumption set: $X = \{x \in L_+ \mid \sum x_s = 1,\ x_{(c,k,h)} = 0 \ \text{if}\ k > \bar{k}\ \}.$

The set X is convex.

Utility function: u: $X \to \Re$: $u(x) = \sum_s x_s\, U(c,h).$

The utility function is concave. This along with convexity of X assures convex preferences and permits attention to be restricted to type-identical allocations. Further, the utility function is continuous. Thus assumptions A15.1-A15.3 in Lucas and Stokey are satisfied.

Aggregate production set Y:

The aggregate production set, where $a = (h,k,n) \in H \times K \times \{0,1, \ldots\},$ is

$Y = \{y \in L_+ \mid \exists$ measure of production units, $\{z_a\}$, such that

(i) For all h, $- \Sigma_{c,k}\, y_s + \Sigma_{k,n}\, n\, z_a = 0;$

(ii) $\Sigma\, c\, y_s \; - \; \Sigma\, h\, f(k,n)\, z_a \leq 0;$

(iii) $- \Sigma\, k\, y_s + \Sigma\, k\, z_{hk} \leq 0 \,\}.$

Comments: The measure z is a production plan. It specifies how many production units of each type a are operated. Constraint (i) states that the number of people working h hours equals the number of people working in plants that are operated h hours. Constraint (ii) is that enough is produced to fulfill the firm's contractual obligations to deliver the consumption good. Constraint (iii) is there is enough capital to fulfill plan z.

Problem: Prove that Y is convex.

Proposition 1: A type-identical optimum exists.

Outline of proof:

The problem is to maximize $u(x)$ subject to $x \in X \cap Y$. One way to show existence is to show that the constraint set is non-empty and compact and that the objective function is continuous.

Comment: The existence can be extended to the case that S is a separable metric space.

Problem: Carry out this extension if you have the mathematical background in functional analysis. Here the continuity of the U function is needed.

Proposition 2: If C, H and K are subsets of $\Re_+$, C is convex (therefore not finite), U is strictly concave and strictly increasing in c, and f is strictly increasing in k, then optimality requires all individuals with the same h to have the same c and k.

Outline of a proof:

Clearly $x_{chk} = 0$ unless $k = \bar{k}$. Otherwise, consumption could be increased given $f(k,n)$ is strictly increasing in k. All individuals with the same h must have the same c. Otherwise, the optimum could be improved upon by giving all with the same h the mean consumption for those with that h. This follows from concavity of U in c given h. Thus, for any optimum, all individuals with the same h have the same c and k.

Comment: All the assumptions for the Second Welfare Theorem (see Stokey and Lucas Chapter 15) are satisfied so that any optimum can be supported as a competitive equilibrium. Showing that there exists a cheaper point is easy provided there is some assumption that guarantees that for the optimum, consumption is strictly positive with positive probability.

Problem: Make a reasonably weak assumption that guarantees that for the optimum, consumption is strictly positive with positive probability.

The Social Planner's Problem

We now assume that f is such that proportional scaling of inputs if feasible results in the same proportional scaling in output. If $(n,k) \in \Re_+^2$, this would imply that f displays constant returns to scale. This assumption permits us to consider only production plans with one person operating a plant technology. This restriction does not reduce the aggregate production possibility set Y. With this assumption, the social planner's problem is

$$
\begin{aligned}
&\max_{x\geq 0} \sum_s U(c,h)\, x_s \\
&s.t. \quad \sum_s x_s\, [c - h\, f(k,1)] \leq 0 \\
&\qquad \sum_s k\ x_s \leq \bar{k} \\
&\qquad \sum_s x_s = 1
\end{aligned}
$$

There are three constraints. Therefore, if an optimum exists, one exists that places all its mass on at most three points. If there were *m* constraints, then one would exist that places mass on at most *m* points.

Note: x_s here represents both an individual's probability of being at point *s* and the fraction of people who are at point *s*, because individuals face identical (not necessarily independent) probability distribution.

Implementing a feasible allocation $\hat{x} = \{\hat{x}_s\}$**:** The *s* must be distributed identically across individuals with distribution $\hat{x}$. Furthermore, the fraction of the people receiving s must be $\hat{x}_s$ for all *s*. One way to achieve this is as follows: Let *i* be the name of an individual for $i \in [0,\lambda]$ and let ε be a random variable uniformly distributed on $[0,\lambda]$. A function $s(i,\varepsilon)$ will be constructed with the property that if $s(i,\varepsilon)$ determines the outcome for any individual *i*, then *i*'s ex-ante distribution of *s* is $\hat{x}$. Further, the fraction of individuals receiving *s* is $\hat{x}_s$ for all *s*.

The first step of the construction is to list the $S = \{s_1, s_2, \ldots\}$. An appropriate function is

$$
s(i,\varepsilon) = s_j \qquad \text{if} \quad (i+\varepsilon) \bmod \lambda \ \in \ \left[\lambda \sum_{l=1}^{j-1} \hat{x}_l,\ \ \lambda \sum_{l=1}^{j} \hat{x}_l\right).
$$

Note: *x mod y* is the positive number remaining after subtracting that maximum number of integer multiples of y from x. Thus it belongs to [0,y).

Problem: Pick reasonable functions for preferences and technology such that the social optimum has the property that all consumers work, though some work short weeks and use little capital while others work long weeks and use large amounts of capital. I work long weeks and like to think of myself as having a large stock of human capital.

Open problem: There are large fixed time costs of some occupations just to keep current. There may be problems with the enforceability of contracts, which makes this an interesting problem.

Open problem: Another important open problem is shift work. In some industries (for example auto) much of the variation in production is related to the number of shifts.

Related problem: Another related, but easier problem, is the number of plants to operate.

(See, "An Equilibrium Analysis With Idle Resources and Varying Capacity Utilization Rates," with T. F. Cooley and G. D. Hansen, *Economic Theory* 6 (1), 35 –50, June 1995 and "Capacity Constraints, Asymmetries, and the Business Cycle," with G. D. Hansen, November 1998.)

Problem: Assume that $U(c,h) = u_c(c) - u_h(h)$ and that u_c is concave. Show that the commodity space $L = \Re^{\#H+2}$ is sufficient to represent this environment as a Debreu economy, where the commodities are c, k, and the $\{x_h\}$.

Motivation for assuming f(k,n) is Cobb-Douglas.

Labor's share of product shows little trend even though the relative price of labor relative to capital has increased dramatically. The rental price of capital has shown no trend.

Problem: Show if $f(k,n)$ is a CES production function and factor markets are competitive, that

(i) if the elasticity exceeds one, labor share falls if the rental price of labor increases relative to the capital rental price;

(ii) if this elasticity is less than one, labor share increases if the relative price of labor services increases.

Motivation for assuming $$U(c,h)=\frac{[c^{\gamma}(1-h)^{1-\gamma}]^{1-\sigma}-1}{1-\sigma}.$$

The length of the average workweek *h* has been pretty stable at about 40 hours over the last 50 years even though the real wage has increased dramatically during this period. This requires unit elasticity of substitution between leisure, 1- *h*, and consumption.

A more general class of utility functions that results in constant workweek length is $U(c,h)=g(h)\,\dfrac{c^{1-\gamma}-1}{1-\gamma}$, where there g is such that *U* is increasing in leisure and concave.

Some implicit assumptions are being made when I said the real wage increased. Actually there is a price for each workweek length. Implicitly I am assuming all these prices increased by the same factor.