Macro Theory III.
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Lecture 13

Hopenhayn's Technology Draw Model

Technology: A technology at a point in time is indexed by $s \in S \subset \Re_+$, and the measure of technologies is $x(ds)$. Technology $0 \in S$. Technology s, if operated using n units of labor services, produces output

$$s\ f(n).$$

For simplicity, $n \in N = [n_{\min}, n_{\max}]$, where $n_{min} > 0$. The function f is increasing and continuous. Further f is convex in the region $[n_{\min}, \bar{n}]$ and concave in the region $[\bar{n}, n_{\max}]$. Technologies are "small" relative to the size of the economy. If a technology is not operated, the technology is lost.

The processes on the s are identical with transition function $Q(s,ds')$. Further, the law of large numbers holds. For purposes of this presentation, the process is a finite state Markov chain with transition probabilities, $q(s,s')$. Thus, $q(s,s') = Q(s,\{s'\})$. These probabilities are such that $q^n(s,0) > 0$ for some n_s. Further, $q(0,0) = 1$, so state $s = 0$ is absorbing. [The $q^n(s_t,s_{t+n})$ are the n stage transition probabilities.] The reason for this assumption is to guarantee the ultimate death of a firm in equilibrium.

Composite output is used for consumption, c, and technology draws, d. The resource cost of a technology draw is $\varphi > 0$.

Let $z_s(A)$ denote the measure of operated technologies of type s using $n \in A \in B(N)$ units of labor. Note that I have not restricted all plants using a technology of type s to have the same input, n. The aggregate resource constraint is

$$c + d\varphi \le \sum_s \int s\ f(n)\ z_s(dn).$$

The number of technologies of type *s* operated must be less than or equal to the number of type *s* technologies. Thus, for all $s \in S$

$$z_s(N) = \int z_s(dn) \le x(s)$$

where *x(s)* is the number of type s technologies.

Technology draw technology: As previously stated, the resource cost of a technology draw is φ. If a draw is made, the probability that the resulting technology is of type s' for use next period is $\pi(s')$. It is assumed that $\pi(0) > 0$. Thus, the law of motion for *x* is

$$x'(s') = d\ \pi(s') + \sum_s q(s,s') \int dz_s \ .$$

Preferences: There is measure one of identical individuals with preferences over infinite consumption streams ordered by

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \ .$$

The function *u* is strictly increasing, strictly concave and continuous. All people own an equal share of every technology and one unit of every date labor.

State variable: The state variable is the vector *x*.

Problem: Add physical capital accumulation to this structure. Note *k* must be part of the state variable.

Finding and characterizing the steady-state:

The steady-state wage is denoted by *w*. The steady-state interest factor is equal to β. This comes from the preference side. The problem is to find the steady-state $(c, d, \{x(s)\}, \{z_s\})$ and plant employments, $\{n(s)\}$.

Given the steady-state prices, the dynamic program facing the operator of a technology is

$$v(s \mid w) = \max\{0, \max_n \{s\ f(n) - w\,n + \beta \sum_{s'} v(s' \mid w)\ q(s,s')\}\} \ .$$

Here, v is the present value of dividends. Given the assumptions, this is a well-behaved dynamic program. The optimal employment function, conditional on a s-technology being operated, is denoted $n(s \mid w)$. This notation emphasizes that these functions depend upon the wage, w.

The first step is to find steady-state w. An equilibrium condition is that the present value of the dividend stream generated by a technology draw must be φ. Thus,

$$\varphi = \beta \sum_{s'} \pi(s')\, v(s' \mid w) .$$

(Note that this implies that the left-hand argument of the max operator above is 0.)

Exercise: Verify that v is positive, decreasing, and continuous in w with $v(s' \mid \infty) = 0$. This guarantees that a unique steady-state wage w exists.

Exercise: An assumption is needed to insure that the steady-state w is strictly positive. Specify an assumption that insures this is the case.

For some s, the value of the technology may be the same whether or not exit occurs. If so, there is a multiplicity of steady states. To be concrete, I assume that firms exit if the value of staying is equal to 0. Let S_o be the set of plant types that are operated.

Let $p(s,s') = q(s,s')$ for $s \in S_0$ and $p(s,s') = \pi(s')$ for $s \notin S_o$. Let $y(s)$ denote the steady-state technology **fractions** that are of type s. They must satisfy for $s' \in S$

$$y'(s') = \sum_{s} p(s,s')\, y(s) . \qquad (1)$$

These equations define an operator T^* taking the space of probability vectors into itself. The space of probability vectors is convex and compact, and the mapping T^* is continuous. Therefore, by Brower's fixed-point theorem, equation (1) has a solution.

Definition: An **ergodic set** (for a Markov chain) is a set of states with the property that the probability of exiting the set is zero and the probability of going from any state to any other state in the set in some finite number of periods is positive.

Definition: A point is in the support if any open set containing the point has positive probability. If the space is discrete, as it is for this problem, the support is the smallest set that has probability one.

Exercise: Show that T^* maps the set of probability measures with support in a given ergodic set into itself. Show that T^* has at least one fixed point given any ergodic set as support.

Lemma: The process with transition probabilities P has a unique invariant distribution.

Proof outline: There is a single ergodic set, namely those states that can be reached from state 0. There are no cyclically moving subsets given $\pi(0) > 0$. By Lemma 11.3 in Stokey and Lucas (page 332), the operator T^* is a contraction operator (the metric is defined by the ℓ_1-norm). The set of probability measures with this metric is complete given that S is finite.

The steady-state employment for operated plants is $n(s) = n(s|w)$, where w is the steady state w. Let y be the unique invariant distribution of the P process. Define ε as the steady state fraction of technologies, which are not operated, i.e.

$$\varepsilon \equiv \sum_{s \notin S_o} y(s) .$$

Each period, measure d of new technologies are drawn, $(1-\varepsilon)$ of which survive to the next period. Thus, in the steady state there is measure $d + (1-\varepsilon)d + (1-\varepsilon)^2 d + (1-\varepsilon)^3 d + \ldots = d/\varepsilon$ of technologies. (Note that if ε technologies were drawn in the steady state (i.e. if $d=\varepsilon$) then the measure of technologies would be one.)

Market clearing in the labor market then requires

$$\sum_{s \in S_o} \frac{d}{\varepsilon}\, y(s)\, n(s) = 1 .$$

This equation determines steady-state d.

The steady-state state, $x(s)$, are $(d/\varepsilon)\, y(s)$ for $s \in S$. Finally, the steady-state z_s are defined by $z_s(A) = x(s)$ if both $s \in S_o$ and $n(s) \in A$, and $z_s(A) = 0$ otherwise for all Borel measurable sets A.

Equilibrium Path

Let x_0 be the initial value of the state. The problem is to find the equilibrium sequence of prices, state, and quantities:

$$\{w_t, r_t, x_{t+1},\ c_t, d_t, \{n_t(s)\}_{s\in S}\}_{t=0}^{\infty}.$$

Price w_t is the date t real wage and r_t is the **real interest factor**.

The dynamic program facing the operator of a firm is

$$v_t(s) = \max\{\, 0, \max\{sf(n) - w_t n + r_t \sum_{s'} q(s,s')v_{t+1}(s')\}\,\}.$$

An approximate equilibrium:

The approach requires that $3T$ equations in $3T$ variables be solved. The variables are $\{w_t, r_t, d_t\}_{t=0}^{T-1}$. At dates t = 0,1, …, T-1, the labor market clears; the inter-temporal marginal rate of substitution between c_t and c_{t+1} equals the interest factor r_t; and the expected present value of a technology draw is φ if $d_t > 0$ and less than or equal to φ otherwise.

The steps in finding these equations are as follows:

1. Solve the dynamic program given the prices $p = \{w_t, r_t\}_{t=0}^{T-1}$. You need $v_T(s)$ to start the backward induction. Set $v_T(s)=v^*(s)$, where $v^*(s)$ denotes the steady state value function; and pick T large enough. You know that you've picked T large enough if increasing it further doesn't change the result.

2. The value of a draw in period t is $V_t = r_t \sum_{s'} \pi(s')v_{t+1}(s')$. The first set of equations is $\max\{-d_t,\ \ d_t(V_t - \varphi),\ \ V_t - \varphi\} = 0$ for t = 0, …, T-1. This says: d_t cannot be negative; if $V_t < \varphi$, then d_t = 0; and finally that $V_t > \varphi$ is not possible.

3. The employment functions are $n(p) = \{n_t(s\,|\,p)\}_{t=0}^{T-1}$. The exit sets are $E(p) = \{E_t(p)\}_{t=0}^{T-1}$. Compute the sequence $x(d,p) = \{x_t(d,p)\}_{t=1}^{T}$ given x_0, and $d = \{d_t\}_{t=0}^{T-1}$. Given the exit set and x_0, find $x(d,p)$.

4. Compute the labor demand functions for each date using $x(d,p)$, $E(p)$ and $n(p)$. These equations are $\sum_{s \notin E_t} n_t(s\,|\,p)\, x_t(s\,|\,d,p) = 1$ for t = 0, …, T-1. This is the second set of T equations.

5. Compute $c_t(d,p) = \sum_{s \notin E_t} s\, f(n_t(s\,|\,p)\, x_t(s\,|\,d,p) - d_t\varphi$. The final set of equations is $\dfrac{\beta\, u'[c_{t+1}(d,p)]}{u'[c_t(d,p)]} = r_t$ for t = 0,1, …, T.

6. After solving this system of equations using numerical methods, check whether x_T is near the steady-state x^*.